IE 310 - Homework IV

January 21, 2022

1 Answers

1) (a) First, convert all the constraints into $\leq$ constraints:

$$\begin{aligned} x_1 + x_2^2 + x_3 &\leq 5 \\ -x_1 - x_2^2 - x_3 &\leq -5 \\ -5x_1^2 + x_2^2 + x_3 &\leq -2 \\ -x_1 &\leq 0 \\ -x_2 &\leq 0 \\ -x_3 &\leq 0 \end{aligned}$$

Lagrangian:

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = x_1^3 - x_2^2 + x_1x_3^2 - \lambda_1(x_1 + x_2^2 + x_3 - 5)$$
$$-\lambda_2(-x_1 - x_2^2 - x_3 + 5) - \lambda_3(-5x_1^2 + x_2^2 + x_3 + 2) - \lambda_4(-x_1) - \lambda_5(-x_2) - \lambda_6(-x_3)$$

KKT Conditions:

i. Stationary Point

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 3x_1^2 + x_3^2 - \lambda_1 + \lambda_2 + 10\lambda_3 x_1 + \lambda_4 = 0 \\ \frac{\partial L}{\partial x_2} &= -2x_2 - 2\lambda_1 x_2 + 2\lambda_2 x_2 - 2\lambda_3 x_2 + \lambda_5 = 0 \\ \frac{\partial L}{\partial x_3} &= 2x_1x_3 - \lambda_1 + \lambda_2 - \lambda_3 + \lambda_6 = 0 \end{aligned}$$

ii. Complementary Slackness

$$\begin{aligned} \lambda_1(x_1 + x_2^2 + x_3 - 5) &= 0 \\ \lambda_2(-x_1 - x_2^2 - x_3 + 5) &= 0 \\ \lambda_3(-5x_1^2 + x_2^2 + x_3 + 2) &= 0 \\ \lambda_4(-x_1) &= 0 \\ \lambda_5(-x_2) &= 0 \\ \lambda_6(-x_3) &= 0 \end{aligned}$$

iii. Dual feasibility

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0$$

iv. Primal feasibility

$$\begin{aligned} x_1 + x_2^2 + x_3 &\leq 5 \\ -x_1 - x_2^2 - x_3 &\leq -5 \\ -5x_1^2 + x_2^2 + x_3 &\leq -2 \\ -x_1 &\leq 0 \\ -x_2 &\leq 0 \\ -x_3 &\leq 0 \end{aligned}$$

(b) First, convert all the constraints into $\leq$ constraints:

$$\begin{aligned} x_1^2 - x_2^2 + x_3^3 &\leq 10 \\ -x_1^3 - x_2^2 - 4x_3^2 &\leq -20 \end{aligned}$$

Lagrangian:

$$\begin{aligned} L(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1^4 + x_2^2 + 5x_1x_2x_3 + \lambda_1(x_1^2 - x_2^2 + x_3^3 - 10) \\ + \lambda_2(-x_1^3 - x_2^2 - 4x_3^2 + 20) \end{aligned}$$

KKT Conditions:

i. Stationary Point

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 4x_1^3 + 5x_2x_3 + 2\lambda_1 x_1 - 3\lambda_2 x_1^2 = 0 \\ \frac{\partial L}{\partial x_2} &= 2x_2 + 5x_1x_3 - 2\lambda_1 x_2 - 2\lambda_2 x_2 = 0 \\ \frac{\partial L}{\partial x_3} &= 5x_1x_2 + 3\lambda_1 x_3^2 - 8\lambda_2 x_3 = 0 \end{aligned}$$

ii. Complementary Slackness

$$\begin{aligned} \lambda_1(x_1^2 - x_2^2 + x_3^3 - 10) &= 0 \\ \lambda_2(-x_1^3 - x_2^2 - 4x_3^2 + 20) &= 0 \end{aligned}$$

iii. Dual feasibility

$$\lambda_1, \lambda_2 \geq 0$$

iv. Primal feasibility

$$\begin{aligned} x_1^2 - x_2^2 + x_3^3 &\leq 10 \\ -x_1^3 - x_2^2 - 4x_3^2 &\leq -20 \end{aligned}$$

(c) First, convert all the constraints into $\leq$ constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 5 \\ x_1 + x_3 &\leq 2 \\ -x_1 &\leq -1 \\ -x_2 &\leq -2 \\ -x_3 &\leq 0 \end{aligned}$$

Lagrangian:

$$\begin{aligned} L(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = {} & x_1^2 + x_2^2 + x_3^2 + \lambda_1(2x_1 + x_2 - 5) + \lambda_2(x_1 + x_3 - 2) \\ & + \lambda_3(-x_1 + 1) + \lambda_4(-x_2 + 2) + \lambda_5(-x_3) \end{aligned}$$

KKT Conditions:

i. Stationary Point

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 \\ \frac{\partial L}{\partial x_2} &= 2x_2 + \lambda_1 - \lambda_4 \\ \frac{\partial L}{\partial x_3} &= 2x_3 + \lambda_2 - \lambda_5 \end{aligned}$$

ii. Complementary Slackness

$$\begin{aligned} \lambda_1(2x_1 + x_2 - 5) &= 0 \\ \lambda_2(x_1 + x_3 - 2) &= 0 \\ \lambda_3(-x_1 + 1) &= 0 \\ \lambda_4(-x_2 + 2) &= 0 \\ \lambda_5(-x_3) &= 0 \end{aligned}$$

iii. Dual feasibility

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$$

iv. Primal feasibility

$$\begin{aligned} 2x_1 + x_2 &\leq 5 \\ x_1 + x_3 &\leq 2 \\ -x_1 &\leq -1 \\ -x_2 &\leq -2 \\ -x_3 &\leq 0 \end{aligned}$$

2) Lagrangian:

$$L(x_1, x_2, \lambda_1) = x_1 - x_2 - \lambda_1(x_1^2 + x_2^2 - 1)$$

KKT Conditions:

