EE386 Digital Signal Processing Lab **Jul-Dec 2021**

Experiment 8 Report

Author: J Karan Tejas *Email:* karantejas.191ee126@nitk.edu.in

Problem 1

(Transfer Function and Pole Zero Plot)

(Solution)

The propagation mechanism of an epidemic, such as the one caused by the SARS-CoV-2 virus, can be modelled, at least in its initial phase, as a process in which each infected individual will eventually transmit the disease to an average of R_o healthy people; these newly infected patients will, in turn, infect R0 healthy individuals each, and so on, creating a pernicious positive feedback in the system. The constant R_o is called the basic reproduction number for a virus. During the initial phase, the infection mechanism can be modelled as first order recursive filter:

$$y[n] = \delta[n] + R_o y[n-1] \tag{0.1}$$

where y[n] is the number of new cases on the n_{th} day, $\delta[n]$ is considered as the input of the system, which is the single case occurring on the day 0, due to which the infection mechanism begins, which is equivalent to an impulse. The transfer of the system is given by :

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - R_o} \tag{0.2}$$

The transfer functions for different values of R_o are calculated and their pole zero plots are plotted.

```
Transfer function H1(z) for Ro = 0.8
   z
-------
z - 0.8

Transfer function H1(z) for Ro = 1
  z
-----
z - 1

Transfer function H1(z) for Ro = 1.3
   z
-------
z - 1.3

Transfer function H1(z) for Ro = 2
  z
-----
z - 2
```

