# Discrete Mathematics

## Logic

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# 3 Propositional Equivalences

**Introduction**

An important type of step used in a mathematical argument is the replacement of a statement with another statement with *the same truth value*.

Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of *mathematical arguments*.

Note that we will use the term "compound proposition" to refer to an expression formed from propositional variables using logical operators, such as $p \land q$.

## 3 Propositional Equivalences

**DEFINITION 1**

A compound proposition that is *always true*, no matter what the truth values of the propositional variables that occur in it, is called a ***tautology***.

A compound proposition that is *always false* is called a ***contradiction***.

**Example 1**

Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$.

**TABLE 1** **Examples of a Tautology and a Contradiction.**

| $p$ | $\neg p$ | $p \vee \neg p$ | $p \wedge \neg p$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

# 3 Logical Equivalences

**DEFINITION 2**

The compound propositions $p$ and $q$ are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that $p$ and $q$ are logically equivalent.

The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

**Example 2**

One way to determine whether two compound propositions are equivalent is to use a truth table.

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

**Example 3**

Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

# 3 Logical Equivalences

| *Equivalence* | *Name* |
|---|---|
| $p \wedge \mathbf{T} \equiv p$<br>$p \vee \mathbf{F} \equiv p$ | Identity laws |
| $p \vee \mathbf{T} \equiv \mathbf{T}$<br>$p \wedge \mathbf{F} \equiv \mathbf{F}$ | Domination laws |
| $p \vee p \equiv p$<br>$p \wedge p \equiv p$ | Idempotent laws |
| $\neg(\neg p) \equiv p$ | Double negation law |
| $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | Commutative laws |

## 3 Logical Equivalences

| | |
|---|---|
| $(p \vee q) \vee r \equiv p \vee (q \vee r)$<br>$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | Associative laws |
| $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive laws |
| $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |
| $p \vee (p \wedge q) \equiv p$<br>$p \wedge (p \vee q) \equiv p$ | Absorption laws |
| $p \vee \neg p \equiv \mathbf{T}$<br>$p \wedge \neg p \equiv \mathbf{F}$ | Negation laws |

# 3 Logical Equivalences

**Constructing New Logical Equivalences**

A proposition in a compound proposition *can be replaced* by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.

**Example 4**

Show that $\neg(\boldsymbol{p} \rightarrow \boldsymbol{q})$ and $\boldsymbol{p} \wedge \neg \boldsymbol{q}$ are logically equivalent.

$$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg(\neg p) \wedge \neg q \\ &\equiv p \wedge \neg q \end{aligned}$$

**Example 5**

Show that $\neg(\boldsymbol{p} \vee (\neg \boldsymbol{p} \wedge \boldsymbol{q}))$ and $\neg \boldsymbol{p} \wedge \neg \boldsymbol{q}$ are logically equivalent.

**Example 6**

Show that $(\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow (\boldsymbol{p} \vee \boldsymbol{q})$ are a tautology.