(a) Stationary Point

$$\frac{\partial L}{\partial x_1} = 1 - 2x_1\lambda_1 = 0$$
$$\frac{\partial L}{\partial x_2} = -1 - 2x_2\lambda_1 = 0$$

(b) Complementary Slackness

$$\lambda_1(x_1^2 + x_2^2 - 1) = 0$$

(c) Dual feasibility

$$\lambda_1 \geq 0$$

(d) Primal feasibility

$$x_1^2 + x_2^2 \leq 1$$

- Case (1):

$$\lambda_1 = 0 \longrightarrow 1 + 0 \neq 0 \quad \text{(contradiction with stationary point)}$$

- Case (2):

$$\lambda_1 > 0 \longrightarrow x_1^2 + x_2^2 - 1 = 0$$
$$x_1 = \frac{1}{2\lambda_1} \quad x_2 = \frac{-1}{2\lambda_1} \quad \text{(from stationary points)}$$
$$x_1^2 = x_2^2$$
$$x_1^2 = \frac{1}{2}$$
$$x_1 = \pm\frac{1}{\sqrt{2}}, \quad x_2 = \pm\frac{1}{\sqrt{2}}$$

x_1 cannot be negative since $\lambda_1 > 0$. Therefore $x_1 = \frac{1}{\sqrt{2}}$. Since $x_1 > 0$, $x_2 = \frac{-1}{\sqrt{2}}$. Optimal solution is $f(x_1, x_2) = \sqrt{2}$.

3) Amount of product 1 $\longrightarrow x_1$ ml
Amount of product 2 $\longrightarrow x_2$ ml
Objective function:

Maximize $z = x_1(30 - x_1) + x_2(50 - 2x_2) - 3x_1 - 5x_2 - 10(x_1 + x_2)$
Subject to:

$$x_1 + x_2 \leq 15$$
$$x_1 \geq 0 \quad \text{(monopolist cannot purchase negative number of raw material)}$$
$$x_2 \geq 0$$

Rearrange the objective function:
Maximize $z = -x_1^2 - 2x_2^2 + 17x_1 + 35x_2$

Lagrangian:

$$L(x_1, x_2, \lambda_1) = -x_1^2 - 2x_2^2 + 17x_1 + 35x_2 - \lambda_1(x_1 + x_2 - 15) - \lambda_2(-x_1) - \lambda_3(-x_2)$$

KKT Conditions:

(a) Stationary Point

$$\frac{\partial L}{\partial x_1} = -2x_1 + 17 - \lambda_1 + \lambda_2$$
$$\frac{\partial L}{\partial x_2} = -4x_2 + 35 - \lambda_1 + \lambda_3$$

(b) Complementary Slackness

$$\lambda_1(x_1 + x_2 - 15) = 0$$
$$\lambda_2(-x_1) = 0$$
$$\lambda_3(-x_2) = 0$$

(c) Dual feasibility

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

(d) Primal feasibility

$$x_1 + x_2 \leq 15$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

i. Case (1):

$$\lambda_1, \lambda_2, \lambda_3 = 0$$
$$-2x_1 + 17 = 0 \longrightarrow x_1 = \frac{17}{2}$$
$$-4x_2 + 35 = 0 \longrightarrow x_2 = \frac{35}{4}$$
$$x_1 + x_2 > 15 \longrightarrow \text{contradiction with primal feasilibity}$$

ii. Case (2):

$$\begin{aligned}
&\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0 \\
&x_1 + x_2 = 15 \\
&-4x_1 - 4x_2 + 69 - 3\lambda_1 = 0 \\
&= 69 - 60 - 3\lambda_1 = 0 \\
&\lambda_1 = 3 \longrightarrow x_1 = 7 \quad x_2 = 8
\end{aligned}$$

$x_1 = 7, x_2 = 8$ satisfy the KKT conditions. $z = 222$ is the value of the objective function.

iii. Case (3):

$$\begin{aligned}
&\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0 \\
&x_1 = 0, \lambda_2 = -17 \longrightarrow \text{(contradiction with dual feasibility)}
\end{aligned}$$

iv. Case (4):

$$\begin{aligned}
&\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0 \\
&x_2 = 0, \lambda_3 = -35 \longrightarrow \text{(contradiction with dual feasibility)}
\end{aligned}$$

v. Case (5):

$$\begin{aligned}
&\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0 \\
&x_1 = 0, x_2 = 15 - 0 = 15 \\
&35 - 60 - \lambda_1 = 0 \longrightarrow \lambda_1 = -25 \quad \text{(contradiction with dual feasibility)}
\end{aligned}$$

vi. Case (6):

$$\begin{aligned}
&\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 \neq 0 \\
&x_2 = 0, x_1 = 15 - 0 = 15 \\
&-30 + 17 - \lambda_1 = 0 \longrightarrow \lambda_1 = -13 \quad \text{(contradiction with dual feasibility)}
\end{aligned}$$

vii. Case (7):

$$\begin{aligned}
&\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0 \\
&x_1 = x_2 = 0 \\
&\lambda_2 = -17, \lambda_3 = -35 \quad \text{(contradiction with dual feasibility)}
\end{aligned}$$

viii. Case (8):

$$\begin{aligned}
&\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0 \\
&x_1 = x_2 = 0 \longrightarrow 0 + 0 - 15 \neq 0 \longrightarrow \text{(contradiction with complementary slackness)}
\end{aligned}$$

In conclusion, profit can be maximized by choosing $x_1 = 7$ and $x_2 = 8$. Value of the objective function becomes $z = 222$.

