Channel Capacity Calculations

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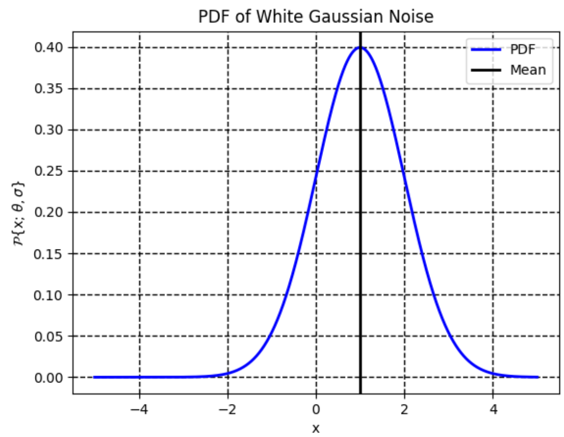


1 Chapter 1 - Introduction

Step 0 in Estimation Theory is to model the data. Before moving further into the details, we first discuss some commmonly used mathematical tools.

A Parameterized Probability Distribution Function (PDF) is useful to analyse the model we create and is therefore a commonly used mathematical tool in the following chapters. A Parameterized PDF is of the form shown in equation 1 where g is some function of the random variabel, x and some parameter, θ. The PDF of a vector is the product of the PDFs of its elements.

$$\mathcal{P}\left\{x;\theta\right\} = g(x,\theta) \tag{1}$$

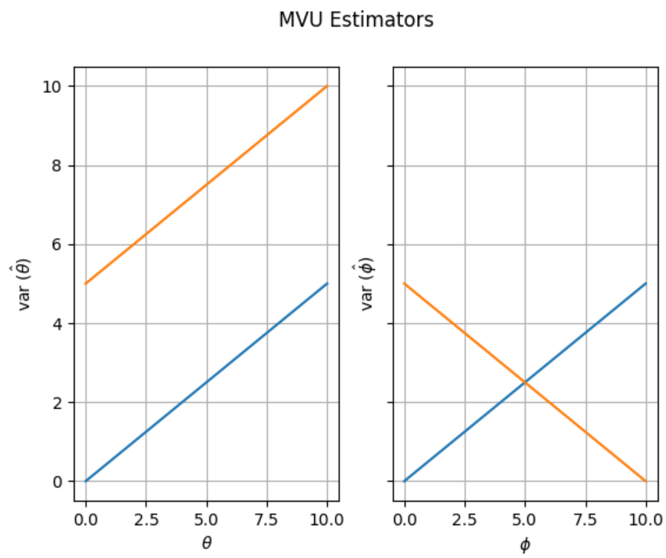
White Gaussian Noise is another commonly used tool to make the problem mathematically solvable. The PDF of White Gaussian Noise if given in equation 2 and the figure is plotted pictorial in figure 1. You can generate this yourself using the python code at this GIT Repo. In fact, most of the figures and problems are documented here.

$$\mathcal{P}\left\{x;\theta,\sigma\right\} = \frac{1}{\sqrt{2\pi\sigma^2}}\, exp\left[\frac{-1}{2\sigma^2}\left(x-\theta\right)^2\right] \tag{2}$$

There are two general classes of estimators:

1. **Classical Estimators:** The Parameters to be estimated are Deterministic as shown in equation 3.

$$\hat{\theta} = g(x) \text{ where } \mathcal{P}\left\{x;\theta\right\} \text{ is the PDF of x} \tag{3}$$



2. **Bayesian Estimators:** The Parameters to be estimated are Random Variables and their moments are to be estimated. These are described in equation 4

$$\hat{\theta} = g(x) \text{ where } \mathcal{P}\{x|\theta\}\,\mathcal{P}\{\theta\} \text{ is the PDF of x} \tag{4}$$

An estimator is just a transform of one random variable to another. An estimator takes realization of a random variable and gives the best guess of the parameter that it was designed to estimate. An estimator boils down to a random variable which has a mean equal the parameter in question and with as low a variance as possible. This needs to be established using Monte Carlo simulations.

2 Chapter 2 - Minimum Variance Unbiased Estimator

An estimator is said to be unbiased when on average the estimator yields the true value for any range of values that the estimated parameter can take. Mathematically, this can be written as shown in equation 5 where $g(\mathbf{x})$ is the estimator.

$$\mathcal{E}(\hat{\theta}) = \int g(\mathbf{x})p(\mathbf{x};\theta)d\mathbf{x} = \theta \qquad \forall \qquad \theta \in (a,b) \tag{5}$$

The same in vector form can be defined as shown in equation 6 if the parameter $\theta = [\theta_1\theta_2\ldots\theta_N]^T$ is estimated using an estimator defined as $\hat{\theta} = \left[\hat{\theta}_1\hat{\theta}_2\ldots\hat{\theta}_N\right]^T$. If we define the expected value of a vector to be the expected value of its components then it can also be written in vector form.

$$\begin{aligned} \mathcal{E}(\hat{\theta}_i) &= \theta_i \qquad \forall \qquad \theta_i \in (a_i, b_i) \\ \mathcal{E}(\hat{\theta}) &= \theta \end{aligned} \tag{6}$$

An unbiased estimator does not imply a good estimator but only that it converges on average to the true value. However, an biased estimator always leads to poor results. Therefore, to judge the quality of an estimator, we would need to define some optimality criterion.

2.1 Minimize the Mean Square Error?

A natural choice for the optimality is the estimtaor that minimizes the **mean square error**. However, these estimators are generally difficult to find. Consider the mean square error are shown in equation 7. One way of finding the estimator could be to assign the first derivative of this equation to 0 which might yield the optimal estimator. However, this result would depend on the parameter θ to be estimated which makes it a chicken or egg problem. Therefore a minimum MSE estimator in most cases is unrealizable.

$$\begin{aligned} \mathrm{mse}(\hat{\theta}) &= \mathcal{E}\left\{(\hat{\theta} - \theta)^2\right\} \\ &= \mathcal{E}\left\{\hat{\theta}^2 + \theta^2 + 2\hat{\theta}\theta\right\} \\ &= \mathcal{E}(\hat{\theta}^2) + \theta^2 + 2\mathcal{E}(\hat{\theta})\theta \end{aligned} \tag{7}$$

2.2 Alternative: Minimum Variance Unbiased Estimator

Another approach is to make sure that the estimator is unbiased and them minimizing its variance. The question now is whether such an estimator exists. In general, the MVU does not always exist as the best MVU **could** depend no the value of the parameter that is to be estimated. This is seen in figure 2 where θ has a realizable MVU estimator and ϕ doesn't. Even if it does exist, sometimes the best estimator cannot be found. Therefore, three possible approaches can be taken:

- Determine Cramer-Rao Lower Bound (CRLB) $\longrightarrow$ Find Estimator that satisfies it
- Apply the Rao-Blackwell-Lehmann-Scheffe (RBLS) Theorem
- Restrict the Estimator to be **Linear** as well as Unbiased. This helps reduce the search space to find an Estimator