IE 310 - Homework I

January 21, 2022

1 Answers

1) First we should define the variables. We can denote the variables as x_1, x_2, x_3, x_4, x_5 representing the units of meals, BoBurger, XL BoBurger, Chicken Sandwich, Salad with chicken and French Fries respectively.

Our objective is to minimize the cost of the meal.

$$
\begin{aligned}
\text{Variables: } & x_1, x_2, x_3, x_4, x_5 \ (units) \\
\text{Cost function: } & f(x_1, x_2, x_3, x_4, x_5) = x_1 + 3x_2 + 2.5x_3 + 3x_4 + x_5 \\
\text{Constraints: } & \\
\text{Sodium: } & 480x_1 + 1170x_2 + 800x_3 + 580x_4 + 160x_5 \leq 1100 \\
\text{Protein: } & 30x_1 + 45x_2 + 15x_3 + 25x_4 + 3x_5 \geq 30 \\
\text{Calories: } & 250x_1 + 770x_2 + 360x_3 + 190x_4 + 230x_5 \geq 600 \\
& 250x_1 + 770x_2 + 360x_3 + 190x_4 + 230x_5 \leq 900 \\
\text{Fat calories: } & 80x_1 + 360x_2 + 145x_3 + 45x_4 + 100x_5 \\
& \leq 0.4 \cdot (250x_1 + 770x_2 + 360x_3 + 190x_4 + 230x_5) \\
\text{Further : } & x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Reformat the fat calories constraint.

$$
\begin{aligned}
80x_1 + 360x_2 + 145x_3 + 45x_4 + 100x_5 &\leq 100x_1 + 308x_2 + 144x_3 + 76x_4 + 92x_5 \\
-20x_1 + 52x_2 + x_3 - 31x_4 + 8x_5 &\leq 0
\end{aligned}
$$

2) Strict inequalities are not allowed in LP models. Generally in LP questions, we are trying to find a minimum or maximum value for our objective function on a **closed region**. If we introduce strict inequalities to the constraints, then feasible set of solution may become no longer closed. In this case we can't find an optimal solution since for each solution there is a better solution in the feasible region, which can be found by coming closer to the boundary created by the strict inequality (dashed line).

3) Since x - $y \geq 1$ we know that x - y is positive. Therefore we can multiply the denominator so that the constraint becomes linear.

$$(x-y) \cdot \frac{x+y}{x-y} \geq (x-y) \cdot 1 \Longrightarrow x+y \geq x-y$$

$$x+y \geq x-y$$

$$y \geq -y$$

$$y \geq 0$$

Linear constraints: $y \geq 0, \quad x-y \geq 1$

4) To derive the equivalent LP transformation, we should define new variables.

$$\min z = |x_1 - 2| + |x_2| + |x_1| + |x_2 + 3|$$

First, notice that we can do the translation below.

$$\begin{aligned} |x_1 - 2| &= max\{x_1 - 2, -x_1 + 2\} \\ |x_2| &= max\{x_2, -x_2\} \\ |x_1| &= max\{x_1, -x_1\} \\ |x_2 + 3| &= max\{x_2 + 3, -x_2 - 3\} \end{aligned}$$

Substitute the variables.

$$\begin{aligned} &max\{x_1 - 2, -x_1 + 2\} = u_1 \\ &max\{x_2, -x_2\} = u_2 \\ &max\{x_1, -x_1\} = u_3 \\ &max\{x_2 + 3, -x_2 - 3\} = u_4 \end{aligned}$$

The problem has now been translated to a linear programming problem that can be solved normally.

$$\begin{aligned} \min\ & z = u_1 + u_2 + u_3 + u_4 \\ \text{s.t.}\ & u_1 \geq x_1 - 2 \\ & u_1 \geq -x_1 + 2 \\ & u_2 \geq x_2 \\ & u_2 \geq -x_2 \\ & u_3 \geq x_1 \\ & u_3 \geq -x_1 \\ & u_4 \geq x_2 + 3 \\ & u_4 \geq -x_2 - 3 \\ & u_1, u_2, u_3, u_4 \geq 0 \end{aligned}$$

5) Result of part (A) should be:

$$x_1 = 23,\ x_2 = 15,\ x_3 = 9,\ x_4 = 0,\ \text{total cost} = \$33225$$

Result of part (B) should be:

$$x_1 = 19,\ x_2 = 18,\ x_3 = 9,\ x_4 = 0,\ x_5 = 10,\ \text{total cost} = \$32400$$

Description of the Solution

I've implemented five main functions in my solution. First two functions are used to solve part (A) and part (B) by brute force. Since decision variables are integers, therefore discrete, we can iterate through the valid range $0 \le x_i \le 30$ for all $i \in \{1, 2, 3, 4\}$. Part (A) takes approximately 1 second to complete, however second part takes much longer than the former. Brute force is enough to find a solution but our algorithm can be improved further.

