Topology

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1 Topological Spaces

1.1 Topologies

Definition 1 (Topology)**.** Given a set X, we say a collection of subsets of X, $\mathcal{T}$, is a topology if

1. $\varnothing, X \in \mathcal{T}$.
2. $\mathcal{T}$ is closed under arbitrary unions
3. $\mathcal{T}$ is closed under finite intersections.

We call a set X equipped with a topology $\mathcal{T}$ a topological space and the members of $\mathcal{T}$ open sets.

Definition 2 (Fineness)**.** Given topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, we say $\mathcal{T}$ is finer than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$ and we define coarser similarly.

We say two topologies are comparable if one is finer or coarser than the other.

For an arbitrary set, X, we always have the discrete topology consisting of all subsets of X and the indiscrete topology $\{\varnothing, X\}$. Hence the discrete topology is finer than every topology and the indiscrete topology is coarser than every other topology.

1.2 Basis of a Topology

Often there is no way to simply describe every possible open set of a topology, so we wish to describe it instead in terms of special open sets that in a sense "make up" the entire topology. We call this collection of "special" open sets a basis.

Definition 3 (Basis)**.** Let X be a set. Then we say a collection, $\mathcal{B}$, of subsets of X is a basis if

1. Every $x \in X$ is contained in a basis element.
2. For any basis elements B_1, B_2, if $x \in B_1 \cap B_2$, then we have a basis element B_3 such that

$$x \in B_3 \subset B_1 \cap B_2.$$

We then define the topology, $\mathcal{T}$, generated by $\mathcal{B}$ to be such that a set U is open precisely when for every point $x \in U$ we have a basis element such that

$$x \in B \subset U.$$

It is trivial to verify that the topology generated by a basis actually satisfies the axioms for a topology. Also as the term may suggest, the basis elements that generate a topology all belong to the topology.

We may alternatively characterise basis sets in a more natural way

Theorem 1. *Let $\mathcal{B}$ be a basis of X that generates $\mathcal{T}$. Then we have*

1. *$\mathcal{T}$ is the intersection of all topologies on X that contain $\mathcal{B}$.*

2. *$\mathcal{T}$ is the set of all unions of collections of basis elements.*

Proof. Let $\mathcal{T}'$ be an arbitrary topology of X that contains $\mathcal{B}$. Consider an arbitrary $U \in \mathcal{T}$.

We have for every $x \in U$ there exists a $B_x \in \mathcal{B} \subset \mathcal{T}'$ such that

$$x \in B_x \subset U.$$

Hence we have

$$U = \cup\{B_x\}_{x \in U}$$

That is, U is a union of sets open in $\mathcal{T}'$, so $U \in \mathcal{T}'$. Hence $\mathcal{T} \subset \mathcal{T}'$.

$\mathcal{T}$ is a topology that contains $\mathcal{B}$ itself, so we have $\mathcal{T}$ is the intersection of all such topologies on X.

The basis elements are open themselves, so any union of basis elements is open. For any $U \in \mathcal{T}$ we have for every $x \in U$ a basis element B_x such that

$$x \in B_x \subset U.$$

So we have

$$U = \cup\{B_x\}_{x \in U}$$

Hence U is union of a collection of basis elements. Hence $\mathcal{T}$ is the set of all unions of collections of basis elements. $\square$

This means that the topology generated by a basis is the minimal (under the partial order of the subset relation) topology that contains the basis.

We now characterise the basis sets that generate a topology.

Theorem 2. Let $\mathcal{T}$ be a topology on X.

Then $\mathcal{B}$ is a basis that generates $\mathcal{T}$ if and only if every member of $\mathcal{B}$ is open and for every point x in every open set U we have a $B \in \mathcal{B}$ such that

$$x \in B \subset U.$$

Proof. The forwards implication is by definition.

So assume that $\mathcal{B}$ satisfies the latter hypothesis.

As X is open, we have that for every $x \in X$ there is a basis element B with $x \in B$. For any $B_1, B_2 \in \mathcal{B} \subset \mathcal{T}$, we have $B_1 \cap B_2$ is open, so we have a $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

Hence $\mathcal{B}$ is a basis.

As $\mathcal{B} \subset \mathcal{T}$, the topology generated by $\mathcal{B}$ is coarser than $\mathcal{T}$. By definition every open set is in the topology generated by the basis. Hence $\mathcal{B}$ is a basis that generates $\mathcal{T}$. $\square$

We now consider how to compare two topologies using bases that generate them.

Theorem 3. Let the basis $\mathcal{B}$ and $\mathcal{B}'$ generate the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the following are equivalent

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.

2. For every $B \in \mathcal{B}$ and every $x \in B$ we have $B' \in \mathcal{B}'$ such that

$$x \in B' \subset B.$$

Proof. Assume that $\mathcal{T}'$ is finer than $\mathcal{T}$. Then every $B \in \mathcal{B}$ is open is $\mathcal{T}'$, so for every $x \in B$ we have $B' \in \mathcal{B}'$ such that

$$x \in B' \subset B.$$

Now assume the latter hypothesis. Say U is open in $\mathcal{T}$. Then for every $x \in U$ we have a $B \in \mathcal{B}$ such that

$$x \in B \subset U.$$

We also have a $B' \in \mathcal{B}'$ such that

$$x \in B' \subset B.$$

Hence we have

$$x \in B' \subset U.$$

Hence U is open in $\mathcal{T}'$. □

Considering the characterisation of the topology generated by a basis as the minimal topology containing the basis, we are motivated to generalise the notion of a basis.

Definition 4 (Subbasis)**.** Let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis of X if for every $x \in X$ we have a $S \in \mathcal{S}$ such that $x \in S$.

We then define the topology generated by $\mathcal{S}$ to be the collection of all unions of finite intersections of $\mathcal{S}$.

1.3 Examples of Topologies

We now define various useful ways to impose topologies on sets with some kind of structure.

Definition 5 (Order Topology)**.** Let X be a totally ordered set. Then we define the order topology as the topology generated by the basis consisting of

1. All open intervals.

2. All intervals of the form $(a, b_0]$ when there is a maximal element b_0.

3. All intervals of the form $[a_0, b)$ when there is a minimal element a_0.

Definition 6 (Product Topology)**.** Let X and Y be topological spaces. Then we define the product topology on $X \times Y$ as the topoology generated by the basis consisting of all products of open sets in X and Y.

Theorem 4. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for the topologies on X and Y, respectively. Then the products of basis elements of X and Y form a basis for $X \times Y$.

Proof. Consider an arbitrary basis element $U \times V$ of $X \times Y$. Then for every $(x, y) \in U \times V$ we have $B_x \in \mathcal{B}$ and $C_x \in \mathcal{C}$ such that

$$(x, y) \in B_x \times C_y \subset U \times V.$$

Hence we have

$$U \times V = \cup\{B_x \times C_y\}_{x \in U, y \in V}.$$

So the products of basis elements generate the product topology. $\square$

Definition 7 (Subspace Topology)**.** Let $Y \subset X$. Then we define the subspace topology on Y to be

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$$