IE 310 - Homework II

January 21, 2022

1 Answers

1) To formulate the mathematical model, let x_i denote the amount in tons to be supplied quarterly from supplier S_i to warehouse W_i, for $i = 1, 2$. $40 - x_1$ denotes the amount supplied from S_1 to W_2 and $60 - x_2$ denotes the amount supplied from S_2 to W_2. We could have represented $(40 - x_1)$ and $(60 - x_2)$ as new variables, but we do not have to since we know their relationship with x_1 and x_2. And also doing so would make our calculation much more longer.

Then each decision variable x_i must be non-negative.

$$x_i \geq 0 \quad \text{for} \quad i = 1, 2$$

We know that the amount supplied from each supplier cannot exceed the specified value. Then:

$$x_1 + (40 - x_1) = 40$$
$$x_2 + (60 - x_2) = 60$$

We have a limit on the amount of chocolate transferred from each warehouse to the production plant:

$$\begin{aligned} x_1 + x_2 &\leq 70 \qquad (1) \\ (40 - x_1) + (60 - x_2) &\leq 70 \\ x_1 + x_2 &\geq 30 \qquad (2) \end{aligned}$$

We also have a limit on the amount of chocolate that can be supplied from each supplier to each warehouse:

$$\begin{aligned} x_1 \leq 30 \quad &\text{and} \quad (40 - x_1) \leq 30 \\ 10 \leq x_1 &\leq 30 \quad (1) \\ x_2 \leq 50 \quad &\text{and} \quad (60 - x_2) \leq 50 \\ 10 \leq x_2 &\leq 50 \quad (2) \end{aligned}$$

$$
\begin{aligned}
\min z &= 2000x_1 + 1700(40 - x_1) + 1600x_2 + 1100(60 - x_2) + 400(x_1 + x_2) + \\
&\quad 800(40 - x_1 + 60 - x_2) \\
&= -100x_1 + 100x_2 + 214000 \\
\text{s.t. } & x_1 + x_2 \leq 70 \\
& x_1 \geq 10 \\
& x_1 \leq 30 \\
& x_2 \geq 10 \\
& x_2 \leq 50 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Let's get rid of the lower bounds on the decision variables. We should treat the upper bounds as functional constraints.

$$
\begin{aligned}
\min z &= -100(x_1' + 10) + 100(x_2' + 10) + 214000 \\
\text{s.t. } & (x_1' + 10) + (x_2' + 10) \leq 70 \\
& (x_1' + 10) \leq 30 \\
& (x_2' + 10) \leq 50 \\
& x_1', x_2' \geq 0
\end{aligned}
$$

Standard form:

$$
\begin{aligned}
\min z &= -100x_1' + 100x_2' + 214000 \\
\text{s.t. } & x_1' + x_2' + S_1 = 50 \\
& x_1' + S_2 = 20 \\
& x_2' + S_3 = 40 \\
& S_1, S_2, S_3, x_1', x_2' \geq 0
\end{aligned}
$$

Let's find a basic feasible solution by setting S_1, S_2 and S_3 as basic variables. Let's construct our simplex tableau. To move from one bfs to another, we will take the most negative coefficient in the objective function and make ratio test to find which basic variable to replace with.

	x_1'	x_2'	S_1	S_2	S_3	
S_1	1	1	1	0	0	50
S_2	$1*$	0	0	1	0	20
S_3	0	1	0	0	1	40
	-100	100	0	0	0	$z - 214000$

Ratio test: $20/1 = 20$, $50/1 = 50$

Pivot is marked with asterisk. After pivoting:

	x_1'	x_2'	S_1	S_2	S_3	
S_1	0	1	1	-1	0	30
x_1'	1	0	0	1	0	20
S_3	0	1	0	0	1	40
	0	100	0	100	0	$z - 212000$

Therefore the answer:

$$z = 212000 \quad x_1' = 20,\ x_2' = 0,\ S_1 = 30,\ S_2 = 0,\ S_3 = 40$$

Which corresponds to:

$$z = 212000 \quad x_1 = 30,\ x_2 = 10$$

Let a_i denote the amount in tons to be supplied quarterly from supplier S_i to warehouse W_i, for $i = 1, 2$. Then our answer becomes:

$$z = 212000 \quad a_{11} = 30,\ a_{12} = 10,\ a_{21} = 10,\ a_{22} = 50$$

2) Our objective function:

$$\min \max \ \{5x_1, 7|x_2 - 10|, 2|x_1 - 2x_3|\}$$

It is equivalent to:

$$\begin{aligned}
&\min z = y \\
&\quad y \geq 5x_1 \\
&\quad y \geq 7|x_2 - 10| \implies y \geq 7x_2 - 70 \text{ and } y \geq -7x_2 + 70 \\
&\quad y \geq 2|x_1 - 2x_3| \implies y \geq 2x_1 - 4x_3 \text{ and } y \geq -2x_1 + 4x_3
\end{aligned}$$

$$\text{To have a standard LP form:} \begin{cases} \text{All decision variables must be non-negative.} \\ \text{All constraints should be equality.} \\ \text{Right hand side of the equations must be non-negative.} \end{cases}$$