Figure 1: Transfer Function for different values of R_o

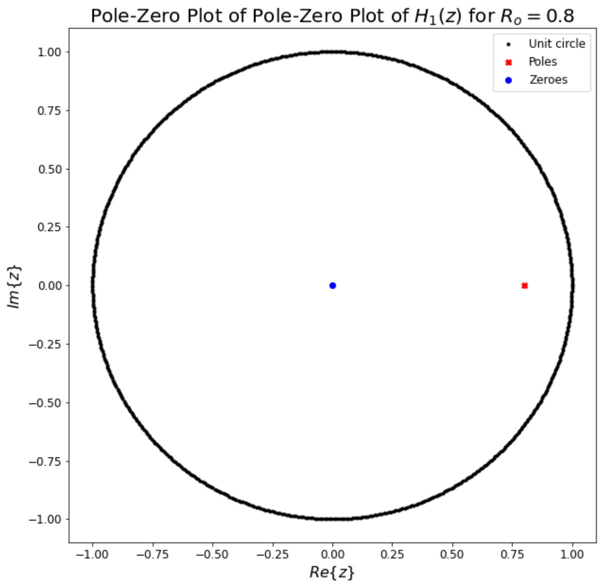

Figure 2: Pole Zero Plot $R_o = 0.8$

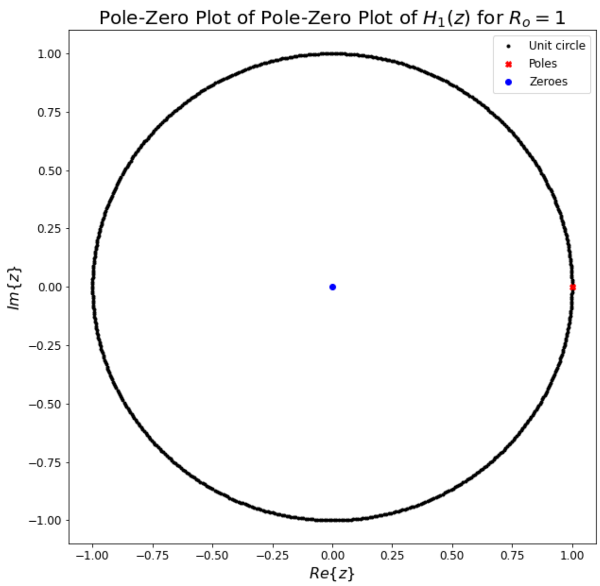

Figure 3: Pole Zero Plot $R_o = 1$

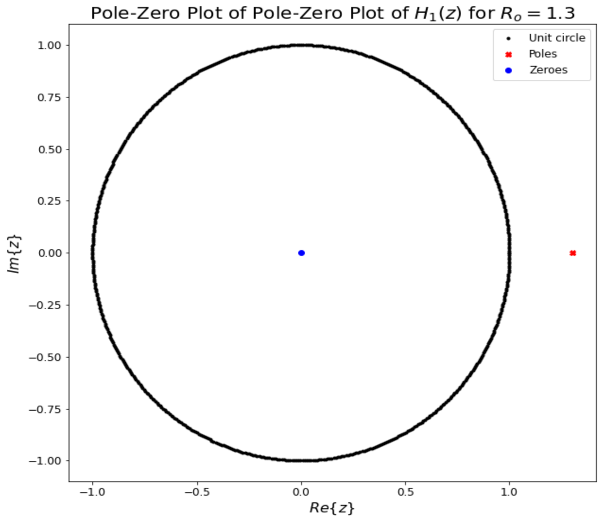

Figure 4: Pole Zero Plot $R_o = 1.3$

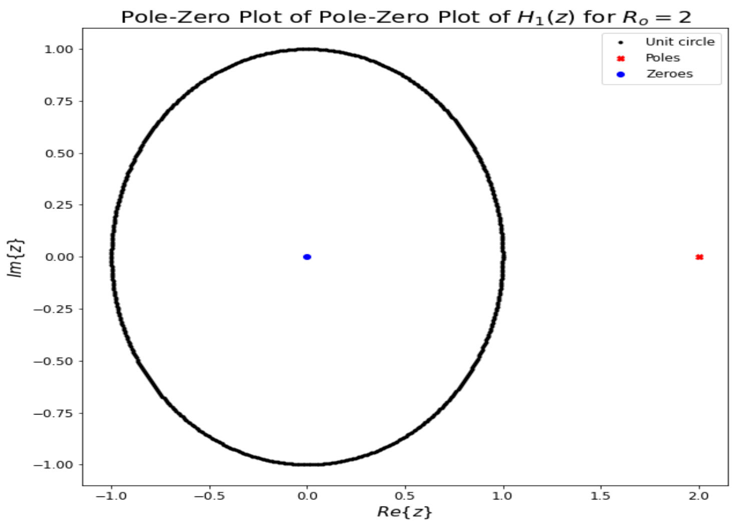

Figure 5: Pole Zero Plot $R_o = 2$

(Difference Equation Solution)

(Solution)

The input of the system is an impulse. Hence for the given system the impulse response is calculated using inverse Z transform and found to be exponential ,i.e, $(R_o)^n$. The impulse response for different R_o are plotted.

