Discrete Mathematics

Logic

Contents

1 Propositional Logic

Propositions

A **proposition** is a declarative sentence that is either **true** or **false**, but not both.

Example

All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1+1 = 2$.
4. $2+2 = 3$.

Example

Consider the following sentences.

1. What time is it?
2. Read this carefully!
3. $x + 1 = 2$.

1 Propositional Logic

Propositional Variables

We use letters to denote propositional variables. The conventional letters used for propositional variables are $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$,…

Truth Value

The **truth value** of a proposition is true, denoted by **T**, if it is a true proposition, and the truth value of a proposition is false, denoted by **F**, if it is a false proposition.

Compound Propositions

Compound propositions are formed from existing propositions using logical operators.

1 Compound Propositions – The Negation

DEFINITION 1

Let p be a proposition. The ***negation*** of p, denoted by $\neg p$, is the statement

"It is not the case that p."

The proposition $\neg p$ is read "**not** p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example

Find the negation of the proposition

"Michael's PC runs Linux"

Example

Find the negation of the proposition

"Vandana's smartphone has at least 32GB of memory"

1 Truth Table

TABLE 1 **The Truth Table for the Negation of a Proposition.**

p	$\neg p$
T	F
F	T

Table 1 displays the **truth table** for the negation of a proposition p.

This table has a row for each of the two possible truth values of a proposition p.

Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

The negation of a proposition can also be considered the result of the operation of the **negation operator** on a proposition.

1 Compound Propositions – The Conjunction

DEFINITION 2

Let p and q be propositions.

The ***conjunction*** of p and q, denoted by $p \wedge q$, is the proposition "p and q."

The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2 displays the truth table of $p \wedge q$.

This table has a row for each of the four possible combinations of truth values of p and q.

The four rows correspond to the pairs of truth values **TT**, **TF**, **FT**, and **FF**.

1 Compound Propositions – The Disjunction

DEFINITION 3

Let p and q be propositions.

The ***disjunction*** of p and q, denoted by $p \vee q$, is the proposition "p or q."

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the connective ***or*** in a disjunction corresponds to one of the two ways the word or is used in English, namely, as an **inclusive or**.

A disjunction is true when at least one of the two propositions is true.

1 Compound Propositions – The Exclusive Or

DEFINITION 4

Let p and q be propositions.

The ***exclusive or*** of p and q, denoted by $p \oplus q$, is the proposition "p xor q" that is true when exactly one of p and q are true and is false otherwise.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1 Compound Propositions – Condition Statements

DEFINITION 5

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be propositions.

The *conditional statement* $\boldsymbol{p \to q}$ is the proposition "if $\boldsymbol{p}$, then $\boldsymbol{q}$".

The *conditional statement* $\boldsymbol{p \to q}$ is false when $\boldsymbol{p}$ is true and $\boldsymbol{q}$ is false, and true otherwise.

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth table for the conditional statement $\boldsymbol{p \to q}$ is shown in Table 5.

Note that the statement $\boldsymbol{p \to q}$ is true when both $\boldsymbol{p}$ and $\boldsymbol{q}$ are true and when $\boldsymbol{p}$ is false (no matter what truth value $\boldsymbol{q}$ has).

1 Compound Propositions – Biconditionals

DEFINITION 6

Let p and q be propositions.

The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q".

The *biconditional statement* $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

$\leftrightarrow$

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The truth table for $p \leftrightarrow q$ is shown in Table 6.

Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise.

1 Truth Tables of Compound Propositions

Example

Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$

TABLE 7 **The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.**

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

1 Precedence of Logical Operators

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\land$	2
$\lor$	3
$\rightarrow$	4
$\leftrightarrow$	5