Differential Equations

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March 22, 2021

1 First Order Equations

1.1 Homogeneous Equations

A homogenous differential equation is one of the form

$$M(x, y)dx + N(x, y)dy = 0.$$

Where M and N are homogenous functions of equal degree. A trivial rearrangement yields

$$\frac{dy}{dx} = f(x, y).$$

Where f is a homogenous function of degree 0. This means that $f(x, y) = f(tx, ty)$, and in particular that $f(x, y) = f(1, \frac{y}{x})$. Letting $z = \frac{y}{x}$, we have $zx = y$, and implicitly differentiating we have $xdz + zdx = dy$, which gives $\frac{dy}{dx} = z + x\frac{dz}{dx}$. Applying these results to the above equation yields

$$z + x\frac{dz}{dx} = f(1, z).$$

This is a separable equation.

Example 1. Solve $(x+y)dx - (x-y)dy = 0$.

We note that the dx and dy coefficients are homogenous of equal degree (degree 1), and so we may rearrange and apply the substitution $z = \frac{y}{x}$, to yield a separable equation.

$$\begin{aligned} x\frac{dz}{dx} + z &= \frac{1+z}{1-z} \\ \frac{1-z}{1+z^2}dz &= \frac{1}{x}dx \\ \log x = \arctan z - \log\sqrt{1+z^2} &+ C \\ \arctan\frac{y}{x} &= \log\sqrt{x^2+y^2} + C \end{aligned}$$

1.2 Exact Equations

We say a differential equation is exact if it is of the form

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

In this case the equation may be re-written as $df = 0$, meaning the solution is $f(x, y) = C$, for some constant C. However in general it may not be apparent from inspection that an equation of the form $Mdx + Ndy = 0$ is exact. It may be trivially shown that the equation is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 2. Solve $(\sin x \tan y + 1)dx - (\cos x \sec^2 y)dy = 0$.

$$\frac{\partial M}{\partial y} = \sin x \sec^2 y \frac{\partial N}{\partial x} = \sin x \sec^2 y$$

Hence the equation is exact, so we have there is a function f such that the LHS is df.

$$\frac{\partial f}{\partial x} = \sin x \tan y + 1$$
$$f = \int (\sin x \tan y + 1) dx$$
$$= -\tan y \cos x + x + g(y)$$

Differentiating with respect to y

$$-\cos x \sec^2 y = -\cos x \sec^2 y + g'(y) g'(y) = 0 g(y) = C$$

Hence we have

$$f = x - \tan y \cos x + C$$

So the solution to the equation is

$$\tan y \cos x - x = C$$

2 Miscellaneous

2.1 Motion of a Pendulum

We model a pendulum as a light rod of length l with one end fixed and to the other attatched a particle of mass m that is released from rest at an angle α from the vertical.

We may obtain a differential equation for its motion in terms of its angle θ

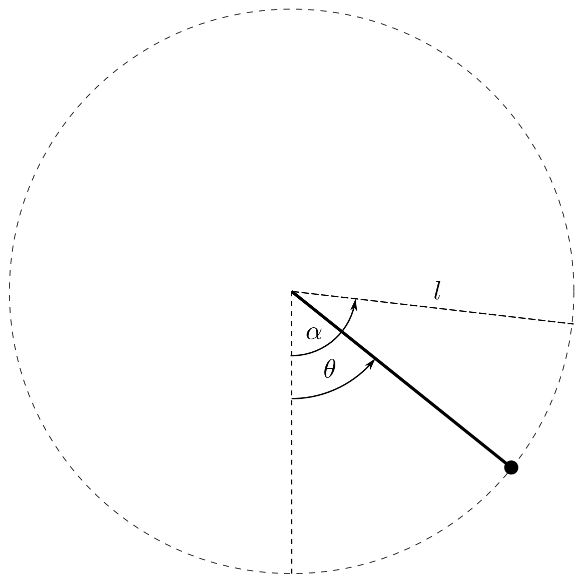

Figure 1

from the vertical by using that the sum of the particles kinetic and gravitational potential energy is constant.

$$\frac{1}{2}m(l\dot{\theta})^2 + mgl(1-\cos\theta) = mgl(1-\cos\alpha).$$

Rearranging and considering the descent of the pendulum to its minimum point yields

$$dt = -\sqrt{\frac{l}{2g}}\frac{d\theta}{\sqrt{cos\theta - cos\alpha}}.$$

We now integrate from $t=0$ to the minimum point of the pendulum

$$\frac{T}{4} = \sqrt{\frac{l}{2g}}\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

We now begin a series of manipulations with the aim of transforming this into a relatively simple expression of an elliptic integral of the first kind. Applying the double angle formula

$$\frac{T}{4} = \sqrt{\frac{l}{2g}} \int_0^\alpha \frac{d\theta}{\sqrt{2}\sqrt{\sin^2 \frac{\alpha}{2} - sin^2 \frac{\theta}{2}}}.$$

Let $k = \sin \frac{\alpha}{2}$. Now we make the substitution $\sin \frac{\theta}{2} = k \sin \phi$. So $\frac{1}{2} \cos \frac{\theta}{2} d\theta = k \cos \phi d\phi$, which written in terms of $d\theta$ yields $d\theta = \frac{2k \cos \phi}{\sqrt{1-k^2 sin^2 \phi}}$. So we have the time period of the pendulum as

$$T = 4\sqrt{\frac{l}{g}} F\left(\sin \frac{\alpha}{2}, \frac{\pi}{2}\right).$$

where $F(k, \phi)$ denotes the elliptic integral of the first kind.