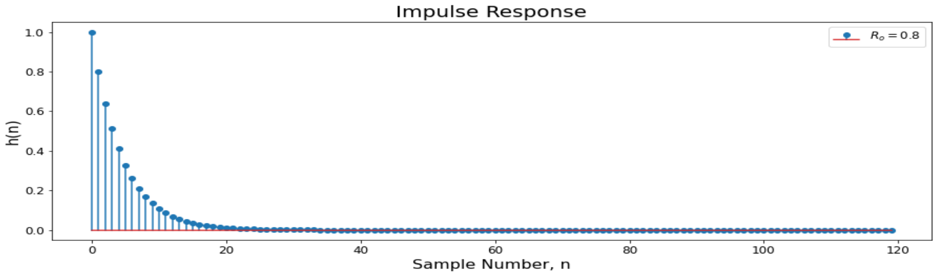

Figure 6: Impulse Response for $R_o = 0.8$

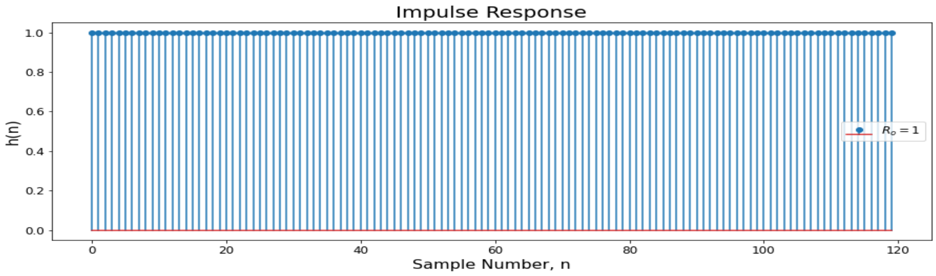

Figure 7: Impulse Response for $R_o = 1$

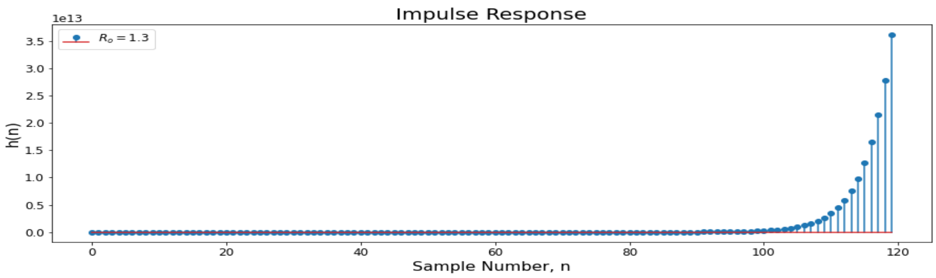

Figure 8: Impulse Response for $R_o = 1.3$

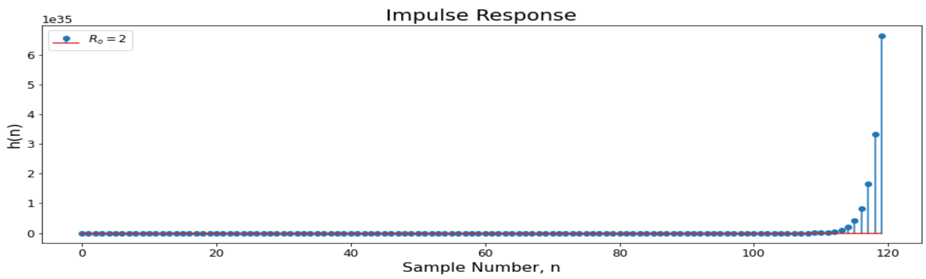

Figure 9: Impulse Response for $R_o = 2$

Effects of parameter R_o on the infection mechanism and the inference from the pole zero plots are:

- For $R_o < 1$, then number of new daily cases becomes 0 as $n \rightarrow \infty$. In the pole zero plots the poles lie within the lie inside the unit circle, hence the system is stable and the response dies out after a long time.
- For $R_0 = 1$, the number of daily cases remains constant. The poles of this system lie on the unit circle. Hence the system is marginally stable and the response is bounded and constant.
- For $R_o > 1$, the number of daily cases exponentially increases to ∞ as $n \rightarrow \infty$. The system is unstable since the poles lie outside the unit circle and the response is unbounded and grows to ∞.

(Reaching 1 Million Cases)
(Solution)
The system takes **16 days** to reach 1 million daily case for $R_o = 2.5$. The number of cases on the 16^{th} day is **2.32 Million**. The number of new daily cases is plotted against the number of days from day 0.

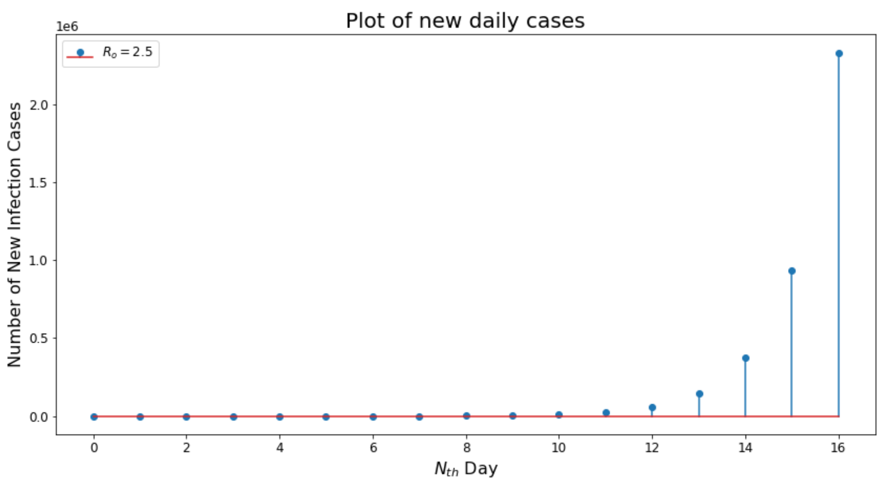

Figure 10: Daily New Cases for $R_o = 2.5$

(Determining R_o for COVID First Wave)
(Solution)
Since the response of the system is exponential, on any given day following the model mentioned

