School of Computing & Engineering Sciences

Bachelor of Informatics & Computer Science

ICS 2106: Probability & Statistics II

End of Semester Examinations

Date: 14th December 2023 **Time: 13:00 - 15:00 Hours**

Instructions:

(i) Answer Question one and any other two questions.

(ii) Show all your workings clearly.

(iii) This paper consists of Two Printed Pages.

Question 1 (30 Marks)

(a) Let X and Y be two jointly continuous random variables with joint probability density function given by

$$f(x,y) = \begin{cases} 2, & y+x \leq 1, \quad x>0, \quad y>0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Obtain the marginal density of X and Y, i.e. $f_1(x)$ and $f_2(y)$. **[5 Marks]**

(ii) Are the random variables X and Y independent? Justify your answer. **[2 Marks]**

(iii) Find the covariance of X and Y, i.e. σ_{xy} **[4 Marks]**

(iv) Obtain the correlation coefficient between X and Y, i.e. ρ_{xy} **[2 Marks]**

(b) Suppose that the joint probability mass function of the random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{x+y}{30}, & x=0,1,2,3, \quad y=0,1,2 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Verify that $f(x,y)$ is a probability mass function. **[3 Marks]**

(ii) Find the $P(X>Y)$. **[3 Marks]**

(iii) Obtain $P(X+Y=4)$. **[2 Marks]**

(c) Suppose X and Y have bivariate normal density with mean vector and covariance matrix given by

$$\vec{\mu} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

(i) Compute $Var(2X + 6Y - 9)$. **[3 Marks]**

(ii) Obtain the conditional distribution of Y given X. **[4 Marks]**

(iii) Find the regression equation of Y on X. **[2 Marks]**

Question 2 (20 Marks)

(a) Consider the joint probability density function of the random variables X and Y given by

$$f(x, y) = \begin{cases} k(2x + 3y), & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find the value of the constant k. **[3 Marks]**

(ii) Evaluate the probability, $P[(X, Y) \in A]$, where A is the region described as $\{(x, y)|0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$. **[3 Marks]**

(iii) Compute the $P(X + Y \leq 1)$. **[3 Marks]**

(b) Let X and Y be continuous random variables with joint density function $f(x, y)$ and marginal densities $f_1(x)$ and $f_2(y)$ respectively. Show that $E(Y) = E[E(Y|X)]$. **[4 Marks]**

(c) Prove that if X and Y are jointly continuous random variables, then

$$E(X) = \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y) dy$$

[4 Marks]

(d) Electronic components of a certain type have a life length X with a probability density function given by

$$f(x) = \begin{cases} \frac{1}{100} e^{-\frac{1}{100}x}, & x > 0, \\ 0, & \text{elsewhere} \end{cases}$$

(Life length is measured in hours). Suppose that two such components operate independently and in series in a certain system, i.e., the system fails when either component fails. Find the density function for Y, the life length of the system.

[3 Marks]

Question 3 (20 Marks)

Let the joint probability mass function of the random variables X and Y be given by the following table:

(x, y)	$f(x, y)$
$(1, 1)$	$\frac{3}{8}$
$(2, 1)$	$\frac{1}{8}$
$(1, 2)$	$\frac{1}{8}$
$(2, 2)$	$\frac{3}{8}$

(a) Obtain the following probability distributions:

(i) $f_{X|Y}(x|y)$. **[3 Marks]**

(ii) $f_{Y|X}(y|x)$. **[3 Marks]**

(b Find the values of the following random variables:

(i) $\mu_{x|y}$, the conditional mean of X given Y. **[3 Marks]**

(ii) $\mu_{y|x}$, the conditional mean of Y given X. **[3 Marks]**

(c) Describe the random variable $\sigma^2_{Y|X}$, the conditional variance of Y given X. **[5 Marks]**

(d) Are X and Y independent? Justify your answer. **[3 Marks]**

Question 4 (20 Marks)

(a) The management at a fast-food outlet is interested in the joint behaviour of the random variables Y_1, defined as the total time between a customer's arrival at the store and the departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of the observed values of Y_1 and Y_2 can be modelled by the probability density function given by

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{otherwise} \end{cases}$$

with time measured in minutes. Find:

(i) $P(Y_1 > 2, Y_2 > 1)$. **[3 Marks]**

(ii) $P(Y_1 \geq 2Y_2)$. **[3 Marks]**

(iii) $P(Y_1 - Y_2 \geq 1)$, (Notice that $Y_1 - Y_2$ denotes the time spent at the service window). **[4 Marks]**

(b) Let the joint $p.d.f$ of the random variables X and Y be given by

$$f(x, y) = \begin{cases} 8xy, & 0 \leq x \leq 1, \quad 0 \leq y \leq x \\ 0, & \text{elsewhere} \end{cases}$$

Find the conditional expected value of:

(i) Y given X, $E(Y|X)$. **[5 Marks]**

(ii) X given Y, $E(X|Y)$. **[5 Marks]**

Question 5 (20 Marks)

(a) Suppose that X_1 and X_2 are independent exponential random variables with parameter $\theta = 1$ so that their joint $p.d.f$ is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} e^{-x_1-x_2}, & 0 \le x_1 < \infty, \quad 0 \le x_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Consider the transformation given by $U = X_1 - X_2$ and $V = X_1 + X_2$. Obtain the joint distribution of U and V using the Jacobian of transformation hence obtain the marginal probability density functions of U and V. **[9 Marks]**

(b) Consider two discrete random variables X and Y whose joint probability mass function is given in tabular form below:

	Y			
X	0	1	2	$f_1(x)$
0	0.10	0.10	0.20	0.40
1	0	0.15	0.05	0.20
3	0.10	0.20	0.10	0.40
$f_2(y)$	0.20	0.45	0.35	1

(i) Obtain the covariance of X and Y, i.e. σ_{xy}. **[5 Marks]**

(ii) Calculate the correlation coefficient between X and Y, i.e. ρ_{xy} and interpret the results. **[6 Marks]**