4) Assume that p is greater than $p - 1$. Proof is the same for the opposite case.

$$\begin{aligned}
\frac{1}{p} &= \frac{p}{1-p} \\
1 - p &= p^2 \\
p^2 + p - 1 &= 0 \\
(p + \frac{1}{2})^2 - \frac{5}{4} &= 0 \\
p + \frac{1}{2} &= \pm\frac{\sqrt{5}}{2} \\
p &= \frac{\pm\sqrt{5} - 1}{2}
\end{aligned}$$

p cannot be negative since it represents a length. $p = \frac{\sqrt{5}-1}{2}$

5) Given function is twice differentiable. As an extra, global minimum will be found using four Bisection Search call for four different interval in which f is convex. Such intervals are found by evaluating the second order derivative.

$$\begin{aligned}
f(x) &= 10 + 0.01x - 0.1x^2 + 0.8cos(3x) \\
f'(x) &= 0.01 - 0.2x - 2.4sin(3x) \\
f''(x) &= -0.2 - 7.2cos(3x) \\
cos(3x) &= -\frac{1}{36} \\
3x &= \arccos(-\frac{1}{36}) + 2\pi k \quad (k \in \mathbb{Z}) \quad (1) \\
3x &= (-\arccos(-\frac{1}{36})) + 2\pi k \quad (k \in \mathbb{Z}) \quad (2) \\
x &= \frac{\arccos(-\frac{1}{36})}{3} + \frac{2\pi k}{3} \quad (k \in \mathbb{Z}) \quad (1) \\
x &= \frac{\arccos(\pi - \frac{1}{36})}{3} + \frac{2\pi k}{3} \quad (k \in \mathbb{Z}) \quad (2) \\
x &= \{..., -4.721, -3.655, -2.627, -1.562, -0.533, \\
&\quad 0.533, 1.562, 2.627, 3.655, 4.721, ...\}
\end{aligned}$$

Function f is convex in the subintervals [-3.655, -2.627], [-1.562, -0.533], [0.533, 1.562] and [2.627, 3.655]. To find the global minimum, BisectionSearch algorithm is applied to all of the subintervals given above. In addition, since [-4, -3.655] and [3.655, 4] are concave we should compare our result with $f(-4)$ and $f(4)$. It might be the case that -4 or 4 is the global minimum for the interval [-4, 4]. Screenshots of the outputs are given below. In each iteration $x = \frac{a+b}{2}$ including the initial x.