above, the daily new cases is given by $(R_o)^n$. To find the R_o for a given day n and the daily new cases y(n), we can use simple one point trick :

$$R_o = \sqrt[n]{y(n)} \tag{0.3}$$

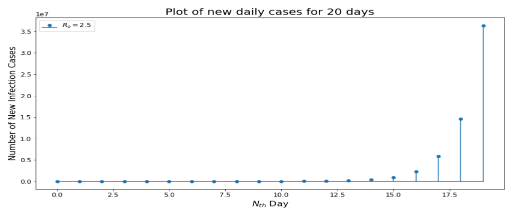

Figure 11: R_o for random days for COVID 19 initial phase

The average of R_o for the days 150 to 240 is **1.0583**. This technique is not a reliable method to find the reproduction since R_o is not constant as shown in the given data. Also the factors mentioned in problem 2,3 and 4,i.e, the quarantine period, social distancing and a finite population is not accounted for. Hence the given model is not accurate. A better way to find R_o is to take average over a given interval to reduce the noise and also include the factors mentioned in the problems 2, 3 and 4.

(Integrator Filter)

(Solution)

Daily new cases are plotted against number of days for 20 days for $R_o = 2.5$:

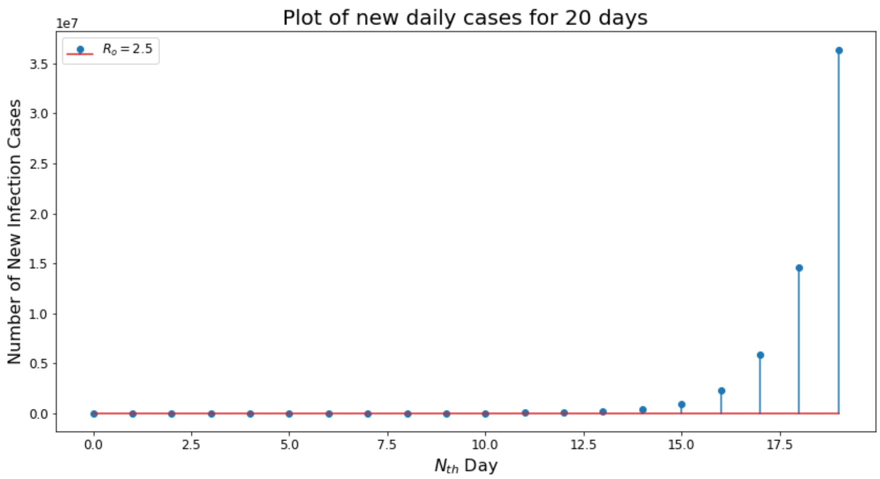

Figure 12: Daily New Cases for $R_o = 2.5$

To find the total number of cases $y_t(n)$ let us consider the new daily cases y(n). The two quantities can be related as :

$$y_t(n) - y_t(n-1) = y(n) \tag{0.4}$$

This transfer function of system with y(n) as the input and $y_t(n)$ as the output can be given as :

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-1} \tag{0.5}$$

The total number of infections are found using *lfilter()* function of scipy library and plotted against number of days for $R_o = 2.5$.

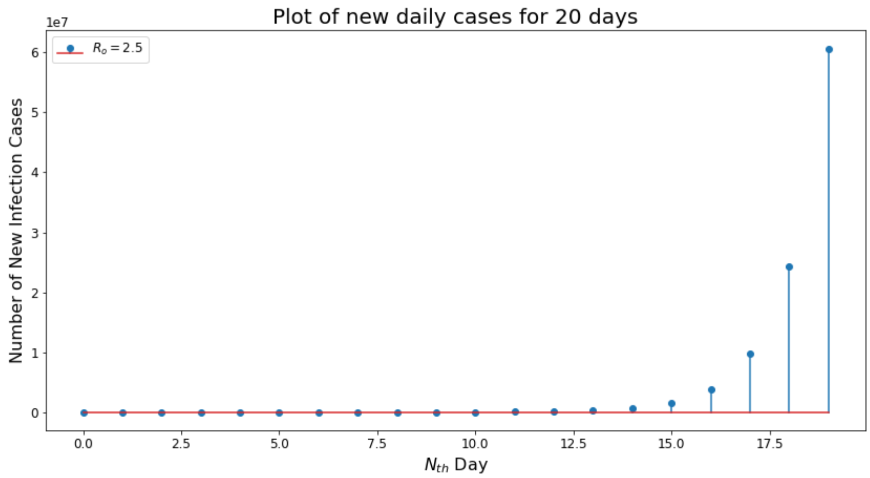

Figure 13: Total Number of Cases for $R_o = 2.5$

$\boxed{\textbf{Problem 2}}$

(Transfer Function)