To optimize the solution, I've designed another function which is a general function to solve LP models by checking all the basic solutions. The main logic behind the algorithm actually comes from the Optimality Criterion:

Theorem 1 (Optimality Criterion)**.** *For the linear programming problem (*) if $c_j \ge 0, j = m+1, ..., n$ then the minimal value of the objective function is $-z_0$ and is attained at the point $(b_1, b_2, ..., b_m, 0, ..., 0)$.*

I created a matrix representing the objective function and constraints. Then I traversed all the basic solutions by changing basic and non-basic variables. I've implemented helper functions like `transpose()` and `Gauss_Jordan()` to reach the solution. You can test the `solve()` with different parameters as well. For example, you can try solving the next question. Corresponding parameters can be found in one of the comment blocks.

Function `solve()` can be used to solve any LP problem for real-valued decision variables. However this specific question states that our variables are integer, therefore we have to modify the solution. If variables were real, solution would be:

$$x_1 = 19.54,\ x_2 = 17.25,\ x_3 = 9.13,\ x_4 = 0,\ \text{total cost} = \$32208.47$$

My two remaining functions utilize the real-valued solutions and try to find integer-valued solutions near the same point. Therefore I find a general solution using `solve()` function, and then call the `integer_solution()` function for part (A) and part (B) separately. For each decision variable x_i, I check all the integers that satisfy $x_i - range <= a <= x_i + range$. Value of the *range* is question-dependent. Questions with the higher numbers will require a higher *range* value. However, it's a risky solution since there is a possibility that integer-valued solution does not lie within the *range* of integers I check. We can try increasing the *range* as far as possible but then it becomes like the brute force solution.

Overall, different techniques I've used was successful at solving the problem. However, for integer programming questions there are better solution techniques like Cutting Plane Method or The Branch and Bound Method.

Comments About the Results

Introduction of the overtime variable x_5 caused the solution to change. However, it's important to notice that solution wouldn't change if decision variables were real-valued. Why did the solution change? Well, because for the integer-valued decision variables, first constraint had an important impact. When we introduced the variable x_5, we can come up with a (x_1, x_2, x_3, x_4) combination whose sums $(20x_1 + 30x_2 + 10x_3 + 25x_4)$ is bigger than 1000, yet minimizes the function. We could minimize our function by decreasing the variables with higher coefficients, and increasing the ones with lower. It turns out that in our solution we can decrease x_1 (multiplied by 690 in objective) and increase x_2 (multiplied by 545 in objective) to minimize our function further. We were not able to do that in part (A) since there was no x_5 in the right hand side of the first constraint.

6)

$$\begin{aligned} -x_1 + 2x_2 \leq 15 &\Longrightarrow R_1 = (0, \frac{15}{2}),\ R_2 = (-15, 0) \\ x_1 + x_2 \leq 12 &\Longrightarrow Q_1 = (0, 12),\ Q_2 = (12, 0) \\ 5x_1 + 3x_2 \leq 45 &\Longrightarrow P_1 = (0, 15),\ P_2 = (9, 0) \end{aligned}$$

Then find the intersection points between the lines.

$$I_1 = (3, 9) \begin{cases} -x_1 + 2x_2 \leq 15 \\ x_1 + x_2 \leq 12 \end{cases}$$

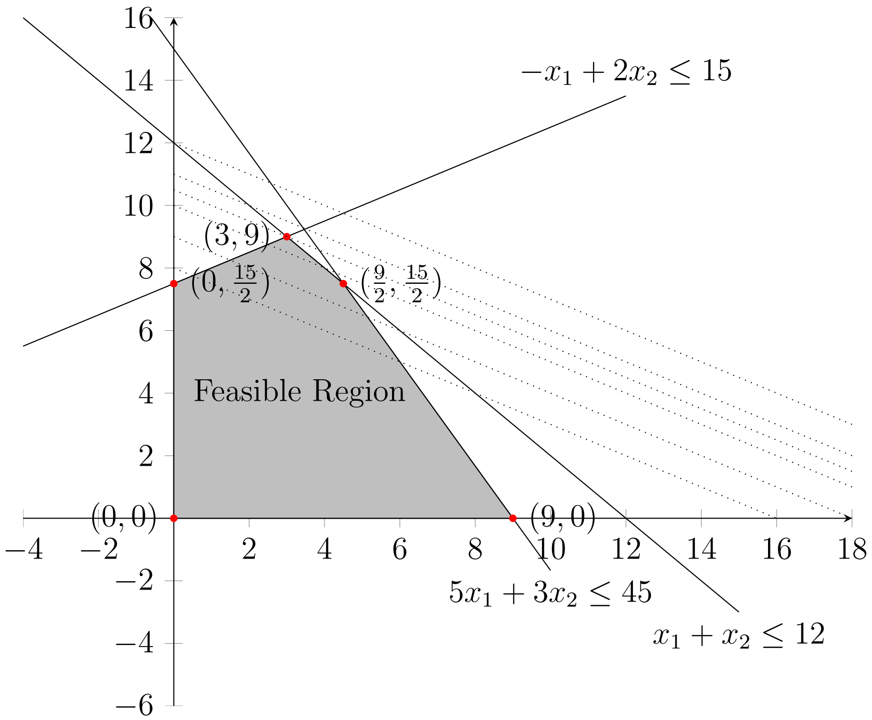
$$I_2 = (\frac{9}{2}, \frac{15}{2}) \begin{cases} 5x_1 + 3x_2 \leq 45 \\ x_1 + x_2 \leq 12 \end{cases}$$

