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Optimal diversification

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In this paper we develop a new approach to portfolio construction that relies solely on the covariance structure of the investment opportunity set. Using this new approach leads to an alternative to the mean–variance efficient set of portfolios as traditionally implemented. We define an "optimally diversified" portfolio as one which is equally correlated to each of its components. We show empirically in this paper that using this "optimally diversified" portfolio construction methodology results in a significant improvement in portfolio performance over traditional mean–variance efficient portfolios as well as naively diversified (eg, equally weighted) portfolios.

1 INTRODUCTION

One of the first principles that any student learns in finance is diversification, a concept that Harry Markowitz developed in his seminal paper in 1952 (see Markowitz 1952, 1959). Therefore, after sixty years of dissemination, we would expect the concept of

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diversification to be embraced and understood by every investor. Yet this is not the case. Particularly following the global financial crisis of 2008 and 2009, when portfolio values plummeted along with most financial indexes, modern portfolio theory has been under assault by critics for resulting in "underdiversified" portfolios, ie, portfolios with too much inherent risk or lack of sufficient risk control (see, for example, Liesching 2010). In this paper, we introduce the definition of an "optimally diversified" portfolio as one which is equally correlated to each of its components. We then develop the portfolio allocation methodology that leads to optimally diversified portfolios and demonstrate empirically and through simulation that this new methodology results in a significant improvement in portfolio performance over traditional portfolio allocation strategies.

The consequences of insufficient diversification can be seen in the portfolios of hundreds of millions of people saving for retirement. Although extreme examples can be found in the large losses in retirement savings of employees and retirees of Lehman Brothers, Bear Stearns and Enron – who frequently had large allocations to company stock in their portfolios – other supposedly diversified approaches failed as well. For example, a number of target-date funds, part of the fastest growing segment of the investment market in America, suffered losses of 20–30% during 2008. This was true even of funds targeting retirement in 2010 (*Economist* 2011).

Exacerbating this situation has been the shift from defined benefit (DB) plans to defined contribution (DC) plans, which has passed nearly all the market risk to employees, who are often insufficiently equipped to make the required investment decisions. Between 1979 and 2009 the share of employees in DB pension plans in America fell from 62% to 7% of the total, while those in DC plans rose from 16% to 67% (*Economist* 2011). The size of the asset pool in American DC schemes, amounting to US$2.8 trillion at the end of 2009, underlines the importance of this topic from a public policy perspective. Although most employers offer a "diversified" plan option, the allocation of those schemes often lacks a quantifiable diversification concept. Improvements in portfolio management utilizing more precise diversification metrics could be of great value for a substantial segment of the US population.

The goal of diversification is to eliminate risk for which the compensation, ie, the expected return, is low. In doing so, we eliminate risk in an investment portfolio without significantly reducing the expected return, thereby increasing the Sharpe ratio of the portfolio. An example of this process is traditional mean–variance optimization. Mean–variance optimization tries to reduce variance that does not result in high expected returns – essentially, portfolio weighting schemes that increase portfolio variance but do not substantially increase the portfolio's expected return are thrown out. Of course, the main difficulty with mean–variance optimization lies in implementation, due to errors in parameter estimation[1] (particularly in estimation of the mean return[2]).

A response to the difficulties of implementing mean–variance optimal portfolios has been a renaissance of naive portfolio construction: not using any information about risk or return. With this approach, the portfolio is equally allocated among the assets in the investment opportunity set (so that if there are n assets, then each asset has a portfolio weight of $1/n$). Papers by Bloomfield *et al* (1977), Jorion (1991), Benartzi and Thaler (2001), Windcliff and Boyle (2004) and DeMiguel *et al* (2009) all consider this approach.[3] For example, Bloomfield *et al* (1977), Jorion (1991) and DeMiguel *et al* (2009) have all shown that the naive equal-weighting allocation method, which allocates a portfolio equally among all of the investable assets without any regard to expected return, risk or any other security or portfolio characteristics, performs surprisingly well relative to many popular portfolio construction strategies, including methods which use Bayesian updating to improve the quality of the estimated expected returns. The trouble with naive portfolio construction is that all information, even potentially useful information, about risk and return is discarded.[4]

In response to the difficulties in implementing mean–variance portfolio optimization, a growing literature in recent years has advocated risk-parity portfolio construction.[5] Although there is not yet a consensus on what constitutes a "risk parity" portfolio – as there are many different ways to define "risk" – most risk-parity portfolios seek to achieve some level of diversification by defining risk in a particular

[1] A huge literature exists on this problem (see, for example, Jorion 1985, 1986; Frost and Savarino 1986; Best and Grauer 1991; Chopra 1993; Michaud 1998; Britten-Jones 1999; Chan *et al* 1999; Pastor 2000; MacKinlay and Pastor 2000; Pastor and Stambaugh 2000; Goldfarb and Iyengar 2003; Ledoit and Wolf 2004; Kan and Zhou 2007; Garlappi *et al* 2006). In fact, DeMiguel *et al* (2009) argue that estimation error in the input parameters destroys most of the value that formal portfolio optimization models can create, even when improved estimators are used.

[2] Minimum-variance investing is a mean–variance based strategy that avoids the difficulty of estimating the mean return. Minimum-variance portfolios are equivalent to mean–variance efficient portfolios where the expected returns are equal across all assets. Thus, a minimum-variance strategy does away with the need to estimate expected returns. See Clarke *et al* (2011, 2013), Scherer (2011), and Shah (2011) for more on minimum-variance investing. We will discuss more about this substrategy of mean–variance optimization later in the paper.

[3] See Lee (2011) for a summary and comparison of risk-parity portfolio construction to naive portfolio construction and mean–variance optimization.

[4] For example, economic and financial theory both suggest that US Treasury securities are less risky than corresponding maturity corporate debt, which, in turn, is less risky than equity of the same company. A naive portfolio construction approach would ignore this type of structural information.

[5] In addition, a number of new products have been created recently by the money management industry utilizing risk-parity portfolio construction (see Johnson 2008).

way and then ensuring that each security or asset class in a portfolio contributes a predefined amount of risk to the overall portfolio. The common theme of risk parity strategies is to conduct portfolio allocations by risk contribution rather than notional amount, as is the approach with mean–variance based portfolio allocation. Examples of this approach include Qian (2005, 2011), Clarke *et al* (2006), Behr *et al* (2008), Martellini (2008), Bailey and de Prado (2012) and Goldberg and Mahmoud (2013).[6]

In this paper, we propose a simple portfolio allocation strategy that builds on the risk-parity approach of portfolio construction. In particular, we propose that investors equalize the correlation between the overall portfolio and each of its constituent parts. We call these portfolios "optimally diversified" because they have two important properties. First, this approach results in a portfolio that is equally exposed to each of its components, in the sense that a one-standard-deviation move of any of the component assets results in a one-standard-deviation move for the portfolio as a whole.[7] Essentially, the probability of any portfolio component causing a one-standard-deviation move in the overall portfolio is equal – no single portfolio holding has a greater chance of causing a portfolio movement than any other holding. Intuitively, this is how a portfolio whose primary goal is to be well-diversified should behave. The second property is that these portfolios are also exactly the same as those which would result from a mean–variance optimizer if the expected returns were computed from estimated volatilities by assuming that all constituents of the portfolio have equal Sharpe ratios. The reason one would hold a nondiversified portfolio is to achieve a superior Sharpe ratio – the cost of being nondiversified (and therefore taking extra risk) is more than made up for by the extra return produced by the over-concentration in an asset. By equalizing the Sharpe ratios of all the portfolio components, this tradeoff is precluded for the mean–variance optimizer, so the optimizer can only produce an optimally diversified portfolio.

We show, through extensive empirical testing using simulated and actual historical datasets, that this approach yields returns which are robust and have a number of attractive properties including higher Sharpe ratios and lower kurtosis relative to a number of alternative investment strategies across a wide variety of market environments. Although this is particularly true in cases where the Sharpe ratios of the underlying components are equal, it also appears to be true, albeit to a lesser extent,

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[6] See also Allen (2010), Foresti and Rush (2010), Levell (2010), Anderson *et al* (2012) and Jurczenko and Teiletche (2013) for industry summaries of the trend toward risk-parity portfolios.

[7] For instance, if the estimated volatility of the S&P 500 is 20%, and the target volatility of the portfolio is 10%, then a 5% increase in the S&P 500 should result in an average 2.5% increase for the portfolio.

even when the assumption of equal Sharpe ratios holds only approximately (as is more likely the case in reality).[8]

The remainder of the paper is structured as follows. In Section 2 we develop our portfolio construction technique, which relies on equalizing the correlations between the overall portfolio and each of its components. In this section we also show that the portfolios constructed under this approach are also mean–variance optimal under a simple but intuitive expected returns model. In Section 3 we show empirically (with both historical and simulated datasets) the improvement produced by this portfolio construction approach as compared with traditional mean–variance portfolio construction and naive diversification. Section 4 concludes.