Values are printed in 8 decimal point precision in order to clearly show the differences between each iteration.

```
Subinterval : (-3.655, -2.627)
Iteration 0 : x = -3.14100000 || f(x) = 8.18200316 || f(x + e) = 8.18206745 || a = -3.65500000 || b = -2.62700000
Iteration 1 : x = -3.39800000 || f(x) = 8.23661799 || f(x + e) = 8.23652003 || a = -3.65500000 || b = -3.14100000
Iteration 2 : x = -3.26950000 || f(x) = 8.15651987 || f(x + e) = 8.15649645 || a = -3.39800000 || b = -3.14100000
Iteration 3 : x = -3.20525000 || f(x) = 8.15512859 || f(x + e) = 8.15514817 || a = -3.26950000 || b = -3.14100000
Iteration 4 : x = -3.23737500 || f(x) = 8.15236726 || f(x + e) = 8.15236503 || a = -3.26950000 || b = -3.20525000
Iteration 5 : x = -3.22131250 || f(x) = 8.15287152 || f(x + e) = 8.15288013 || a = -3.23737500 || b = -3.20525000
Iteration 6 : x = -3.22934375 || f(x) = 8.15240165 || f(x + e) = 8.15240482 || a = -3.23737500 || b = -3.22131250
Iteration 7 : x = -3.23335938 || f(x) = 8.15233020 || f(x + e) = 8.15233067 || a = -3.23737500 || b = -3.22934375
Iteration 8 : x = -3.23536719 || f(x) = 8.15233519 || f(x + e) = 8.15233431 || a = -3.23737500 || b = -3.23335938
Iteration 9 : x = -3.23436328 || f(x) = 8.15232931 || f(x + e) = 8.15232910 || a = -3.23536719 || b = -3.23335938
Iteration 10: x = -3.23386133 || f(x) = 8.15232891 || f(x + e) = 8.15232903 || a = -3.23436328 || b = -3.23335938
Iteration 11: x = -3.23411230 || f(x) = 8.15232890 || f(x + e) = 8.15232885 || a = -3.23436328 || b = -3.23386133
Iteration 12: x = -3.23398682 || f(x) = 8.15232885 || f(x + e) = 8.15232889 || a = -3.23411230 || b = -3.23386133
Iteration 13: x = -3.23404956 || f(x) = 8.15232886 || f(x + e) = 8.15232886 || a = -3.23411230 || b = -3.23398682
BisectionSearch has finished.
```

```
Subinterval : (-1.562, -0.533)
Iteration 0 : x = -1.04750000 || f(x) = 9.07979970 || f(x + e) = 9.07982147 || a = -1.56200000 || b = -0.53300000
Iteration 1 : x = -1.30475000 || f(x) = 9.24386863 || f(x + e) = 9.24372822 || a = -1.56200000 || b = -1.04750000
Iteration 2 : x = -1.17612500 || f(x) = 9.10900968 || f(x + e) = 9.10894371 || a = -1.30475000 || b = -1.04750000
Iteration 3 : x = -1.11181250 || f(x) = 9.08025250 || f(x + e) = 9.08022953 || a = -1.17612500 || b = -1.04750000
Iteration 4 : x = -1.07965625 || f(x) = 9.07642752 || f(x + e) = 9.07642682 || a = -1.11181250 || b = -1.04750000
Iteration 5 : x = -1.06357813 || f(x) = 9.07721015 || f(x + e) = 9.07722066 || a = -1.07965625 || b = -1.04750000
Iteration 6 : x = -1.07161719 || f(x) = 9.07659328 || f(x + e) = 9.07659818 || a = -1.07965625 || b = -1.06357813
Iteration 7 : x = -1.07563672 || f(x) = 9.07645406 || f(x + e) = 9.07645616 || a = -1.07965625 || b = -1.07161719
Iteration 8 : x = -1.07764648 || f(x) = 9.07642672 || f(x + e) = 9.07642741 || a = -1.07965625 || b = -1.07563672
Iteration 9 : x = -1.07865137 || f(x) = 9.07642360 || f(x + e) = 9.07642360 || a = -1.07965625 || b = -1.07764648
Iteration 10: x = -1.07814893 || f(x) = 9.07642428 || f(x + e) = 9.07642462 || a = -1.07865137 || b = -1.07764648
Iteration 11: x = -1.07840015 || f(x) = 9.07642372 || f(x + e) = 9.07642389 || a = -1.07865137 || b = -1.07814893
Iteration 12: x = -1.07852576 || f(x) = 9.07642360 || f(x + e) = 9.07642369 || a = -1.07865137 || b = -1.07840015
Iteration 13: x = -1.07858856 || f(x) = 9.07642359 || f(x + e) = 9.07642363 || a = -1.07865137 || b = -1.07852576
BisectionSearch has finished.
```

```
Subinterval : (0.533, 1.562)
Iteration 0 : x = 1.04750000 || f(x) = 9.10074970 || f(x + e) = 9.10073001 || a = 0.53300000 || b = 1.56200000
Iteration 1 : x = 1.30475000 || f(x) = 9.26996363 || f(x + e) = 9.27010609 || a = 1.04750000 || b = 1.56200000
Iteration 2 : x = 1.17612500 || f(x) = 9.13253218 || f(x + e) = 9.13260022 || a = 1.04750000 || b = 1.30475000
Iteration 3 : x = 1.11181250 || f(x) = 9.10248875 || f(x + e) = 9.10251378 || a = 1.04750000 || b = 1.17612500
Iteration 4 : x = 1.07965625 || f(x) = 9.09802065 || f(x + e) = 9.09802342 || a = 1.04750000 || b = 1.11181250
Iteration 5 : x = 1.06357813 || f(x) = 9.09848171 || f(x + e) = 9.09847326 || a = 1.04750000 || b = 1.07965625
Iteration 6 : x = 1.07161719 || f(x) = 9.09802562 || f(x + e) = 9.09802279 || a = 1.06357813 || b = 1.07965625
Iteration 7 : x = 1.07563672 || f(x) = 9.09796680 || f(x + e) = 9.09796677 || a = 1.07161719 || b = 1.07965625
Iteration 8 : x = 1.07764648 || f(x) = 9.09797965 || f(x + e) = 9.09798102 || a = 1.07563672 || b = 1.07965625
Iteration 9 : x = 1.07664160 || f(x) = 9.09796970 || f(x + e) = 9.09797038 || a = 1.07563672 || b = 1.07764648
Iteration 10: x = 1.07613916 || f(x) = 9.09796737 || f(x + e) = 9.09796769 || a = 1.07563672 || b = 1.07664160
Iteration 11: x = 1.07588794 || f(x) = 9.09796686 || f(x + e) = 9.09796701 || a = 1.07563672 || b = 1.07613916
Iteration 12: x = 1.07576233 || f(x) = 9.09796677 || f(x + e) = 9.09796684 || a = 1.07563672 || b = 1.07588794
Iteration 13: x = 1.07569952 || f(x) = 9.09796677 || f(x + e) = 9.09796679 || a = 1.07563672 || b = 1.07576233
BisectionSearch has finished.
```

```
Subinterval : (2.627, 3.655)
Iteration 0 : x = 3.14100000 || f(x) = 8.24482316 || f(x + e) = 8.24476095 || a = 2.62700000 || b = 3.65500000
Iteration 1 : x = 3.39800000 || f(x) = 8.30457799 || f(x + e) = 8.30467799 || a = 3.14100000 || b = 3.65500000
Iteration 2 : x = 3.26950000 || f(x) = 8.22190987 || f(x + e) = 8.22193537 || a = 3.14100000 || b = 3.39800000
Iteration 3 : x = 3.20525000 || f(x) = 8.21923359 || f(x + e) = 8.21921607 || a = 3.14100000 || b = 3.26950000
Iteration 4 : x = 3.23737500 || f(x) = 8.21711476 || f(x + e) = 8.21711907 || a = 3.20525000 || b = 3.26950000
Iteration 5 : x = 3.22131250 || f(x) = 8.21729777 || f(x + e) = 8.21729123 || a = 3.20525000 || b = 3.23737500
Iteration 6 : x = 3.22934375 || f(x) = 8.21698853 || f(x + e) = 8.21698743 || a = 3.22131250 || b = 3.23737500
Iteration 7 : x = 3.23335938 || f(x) = 8.21699739 || f(x + e) = 8.21699900 || a = 3.22934375 || b = 3.23737500
Iteration 8 : x = 3.23135156 || f(x) = 8.21697937 || f(x + e) = 8.21697963 || a = 3.22934375 || b = 3.23335938
Iteration 9 : x = 3.23034766 || f(x) = 8.21698055 || f(x + e) = 8.21698013 || a = 3.22934375 || b = 3.23135156
Iteration 10: x = 3.23084961 || f(x) = 8.21697911 || f(x + e) = 8.21697903 || a = 3.23034766 || b = 3.23135156
Iteration 11: x = 3.23110059 || f(x) = 8.21697903 || f(x + e) = 8.21697912 || a = 3.23084961 || b = 3.23135156
Iteration 12: x = 3.23097510 || f(x) = 8.21697902 || f(x + e) = 8.21697902 || a = 3.23084961 || b = 3.23110059
Iteration 13: x = 3.23091235 || f(x) = 8.21697905 || f(x + e) = 8.21697901 || a = 3.23084961 || b = 3.23097510
BisectionSearch has finished.
```

Minimum value of the function in the interval [-4, 4] is achieved approximately at $x_0 = -3.234$ and $f(x_0) = 8.152$.

