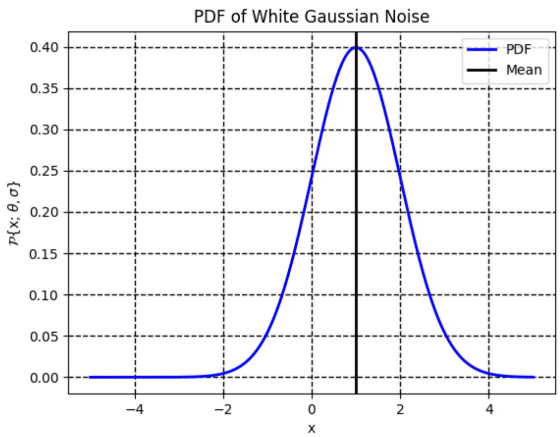
Channel Capacity Calculations

Karan Jayachandra

Personal Archive

November 1, 2022



1 Chapter 1 - Introduction

Step 0 in Estimation Theory is to model the data. Before moving further into the details, we first discuss some commmonly used mathematical tools.

A Parameterized Probability Distribution Function (PDF) is useful to analyse the model we create and is therefore a commonly used mathematical tool in the following chapters. A Parameterized PDF is of the form shown in equation 1 where g is some function of the random variabel, x and some parameter, θ. The PDF of a vector is the product of the PDFs of its elements.

$$\mathcal{P}\left\{x;\theta\right\} = g(x,\theta) \tag{1}$$

White Gaussian Noise is another commonly used tool to make the problem mathematically solvable. The PDF of White Gaussian Noise if given in equation 2 and the figure is plotted pictorial in figure 1. You can generate this yourself using the python code at this GIT Repo. In fact, most of the figures and problems are documented here.

$$\mathcal{P}\left\{x;\theta,\sigma\right\} = \frac{1}{\sqrt{2\pi\sigma^2}}\, exp\left[\frac{-1}{2\sigma^2}\left(x-\theta\right)^2\right] \tag{2}$$

There are two general classes of estimators:

1. **Classical Estimators:** The Parameters to be estimated are Deterministic as shown in equation 3.

$$\hat{\theta} = g(x) \text{ where } \mathcal{P}\left\{x;\theta\right\} \text{ is the PDF of x} \tag{3}$$

2. **Bayesian Estimators:** The Parameters to be estimated are Random Variables and their moments are to be estimated. These are described in equation 4

$$\hat{\theta} = g(x) \text{ where } \mathcal{P}\left\{x|\theta\right\}\mathcal{P}\left\{\theta\right\} \text{ is the PDF of x} \tag{4}$$

An estimator is just a transform of one random variable to another. An estimator takes realization of a random variable and gives the best guess of the parameter that it was designed to estimate. An estimator boils down to a random variable which has a mean equal the parameter in question and with as low a variance as possible. This needs to be established using Monte Carlo simulations.

2 Chapter 2 - Minimum Variance Unbiased Estimator

An estimator is said to be unbiased when on average the estimator yields the true value for any range of values that the estimated parameter can take. Mathematically, this can be written as shown in equation 5 where $g(\mathbf{x})$ is the estimator.

$$\mathcal{E}(\hat{\theta}) = \int g(\mathbf{x})p(\mathbf{x};\theta)d\mathbf{x} = \theta \qquad \forall \qquad \theta \in (a,b) \tag{5}$$

An unbiased estimator does not imply a good estimator but only that it converges on average to the true value. However, an biased estimator always leads to poor results. Therefore, to judge the quality of an estimator, we would need to define some optimality criterion.

2.1 Minimum Variance Criterion

A natural choice for the optimality is the estimtaor that minimizes the mean square error. However, these estimators are generally difficult to find. Consider the mean square error are shown in equation 6.

$$\begin{aligned}
\text{mse}(\hat{\theta}) &= \mathcal{E}\left\{(\hat{\theta}-\theta)^2\right\} \\
&= \mathcal{E}\left[\left\{(\hat{\theta}-\mathcal{E}(\hat{\theta})) + (\mathcal{E}(\hat{\theta})-\theta)\right\}^2\right] \\
&= \text{var}(\hat{\theta}) + \left[\mathcal{E}(\hat{\theta})-\theta\right]^2
\end{aligned} \tag{6}$$