Discrete Mathematics

Logic

Rules of Inference

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6 Rules of Inference

Introduction

Proofs in mathematics are *valid arguments* that establish the truth of mathematical statements.

By an argument, we mean a sequence of statements that end with a *conclusion*.

By valid, we mean that the conclusion must follow from the truth of the preceding statements, or *premises*, of the argument.

That is, an argument is valid if and only if it is impossible for *all the premises to be true* and *the conclusion to be false*.

To deduce new statements from statements we already have, we use *rules of inference* which are templates for constructing valid arguments.

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Valid Arguments in Propositional Logic

Consider the following argument involving a sequence of propositions:

"*If you have a current password, then you can log onto the network.*"

"*You have a current password.*"

Therefore,

"**You can log onto the network**."

We would like to determine whether the conclusion "**You can log onto the network**" must be *true* when the premises "*If you have a current password, then you can log onto the network*" and "*You have a current password*" are *both true.*

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Argument Forms

Use $\boldsymbol{p}$ to represent "*You have a current password*", and $\boldsymbol{q}$ to represent "*You can log onto the network*"

Then, the argument has the form

$$\begin{array}{l} \quad p \rightarrow q \\ \quad \underline{p \qquad\quad} \\ \therefore q \end{array}$$

where ∴ is the symbol that denotes "therefore."

When p and q are propositional variables, the statement $((\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge \boldsymbol{p}) \rightarrow \boldsymbol{q}$ is a *tautology* .

This form of argument is **valid** because whenever all its premises are *true*, the conclusion must also be *true*.

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DEFINITION 1

An *argument* in propositional logic is a sequence of propositions.

All but the final proposition are called *premises* and the final proposition is called the *conclusion*.

An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions.

An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

The argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, when $\boldsymbol{(p_1 \land p_2 \land \cdots \land p_n) \to q}$ is a **tautology**.

6 Rules of Inference for Propositional Logic

Rule Of Inference

We can always use a **truth table** to show that an argument form is valid.

Instead, we can first establish the validity of some relatively **simple argument forms**, called rules of inference.

The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens**.

This tautology leads to the following valid argument form:

$$\begin{array}{rl} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array}$$

6 Rules of Inference for Propositional Logic

EXAMPLE 1

Suppose that the conditional statement "*If it snows today, then we will go skiing*" and its hypothesis, "*It is snowing today,*" are true.

Then, by modus ponens, it follows that the conclusion of the conditional statement, "**We will go skiing**," is true.

EXAMPLE 2

Determine whether the *argument* given here is *valid* and determine whether its *conclusion* must be *true* because of the **validity** of the argument:

"If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$."

6 TABLE 1 - Rules of Inference

Rule of Inference	*Tautology*	*Name*
p $p \rightarrow q$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism

6 TABLE 1 - Rules of Inference (cont.)

Rule of Inference	*Tautology*	*Name*
p $\therefore\ p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$ $\therefore\ p$	$(p \wedge q) \rightarrow p$	Simplification
p q $\therefore\ p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore\ q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

6 Using Rules of Inference to Build Arguments

Building Arguments

When there are many premises, *several rules of inference* are often needed to show that an argument is valid.

The steps of arguments are displayed on separate lines, with *the reason* for each step explicitly stated.

EXAMPLE 6

Show that the premises "*It is not sunny this afternoon and it is colder than yesterday,*" "*We will go swimming only if it is sunny,*" "*If we do not go swimming, then we will take a canoe trip,*" and "*If we take a canoe trip, then we will be home by sunset*" lead to the conclusion "**We will be home by sunset**."

6 Building Arguments

SOLUTION

Let p be the proposition "*It is sunny this afternoon,*"

q the proposition "*It is colder than yesterday,*"

r the proposition "*We will go swimming,*"

s the proposition "We will take a canoe trip,"

and t the proposition "*We will be home by sunset.*"

Then the premises become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$.

The conclusion is simply t.

We need to give a valid argument with premises and conclusion.

6 Building Arguments (cont.)

SOLUTION

We *construct* an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

6 Fallacies

Fallacies

Several common fallacies arise in *incorrect arguments*.

These fallacies resemble rules of inference, but are based on **contingencies** rather than tautologies.

These are discussed here to show the distinction between correct and incorrect reasoning.

Fallacy of Affirming the Conclusion

The proposition $((p \to q) \land q) \to p$ is not a *tautology*.

However, there are *many incorrect arguments* that treat this as a tautology.

In other words, they treat the argument with premises $p \to q$ and q and conclusion p as a *valid argument form*, which it is not.

6 Fallacies (cont.)

EXAMPLE 10

Is the following argument valid?

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.

Therefore, *you did every problem in this book.*

SOLUTION

Let $\boldsymbol{p}$ be the proposition "*You did every problem in this book.*"

Let $\boldsymbol{q}$ be the proposition "*You learned discrete mathematics.*"

Then this argument is of the form: if $\boldsymbol{p} \rightarrow \boldsymbol{q}$ and $\boldsymbol{q}$, then $\boldsymbol{p}$.

This is an example of *an incorrect argument* using *the fallacy of affirming the conclusion.*