2 EQUATING CORRELATION WEIGHTS

In addition to the problems arising from parameter estimation with using standard mean-variance efficient strategies in practice (see Appendix A for a simple example), they are also clearly deficient in a second way, which is immediately obvious to practitioners: they tend to place highly leveraged bets on just a handful of investable assets, for example going 500% long one asset and 400% short another.[9]

We propose a new method of portfolio diversification, based solely on second-moment criteria, which consists of equalizing the correlations[10] between each asset and the overall portfolio.[11] As discussed earlier, this method has two important properties. First, it equalizes exposures in an intuitive way: a one-standard-deviation move of any of the investable assets results on average in a one-standard-deviation move for

[8] Other authors (eg, Dalio 2005) have found that over certain time periods it is plausible that the Sharpe ratios of various asset classes are broadly similar (and are not reliably known). One investment universe in which this is likely to occur is when investments are determined at the asset-class level, as the idiosyncratic risk of individual companies/securities has already been diversified away.

[9] The underlying (numerical) causes of this extreme leveraging are discussed in Green and Hollifield (1992), who find that the outsized weights which typically result from mean–variance (or minimum-variance) optimization are generally related to the dominance of the first principal component of returns, and suggest that if one eliminates/ignores exposure to this factor (by adjusting the eigenvectors/values) the resulting implied portfolio weights are generally well-behaved. Nevertheless, we view this as more of a numerical fix than an economic one.

[10] To keep the exposition and results as straightforward as possible, in this paper we rely on the standard textbook definition of correlation; it is not our goal here to advance correlation estimation techniques. Engle (2009) provides a good overview over the current state of the art of estimating correlations. These techniques include probability weighted estimates (eg, exponential smoothing), conditional correlations and potentially nonlinear inter-dependencies (eg, copulas) which are beyond the scope of this paper.

[11] Technically, since we rely on central second moments, our method relies indirectly on first-moment estimates. In practice, however, this introduces only a second-order increase in the standard errors and can be disregarded.

the portfolio as a whole. Second, it equates to running mean–variance optimization if we assume that the Sharpe ratios of the investable assets are all equal (conditional on exact estimation of the covariance matrix; see Appendix B for details).

As discussed earlier, economic theory would argue that this case could certainly be present in reality, particularly if the assets consist of asset-class-level indexes which are largely immune from idiosyncratic risks; the idea being that the reward for the risk taken in each larger asset class can be thought of as being approximately identical. However, in Section 4 we will also analyze scenarios in which the equal Sharpe ratio assumption is violated to varying degrees. It is not our belief that Sharpe ratios will be perfectly equal for all asset classes over any time horizon but, given the difficulty in estimating a Sharpe ratio at any point in time, it represents a reasonable approach (similar to the approach used in minimum-variance investing, which equalizes the expected returns of all assets due to the difficulty in estimating first moments).

To illustrate the mechanics of the equal-correlation allocation, consider the two-asset case in which the portfolio is composed of two assets, a_1 and a_2. Then the idea is to choose weights for these assets, w_1 and w_2 such that $\mathrm{Corr}(a_1, p) = \mathrm{Corr}(a_2, p)$. Thus,

$$\begin{aligned}\mathrm{Corr}(a_1, p) &= \frac{\mathrm{Cov}(a_1, p)}{\sigma_1 \sigma_p} \\ &= \frac{\mathrm{Cov}(a_1, w_1 a_1 + w_2 a_2)}{\sigma_1 \sigma_p} \\ &= \frac{\mathrm{Cov}(a_1, w_1 a_1) + \mathrm{Cov}(a_1, w_2 a_2)}{\sigma_1 \sigma_p} \\ &= \frac{w_1 \sigma_1^2 + w_2 \sigma_1 \sigma_2 \rho_{12}}{\sigma_1 \sigma_p} \\ &= \frac{w_1 \sigma_1 + w_2 \sigma_2 \rho_{12}}{\sigma_p}. \end{aligned} \tag{2.1}$$

Equalizing the correlations from the two assets gives us

$$w_1 \sigma_1 + w_2 \sigma_2 \rho_{12} = w_2 \sigma_2 + w_1 \sigma_1 \rho_{12}. \tag{2.2}$$

Imposing the budget constraint ($w_1 + w_2 = 1$) yields the allocations

$$w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \qquad w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}. \tag{2.3}$$

Moving to the general case of n assets, the equal-correlation allocation is defined as the requirement that $\mathrm{Corr}(a_i, p) = \mathrm{Corr}(a_j, p)\ \forall i, j$. This yields a system of n equations and $n + 1$ unknowns of the form

$$\mathrm{Corr}(a_i, p) = \frac{\sum_{j=1}^{n} \mathrm{Cov}(a_i, a_j) w_j}{\sigma_i \sigma_p} = \lambda, \tag{2.4}$$

which becomes solvable with the addition of the budget constraint (ie, that $\sum w_i = 1$).[12]

A second way to view the equal-correlation allocation is in terms of "normalized betas". Defining the β of the portfolio to asset i in the usual way as

$$\beta_i = \frac{\mathrm{Cov}(a_i, p)}{\sigma_i^2} = \frac{\sum_{j=1}^{n} \mathrm{Cov}(a_i, a_j) w_j}{\sigma_i^2}. \tag{2.5}$$

The "normalized beta", which is the return for the portfolio associated with a one-standard-deviation move for a_i is $\beta_i \sigma_i$. Comparing (2.4) and (2.5), it becomes clear that the concept of equalizing the correlations is equivalent to the concept of equalizing the normalized betas.

Yet a third way to view the equal-correlation allocation is in terms of mean–variance optimization. As is well-known, the relative intra-risky-asset weights in the mean–variance efficient portfolio are given by $w = \Sigma^{-1}\mu$, where μ is the vector of expected returns and Σ is the variance-covariance matrix of the asset returns. This can be rewritten as the system of equations

$$\frac{\sum_{j=1}^{n} \mathrm{Cov}(a_i, a_j) w_j}{\mu_i} = \lambda. \tag{2.6}$$

Viewed in this way it becomes obvious that the intra-risky-asset weights for the equal-correlation allocation are equal to the mean–variance efficient weights in the special case that the vector of expected returns, μ, is proportional to the vector of volatilities, σ. Put another way, the equal-correlation strategy is equivalent to the mean–variance efficient strategy in the case where all of the assets have (or are assumed to have) the same Sharpe ratio. This has the added benefit of providing a simple analytic solution for the equal-correlation allocation: $w = \Sigma^{-1}\sigma$, where σ is the vector of volatilities.

3 EMPIRICAL AND NUMERICAL RESULTS

In the preceding sections we have shown that the equal-correlation strategy has several attractive factors to recommend it from a theoretical perspective. In this section we focus on validating this empirically. With this in mind, we compare the equal-correlation strategy with a number of alternative investment strategies across a wide variety of market environments. First, we rely on estimates of realized first and second moments from Perold and Stafford (2010) to generate simulated historical data for the Harvard Management Company. By simulating the data we are able to draw robust conclusions about the performance of the strategies when investing across multiple asset classes. Next, we examine the performance of the strategies on actual realized

[12] Note that, unlike most risk-parity allocation schemes, this procedure does not target a volatility.

historical data in order to provide a real-world example. Finally, using a time series of daily historical returns for asset-class-level indexes, we generate a large number of bootstrapped histories. This provides additional robustness for our conclusions as the bootstrapping approach allows us to see the impacts of the nonnormality more typically faced by investors.

As is well-known, every set of asset weights can be thought of as being equivalent to a mean–variance efficient allocation subject to a particular set of assumptions about the means and covariance matrix. In Table 1 on the facing page, we apply this result to show the various assumptions under which each of a number of popular investment alternatives are equivalent. For example, the asset allocations made using the minimum-variance strategy are equivalent to those which result from running the mean–variance efficient strategy in combination with the assumption that the expected returns are equal across all assets. If this assumption is valid (ie, satisfied by the data generating process) then we should expect that the minimum-variance strategy will probabilistically dominate[13] the mean–variance efficient strategy as it will essentially be estimating the means with zero estimation error. Using this same approach, we can see that the equal-weight strategy is equivalent to running the mean–variance efficient strategy in combination with the assumption that the ratio of each asset's expected return to the sum of its row of the covariance matrix is equal across all assets. Viewed in this light, it becomes clear that, despite its generally strong empirical performance, the equal-weight strategy has comparatively little to recommend it from a theoretical perspective, ie, it is difficult to argue from a fundamental theoretical perspective that each asset should have an identical ratio of expected return to the sum of its row of the covariance matrix.[14]

In the empirical analysis that follows, we compare the performance of the seven different asset allocation strategies shown in Table 2 on page 68. In addition to running the equal-weight strategy, we run both the mean–variance efficient and the equal-correlation strategies using three different sets of weights:

(1) the unconstrained implementation,

(2) a long-only implementation with no limits on individual positions, and

(3) a long-short implementation with only a 250% limit on the gross exposure.

[13] By this we mean that it would be preferred by a risk-averse investor. Although a proof is beyond the scope of this paper, we conjecture that for symmetric return distributions the conditions for second order stochastic dominance will be satisfied.

[14] As an empirical aside, the assumption implicit in the equal-correlation strategy, namely that all assets have the same Sharpe ratio, is much more likely to be satisfied (or nearly satisfied) in the market. This is especially true at the asset class level because asset classes have substantially lower idiosyncratic risk than individual assets.