$$I_3 = (\frac{45}{13}, \frac{120}{13}) \begin{cases} -x_1 + 2x_2 \leq 15 \\ 5x_1 + 3x_2 \leq 45 \end{cases}$$

Notice that I_3 won't be in feasible region.

Now, plot the graph, mark the intersection points and draw the contour lines corresponding to $10x_1 + 20x_2 = c$ for different c values.

As seen from the graph, $(3, 9)$ is the **optimal solution**. In addition, we can solve



the problem by checking each corner points to maximize z.

$$\begin{aligned}
\max z &= 10x_1 + 20x_2 \\
(0,0) &\implies z = 0 \\
(0, \frac{15}{2}) &\implies z = 150 \\
(3,9) &\implies z = 210 \\
(\frac{9}{2}, \frac{15}{2}) &\implies z = 195 \\
(9,0) &\implies z = 90
\end{aligned}$$

Answer: $(3, 9)$ is the optimal point to maximize z.

$$x_1 = 3, \quad x_2 = 9, \quad z = 210$$

7) Let's start with defining the decision variables:

$$\begin{aligned}
A &= \$ \text{ amount to invest in bond/security A} \\
B &= \$ \text{ amount to invest in bond/security B} \\
C &= \$ \text{ amount to invest in bond/security C} \\
D &= \$ \text{ amount to invest in bond/security D} \\
E &= \$ \text{ amount to invest in bond/security E}
\end{aligned}$$

Objective function: Maximize the profit after tax earnings given that the tax rate is 50 percent. We should maximize $f(A,B,C,D,E)$ function given below.

$$f(A,B,C,D,E) = (\frac{4.3}{100})A + (\frac{1}{2})\cdot(\frac{5.4}{100})B + (\frac{1}{2})\cdot(\frac{5.0}{100})C + (\frac{1}{2})\cdot(\frac{4.4}{100})D + (\frac{4.5}{100})E$$
$$f(A,B,C,D,E) = 0.043A + 0.027B + 0.025C + 0.022D + 0.045E$$

Constraints:

(a) The bank has defined him a target so that he should be investing \$10 million.

(b) The portfolio must total at least \$4 million in government and agency bonds.

(c) The average rating of a portfolio does not exceed 1.4 on the bank's quality scale.

(d) The average years to maturity of the portfolio does not exceed 5 year.

(e) Amount of money to invest cannot be negative.

(a.) $A + B + C + D + E \leq 10\,000\,000$

(b.) $B + C + D \geq 4\,000\,000$

(c.) $\dfrac{2A + 2B + C + D + 5E}{A + B + C + D + E} \leq 1.4$

$2A + 2B + C + D + 5E \leq 1.4A + 1.4B + 1.4C + 1.4D + 1.4E$

$0.6A + 0.6B - 0.4C - 0.4D + 3.6E \leq 0$

(d.) $\dfrac{9A + 15B + 4C + 3D + 2E}{A + B + C + D + E} \leq 5$

$9A + 15B + 4C + 3D + 2E \leq 5A + 5B + 5C + 5D + 5E$

$4A + 10B - C - 2D - 3E \leq 0$

(e.) $A, B, C, D, E \geq 0$

For the second problem, we should introduce a new variable called F, that represents amount of borrowed money. We should subtract 50 percent tax from the money.

$$F = \$ \text{ amount of borrowed money}$$

After the taxes, ratio we should pay becomes $0.055 \cdot 0.5 = 0.0275$.
Maximize the objective function $f(A,B,C,D,E,F)$ given below.

$$f(A,B,C,D,E,F) = 0.043A + 0.027B + 0.025C + 0.022D + 0.045E - 0.0275F$$

We should modify the first and last constraints, also add a new constraint:

(a.) $A + B + C + D + E \leq 10\,000\,000 + F$

(Total money to invest should be less than or equal to 10 million plus borrowed money)

(b.) $B + C + D \geq 4\,000\,000$

(c.) $0.6A + 0.6B - 0.4C - 0.4D + 3.6E \leq 0$

(d.) $4A + 10B - C - 2D - 3E \leq 0$

(e.) $F \leq 1\,000\,000$ (up to 1 million)

(f.) $A, B, C, D, E, F \geq 0$