Introduce slack and surplus variables for the inequalities. Multiply RHS with -1 if it's negative. Standard LP form:

$$\begin{aligned}
&\min z = y \\
&\text{s.t. } x_1 + x_2 + x_3 = 50 \\
&\qquad x_1 + 3x_2 - 2x_3 - S_1 = 10 \\
&\qquad -3x_1 + 2x_2 + -x_3 + S_2 = 22 \\
&\qquad y - S_3 - 5x_1 = 0 \\
&\qquad -y + S_4 + 7x_2 = 70 \\
&\qquad y - S_5 + 7x_2 = 70 \\
&\qquad y - S_6 - 2x_1 + 4x_3 = 0 \\
&\qquad y - S_7 + 2x_1 - 4x_3 = 0 \\
&\qquad x_1, x_2, x_3, y \geq 0 \\
&\qquad S_i \geq 0 \text{ for } \forall i \in \{1, 2, 3, 4, 5, 6, 7\}.
\end{aligned}$$

3) Let's start by constructing the simplex tableau. At each step we should choose the most negative value from the objective function. The we should compute the ratio for positive values in the same column.

Basic variables are listed on the left side of the tableau in each iteration. Pivot elements are marked with asterisk (*).

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_1	1	0	0	$\frac{1}{4}$	-8	-1	9	0
x_2	0	1	0	$\frac{1}{4}*$	-12	$-\frac{1}{2}$	3	0
x_3	0	0	1	0	0	6	0	1
	0	0	0	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	z

$\frac{-3}{4}$ is the most negative value in the objective function. We choose x_4 as basic variable.

$$0/(\frac{1}{4}) = 0$$

We can either replace it with x_2 or x_1. I replaced it with x_2.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_1	1	-1	0	0	$4*$	$-\frac{1}{2}$	6	0
x_4	0	4	0	1	-48	-2	12	0
x_3	0	0	1	0	0	6	0	1
	0	3	0	0	-16	-2	15	z

-16 is the most negative value in the objective function. We choose x_5 as basic variable. There is only one positive value in the corresponding column. We make x_1 nonbasic and x_5 basic variable.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	$\frac{1}{4}$	$\frac{-1}{4}$	0	0	1	$-\frac{1}{8}$	$\frac{3}{2}$	0
x_4	12	-8	0	1	0	-8	84	0
x_5	0	0	1	0	0	$6*$	0	1
	4	-1	0	0	0	-4	39	z

-4 is the most negative value in the objective function. We choose x_6 as basic variable. There is only one positive value in the corresponding column. We make x_5 nonbasic and x_6 basic variable.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{48}$	0	1	0	$\frac{3}{2}$	$\frac{1}{48}$
x_4	12	-8	$\frac{4}{3}$	1	0	0	84	$\frac{4}{3}$
x_6	0	0	$\frac{1}{6}$	0	0	1	0	$\frac{1}{6}$
	4	-1	4	0	0	0	39	$z + \frac{2}{3}$

-1 is the most negative value in the objective function. We choose x_2 as basic variable. But notice that there is no positive value in the column. By the Theorem (Unbounded), we conclude that the objective function is unbounded below.

Theorem (Unbounded)
For the linear programming problem (*), if there is an index $s, m+1 \leq s \leq n$, such that $c_s \leq 0$ and $a_{i,s} \leq 0$ for all $i = 1, 2, ..., m$, then the objective function is unbounded below. $z \to -\infty$

4) First let's show the problem in standard form by introducing slack and surplus variables.

$$\begin{aligned}
&\min\ z = 3x_1 + 2x_2 + 4x_3 + 8x_4 \\
&\quad \text{s.t.} \\
&\qquad x_1 - 2x_2 + 3x_3 + 6x_4 - S_1 = 8 \\
&\qquad -2x_1 + 5x_2 + 3x_3 - 5x_4 + S_2 = 3 \\
&\qquad x_1, x_2, x_3, x_4, S_1, S_2 \geq 0
\end{aligned}$$

Now we should introduce an artificial variable.

$$\begin{aligned}
&\min\ z = 3x_1 + 2x_2 + 4x_3 + 8x_4 + Mx_5 \\
&\quad \text{s.t.} \\
&\qquad x_1 - 2x_2 + 3x_3 + 6x_4 - S_1 + x_5 = 8 \\
&\qquad -2x_1 + 5x_2 + 3x_3 - 5x_4 + S_2 = 3 \\
&\qquad x_1, x_2, x_3, x_4, x_5, S_1, S_2 \geq 0
\end{aligned}$$

M represents a "very large" positive number. Subtract M times first constraint from objective function to pivot basic variables x_5 and S_2.

$$z - 8M = (3-M)x_1 + (2+2M)x_2 + (4-3M)x_3 + (4-6M)x_4 + MS_1$$

Now, let's build our tableau.