6) In order to solve the problem, Steepest Descent Algorithm and Bisection Search Method are implemented. Helper functions are written to calculate length of a vector, take derivative using limit definition, calculate gradient and find result of the given function. In order to use Bisection Method which was written for one-variable functions, x_1 is replaced by $x_1 + \alpha d^{(k)}$ and x_2 is replaced by $x_2 + \alpha d^{(k)}$ for fixed $x^{(k)}$ and $d^{(k)}$ values. Then main function becomes a one-variable function dependent on α. Optimum α value is found using Bisection Search Method. For the initial points, $x_1 = 5$ and $x_2 = 30$ are selected in order to minimize first term of the function. Find the iterations and values below.

```
Iteration 0 : x1 = 5.0000 || x2 = 25.0000 || d[0] = -7.0000 || d[1] =  2.0000  || a = 0.0188 || f(x1, x2) = -24.0000
Iteration 1 : x1 = 4.8681 || x2 = 25.0377 || d[0] = 0.0433  || d[1] =  0.6441  || a = 0.3216 || f(x1, x2) = -24.7450
Iteration 2 : x1 = 4.8820 || x2 = 25.2448 || d[0] = 4.8713  || d[1] =  -0.3271 || a = 0.0054 || f(x1, x2) = -24.8160
Iteration 3 : x1 = 4.9084 || x2 = 25.2430 || d[0] = 0.0755  || d[1] =  0.6216  || a = 0.8263 || f(x1, x2) = -24.8774
Iteration 4 : x1 = 4.9707 || x2 = 25.7567 || d[0] = 7.7826  || d[1] =  -0.9447 || a = 0.0048 || f(x1, x2) = -25.0525
Iteration 5 : x1 = 5.0082 || x2 = 25.7521 || d[0] = 0.1837  || d[1] =  0.5600  || a = 0.5273 || f(x1, x2) = -25.1911
Iteration 6 : x1 = 5.1050 || x2 = 26.0474 || d[0] = -4.3586 || d[1] =  1.4297  || a = 0.0083 || f(x1, x2) = -25.2743
Iteration 7 : x1 = 5.0688 || x2 = 26.0593 || d[0] = 0.1893  || d[1] =  0.5346  || a = 0.3504 || f(x1, x2) = -25.3705
Iteration 8 : x1 = 5.1351 || x2 = 26.2466 || d[0] = -3.5426 || d[1] =  1.2545  || a = 0.0077 || f(x1, x2) = -25.4230
Iteration 9 : x1 = 5.1078 || x2 = 26.2563 || d[0] = 0.1678  || d[1] =  0.5232  || a = 0.5685 || f(x1, x2) = -25.4816
Iteration 10: x1 = 5.2032 || x2 = 26.5537 || d[0] = -4.2957 || d[1] =  1.3779  || a = 0.0080 || f(x1, x2) = -25.5602
Iteration 11: x1 = 5.1688 || x2 = 26.5648 || d[0] = 0.1550  || d[1] =  0.5015  || a = 0.6803 || f(x1, x2) = -25.6496
Iteration 12: x1 = 5.2743 || x2 = 26.9060 || d[0] = -4.4913 || d[1] =  1.3885  || a = 0.0079 || f(x1, x2) = -25.7352
Iteration 13: x1 = 5.2390 || x2 = 26.9169 || d[0] = 0.0514  || d[1] =  0.4941  || a = 0.5778 || f(x1, x2) = -25.8321
Iteration 14: x1 = 5.2687 || x2 = 27.2024 || d[0] = 5.1367  || d[1] =  -0.5348 || a = 0.0051 || f(x1, x2) = -25.9074
Iteration 15: x1 = 5.2949 || x2 = 27.1996 || d[0] = 0.0290  || d[1] =  0.4762  || a = 0.2885 || f(x1, x2) = -25.9715
Iteration 16: x1 = 5.3033 || x2 = 27.3370 || d[0] = 3.4398  || d[1] =  -0.2093 || a = 0.0054 || f(x1, x2) = -26.0057
Iteration 17: x1 = 5.3219 || x2 = 27.3359 || d[0] = 0.0148  || d[1] =  0.4683  || a = 0.2003 || f(x1, x2) = -26.0364
Iteration 18: x1 = 5.3249 || x2 = 27.4296 || d[0] = 2.7898  || d[1] =  -0.0880 || a = 0.0056 || f(x1, x2) = -26.0592
Iteration 19: x1 = 5.3404 || x2 = 27.4292 || d[0] = 0.0036  || d[1] =  0.4631  || a = 0.1553 || f(x1, x2) = -26.0801
Iteration 20: x1 = 5.3410 || x2 = 27.5011 || d[0] = 2.4099  || d[1] =  -0.0184 || a = 0.0056 || f(x1, x2) = -26.0972
Iteration 21: x1 = 5.3544 || x2 = 27.5010 || d[0] = 0.0367  || d[1] =  0.4508  || a = 0.3881 || f(x1, x2) = -26.1131
Iteration 22: x1 = 5.3686 || x2 = 27.6759 || d[0] = 3.8118  || d[1] =  -0.3099 || a = 0.0053 || f(x1, x2) = -26.1546
Iteration 23: x1 = 5.3887 || x2 = 27.6743 || d[0] = 0.0276  || d[1] =  0.4390  || a = 0.2949 || f(x1, x2) = -26.1914
Iteration 24: x1 = 5.3968 || x2 = 27.8037 || d[0] = 3.2162  || d[1] =  -0.2020 || a = 0.0053 || f(x1, x2) = -26.2210