TABLE 1 Conditions for equivalence of strategies.

		Strategy B		
		Mean–variance efficient	**Minimum variance**	**Equal correlation**
Strategy A	**Minimum variance**	The means are equal across all assets		
	Equal correlation	The Sharpe ratios are equal across all assets	The variances are equal across all assets	
	Equal weight	The ratio of each asset's expected return to the sum of its row of the covariance matrix is equal across all assets	The row sums of the covariance matrix are equal across all assets	The ratio of each asset's volatility to the sum of its row of the covariance matrix is equal across all assets

The asset weights proposed by Strategy A are equal to the asset weights proposed by Strategy B if and only if the conditions listed are satisfied.

TABLE 2 Investment strategies.

1	Equal weight	
2	Mean–variance efficient	Unconstrained
3		"Long-only, with no individual position limits"
4		"Long–short with gross exposure up to 250%, no limits on individual positions"
5	Equal correlation	Unconstrained
6		"Long-only, with no individual position limits"
7		"Long–short with gross exposure up to 250%, no limits on individual positions"

These strategies were chosen to closely mimic the actual investment approaches used by practitioners.[15]

3.1 Multi-asset class simulations

We first evaluate the performance of the various allocation strategies in a large investable universe consisting of thirteen different asset classes, including domestic equity, foreign equity, emerging markets, private equity, absolute return, high yield, commodities, natural resources, real estate, domestic bonds, foreign bonds, inflation-indexed and cash.

Because many of these asset classes are illiquid and/or not publicly traded, obtaining a reliable historical time series is not practical, therefore we simulate them. Although we retain the assumption of lognormally distributed returns (for now), to be as realistic as possible we took the parameters – expected returns, volatilities, and correlations (shown in Table 3 on page 70) – from a Harvard case study of the Harvard Management Company (Perold and Stafford 2010). We use these to generate simulated historical return data, allowing us to draw robust conclusions about the performance of the strategies when investing across multiple asset classes. As seen in Table 3 on page 70, the correlations among these thirteen assets seem *a priori* relatively rich, ranging from -0.10 between domestic bonds and emerging markets to 0.85 between domestic and foreign equity.[16]

[15] For simplicity, all of our tests assume zero transaction costs, which favors the other strategies because the equal-correlation strategy is frequently the strategy with the lowest turnover.

[16] As originally described in the Harvard case study, the returns represent real returns, ie, returns in excess of realized inflation. For the purposes of the simulation, we abstract from this point, treating "cash" as described in Table 3 on page 70 as just another investable asset (eg, short-term Treasuries) and assuming an interest rate of zero for the risk free asset positions taken in a handful of cases when strategies deviate slightly from 100% net risky asset exposures.

The principal reason for choosing this particular level of aggregation (ie, asset-class level indexes) is that, based on economic theory, we expect the Sharpe ratios to be fairly equal, as the idiosyncratic risk of individual companies/securities has already been diversified away. Obviously we can see this in Table 3 on the next page, as we get to choose the parameters, but the goal is for this to serve as realistic proxy for the actual investment environment.

We start by noting that the Sharpe ratios for the thirteen assets range from 0.21 to 0.50, which, given the previously discussed standard errors in estimating expected returns, are well within the likely range of statistical equality. As the equal-weight strategy is not "optimal" in this case, we should expect the equal-correlation strategies, and to a lesser extent the equal-weight strategy, to dominate the mean–variance efficient strategies due to the substantial measurement error in the mean.

The principal results, shown in Table 4 on page 72, represent the means and standard deviations of the performance statistics across 500 simulation runs, all out-of-sample backtests, each consisting of twenty years of simulated, daily, lognormally distributed data (5040 days). Estimates of means and variances used by the allocation algorithms were obtained using a rolling two-year (104-week) window.[17] The constraints were implemented using the bootstrapping procedure described in Appendix C.

The equal-weight strategy delivers results which can only be characterized as stable and strong. This is not particularly surprising as it does not suffer from any estimation error. Instead, the pitfall of equal weight comes from the potential for model error. The assumptions that it makes regarding the relationship between expected returns and the covariance matrix (see Table 1 on page 67) are only satisfied for a relatively small region of the parameter space. This region is much larger for the equal-correlation strategy as it only requires equality of the Sharpe ratios of the portfolio components. Put another way, so long as the assumption that the Sharpe ratios are equal is not completely violated, the equal-correlation strategy creates a portfolio superior to that of the equal-weight strategy by incorporating (generally fairly precise) estimates of the covariance matrix. Although the mean–variance efficient strategy has the potential to deliver the highest Sharpe ratio, the estimation error of the mean completely negates this, putting it behind both equal-correlation and equal-weight strategies.

The Sharpe ratios for the equal-correlation strategies are economically significantly higher than for the equal-weight and the mean–variance efficient strategies. In turn, equal weight itself does much better than the mean–variance efficient strategies. Turning to Table 5 on page 73, which makes these comparisons statistically robust

[17] As before, we use a two-year (out-of-sample) window to estimate the means and covariances in order to account for the reality that structural changes likely tend to affect the returns and covariances of the assets over time.

TABLE 3 Parameters for simulations based on the Harvard Management Company investment universe. [Table continues on the next page.]

	Annualized real return (%)	Excess volatility (%)	Sharpe ratio	Domestic equity	Foreign equity	Emerging markets	Private equity	Absolute return
Domestic equity	5.75	15.5	0.37	1.00	0.85	0.75	0.80	0.60
Foreign equity	6.25	16.0	0.39	0.85	1.00	0.80	0.65	0.60
Emerging markets	7.00	19.0	0.37	0.75	0.80	1.00	0.60	0.60
Private equity	6.75	20.0	0.34	0.80	0.65	0.60	1.00	0.60
Absolute return	5.00	11.0	0.45	0.60	0.60	0.60	0.60	1.00
High yield	4.75	14.0	0.34	0.60	0.60	0.60	0.60	0.70
Commodities	4.50	21.0	0.21	0.30	0.35	0.40	0.15	0.10
Natural resources	5.00	10.0	0.50	0.10	0.15	0.20	0.10	0.10
Real estate	6.00	15.0	0.40	0.40	0.40	0.40	0.20	0.20
Domestic bonds	1.75	5.5	0.32	0.00	0.00	−0.10	0.00	0.20
Foreign bonds	2.25	5.8	0.39	0.00	0.20	0.00	0.10	0.20
Inflation-indexed	2.25	5.1	0.44	0.15	0.10	0.00	0.10	0.20
Cash	1.00	3.5	0.29	0.10	0.10	0.00	0.10	0.10

TABLE 3 Continued.

	High yield	Commodities	Natural resources	Real estate	Domestic bonds	Foreign bonds	Inflation-indexed	Cash
Domestic equity	0.60	0.30	0.10	0.40	0.00	0.00	0.15	0.10
Foreign equity	0.60	0.35	0.15	0.40	0.00	0.20	0.10	0.10
Emerging markets	0.60	0.40	0.20	0.40	−0.10	0.00	0.00	0.00
Private equity	0.60	0.15	0.10	0.20	0.00	0.10	0.10	0.10
Absolute return	0.70	0.10	0.10	0.20	0.20	0.20	0.20	0.10
High yield	1.00	0.20	0.10	0.20	0.10	0.20	0.20	0.10
Commodities	0.20	1.00	0.10	0.00	0.10	0.10	0.40	0.00
Natural resources	0.10	0.10	1.00	0.10	0.00	0.10	0.00	0.00
Real estate	0.20	0.00	0.10	1.00	0.00	0.20	0.20	0.00
Domestic bonds	0.10	0.10	0.00	0.00	1.00	0.70	0.60	0.30
Foreign bonds	0.20	0.10	0.10	0.20	0.70	1.00	0.50	0.30
Inflation-indexed	0.20	0.40	0.00	0.20	0.60	0.50	1.00	0.40
Cash	0.10	0.00	0.00	0.00	0.30	0.30	0.40	1.00

This table shows the parameters used in the Harvard simulations taken from Perold and Stafford (2010). These parameters are represented to be the capital market assumptions frequently used by the Harvard Management Company in in-house allocation analyses.

TABLE 4 Results from the historical backtest of the Harvard Investment Company using simulated data.