(Solution)

In the given system, the number of daily cases depends not only the number of cases a day ago but also on the number of cases up to 14 cases because the given infection effectively last up to 14 days for a given infected person. Hence the system can be modelled as :

$$y[n] = \delta[n] + \sum_{k=1}^{M} a_k y(n-k) \tag{0.6}$$

where a_k is the number of people, a person who was affected k days ago, is going to infect.

For the given system the transfer function is given by :

$$H(z) = \frac{1}{1 - \sum_{k=1}^{M} a_k z^{-k}} \tag{0.7}$$

The plots for the daily new cases and cumulative cases for 100 days are found using the *lfilter()* function of the scipy library and the integrator of the first problem and are plotted against the number of days.

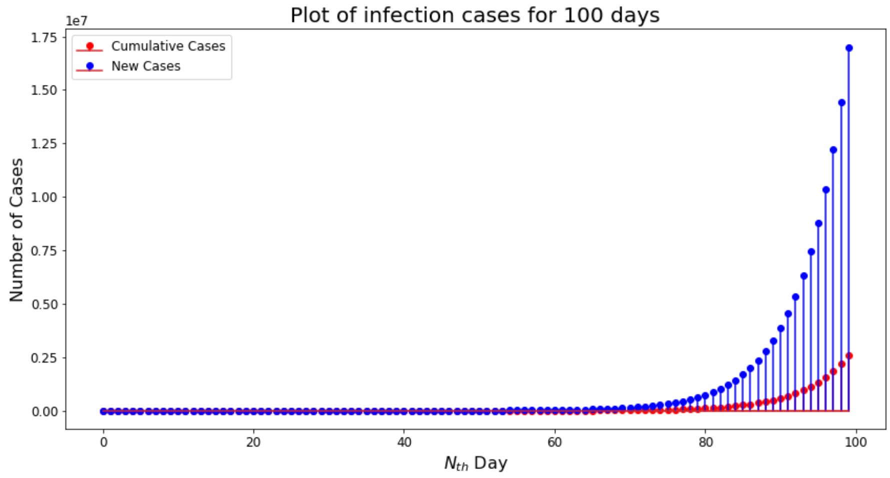

Figure 14: Number of Infection Cases

(Comparison with First System and Reaching 1 Million Cases)
(Solution)
The system takes **94 days** to reach 1 million daily new case while it takes **16 days** to reach 1 million for the first system wih $R_o = 2.5$. Even though the area under the curve of the Reproduction Rate $a_k = 2.5$, the second system grows slowly. This can be understood from the pole zero plot of the system.

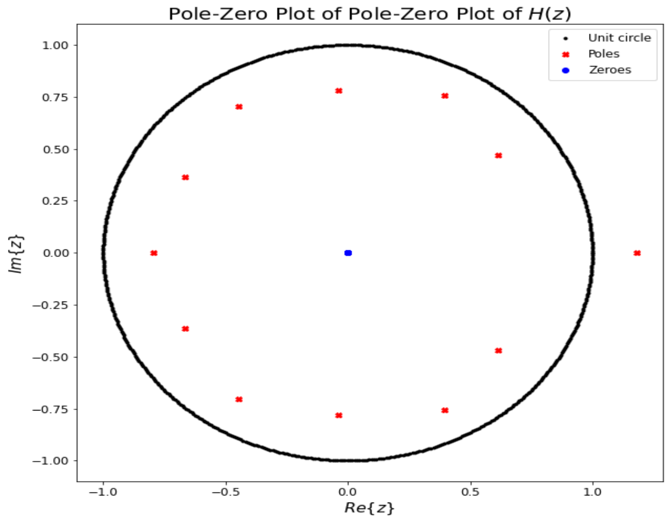

Figure 15: Pole Zero Plot

In the pole zero plot, we can see 11 poles are stable except a single pole which lies outside and causes the system to be unstable. But the value is much smaller than 2.5 and hence it has a smaller growth rate.