	x_1	x_2	x_3	x_4	x_5	S_1	S_2	
x_5	1	-2	3	6	1	-1	0	8
S_2	-2	5	3	-5	0	0	1	3
	$3-M$	$2+2M$	$4-3M$	$8-6M$	0	M	0	$-8M$

$8-6M$ is the most negative number. We should make x_4 basic variable. We should swap it with x_5 since it is the only positive number available in that column. Pivot is marked with asterisk (*) above. Below you can find the tableau after pivoting operations.

	x_1	x_2	x_3	x_4	x_5	S_1	S_2	
x_4	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{4}{3}$
S_2	$-\frac{7}{6}$	$\frac{10}{3}$	$\frac{11}{2}$	0	$\frac{5}{6}$	$-\frac{5}{6}$	1	$\frac{29}{3}$
	$\frac{5}{3}$	$\frac{14}{3}$	0	0	$M-\frac{4}{3}$	$\frac{4}{3}$	0	$-\frac{32}{3}$

There is no negative coefficient left in the objective function. Therefore the answer:

$$z = \frac{32}{3} \quad x_1 = 0, x_2 = 0, x_3 = 0, x_4 = \frac{4}{3}, S_1 = 0, S_2 = \frac{29}{3}$$

5) I've implemented the linear equation solver in Python. I've written different methods to make my work easier by dividing the task into sub-tasks. Here you can find the list of my methods and their purposes:

- **interchange_row()** = Interchanges two rows with each other in a matrix.
- **multiply_row()** = Multiplies a row with a constant.
- **identity_matrix()** = Returns a $n \times n$ identity matrix.
- **rank()** = Returns the rank of a matrix in row-reduced echelon form.
- **isZero()** = Returns true if the given number is likely to be zero, false otherwise. It must be used since floating point precision may not be flawless in programming languages.
- **Gauss_Jordan()** = Implements the Gauss Jordan method on the given matrix with the help of elementary row operations.
- **inverse_matrix()** = Returns the inverse of an invertible matrix by the same logic applied in Gauss Jordan method.
- **solve()** = Takes two parameters, A and b. It calculates the ranks of $\mathbf{A}$ and $\mathbf{A}|\mathbf{b}$. Assuming n is the length of the matrix:

$$\begin{aligned}
\text{rank(A)} = n &\longrightarrow \text{return unique solution and inverse of A} \\
\text{rank(A)} = \text{rank(A|b)} < n &\longrightarrow \text{return arbitrary solution and arbitrary values} \\
\text{rank(A)} < \text{rank(A|b)} &\longrightarrow \text{Inconsistent problem}
\end{aligned}$$

- **main()** = Takes a filename as parameter and reads input from it. It creates the corresponding matrix and vector b. Then it calls the function **solve()**, and according to the flags set, it prints out the solution.

How to Run?

Within the main section of the program, **main()** function is called eight times.

```
main("test1.txt")
main("test2.txt")
main("test3.txt")
main("Data1.txt")
main("Data2.txt")
main("Data3.txt")
main("Data4.txt)
main("input.txt")
```

First three calls are made to three different test files I've prepared. Following four calls are made to the input files provided by you. Last call is made for you to test the code the with an arbitrary input you want. You can change the content of the *input.txt* file (actually you can change content of the any file you want). Expected outputs for the first seven files are given within the program as a block of comments. I've written the **main()** function so that each value will have two decimal points. If you want better precision, modify the associated parts within the function.

(a) Expected output for *test1.txt*:
Unique solution: x1 = 1.00, x2 = 2.00, x3 = 3.00
Inverted A: 1.00 2.00 -1.00
0.50 -0.50 0.00
-0.50 -1.50 1.00

(b) Expected output for *test2.txt*:
Inconsistent problem

(c) Expected output for *test3.txt*:
Arbitrary variables: x2 = 0.00
Arbitrary solution: x1 = -8.00, x2 = 0.00, x3 = -6.00

(d) Expected output for *Data1.txt*:
Arbitrary variables: x3 = 0.00
Arbitrary solution: x1 = 6.60, x2 = 1.80, x3 = 0.00

(e) Expected output for *Data2.txt*:
Unique solution: x1 = 1.00, x2 = -0.50, x3 = 1.50
Inverted A: 0.50 0.17 0.33
1.00 0.40 0.20
0.00 0.13 0.07

(f) Expected output for *Data3.txt*:
Inconsistent problem

(g) Expected output for *Data4.txt*:
Unique solution: 1 = 1.60, x2 = 0.28, x3 = 0.77, x4 = -0.69, x5 = 0.97, x6 = -1.04
Inverted A: 0.21 -0.09 0.04 0.05 -0.05 -0.16
-0.00 -0.05 -0.04 0.10 -0.07 0.17
-0.05 -0.03 0.07 -0.03 0.09 -0.06
-0.07 0.09 -0.02 -0.03 -0.04 0.08
0.08 -0.06 -0.03 0.03 -0.01 0.05
-0.14 0.12 -0.02 -0.07 0.07 -0.02