	Equal weight	Mean–variance efficient			Equal correlation		
		Unconstrained	A	B	Unconstrained	A	B
Average annualized arithmetic excess return	3.5% (1.7%)	5.9% (1364.9%)	2.6% (1.6%)	2.8% (2.3%)	2.2% (1.0%)	2.2% (0.9%)	2.2% (1.0%)
Average annualized geometric excess return	3.3% (1.8%)	6081648.5% (1695695.4%)	2.4% (1.6%)	2.4% (2.4%)	2.1% (1.0%)	2.2% (0.9%)	2.1% (1.0%)
Average annualized volatility (of excess returns)	7.8% (0.1%)	2040.3% (5847.5%)	7.2% (0.6%)	10.3% (0.9%)	4.5% (0.1%)	3.8% (0.1%)	4.3% (0.1%)
Sharpe ratio (of excess returns)	0.45 (0.22)	0.06 (0.24)	0.36 (0.22)	0.28 (0.23)	0.49 (0.22)	0.58 (0.22)	0.51 (0.22)
Best single-day return	1.8% (0.2%)	4773.1% (18367.2%)	2.6% (0.4%)	4.0% (0.7%)	1.1% (0.1%)	1.0% (0.1%)	1.0% (0.1%)
Worst single-day return	−1.8% (0.2%)	−100.0% (19418.7%)	−2.7% (0.5%)	−4.1% (0.8%)	−1.1% (0.1%)	−1.0% (0.1%)	−1.0% (0.1%)
Skewness	−0.02 (0.03)	3.33 (42.37)	−0.02 (0.11)	−0.02 (0.12)	−0.01 (0.04)	−0.01 (0.04)	−0.01 (0.04)
Excess kurtosis	0.01 (0.07)	2410.98 (1503.93)	2.57 (0.71)	2.96 (0.93)	0.13 (0.10)	0.15 (0.12)	0.07 (0.08)

This table shows the means and standard deviations across 500 simulation runs for a variety of performance statistics for each of the allocation strategies. Each of the 500 simulation runs are out-of-sample backtests, consisting of 20 years of simulated, daily, lognormally distributed data (5040 days). The means and the covariance matrix used for the simulation were taken from Perold and Stafford (2010). Estimates of means and variances used by the allocation algorithms were obtained using a rolling two-year (104-week) window. The constraints were implemented using the bootstrapping procedure described in the Appendix. The column headings "A" and "B" are labels for "Long-only, with no individual position limits" and "Long–short with gross exposure up to 250%, no limits on individual positions", respectively.

TABLE 5 t-tests from the historical backtest of the Harvard Investment Company using simulated data.

(a) Mean–variance efficient

		Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
A	Mean	−0.39						
	SD	0.31						
	t-statistic	28.31						
	p-value	0.00						
B	Mean	−0.09	0.31					
	SD	0.19	0.32					
	t-statistic	10.11	21.47					
	p-value	0.00	0.00					
C	Mean	−0.17	0.22	−0.08				
	SD	0.23	0.32	0.19				
	t-statistic	16.24	15.68	9.63				
	p-value	0.00	0.00	0.00				

(b) Equal correlation

		Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
A	Mean	0.05	0.44	0.13	0.22			
	SD	0.19	0.32	0.22	0.26			
	t-statistic	5.44	30.61	13.37	18.80			
	p-value	0.00	0.00	0.00	0.00			
B	Mean	0.13	0.52	0.22	0.30	0.08		
	SD	0.18	0.32	0.21	0.24	0.12		
	t-statistic	15.74	36.57	23.23	27.55	15.59		
	p-value	0.00	0.00	0.00	0.00	0.00		
C	Mean	0.06	0.46	0.15	0.23	0.02	−0.07	
	SD	0.19	0.32	0.22	0.26	0.03	0.11	
	t-statistic	7.51	31.77	15.10	20.39	13.72	13.28	
	p-value	0.00	0.00	0.00	0.00	0.00	0.00	

This table shows the sample mean and sample standard deviations (SD) of differences between the Sharpe ratios of the allocation strategies across the 500 backtests. Also shown are the t-statistics and p-values associated with the paired two-sample, two-tailed t-tests. Note that the t-statistic is equal to the sample mean divided by the sample standard deviation, multiplied by the square root of the number of observations (in this case 500). The column and row headings "A", "B" and "C" are labels for "Unconstrained", "Long-only, with no individual position limits" and "Long–short with gross exposure up to 250%, no limits on individual positions", respectively.

by conducting a series of paired two-sample two-tailed t-tests, we can see that all of these differences are also (rather extremely) statistically significant.

Despite this relative triumph, it is both important and interesting to note that the standard deviations of the Sharpe ratios over the 500 runs are very high and range from between 50% of the value of the Sharpe ratio (for the equal-correlation and equal-weight strategies) to 80% of the value of the Sharpe ratio for the constrained mean–variance efficient strategies. (We exclude as a straw-man the unconstrained case, which returns −100%.) This wide variation in Sharpe ratios across simulation runs is by no means a surprise. Indeed it is driven by the high standard deviation of the expected returns, which range from roughly 40% of the expected return itself for the equal-correlation and equal-weight strategies to roughly 60–80% for the constrained mean–variance efficient strategies. This matters. It means that over any finite period the equal-correlation strategy in this setup is more likely to "win" than the other strategies but the likelihood of "losing" in one particular outcome of reality is still substantial.

3.2 Actual historical data (cross-asset-class investing)

Having shown the robustness of the equal-correlation strategy in a simulated setting, we next test the strategies on actual historical data – which allows us to include realistic nonnormality present in historical returns (eg, skewness and kurtosis). Although realized history can be thought of as only a single draw from the set of all possible histories, we hold that, given that it actually occurred, this is an important case, and one which can potentially provide useful intuition, as it is the draw that most of us are by far the most familiar with.[18]

Constructing a relatively long time series of daily data across multiple asset classes is not a trivial task. Many commonly used indexes are either not available on a daily basis, not available on a total returns basis, or only available for the recent past. Nevertheless, we were able to construct a time series covering the period from December 31, 1990 to January 2, 2012. As shown in Table 6 on the facing page, which also displays summary statistics for the time series, the seven assets we include in the investable universe are Datastream Developed and Emerging Market Indices, Datastream US Total 10yr+ Government, Barclays NA Long Credit 10yr Investment Grade Total Return, and the Dow Jones/UBS Agriculture, Energy and Precious Metals indexes.

Reflecting the difficulties in estimating means precisely, the Sharpe ratios of these assets are bit more spread out (−0.08 to 0.59) than in the Harvard Management Company case. The correlations also cover a broad spectrum, varying from −0.32

[18] In addition, realized histories contain the full set of real-world phenomena present in the markets, such as time-varying parameters, serial correlation and measurement error.

TABLE 6 Summary statistics for realized historical data.

	A	B	C	D	E	F	G
Average annualized arithmetic excess return	5.0%	5.8%	5.9%	−0.3%	0.3%	4.6%	5.8%
Average annualized geometric excess return	3.9%	3.4%	5.6%	−0.2%	−1.2%	0.0%	4.2%
Average annualized volatility (of excess returns)	15.1%	18.1%	10.0%	4.3%	17.6%	30.2%	18.4%
Sharpe ratio (of excess returns)	0.33	0.32	0.59	−0.08	0.02	0.15	0.32
Best single-day return	0.09	0.09	0.04	0.03	0.07	0.11	0.09
Worst single-day return	−0.06	−0.09	−0.03	−0.03	−0.07	−0.26	−0.09
Skewness	−0.20	−0.61	−0.07	−0.09	−0.03	−0.44	−0.16
Excess kurtosis	7.69	7.49	2.40	21.68	3.28	9.39	7.01
Number of observations	5480	4435	5480	2109	5480	5480	5480

Correlations of excess returns

	A	B	C	D	E	F	G
A	1.00	0.68	−0.20	0.62	0.23	0.16	0.13
B	0.68	1.00	−0.19	0.48	0.24	0.20	0.20
C	−0.20	−0.19	1.00	−0.32	−0.13	−0.12	−0.02
D	0.62	0.48	−0.32	1.00	0.23	0.19	0.09
E	0.23	0.24	−0.13	0.23	1.00	0.24	0.27
F	0.16	0.20	−0.12	0.19	0.24	1.00	0.23
G	0.13	0.20	−0.02	0.09	0.27	0.23	1.00

This table shows summary statistics for the returns streams used for the actual historical and boostrapped historical backtests. The column and row headings have been abbreviated as follows.
"A", Datastream Developed Market.
"B", Datastream Emerging Markets.
"C", Datastream US Total 10yr+ Government Index.
"D", Barclays NA Long Credit 10yr Investment Grade Total Return Index.
"E", DJ-UBS Agriculture Index.
"F", DJ-UBS Energy Index.
"G", DJ-UBS Precious Metals Index.

TABLE 7 Results from the backtest using realized historical data.

	Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
Average annualized arithmetic excess return	5.2%	633.1%	12.0%	15.6%	2.9%	3.3%	2.9%
Average annualized geometric excess return	4.8%	NA	11.6%	14.2%	2.8%	3.2%	2.8%
Average annualized volatility (of excess returns)	9.7%	2325.3%	14.7%	21.6%	5.9%	6.1%	5.9%
Sharpe ratio (of excess returns)	0.53	0.27	0.82	0.72	0.50	0.55	0.50
Best single-day return	3.5%	9996.4%	7.0%	8.8%	1.8%	1.8%	1.8%
Worst single-day return	−4.2%	−100.0%	−6.3%	−8.6%	−1.8%	−2.1%	−1.8%
Skewness	−0.38	67.14	−0.38	−0.32	−0.12	−0.10	−0.13
Excess kurtosis	3.83	4656.27	6.16	3.73	1.34	1.60	1.33

This table shows performance statistics for each of the allocation strategies from a backtest run on historical data. Estimates of means and variances used by the allocation algorithms were obtained using a rolling two-year (104-week) window. The constraints were implemented using the bootstrapping procedure described in the appendix. The column and row headings "A", "B" and "C" are labels for "Unconstrained", "Long-only, with no individual position limits" and "Long–short with gross exposure up to 250%, no limits on individual positions", respectively.