(Estimating a_k)

(Solution)

To estimate the values of a_k, we can take $2 \times M$ consecutive days. Each M consecutive days will give an equation of the form :

$$y[n] = \delta[n] + \sum_{k=1}^{M} a_k y(n-k) \tag{0.8}$$

Hence $2 \times M$ consecutive days will give M equations. This forms a system of linear equations of M variables (a_k is variable, since we already know number of daily new cases from the data). This system of equations can be expressed as the given form and solved by taking matrix inverse.

$$Y^{'}A = Y \tag{0.9}$$

$$A = Y(Y^{'})^{-1} \tag{0.10}$$

The reduce the effect of noise in the data we can take multiple sets of M equations and take the average of all these sets for each a_k.

A better way to find each value of a_k is to use linear regression of M dimensions. To reduce the error we can take mean square error as the loss function and use gradient descent to find the direction of minimum loss and move the vector A in the calculated direction. This can be done for multiple points and the error can be reduced.

Problem 3

(Social Distancing)

(Solution)

In the new system, people are aware of the infection and take due measures to reduce the spread. Hence they follow social distancing which reduces the spreading rate by ρ. The new system can be modelled as :

$$y[n] = \delta[n] + \sum_{k=1}^{M}(1-\rho)a_k y(n-k) \tag{0.11}$$

The value of ρ indicates the effectiveness of the social-distancing of an individual and $\rho\epsilon(0,1]$. If $\rho = 0$, then the individual is completely ignoring social distancing and the growth rate is as normal. But if $\rho = 1$, indicates that the social interaction is 0 and hence there is no spreading of the infection due to the individual. The transfer function of the this system is given by:

$$H(z) = \frac{1}{1-\sum_{k=1}^{M}(1-\rho)a_k z^{-k}} \tag{0.12}$$

(Plotting Number of Cases)

(Solution)

The plots for the daily new cases for 100 days are found using the *lfilter()* function of the scipy library and are plotted against the number of days for $\rho = \{0.25, 0.5, 0.75\}$ and the cumulative cases is found using the integrator of the first problem.

