Topology

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1 Topological Spaces

1.1 Topological Space Axioms

Definition 1 (Topological Space)**.** A topological space is an ordered pair $(X, \mathcal{T})$, where $\mathcal{T} \subset \mathcal{P}(X)$, satisfying:

1. $\varnothing$ and X are open sets.
2. The union of any family of open sets is open.
3. The intersection of any finite family of open sets is open.

Where we say a subset of X is open if it belongs to $\mathcal{T}$.

In the extreme case of $\mathcal{T}$ containing every subset of X we call $\mathcal{T}$ the discrete topology. On the other extreme of $\mathcal{T} = \{\varnothing, X\}$, we call $\mathcal{T}$ the indiscrete topology.

Definition 2 (Continuity)**.** Let S and T be topological spaces and $f : S \to T$. We say f is continuous if the preimage of every open set under f is open.

Particularly, a continuous bijection with continuous inverse is an isomorphism between topological spaces.

It should be noted that merely being a continuous bijection does not guarantee that the inverse is continuous. For example

Theorem 1. *Let $f : A \to B$ and $g : B \to C$ be continuous maps between topological spaces. Then the map $g \circ f$ is continuous.*

Proof. Consider an arbitrary open set $U \in \mathcal{T}_C$. Then we have

$$
\begin{aligned}
&(g \circ f)^{-1}(U) \\
&= f^{-1} \circ g^{-1}(U)
\end{aligned}
$$

By the continuity of g, we have $g^{-1}(U)$ is open, and so by the continuity of f we have $f^{-1} \circ g^{-1}(U)$ is open. Hence $g \circ f$ is continuous. $\square$

Definition 3 (Subspace Topology)**.** Let $(X, \mathcal{T})$ be a topological space. Then for a subset $S \subset X$, we say the subspace topology, or the induced topology, is the set of intersections of open sets in X with S.

This collection of sets may be trivially verifed to obey the axioms of a topology.

1.2 Metric Spaces

Definition 4 (Metric Space)**.** A metric space is an ordered pair (M, d) where $d : M^2 \to \mathbb{R}$ satisfying

1. Definiteness:

$$d(x, y) = 0 \iff x = y$$

2. Symmetry:

$$d(x, y) = d(y, x)$$

3. Triangle Inequality:

$$d(x, y) + d(y, z) \geq d(x, z)$$

A trivial but noteworthy consequence of these axioms is that the distance between any distinct points is positive.

Definition 5 (Open Ball)**.** Given a metric space (M, d). We define the open ball

$$B_\delta(x) = \{y \in M : d(x, y) < \delta\}$$

We may generalise the definition of continuous functions from real analysis to metric spaces, noting that $\mathbb{R}$ is a metric space under the metric $d(x, y) = |x - y|$.

Definition 6 (Continuity)**.** Let M and N be metric spaces with metrics d_M and d_N, respectively.

Then the function $f : M \to N$ is continuous at $a \in M$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $d(x, a) < \delta$ we have $d(f(a), f(x)) < \epsilon$.

Definition 7 (Metric Topology)**.** Let (M, d) be a metric space. Then we may equip it with the metric topology defined such that a set, U, is open if for every point $x \in U$ there is an open ball centered at x contained in U.

This definition coincides in the case of $\mathbb{R}^n$ with our intuitive notions of openness, however we still must verify that the sets specified do indeed satisfy the axioms for a topology. However this is trivial, so we omit the proof.

Having now also turned every metric space into a topological space, we note that there are now two separate possible definitions of continuity of a function between metric spaces. One in terms of pre-images of open sets and the standard epsilon-delta definition. However this is not an issue, as we will now show the two seemingly distinct definitions coincide.

Theorem 2. Let $f : M \to N$ be a function between metric spaces. Then f is topologically continuous precisely when it satisfies epsilon-delta continuity.

Proof. First assume that f is epsilon-delta continuous. And so consider an arbitrary open set U, and an arbitrary $x \in f^{-1}(U)$.

Hence we have $f(x) \in U$, and so by the openness of U, we have an $\epsilon > 0$ such that $B_\epsilon f(x) \subset U$. By the epsilon-delta continuity of f we have a $\delta > 0$ such that

$$\begin{aligned} p \in B_\delta(x) &\implies f(p) \in B_\epsilon f(x) \\ &\implies f(p) \in U \\ &\implies p \in f^{-1}(U) \end{aligned}$$

So we have that $B_\delta(x)$ is contained in $f^{-1}(U)$, hence $f^{-1}(U)$ is open and so f is topologically continuous.

Now assume that f is topologically continuous. Consider an arbitrary $x \in M$ and $\epsilon > 0$. Then by topological continuity we have that $f^{-1}(B_\epsilon f(x))$ is open and so we have a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon f(x))$, yielding $f(B_\delta(x)) \subset B_\epsilon f(x)$, hence f is epsilon-delta continuous. $\square$

A variety of structures from different areas of mathematics lend themselves naturally to metrisation. From linear algebra, any inner product space is a metric space under the metric $d(x, y) = \|x - y\|$. From graph theory, any connected graph may be metrised using the distance function.

1.3 Bases

Definition 8 (Basis)**.** Let $(X, \mathcal{T})$ be a topological space. Then a collection of open subsets $\mathcal{B}$ is a basis if every open subset of X is the union of members of $\mathcal{B}$.

The main usefulness of bases is that often times a statement can be shown to hold for all open sets by just showing it holds for all elements of a basis. For example to show continuity, we need only show the pre-image of any basis element is open.

2 Topological Properties