(between the Datastream US Total 10yr+ Government Index and the Barclays NA Long Credit 10yr Investment Grade Total Return Index) to 0.68 (between the Datastream Developed Market and the Datastream Emerging Markets Index). An additional indication of the richness of this data can be seen in the skewness and kurtosis of the series (which ranges from 2.4 for the Datastream US Total 10yr+ Government Index to 21.7 for the Barclays NA Long Credit 10yr Investment Grade Total Return Index).

As seen in Table 7, the results are, on first reading, not particularly favorable for our case. Although the realized Sharpe ratios for the equal-weight and equal-correlation strategies are objectively attractive (approximately 0.50), they substantially underperform the constrained mean–variance efficient strategies (which have Sharpe ratios ranging from 0.72 to 0.82). Nevertheless, the equal-correlation strategies still have several attractive characteristics: the equal-correlation strategy is both substantially less volatile and less kurtotic than either the mean–variance efficient strategies or even the equal-weight strategy. It is intriguing to note that the kurtosis is significantly lower (by a factor of 2.5 to 3.5) for the equal-correlation strategy (compared with the mean–variance efficient strategy) as the equal-correlation strategy does not directly optimize this measure. The realized skewness is also substantially lower. As we will see in the following subsection, this is a persistent advantage and not just an effect of the single run.

In the context of the analysis of the historical data we also focus on two periods with significant up and down markets: the "up market" from August 2005 through March 2007, and the "down market" from August 2007 through March 2009. The summary statistics for and results from these subperiods are shown in Table 8 on the next page and Table 9 on page 80. From August 2005 to March 2007 Datastream Developed Markets (Equities) and Datastream Emerging Markets (Equities) realized an average annualized arithmetic excess return of 14.5% and 30.2% respectively, while they lost −35.1% and −30.9% between August 2007 and March 2009. The down market can also be characterized by a significantly higher volatility than the up market (28% versus 10% for developed equities). Reflective of the even greater difficulties of estimating means precisely for such short and extreme periods, the Sharpe ratios are significantly more spread out than for the full period, and range from −0.54 to 1.89 for the up market and from −1.25 to 1.06 for the down market. It is interesting to note that three (Datastream US Total 10yr+ Government Index, DJ-UBS Agriculture Index and DJ-UBS Energy Index) of the seven asset classes lose during the up market for equities while only two asset classes are up in the down market for equities (Datastream US Total 10yr+ Government Index and DJ-UBS Precious Metals Index).

As in the full history, the highest correlation is again between the Datastream Developed Market and the Datastream Emerging Markets Index with a value of 0.76 in the up market and an even higher value of 0.81 for the down market (compared with 0.68 for the entire time period of nineteen years). The lowest correlation in the up market is between the Datastream US Total 10yr+ Government Index and Datastream Emerging Markets (−0.04) while the lowest correlation in the down market is realized between the Datastream US Total 10yr+ Government Index and the Datastream Developed Market (−0.36). In general, correlations are higher in the down market with the two exceptions of government bonds, which provide more negative correlations to the other asset classes, and gold, providing in general lower correlations to the other asset classes.

Although the Sharpe ratios of the individual assets have a wide range, the equal-correlation strategy still achieves superior results in the down-market scenario, with Sharpe ratios from −0.07 to +0.09. The equal-weight strategy achieves only a Sharpe ratio of −0.72 while the mean–variance efficient strategy achieves a Sharpe ratio of 0.54 in the unconstrained case (significantly shorting most assets realizing their negative return expectations) and −0.54 and −0.55 in the constrained implementations.

TABLE 8 Summary statistics for realized historical data. [Table continues on next page.]

(a) Summary statistics for realized historical data for the period August 2005–March 2007

	A	B	C	D	E	F	G
Average annualized arithmetic excess return	14.5%	30.2%	−3.4%	1.3%	−3.1%	−15.4%	23.6%
Average annualized geometric excess return	16.3%	36.3%	−2.5%	1.1%	−2.5%	−23.9%	29.7%
Average annualized volatility (of excess returns)	9.8%	15.9%	6.3%	1.1%	16.4%	33.5%	23.9%
Sharpe ratio (of excess returns)	1.47	1.89	−0.54	1.18	−0.19	−0.46	0.99
Best single-day return	0.02	0.03	0.01	0.00	0.04	0.10	0.05
Worst single-day return	−0.02	−0.05	−0.01	0.00	−0.03	−0.05	−0.09
Skewness	−0.32	−1.25	0.10	−0.51	0.13	0.54	−0.89
Excess kurtosis	1.44	4.24	−0.11	3.18	0.76	1.38	3.87
Number of observations	435	435	435	435	435	435	435

Correlations of excess returns

	A	B	C	D	E	F	G
A	1.00	0.76	0.03	0.32	0.18	0.15	0.38
B	0.76	1.00	−0.04	0.35	0.20	0.15	0.40
C	0.03	−0.04	1.00	−0.03	−0.02	0.03	0.00
D	0.32	0.35	−0.03	1.00	0.07	0.01	0.20
E	0.18	0.20	−0.02	0.07	1.00	0.25	0.32
F	0.15	0.15	0.03	0.01	0.25	1.00	0.29
G	0.38	0.40	0.00	0.20	0.32	0.29	1.00

Most convincing are the worst single day returns for the equal-correlation strategy which range between −1.5% and −2.3%, while this range is significantly wider for the mean–variance efficient strategy – ranging between −7.8% for the long-only implementation to −47.6% for the unconstrained implementation. Even the equal-weight strategy loses −4.0% on its worst day. The skewness is generally positive compared

TABLE 8 Continued.

(b) Summary statistics for realized historical data for the period August 2007–March 2009

	A	B	C	D	E	F	G
Average annualized arithmetic excess return	−35.1%	−30.9%	14.6%	−4.9%	−7.7%	−37.1%	15.7%
Average annualized geometric excess return	−35.7%	−36.0%	14.9%	−4.3%	−12.2%	−40.7%	13.6%
Average annualized volatility (of excess returns)	28.1%	32.8%	13.8%	7.8%	29.5%	40.4%	31.2%
Sharpe ratio (of excess returns)	−1.25	−0.94	1.06	−0.63	−0.26	−0.92	0.50
Best single-day return	0.09	0.09	0.04	0.03	0.07	0.08	0.09
Worst single-day return	−0.06	−0.09	−0.03	−0.03	−0.07	−0.08	−0.07
Skewness	0.00	−0.33	0.21	0.02	−0.35	−0.06	0.30
Excess kurtosis	4.33	3.84	1.81	7.21	1.39	0.97	2.98
Number of observations	435	435	435	435	435	435	435

Correlations of excess returns

	A	B	C	D	E	F	G
A	1.00	0.81	−0.36	0.65	0.43	0.42	0.16
B	0.81	1.00	−0.28	0.52	0.44	0.40	0.18
C	−0.36	−0.28	1.00	−0.33	−0.26	−0.28	0.01
D	0.65	0.52	−0.33	1.00	0.28	0.23	0.08
E	0.43	0.44	−0.26	0.28	1.00	0.60	0.43
F	0.42	0.40	−0.28	0.23	0.60	1.00	0.35
G	0.16	0.18	0.01	0.08	0.43	0.35	1.00

This table shows summary statistics for the returns streams used for the actual historical and boostrapped historical backtests. The column and row headings have been abbreviated as follows.
“A”, Datastream Developed Market.
“B”, Datastream Emerging Markets.
“C”, Datastream US Total 10yr+ Government Index.
“D”, Barclays NA Long Credit 10yr Investment Grade Total Return Index.
“E”, DJ-UBS Agriculture Index.
“F”, DJ-UBS Energy Index.
“G”, DJ-UBS Precious Metals Index.

TABLE 9 Results from the backtest using realized historical data. [Table continues on next page.]

(a) Results from the backtest using realized historical data for the period August 2005–March 2007

	Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
Average annualized arithmetic excess return	7.4%	6.6%	24.0%	27.3%	0.4%	3.1%	0.3%
Average annualized geometric excess return	7.2%	−3.8%	25.6%	28.7%	0.3%	3.1%	0.3%
Average annualized volatility (of excess returns)	9.8%	45.4%	15.1%	19.9%	3.8%	4.0%	3.8%
Sharpe ratio (of excess returns)	0.75	0.15	1.59	1.37	0.10	0.79	0.09
Best single-day return	2.0%	23.8%	3.2%	4.6%	1.0%	1.0%	1.0%
Worst single-day return	−2.5%	−22.1%	−5.1%	−6.7%	−0.8%	−0.8%	−0.8%
Skewness	−0.21	−0.40	−1.20	−1.36	0.04	0.09	0.04
Excess kurtosis	0.83	26.48	4.84	5.01	2.45	2.20	2.38

with the negative values for all other strategies with the exception of the unconstrained mean–variance implementation. The volatilities for the equal-correlation strategy are close to their long-term volatilities of approximately 6.0% while even equal weight deviates significantly from its long-term mean (16.7% versus 9.7%). The volatilities of the mean–variance efficient strategy are in general at elevated levels (particularly in the long-only implementation).