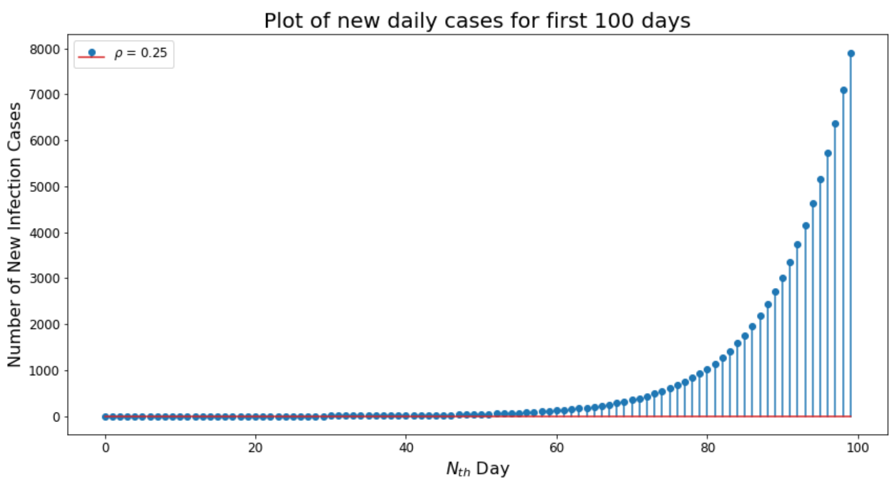

Figure 16: Number of Infection Cases

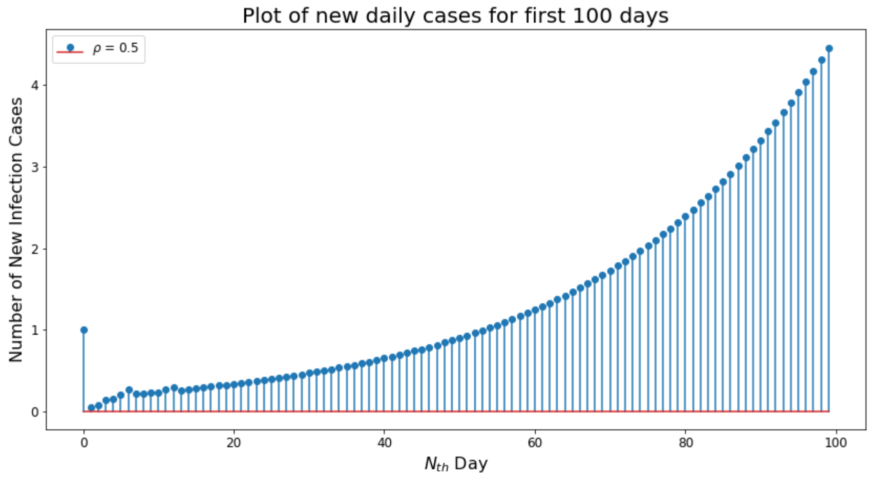

Figure 17: Number of Infection Cases

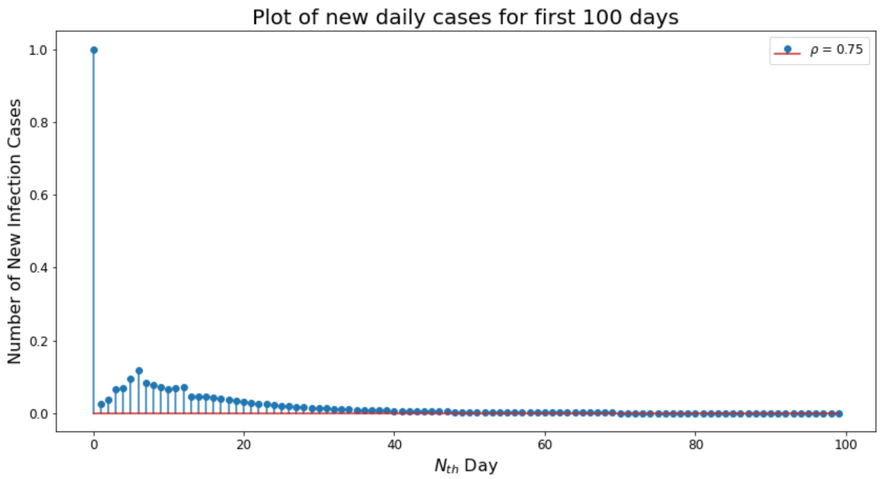

Figure 18: Number of Infection Cases

The total number of cases for $\rho = \{0.25, 0.5, 0.75\}$ after 100 days is {77744.79,133.89,2.53} respectively.

(Effect of Social Distancing)
(Solution)

For $\rho = 0.25$, we can see that even though the rate is reduced but it remains exponential and reaches upto 8000 daily cases on the 100_{th} day. When $\rho = 0.5$, the rate drastically reduces. The rate at which grows is not completely exponential but is partly linear because the stable components are dominant till for smaller values of n and the number of new cases on 100_{th} day is only just above 8. But still, it is not bounded and the number of daily new cases is increasing as n is increasing. But when $\rho = 0.75$, the daily new cases completely reduces. It reduces to 0 as $n \to \infty$. This is because all the poles are inside the unit circle in the new model and hence it is a completely stable system. From the above three cases we can see the effectiveness of social distancing. As every individual reduces social interaction, the spread of the infection decreases effectively and hence social distancing can be very good measure to reduce the total number of infections.

Problem 4

(Logistic Model)

(Solution)

In the case of a viral epidemic, as more and more people contract the disease and achieve immunity, the rate of transmission for the infection progressively decreases. If the rate of diffusion is assumed to be inversely proportional to the fraction of healthy people in a population, the evolution of the cumulative number of infections $y_t[n] = \sum_{k=0}^{n} y[k]$ since the beginning of the disease can be modelled by a logistic function :

$$y_t(n) = \frac{K}{1 + (K(R_o - 1) - R_o)R_o^{-(n+1)}} - \frac{1}{R_o - 1} \tag{0.13}$$

where k is population size. The notable thing about the logistic function is that it has a clear inflection point, after which the epidemic starts to level out; this corresponds to the moment in which the implicit reproduction number becomes less than one. It would be useful to detect the inflexion point because in that case some of the more restrictive measures could start to be relaxed gradually.