Iteration 25: x1 = 5.4138 || x2 = 27.8027 || d[0] = 0.0765  || d[1] =  0.4191  || a = 7.5690 || f(x1, x2) = -26.2478
Iteration 26: x1 = 5.9930 || x2 = 30.9752 || d[0] = 11.6376 || d[1] =  -2.1247 || a = 0.0039 || f(x1, x2) = -26.9717
Iteration 27: x1 = 6.0383 || x2 = 30.9670 || d[0] = 0.2552  || d[1] =  0.1337  || a = 0.0088 || f(x1, x2) = -27.2261
Iteration 28: x1 = 6.0405 || x2 = 30.9681 || d[0] = -0.1064 || d[1] =  0.2051  || a = 0.0137 || f(x1, x2) = -27.2265
Iteration 29: x1 = 6.0391 || x2 = 30.9709 || d[0] = 0.2552  || d[1] =  0.1334  || a = 0.0088 || f(x1, x2) = -27.2269
Iteration 30: x1 = 6.0413 || x2 = 30.9721 || d[0] = -0.1065 || d[1] =  0.2048  || a = 0.0137 || f(x1, x2) = -27.2272
Iteration 31: x1 = 6.0398 || x2 = 30.9749 || d[0] = 0.2552  || d[1] =  0.1330  || a = 0.0088 || f(x1, x2) = -27.2276
Iteration 32: x1 = 6.0421 || x2 = 30.9761 || d[0] = -0.1067 || d[1] =  0.2045  || a = 0.0135 || f(x1, x2) = -27.2280
Iteration 33: x1 = 6.0406 || x2 = 30.9788 || d[0] = 0.2514  || d[1] =  0.1335  || a = 0.0088 || f(x1, x2) = -27.2283
Iteration 34: x1 = 6.0428 || x2 = 30.9800 || d[0] = -0.1043 || d[1] =  0.2037  || a = 0.0138 || f(x1, x2) = -27.2287
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Iteration 34: x1 = 6.0428 || x2 = 30.9800 || d[0] = -0.1043 || d[1] =  0.2037  || a = 0.0138 || f(x1, x2) = -27.2287
Iteration 35: x1 = 6.0414 || x2 = 30.9828 || d[0] = 0.2555  || d[1] =  0.1323  || a = 0.0086 || f(x1, x2) = -27.2290
Iteration 36: x1 = 6.0436 || x2 = 30.9840 || d[0] = -0.1008 || d[1] =  0.2027  || a = 0.0143 || f(x1, x2) = -27.2294
Iteration 37: x1 = 6.0422 || x2 = 30.9869 || d[0] = 0.2617  || d[1] =  0.1308  || a = 0.0085 || f(x1, x2) = -27.2298
Iteration 38: x1 = 6.0444 || x2 = 30.9880 || d[0] = -0.0986 || d[1] =  0.2019  || a = 0.0146 || f(x1, x2) = -27.2301
Iteration 39: x1 = 6.0429 || x2 = 30.9909 || d[0] = 0.2657  || d[1] =  0.1296  || a = 0.0083 || f(x1, x2) = -27.2305
Iteration 40: x1 = 6.0452 || x2 = 30.9920 || d[0] = -0.0944 || d[1] =  0.2009  || a = 0.0152 || f(x1, x2) = -27.2309
Iteration 41: x1 = 6.0437 || x2 = 30.9950 || d[0] = 0.2729  || d[1] =  0.1279  || a = 0.0082 || f(x1, x2) = -27.2312
Iteration 42: x1 = 6.0460 || x2 = 30.9961 || d[0] = -0.0918 || d[1] =  0.2000  || a = 0.0155 || f(x1, x2) = -27.2316
Iteration 43: x1 = 6.0445 || x2 = 30.9992 || d[0] = 0.2754  || d[1] =  0.1272  || a = 0.0082 || f(x1, x2) = -27.2320
Iteration 44: x1 = 6.0468 || x2 = 31.0002 || d[0] = -0.0934 || d[1] =  0.1999  || a = 0.0152 || f(x1, x2) = -27.2324
Iteration 45: x1 = 6.0454 || x2 = 31.0032 || d[0] = 0.2709  || d[1] =  0.1277  || a = 0.0082 || f(x1, x2) = -27.2327
Iteration 46: x1 = 6.0476 || x2 = 31.0043 || d[0] = -0.0912 || d[1] =  0.1992  || a = 0.0155 || f(x1, x2) = -27.2331
Iteration 47: x1 = 6.0462 || x2 = 31.0074 || d[0] = 0.2740  || d[1] =  0.1268  || a = 0.0082 || f(x1, x2) = -27.2335
Iteration 48: x1 = 6.0484 || x2 = 31.0084 || d[0] = -0.0929 || d[1] =  0.1992  || a = 0.0152 || f(x1, x2) = -27.2339
Iteration 49: x1 = 6.0470 || x2 = 31.0114 || d[0] = 0.2698  || d[1] =  0.1273  || a = 0.0082 || f(x1, x2) = -27.2342
Iteration 50: x1 = 6.0492 || x2 = 31.0125 || d[0] = -0.0907 || d[1] =  0.1985  || a = 0.0155 || f(x1, x2) = -27.2346
Iteration 51: x1 = 6.0478 || x2 = 31.0155 || d[0] = 0.2728  || d[1] =  0.1263  || a = 0.0082 || f(x1, x2) = -27.2350
Iteration 52: x1 = 6.0500 || x2 = 31.0166 || d[0] = -0.0925 || d[1] =  0.1985  || a = 0.0152 || f(x1, x2) = -27.2353
Iteration 53: x1 = 6.0486 || x2 = 31.0196 || d[0] = 0.2687  || d[1] =  0.1268  || a = 0.0082 || f(x1, x2) = -27.2357
Iteration 54: x1 = 6.0508 || x2 = 31.0206 || d[0] = -0.0904 || d[1] =  0.1977  || a = 0.0155 || f(x1, x2) = -27.2361
