Discrete Mathematics

Relations

Relations and Their Properties

Contents

1 Relations and Their Properties

DEFINITION 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

We use the notation $a\ R\ b$ to denote that $(a,b) \in R$. Moreover, when (a,b) belongs to R, a is said to be **related to** b by R.

1 Relations and Their Properties

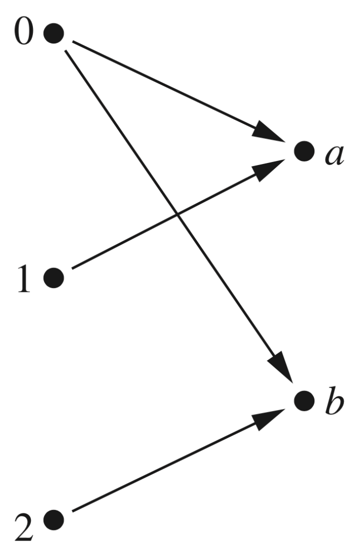
EXAMPLE 1

Let A be the set of students in your school, and let B be the set of courses.

Let R be the relation that consists of those pairs (a,b), where a is a student enrolled in course b.

EXAMPLE 3

Let $A = \{0,1,2\}$ and $B = \{a,b\}$. Then $\{(0,a), (0,b), (1,a), (2,b)\}$ is a relation from A to B.



R	a	b
0	×	×
1	×	
2		×

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Functions As Relations

Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A.

The graph of f is the set of ordered pairs (a,b) such that $b=f(a)$.

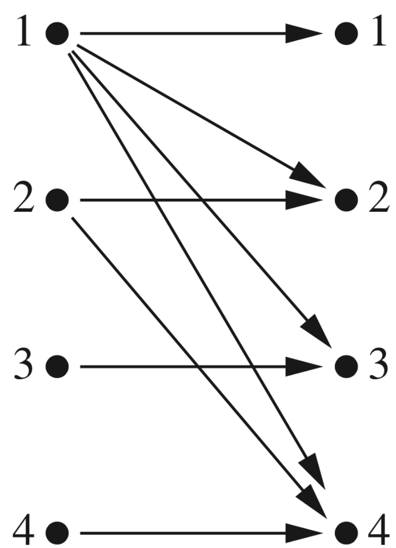
Because the graph of f is a subset of $A \times B$, it is a relation from A to B.

Moreover, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph.

1 Relations on a Set

DEFINITION 2 A *relation on a set* A is a relation from A to A.

EXAMPLE 4 Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

FIGURE 2 Displaying the Ordered Pairs in the Relation R from Example 4.

1 Properties of Relations

DEFINITION 3 A relation R on a set A is called *reflexive* if $(a,a) \in R$ for every element $a \in A$.

EXAMPLE 7 Consider the following relations on {1, 2, 3, 4}. Which of these relations are reflexive?

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R_6 = \{(3, 4)\}.$$

1 Properties of Relations

DEFINITION 4

A relation R on a set A is called *symmetric* if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a, b \in A$.

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Using quantifiers, we see that the relation R on the set A is *symmetric* if $\forall a \forall b\, ((a, b) \in R \rightarrow (b, a) \in R)$.

Similarly, the relation R on the set A is *antisymmetric* if $\forall a \forall b\, (((a, b) \in R \land (b, a) \in R) \rightarrow (a = b))$.

That is, a relation is *symmetric* if and only if a is related to b implies that b is related to a.

A relation is *antisymmetric* if and only if there are no pairs of distinct elements a and b with a related to b and b related to a.

1 Properties of Relations

DEFINITION 5

A relation R on a set A is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$.

Using quantifiers we see that the relation R on a set A is *transitive* if we have $\forall a \forall b \forall c \, (((a, b) \in R \land (b, c) \in R) \rightarrow (a, c) \in R)$.