The up-market scenario is *ex ante* more challenging for the equal-correlation strategy due to the even wider range of the Sharpe ratios of the individual assets than in the down-market scenario, due largely to the fact that three asset classes had negative returns. Nevertheless, the equal-correlation strategy does not blow up even in such a challenging environment. The equal-correlation strategy realizes Sharpe ratios between 0.09 and 0.79 with worst single days of −0.8% and a positive skewness. The Sharpe ratios of the equal-weight strategy (0.75) and of the mean–variance efficient strategy (0.15 to 1.59) are significantly higher, but their worst single day returns are

TABLE 9 Continued.

(b) Results from the backtest using realized historical data for the period August 2007–March 2009

	Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
Average annualized arithmetic excess return	−12.0%	75.1%	−12.7%	−14.6%	0.6%	−0.5%	0.4%
Average annualized geometric excess return	−12.5%	−9.1%	−14.4%	−16.6%	0.4%	−0.7%	0.2%
Average annualized volatility (of excess returns)	16.7%	137.9%	23.6%	26.5%	6.2%	7.2%	6.2%
Sharpe ratio (of excess returns)	−0.72	0.54	−0.54	−0.55	0.09	−0.07	0.06
Best single-day return	3.6%	86.4%	8.3%	6.7%	1.7%	2.8%	1.7%
Worst single-day return	−4.0%	−47.6%	−7.8%	−8.3%	−1.5%	−2.3%	−1.5%
Skewness	−0.42	3.89	−0.40	−0.35	0.30	0.37	0.29
Excess kurtosis	2.49	41.94	8.72	4.09	2.63	5.92	2.70

This table shows performance statistics for each of the allocation strategies from a backtest run on historical data. Estimates of means and variances used by the allocation algorithms were obtained using a rolling two-year (104-week) window. The constraints were implemented using the bootstrapping procedure described in the appendix. The column headings “A”, “B” and “C” are labels for “Unconstrained”, “Long-only, with no individual position limits” and “Long–short with gross exposure up to 250%, no limits on individual positions”, respectively.

significantly worse, with −2.5% for the equal-weight strategy and a range from −5% to −22% for the mean–variance efficient strategy.

Despite these relatively mixed results, it is important to emphasize that these performance statistics are subject to precisely the same problems of measurement error as the means themselves. Therefore, in the next section, we generate bootstrapped data from these historical returns in order to obtain a higher level of confidence in evaluating the different strategies.

3.3 Bootstrapped historical data

Although measuring the performance of various allocation strategies using actual historical data is somewhat helpful, it represents only one “run” of history and gives very little sense of the robustness of the results. Therefore it is a crucial second step to run the same analyses on bootstrapped historical data in order to generate return patterns for the portfolio components. This combines the strength of realistic return distributions[19] with the ability to draw strong statistical inferences.

Table 10 on the facing page does exactly this by looking at the performance of each of the allocation strategies across 500 different simulation runs (where each of the 500 simulation runs is an out-of-sample backtest, consisting of nineteen years of bootstrapped historical data (4958 days)). We find these results to be an extremely compelling recommendation of the equal-correlation strategy. The Sharpe ratios for all three implementations of the equal-correlation strategy are roughly 0.7 which is economically far greater than those of the equal-weight and mean–variance efficient strategies (0.4 to 0.2). Moreover, as seen in Table 11 on page 84, which shows the results of paired two-tailed t-tests for equivalence of the Sharpe ratios of the various strategies, these results are also extremely statistically significant.

One interesting result, as we have noted previously, is that in addition to having both higher Sharpe ratios and lower volatilities, the equal-correlation strategies also have markedly lower average kurtosis than the other strategies. Although it is not entirely clear what is driving this effect, it is nevertheless yet another attractive property of the strategy.

How do we resolve the relatively weak result from the previous section with the extremely strong results of this section? The answer, unsurprisingly, has to do with the estimation error of the mean. The Sharpe ratios are themselves functions of the first moments of the returns, and therefore have high standard errors (as clearly seen in Table 10 on the facing page). What investors should take away is that although the equal-correlation strategy is not a magic alpha-generating machine, it does substantially improve investors' odds – and it probabilistically dominates the other strategies considered.

4 CONCLUSION

In this paper we propose a method of portfolio diversification, based solely on second-moment criteria, which consists of equalizing the correlations between each asset and the overall portfolio. This method can be motivated in two additional ways. First, it equalizes exposures in an intuitive way: a one-standard-deviation move of any of the investable assets results in a one-standard-deviation move for the portfolio as a whole. Second, it equates to running mean–variance optimization if we assume that the Sharpe ratios of the investable assets are equal.

[19] Note we use a jackknife bootstrap technique here, which does not account for any autocorrelation that may be present in the data (though this may wash out the effect of parameter changes over time). This is primarily to simplify the analysis. Using a block-bootstrap technique which accounts for the relatively small autocorrelation in the data does not qualitatively affect any of our conclusions.

TABLE 10 Results from the backtest using bootstrapped historical data.

	Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
Average annualized arithmetic excess return	4.3% (2.3%)	57.9% (1936.5%)	4.8% (3.2%)	5.5% (5.4%)	4.9% (1.6%)	4.9% (1.6%)	4.9% (1.6%)
Average annualized geometric excess return	3.8% (2.4%)	NA NA	4.0% (3.3%)	3.0% (5.6%)	4.7% (1.7%)	4.7% (1.7%)	4.7% (1.7%)
Average annualized volatility (of excess returns)	10.5% (0.2%)	2145.9% (8185.9%)	13.4% (1.2%)	22.9% (1.9%)	7.2% (0.1%)	7.2% (0.1%)	7.2% (0.1%)
Sharpe ratio (of excess returns)	0.41 (0.22)	0.06 (0.24)	0.36 (0.24)	0.24 (0.24)	0.68 (0.22)	0.68 (0.22)	0.68 (0.22)
Best single-day return	4.4% (0.3%)	5420.5% (28236.2%)	7.3% (1.6%)	12.2% (3.5%)	2.9% (0.3%)	3.0% (0.4%)	2.9% (0.3%)
Worst single-day return	−4.6% (0.5%)	−100.0% (22657.1%)	−7.6% (3.6%)	−12.5% (5.5%)	−2.8% (0.3%)	−2.8% (0.3%)	−2.8% (0.3%)
Skewness	−0.27 (0.18)	3.79 (42.34)	−0.30 (0.83)	−0.14 (0.70)	−0.14 (0.14)	−0.14 (0.14)	−0.14 (0.14)
Excess kurtosis	4.80 (0.85)	2367.35 (1552.04)	12.67 (21.20)	10.41 (16.99)	2.97 (0.56)	3.02 (0.65)	2.98 (0.56)

This table shows the means and standard deviations across 500 simulation runs for a variety of performance statistics for each of the allocation strategies. Each of the 500 simulation runs are out-of-sample backtests, consisting of 19 years of simulated, daily, lognormally distributed data (4958 days). The returns used in the simulation were bootstrapped by sampling with replacement from the daily returns used in the actual historical backtest. Estimates of means and variances used by the allocation algorithms were obtained using a rolling two-year (104-week) window. The constraints were implemented using the bootstrapping procedure described in the appendix.

In addition to these theoretical motivations, we have also demonstrated empirically that employing this "optimally diversified" portfolio construction methodology results in a significant improvement in portfolio performance over traditional mean–variance efficient portfolios as well as naively diversified (ie, equally weighted) portfolios. The superiority of the equal-correlation strategy versus the other two strategies is based on an advantageous trade-off between estimation error and relying on potentially invalid assumptions (ie, model error).

As our equal-correlation allocation implicitly assumes that the Sharpe ratios are equal across assets, a fair critique would be: does the data support this? The problem is that as any estimate of an asset's Sharpe ratio depends crucially on the expected return, the Sharpe ratios themselves cannot be precisely estimated. Instead, investors must rely on economic theory to determine situations in which Sharpe ratios are fairly equal. One investment universe in which this is likely to occur is when invest-

TABLE 11 t-tests from bootstrapped historical data.