Iteration 55: x1 = 6.0494 || x2 = 31.0237 || d[0] = 0.2719  || d[1] =  0.1259  || a = 0.0082 || f(x1, x2) = -27.2364
Iteration 56: x1 = 6.0516 || x2 = 31.0247 || d[0] = -0.0924 || d[1] =  0.1978  || a = 0.0152 || f(x1, x2) = -27.2368
Iteration 57: x1 = 6.0502 || x2 = 31.0277 || d[0] = 0.2683  || d[1] =  0.1263  || a = 0.0082 || f(x1, x2) = -27.2372
Iteration 58: x1 = 6.0524 || x2 = 31.0288 || d[0] = -0.0905 || d[1] =  0.1971  || a = 0.0155 || f(x1, x2) = -27.2375
Iteration 59: x1 = 6.0510 || x2 = 31.0318 || d[0] = 0.2719  || d[1] =  0.1253  || a = 0.0080 || f(x1, x2) = -27.2379
Iteration 60: x1 = 6.0532 || x2 = 31.0328 || d[0] = -0.0858 || d[1] =  0.1959  || a = 0.0163 || f(x1, x2) = -27.2382
Iteration 61: x1 = 6.0518 || x2 = 31.0360 || d[0] = 0.2801  || d[1] =  0.1232  || a = 0.0079 || f(x1, x2) = -27.2386
Iteration 62: x1 = 6.0540 || x2 = 31.0370 || d[0] = -0.0832 || d[1] =  0.1950  || a = 0.0167 || f(x1, x2) = -27.2390
Iteration 63: x1 = 6.0526 || x2 = 31.0402 || d[0] = 0.2848  || d[1] =  0.1219  || a = 0.0077 || f(x1, x2) = -27.2394
Iteration 64: x1 = 6.0548 || x2 = 31.0412 || d[0] = -0.0787 || d[1] =  0.1938  || a = 0.0176 || f(x1, x2) = -27.2397
Iteration 65: x1 = 6.0534 || x2 = 31.0446 || d[0] = 0.2945  || d[1] =  0.1197  || a = 0.0076 || f(x1, x2) = -27.2401
Iteration 66: x1 = 6.0556 || x2 = 31.0455 || d[0] = -0.0757 || d[1] =  0.1929  || a = 0.0182 || f(x1, x2) = -27.2405
Iteration 67: x1 = 6.0543 || x2 = 31.0490 || d[0] = 0.3003  || d[1] =  0.1182  || a = 0.0074 || f(x1, x2) = -27.2409
Iteration 68: x1 = 6.0565 || x2 = 31.0499 || d[0] = -0.0708 || d[1] =  0.1916  || a = 0.0195 || f(x1, x2) = -27.2413
Iteration 69: x1 = 6.0551 || x2 = 31.0536 || d[0] = 0.3121  || d[1] =  0.1155  || a = 0.0072 || f(x1, x2) = -27.2417
Iteration 70: x1 = 6.0574 || x2 = 31.0544 || d[0] = -0.0678 || d[1] =  0.1906  || a = 0.0202 || f(x1, x2) = -27.2421
Iteration 71: x1 = 6.0560 || x2 = 31.0583 || d[0] = 0.3188  || d[1] =  0.1139  || a = 0.0071 || f(x1, x2) = -27.2425
Iteration 72: x1 = 6.0583 || x2 = 31.0591 || d[0] = -0.0623 || d[1] =  0.1892  || a = 0.0219 || f(x1, x2) = -27.2429
Iteration 73: x1 = 6.0569 || x2 = 31.0632 || d[0] = 0.3337  || d[1] =  0.1104  || a = 0.0069 || f(x1, x2) = -27.2434
Iteration 74: x1 = 6.0592 || x2 = 31.0640 || d[0] = -0.0590 || d[1] =  0.1881  || a = 0.0230 || f(x1, x2) = -27.2438
Iteration 75: x1 = 6.0578 || x2 = 31.0683 || d[0] = 0.3422  || d[1] =  0.1084  || a = 0.0068 || f(x1, x2) = -27.2443
Iteration 76: x1 = 6.0602 || x2 = 31.0691 || d[0] = -0.0533 || d[1] =  0.1867  || a = 0.0254 || f(x1, x2) = -27.2447
Iteration 77: x1 = 6.0588 || x2 = 31.0738 || d[0] = 0.3632  || d[1] =  0.1039  || a = 0.0066 || f(x1, x2) = -27.2452
Iteration 78: x1 = 6.0612 || x2 = 31.0745 || d[0] = -0.0502 || d[1] =  0.1856  || a = 0.0268 || f(x1, x2) = -27.2457
Iteration 79: x1 = 6.0599 || x2 = 31.0795 || d[0] = 0.3730  || d[1] =  0.1014  || a = 0.0065 || f(x1, x2) = -27.2462
Iteration 80: x1 = 6.0623 || x2 = 31.0801 || d[0] = -0.0433 || d[1] =  0.1837  || a = 0.0307 || f(x1, x2) = -27.2466
Iteration 81: x1 = 6.0610 || x2 = 31.0858 || d[0] = 0.4025  || d[1] =  0.0951  || a = 0.0063 || f(x1, x2) = -27.2472
Iteration 82: x1 = 6.0635 || x2 = 31.0864 || d[0] = -0.0400 || d[1] =  0.1826  || a = 0.0329 || f(x1, x2) = -27.2477
Iteration 83: x1 = 6.0622 || x2 = 31.0924 || d[0] = 0.4161  || d[1] =  0.0919  || a = 0.0062 || f(x1, x2) = -27.2483
Iteration 84: x1 = 6.0648 || x2 = 31.0929 || d[0] = -0.0321 || d[1] =  0.1805  || a = 0.0399 || f(x1, x2) = -27.2489
Iteration 85: x1 = 6.0635 || x2 = 31.1001 || d[0] = 0.4620  || d[1] =  0.0822  || a = 0.0060 || f(x1, x2) = -27.2496
Iteration 86: x1 = 6.0663 || x2 = 31.1006 || d[0] = -0.0287 || d[1] =  0.1793  || a = 0.0436 || f(x1, x2) = -27.2502