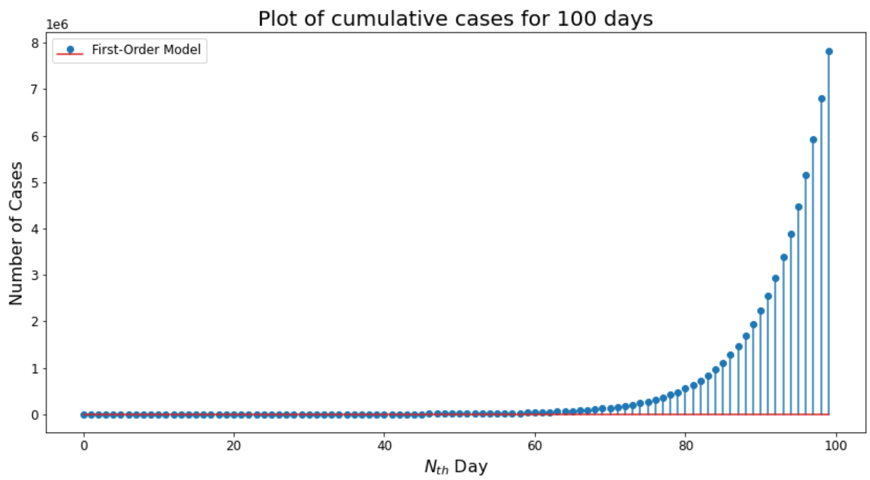

Figure 19: Cumulative Cases for First Order Model

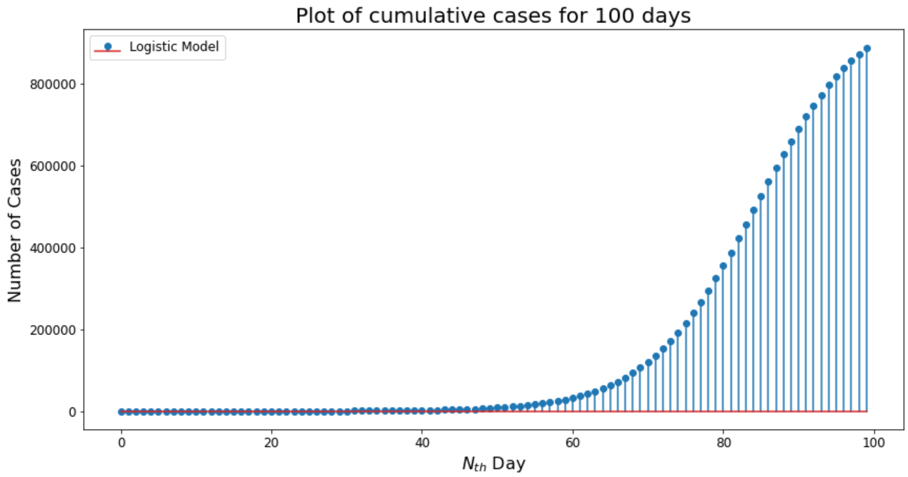

Figure 20: Cumulative Cases for Logistic Model

We can see that the logistic model with $K = 10^6$ and $R_o = 1.15$ reaches up to 8×10^5 total number of cases by 100 days while the first model reaches up to 8×10^6 total cases in the same time, which is about 10 times. In the logistic model we can see that rate of growth is decreasing as n increasing after approximately 90 days. Hence with a limited population the rate of growth of the cases reduces and hence the first model is not very accurate and the logistic model gives a better idea. The first model is the approximate of the logistic model as the population size is very large and n is considered for a limited number.

(Inflection Model)

(Solution)

The inflection point corresponds to the global maximum of the first derivative of the logistic function. In order to find the inflection points we can use 2 methods namely, using the global maximum of the first derivative and the zero-crossing of the second derivative. The first derivative can be approximated by simple 2-tap FIR filter of the form $D_1(z) = 1 - z^{-1}$ while the zero-crossing of the second derivative can be approximated with the FIR filter of the form $D_2(z) = 1 - 2z^{-1} + z^{-2}$. The absolute maximum is found using *find_peaks()* function from the scipy library and zero crossing of the second derivative is point where the second derivative changes the sign of the magnitude. The point of inflection point found using both methods is **85 days**.

Code Repository

The code, input and output of all the problems is in the following repository : `https://github.com/KaranTejas/DSP-Lab/tree/main/Experiment8`.