(a) Mean–variance efficient

		Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
A	Mean	−0.35						
	SD	0.32						
	t-statistic	24.22						
	p-value	0.000						
B	Mean	−0.05	0.30					
	SD	0.23	0.33					
	t-statistic	4.96	19.90					
	p-value	0.000	0.000					
C	Mean	−0.17	0.18	−0.12				
	SD	0.28	0.33	0.17				
	t-statistic	13.25	11.99	15.61				
	p-value	0.000	0.000	0.000				

(b) Equal correlation

		Equal weight	Mean–variance efficient A	Mean–variance efficient B	Mean–variance efficient C	Equal correlation A	Equal correlation B	Equal correlation C
A	Mean	0.27	0.61	0.32	0.44			
	SD	0.13	0.32	0.21	0.27			
	t-statistic	46.48	43.09	33.17	36.24			
	p-value	0.000	0.000	0.000	0.000			
B	Mean	0.27	0.61	0.32	0.44	0.00		
	SD	0.13	0.32	0.21	0.27	0.01		
	t-statistic	46.36	43.08	33.17	36.27	0.67		
	p-value	0.000	0.000	0.000	0.000	0.506		
C	Mean	0.27	0.61	0.32	0.44	0.00	0.00	
	SD	0.13	0.32	0.21	0.27	0.00	0.01	
	t-statistic	46.48	43.08	33.15	36.22	2.45	0.03	
	p-value	0.000	0.000	0.000	0.000	0.014	0.974	

This table shows the sample mean and sample standard deviations of differences between the Sharpe ratios of the allocation strategies across the 500 backtests performed using bootstrapped samples from the actual historical dataset. Also shown are the t-statistics and p-values associated with the paired two-sample, two-tailed t-tests. Note that the t-statistic is equal to the sample mean divided by the sample standard deviation, multiplied by the square root of the number of observations (in this case 500).

ments are determined at the asset-class level, as the idiosyncratic risk of individual companies/securities has already been diversified away. This is supported by our findings, which show that it performs very well in a variety of realistic asset-class-level scenarios.

Despite our conclusion that equal-correlation is a superior strategy, we also conclude that the equal-weight strategy delivers results that can only be characterized as stable and strong.[20] This is not particularly surprising as it does not suffer from any estimation error. Instead, the pitfall for equal weight comes from the potential for model error. The assumptions that it makes regarding the relationship between expected returns and the covariance matrix are only satisfied for a relatively small region of the parameter space. This region is much larger for the equal-correlation strategy as it only requires equality of the Sharpe ratios of the portfolio components. Put another way, so long as the assumption that the Sharpe ratios are equal is not completely violated, the equal-correlation strategy creates a portfolio superior to that of the equal-weight strategy by incorporating generally fairly precise estimates of the covariance matrix. Although the mean–variance efficient strategy has the potential to deliver the highest Sharpe ratio, the estimation error of the mean completely negates this, putting it behind both equal-correlation and equal-weight strategies. One topic we have not explored in this paper, but which could prove interesting for future work, is whether there is an optimal "mixing" strategy between the equal-correlation strategy and other allocation strategies similar to the mean–variance efficient strategy.

APPENDIX A. THE PRACTICAL IMPACT OF ESTIMATION ERRORS

While it is easy to analytically compare the differences between standard errors on estimated means and volatilities, the numerical impact of these is sometimes obscured. Table A.1 on the next page illustrates the problem by showing just how large a difference there is between the precision of estimates of the mean and estimates of the variance for lognormal data (using annual parameters of $\mu = 0.10$, $\sigma = 0.18$).

As seen in the table, even when conditions are ideal and we have access to 400 years (!) of data on a stationary returns distribution, the standard error on the estimated return is still nearly 8%, leading to a 95% confidence region that is nearly 30% of the magnitude of the expected return. The situation becomes particularly acute when we consider the problem frequently faced in evaluating hedge funds, which may only provide two years of data. In this case the standard error of the estimated mean is

[20] We should emphasize that the equal-weight strategy is an excellent candidate for unsophisticated investors. The strong and stable results for the equal-weight strategy have significant implications for the unsophisticated investor. This simple allocation algorithm can be easily implemented by individual investors, enabling them to save substantial fees for advisors who are often not worth their pay relative to the naive but relatively powerful equal-weight allocation.

TABLE A.1 Example of standard errors for mean and volatility estimates.

Years of history	Standard error of estimated mean	Standard error of estimated volatility	Standard error of estimated mean (%)	Standard error of estimated volatility (%)
Monthly (12 observations per year)				
1	0.175	0.036	150.1	19.8
2	0.126	0.026	108.1	14.2
3	0.102	0.021	87.3	11.5
5	0.079	0.016	68.3	9.0
10	0.056	0.012	48.5	6.4
25	0.036	0.007	30.8	4.0
50	0.025	0.005	21.7	2.9
100	0.018	0.004	15.4	2.0
200	0.013	0.003	10.9	1.4
400	0.009	0.002	7.7	1.0
Weekly (52 observations per year)				
1	0.179	0.018	154.0	9.7
2	0.127	0.012	109.2	6.9
3	0.104	0.010	89.1	5.6
5	0.080	0.008	69.0	4.4
10	0.057	0.006	48.9	3.1
25	0.036	0.004	30.9	2.0
50	0.025	0.002	21.8	1.4
100	0.018	0.002	15.5	1.0
200	0.013	0.001	10.9	0.7
400	0.009	0.001	7.7	0.5
Daily (252 observations per year)				
1	0.180	0.008	154.4	4.4
2	0.127	0.006	109.4	3.1
3	0.104	0.005	89.3	2.6
5	0.080	0.004	69.1	2.0
10	0.057	0.003	48.9	1.4
25	0.036	0.002	30.9	0.9
50	0.025	0.001	21.9	0.6
100	0.018	0.001	15.5	0.4
200	0.013	0.001	10.9	0.3
400	0.009	0.000	7.7	0.2

This table shows the standard errors of the estimated mean and volatility (in levels and as percentages of the true values) for lognormally distributed data generated using 0.10 and 0.18 as the (annualized) mean and volatility of the underlying normal distribution.

more than 100% of the expected return, leading to a confidence interval that is more than 400% of the magnitude of the expected return.[21]

Sampling at higher frequencies (ie, daily rather than monthly) does not resolve the problem; the standard error of the estimated mean is unchanged by the sampling frequency. The asymptotic standard error of the estimated mean is $\sqrt{(\mathrm{Var}[X])/N}$, where N is the number of years of observations.

By contrast, the second moments (eg, volatility and correlations) of the asset returns can be much more precisely estimated, even over relatively short time periods.[22] For example, with just two years of monthly data, the standard error of the estimated volatility is only 14% (of the volatility) compared with more than 100% of the estimated mean. Increasing the sampling frequency by using weekly or daily data reduces the standard error of the estimated volatility even more (to 7% and 3% respectively), so that with just two years of daily data the estimation error of the volatility is only 3%, resulting in a 95% confidence interval that is only approximately 12% of the level of the volatility.

The crucial difference in this case is that sampling at higher frequencies provides additional information about these moments: the asymptotic standard error of the estimated volatility is $\sqrt{(\mathrm{Var}[X])/(2 \times N \times f)}$, where f is the number of observations per year (this point is emphatically made in Merton (1980)). The effects of this can be seen clearly in Table A.1 on the facing page: while the standard errors for the estimated mean are essentially unchanged across sampling frequency, the standard errors of the volatility shrink dramatically.

This issue is of no small concern. The standard errors in these parameter estimates translate directly into standard errors in the weights in traditional mean–variance analysis.[23] This can be easily seen in Table A.2 on the next page, which shows the impact of these estimation errors on the allocations from the (unconstrained) mean–variance efficient strategy: with five years of data the 25th/75th percentiles range from 18% to 61%, while the 5th/95th percentiles range from −13% to 205% of the portfolio. Even with 100 years of data, the 25th/75th percentiles range from 32% to 42%.

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[21] Note that these specific quantitative impacts apply only for the numbers and specific assumptions being used in this example. Different assumptions, such as using different estimation periods, will lead to results that may not be as significant.

[22] This assumes that higher frequency observations are possible. Of course, once the frequency attains a certain level, the precision is affected by measurement noise that is created by the technology that is needed to observe the markets at a higher and higher frequency.

[23] Note that portfolio managers use a wide variety of methods to estimate moments. For simplicity and pedagogical reasons, we have used a highly stylized approach – the "textbook approach" – for moment estimation. The impact of the standard errors in parameter estimates can vary depending on the specific estimation approach used.

TABLE A.2 Example of impact of estimation error on portfolio weights. [Table continues on next page.]

Years of history	Standard error of estimated mean of asset 1	Standard error of estimated mean of asset 2	Standard error of estimated volatility of asset 1	Standard error of estimated volatility of asset 2	Standard error of estimated correlation between assets 1 and 2	1st percentile weight of asset 1 (%)	5th percentile weight of asset 1 (%)	25th percentile weight of asset 1 (%)	50th percentile weight of asset 1 (%)	75th percentile weight of asset 1 (%)	95th percentile weight of asset 1 (%)	99th percentile weight of asset 1 (%)
Monthly (12 observations per year)												
1	0.170	0.061	0.017	0.017	0.286	−525	−87	0	27	62	299	1459
2	0.123	0.045	0.012	0.012	0.201	−343	−55	7	30	62	233	1065
3	0.100	0.038	0.010	0.010	0.161	−208	−32	11	31	61	210	1177
5	0.077	0.031	0.007	0.007	0.127	−71	−13	18	35	61	205	793
10	0.057	0.024	0.005	0.005	0.090	−19	4	22	36	54	125	384
25	0.036	0.016	0.003	0.003	0.056	6	15	27	36	48	74	120
50	0.026	0.012	0.002	0.002	0.039	15	21	30	37	45	59	78
100	0.018	0.008	0.002	0.002	0.027	21	25	32	37	42	52	61
200	0.013	0.006	0.001	0.001	0.020	25	28	33	36	40	46	52
400	0.009	0.004	0.001	0.001	0.014	28	31	34	37	39	44	47
Weekly (52 observations per year)												
1	0.172	0.059	0.008	0.008	0.136	−607	−88	2	28	63	311	1408
2	0.123	0.045	0.006	0.006	0.096	−254	−47	9	32	64	265	1480
3	0.100	0.039	0.005	0.005	0.077	−211	−29	12	32	61	228	1081
5	0.077	0.032	0.004	0.004	0.060	−84	−11	17	35	60	191	940
10	0.056	0.024	0.002	0.002	0.043	−20	4	22	37	56	126	387
25	0.035	0.016	0.002	0.002	0.027	8	16	28	37	49	79	114
50	0.025	0.011	0.001	0.001	0.019	15	21	30	37	45	59	73
100	0.018	0.008	0.001	0.001	0.013	22	26	32	36	42	52	61
200	0.013	0.006	0.001	0.001	0.010	25	28	33	37	41	46	51
400	0.009	0.004	0.000	0.000	0.007	28	31	34	37	39	43	46

TABLE A.2 Continued.