Iteration 87: x1 = 6.0650 || x2 = 31.1084 || d[0] = 0.4822  || d[1] =  0.0775  || a = 0.0059 || f(x1, x2) = -27.2509
Iteration 88: x1 = 6.0679 || x2 = 31.1089 || d[0] = -0.0193 || d[1] =  0.1767  || a = 0.0584 || f(x1, x2) = -27.2517
Iteration 89: x1 = 6.0667 || x2 = 31.1192 || d[0] = 0.5610  || d[1] =  0.0611  || a = 0.0057 || f(x1, x2) = -27.2526
Iteration 90: x1 = 6.0699 || x2 = 31.1196 || d[0] = -0.0152 || d[1] =  0.1751  || a = 0.0672 || f(x1, x2) = -27.2535
Iteration 91: x1 = 6.0689 || x2 = 31.1313 || d[0] = 0.6003  || d[1] =  0.0524  || a = 0.0056 || f(x1, x2) = -27.2545
Iteration 92: x1 = 6.0723 || x2 = 31.1316 || d[0] = -0.0037 || d[1] =  0.1719  || a = 0.1113 || f(x1, x2) = -27.2556
Iteration 93: x1 = 6.0719 || x2 = 31.1508 || d[0] = 0.7724  || d[1] =  0.0168  || a = 0.0054 || f(x1, x2) = -27.2572
Iteration 94: x1 = 6.0760 || x2 = 31.1509 || d[0] = 0.0015  || d[1] =  0.1693  || a = 0.1488 || f(x1, x2) = -27.2588
Iteration 95: x1 = 6.0763 || x2 = 31.1760 || d[0] = 0.8864  || d[1] =  -0.0078 || a = 0.0053 || f(x1, x2) = -27.2610
Iteration 96: x1 = 6.0809 || x2 = 31.1760 || d[0] = 0.0171  || d[1] =  0.1641  || a = 0.5767 || f(x1, x2) = -27.2631
Iteration 97: x1 = 6.0908 || x2 = 31.2707 || d[0] = 1.7106  || d[1] =  -0.1785 || a = 0.0050 || f(x1, x2) = -27.2711
Iteration 98: x1 = 6.0993 || x2 = 31.2698 || d[0] = 0.0513  || d[1] =  0.1500  || a = 0.3016 || f(x1, x2) = -27.2784
Iteration 99: x1 = 6.1148 || x2 = 31.3150 || d[0] = -1.0851 || d[1] =  0.3711  || a = 0.0056 || f(x1, x2) = -27.2822
Iteration 100: x1 = 6.1087 || x2 = 31.3171 || d[0] = 0.0382  || d[1] =  0.1488  || a = 1.6297 || f(x1, x2) = -27.2859
Iteration 101: x1 = 6.1709 || x2 = 31.5596 || d[0] = -2.3319 || d[1] =  0.5980  || a = 0.0057 || f(x1, x2) = -27.3047
Iteration 102: x1 = 6.1576 || x2 = 31.5630 || d[0] = 0.0017  || d[1] =  0.1366  || a = 0.1539 || f(x1, x2) = -27.3220
Iteration 103: x1 = 6.1578 || x2 = 31.5840 || d[0] = 0.7299  || d[1] =  -0.0091 || a = 0.0053 || f(x1, x2) = -27.3235
Iteration 104: x1 = 6.1617 || x2 = 31.5840 || d[0] = 0.0095  || d[1] =  0.1334  || a = 0.3239 || f(x1, x2) = -27.3249
Iteration 105: x1 = 6.1647 || x2 = 31.6271 || d[0] = 1.0424  || d[1] =  -0.0743 || a = 0.0051 || f(x1, x2) = -27.3278
Iteration 106: x1 = 6.1701 || x2 = 31.6268 || d[0] = 0.0200  || d[1] =  0.1280  || a = 2.4265 || f(x1, x2) = -27.3306
Iteration 107: x1 = 6.2187 || x2 = 31.9375 || d[0] = 2.5862  || d[1] =  -0.4047 || a = 0.0048 || f(x1, x2) = -27.3515
Iteration 108: x1 = 6.2311 || x2 = 31.9355 || d[0] = 0.0246  || d[1] =  0.1026  || a = 2.7767 || f(x1, x2) = -27.3675
Iteration 109: x1 = 6.2994 || x2 = 32.2204 || d[0] = -2.0257 || d[1] =  0.4854  || a = 0.0056 || f(x1, x2) = -27.3827
Iteration 110: x1 = 6.2881 || x2 = 32.2231 || d[0] = 0.0089  || d[1] =  0.0829  || a = 0.6110 || f(x1, x2) = -27.3952
Iteration 111: x1 = 6.2935 || x2 = 32.2738 || d[0] = 0.8890  || d[1] =  -0.0953 || a = 0.0050 || f(x1, x2) = -27.3973
Iteration 112: x1 = 6.2979 || x2 = 32.2733 || d[0] = 0.0267  || d[1] =  0.0755  || a = 0.2477 || f(x1, x2) = -27.3993
Iteration 113: x1 = 6.3046 || x2 = 32.2920 || d[0] = -0.5073 || d[1] =  0.1796  || a = 0.0053 || f(x1, x2) = -27.4001
Iteration 114: x1 = 6.3019 || x2 = 32.2929 || d[0] = 0.0151  || d[1] =  0.0763  || a = 13.1091 || f(x1, x2) = -27.4009
Iteration 115: x1 = 6.5003 || x2 = 33.2933 || d[0] = -0.0698 || d[1] =  0.0138  || a = 0.0050 || f(x1, x2) = -27.4405
Iteration 116: x1 = 6.4999 || x2 = 33.2934 || d[0] = -0.0013 || d[1] =  0.0002  || a = 0.0043 || f(x1, x2) = -27.4406
Optimal solution -->  x1: 6.4999, x2: 33.2934 || f(x): -27.4406
```

Minimum value of the function is achieved approximately at $x_1 = 6.4999$, $x_2 = 33.2934$, and $f(x) = -27.4406$.