Years of history	Standard error of estimated mean of asset 1	Standard error of estimated mean of asset 2	Standard error of estimated volatility of asset 1	Standard error of estimated volatility of asset 2	Standard error of estimated correlation between assets 1 and 2	1st percentile weight of asset 1 (%)	5th percentile weight of asset 1 (%)	25th percentile weight of asset 1 (%)	50th percentile weight of asset 1 (%)	75th percentile weight of asset 1 (%)	95th percentile weight of asset 1 (%)	99th percentile weight of asset 1 (%)
Daily (252 observations per year)												
1	0.169	0.060	0.004	0.004	0.060	−499	−74	2	27	61	275	1347
2	0.122	0.044	0.002	0.002	0.043	−283	−46	9	31	61	254	1304
3	0.101	0.038	0.002	0.002	0.035	−178	−33	12	34	62	224	1148
5	0.077	0.031	0.002	0.002	0.027	−59	−12	18	34	59	177	743
10	0.056	0.024	0.001	0.001	0.019	−17	3	22	36	55	121	349
25	0.036	0.016	0.001	0.001	0.012	8	16	28	37	49	74	116
50	0.026	0.011	0.000	0.000	0.009	15	22	30	37	45	60	74
100	0.018	0.008	0.000	0.000	0.006	21	26	32	37	42	52	59
200	0.013	0.006	0.000	0.000	0.004	26	29	33	37	41	47	52
400	0.009	0.004	0.000	0.000	0.003	29	31	34	37	39	44	47

This table shows the impact of estimation errors of the estimated mean and volatility (in levels and as percents of the true values) for a lognormally distributed data generated using 0.10 and 0.18 as the (annualized) mean and volatility (of the underlying normal distribution) for asset 1, 0.04 and 0.08 as the mean and volatility for asset 2, and 0.20 as the correlation between them. Weights for percentiles of asset 2 are not shown as, due to the budget constraint, they are simply the mirror of the percentiles for asset 1 (eg, if the 99th percentile weight for asset 1 is 150%, then the 1st percentile weight for asset 2 would be −50%). Each row of the table is the result of 5000 simulated histories.

We are not the first to make this point. Britten-Jones (1999) derives an analytical expression for these standard errors by mapping them into a standard regression framework. Best and Grauer (1991) present analytical and numeric results for the sensitivity of mean–variance efficient portfolios to changes (eg, estimation errors) in the asset means. In addition to dealing with the "unconstrained" case, they also consider the "long only" case where short sales are prohibited.

APPENDIX B. EQUAL CORRELATIONS IN A MEAN–VARIANCE EFFICIENT FRAMEWORK

The equal-correlation strategy is equivalent to the mean–variance efficient strategy in the case where all of the assets have the same Sharpe ratio (ie, it is equivalent to assuming that all assets have the same Sharpe ratio). To see this, start by considering the system of equations that equalizes the correlations in the three-asset case. We start with the two equalities

$$\left.\begin{aligned} w_1\sigma_1\rho_{11} + w_2\sigma_2\rho_{12} + w_3\sigma_3\rho_{13} &= w_1\sigma_1\rho_{12} + w_2\sigma_2\rho_{22} + w_3\sigma_3\rho_{23}, \\ w_1\sigma_1\rho_{12} + w_2\sigma_2\rho_{22} + w_3\sigma_3\rho_{23} &= w_1\sigma_1\rho_{13} + w_2\sigma_2\rho_{23} + w_3\sigma_3\rho_{33}. \end{aligned}\right\} \tag{B.1}$$

Note that (assuming this system of equations has a solution) this can be rewritten as:

$$\left.\begin{aligned} w_1\sigma_1\rho_{11} + w_2\sigma_2\rho_{12} + w_3\sigma_3\rho_{13} &= \kappa, \\ w_1\sigma_1\rho_{12} + w_2\sigma_2\rho_{22} + w_3\sigma_3\rho_{23} &= \kappa, \\ w_1\sigma_1\rho_{13} + w_2\sigma_2\rho_{23} + w_3\sigma_3\rho_{33} &= \kappa, \end{aligned}\right\} \tag{B.2}$$

where κ is a unique constant. Now turn to the solution to the mean–variance optimization problem. The objective function for the three-asset case is

$$\begin{aligned} & w_1\mu_1 + w_1\mu_1 + w_1\mu_1 \\ & \quad - \frac{\gamma}{2}(w_1^2\sigma_1^2\rho_{11} + w_2^2\sigma_2^2\rho_{22} + w_3^2\sigma_3^2\rho_{33} \\ & \qquad + w_1w_2\sigma_1\sigma_2\rho_{12} + w_1w_3\sigma_1\sigma_3\rho_{13} + w_2w_3\sigma_2\sigma_3\rho_{23}), \end{aligned} \tag{B.3}$$

where γ is the coefficient of risk aversion. Taking first-order conditions of this objective function with respect to each of the three weights gives us

$$\left.\begin{aligned} w_1\sigma_1^2\rho_{11} + w_2\sigma_1\sigma_2\rho_{12} + w_3\sigma_1\sigma_3\rho_{13} &= \frac{\mu_1}{\gamma}, \\ w_1\sigma_1\sigma_2\rho_{12} + w_2\sigma_2^2\rho_{22} + w_3\sigma_2\sigma_3\rho_{23} &= \frac{\mu_2}{\gamma}, \\ w_1\sigma_1\sigma_3\rho_{13} + w_2\sigma_2\sigma_3\rho_{23} + w_3\sigma_3^2\rho_{33} &= \frac{\mu_3}{\gamma}. \end{aligned}\right\} \tag{B.4}$$

Dividing both sides of each of these three equations by σ_1, σ_2 and σ_3, respectively, gives us

$$\left.\begin{aligned} w_1\sigma_1\rho_{11} + w_2\sigma_2\rho_{12} + w_3\sigma_3\rho_{13} &= \frac{\mu_1}{\gamma\sigma_1},\\ w_1\sigma_1\rho_{12} + w_2\sigma_2\rho_{22} + w_3\sigma_3\rho_{23} &= \frac{\mu_2}{\gamma\sigma_1},\\ w_1\sigma_1\rho_{13} + w_2\sigma_2\rho_{23} + w_3\sigma_3\rho_{33} &= \frac{\mu_3}{\gamma\sigma_3}. \end{aligned}\right\} \tag{B.5}$$

Now we note that in the case

$$\frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2} = \frac{\mu_3}{\sigma_3},$$

that is, when all three Sharpe ratios are equal, the solutions to the equal-correlation problem and the mean–variance optimization problem are identical, ie, systems (B.2) and (B.5) are identical.

A corollary of this is that the strategy is equivalent to the minimum-variance strategy in the case where all assets have the same volatility.

APPENDIX C. IMPOSING CONSTRAINTS

Assuming that the correlation matrix is of full rank, the equal-correlation approach yields an exactly identified system. Thus, in order to impose constraints, such as individual position limits or aggregate gross/net portfolio limits, it is necessary to specify a loss function that explicitly trades off the differences in correlations across assets. One naive loss function is simply the sum of the squared differences between the chosen correlations and the equalized correlations.

Unfortunately, this approach ignores the fact that some correlations (and thus some asset weights) are estimated more precisely than others. To incorporate this additional information, we treat the unconstrained asset weights as estimates, and sample with replacement from the historical returns data (ie, the two year rolling sample) providing us with an estimate of the variance-covariance matrix of these estimated weights. We then use a distance function which imposes greater penalties for asset weight deviations in assets whose unconstrained asset weights are more precisely estimated and smaller penalties for asset weight deviations in assets whose unconstrained asset weights are less precisely estimated. Specifically, we minimize the loss function $Q(\boldsymbol{w}) = (\boldsymbol{w} - \boldsymbol{w}_{\mathrm{uc}})' V^{-1} (\boldsymbol{w} - \boldsymbol{w}_{\mathrm{uc}})$, subject to the imposed constraints (eg, $\pm 30\%$ position limits), $\boldsymbol{w}_{\mathrm{uc}}$ is the vector of unconstrained asset weights and V is the estimated variance–covariance matrix of those (unconstrained) weights.

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