CSC492: Combinatorial Volumes of the Polar-duals of Matroid polytopes

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CSC492: Combinatorial Volumes of the Polar-duals of Matroid polytopes

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Abstract

We develop a combinatorial approach toward computing volumes of polar duals of matroid polytopes.

CONTENTS

CHAPTER 1

PRELIMINARIES

1.1 MATROIDS

This section covers the introduction to matroids. Matroids are the core mathematical objects that we will be dealing with in this paper. Our main reference for this section is Oxley [Oxl11].

Definition 1.1.1. Let the *ground set E* be a non-empty finite set.

The set of subsets of the set E, denoted by $\mathcal{B}$ is called a *collection of bases* if and only if $\mathcal{B}$ is non-empty and follows the *basis exchange axiom*, which is defined as follows:

$$\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \setminus B_2, \exists y \in B_2 \setminus B_1 : (B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B} \tag{1.1.1}$$

The pair of *ground set* and *collection of bases*, $(E, \mathcal{B})$ is called a *matroid*.

Next, we create our first matroid, denoted by the symbol $\mathsf{M}_\square$.

Example 1.1.2. Let $E = \{1, 2, 3\}$ and $\mathcal{B} = \{\{1, 2\}, \{2, 3\}\}$. When we set E as the ground set, the set $\mathcal{B}$ is both non-empty and satisfies the basis exchange axiom. Thus, as per Definition 1.1.1, the pair $(E, \mathcal{B})$ is a matroid. This matroid will now be referred to as $\mathsf{M}_\square$.

In the context of matroids, the term *independent* is formally defined as follows:

Definition 1.1.3. For every matroid, the elements of the collection of bases are *independent*.

Example 1.1.4. For $\mathsf{M}_\square$, {1,2}, {2,3}, $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$ are the only independent sets as per Definition 1.1.3.

The sets that are not independent are *dependent*.

Definition 1.1.5. For every matroid, a *dependent* set is a set of elements from the ground set that is not a subset of any of the bases.

In the following example, we find all the subsets of the ground set from the matroid $\mathsf{M}_\square$ that are dependent.

Example 1.1.6. By Definition 1.1.5 all the dependent subsets of E in $\mathsf{M}_\square$ are $\{1, 3\}$ and $\{1, 2, 3\}$.

The concept of *rank* in matroid theory is formally defined as follows:

Definition 1.1.7. The *rank* of a subset X is the size of the largest independent set contained in X. [Devon]

In the next example, we will compute the rank of all the subsets of the ground set of the matroid $\mathsf{M}_\square$.

Example 1.1.8. The following table contains the rank of all the subsets of the ground set E of the matroid $\mathsf{M}_\square$:

Subsets of E	$\mathrm{rk}_{\mathsf{M}_\square}$
$\emptyset$	0
$\{1\}$	1
$\{2\}$	1
$\{1,2\}$	2
$\{3\}$	1
$\{1,3\}$	1
$\{2,3\}$	2
$\{1,2,3\}$	2

Table 1.1.1: Subsets and their rank

The *dual* of a matroid $\mathsf{M}_{(E,\mathcal{B})}$ is defined as follows:

Definition 1.1.9. The *dual* operation on a matroid $(E,\mathcal{B})$ denoted by * transforms the collection of bases, $\mathcal{B}$ to be defined as $\{E - B : B \in \mathcal{B}\}$.

Consider the dual of $\mathsf{M}_\square$, which will be denoted by $\mathsf{M}^*_\square$.

Example 1.1.10. As per Definition 1.1.9 $\mathsf{M}^*_\square$ has the same ground set as $\mathsf{M}_\square$ and the collection of bases for $\mathsf{M}^*_\square$ is { $\{1\}$, $\{3\}$ }.

Circuits in matroid are defined as follows:

Definition 1.1.11. For every matroid M, any minimally dependent set is a *circuit*. The collection of all *circuits* in matroid M is represented as $\mathcal{C}(\mathsf{M})$.

In other words, removing one element from a circuit X should not change $\mathrm{rk}_{\mathsf{M}}(X)$ but removing 2 elements should reduce $\mathrm{rk}_{\mathsf{M}}(X)$.

Circuits enable us to label elements of the ground set of the matroids as *loops* and *co-loops*.

Definition 1.1.12. If a single element e from the ground set of a matroid M forms a circuit M, it is called a *loop*. Alternatively, if an element of the ground set is not in any circuits it is called a *co-loop*.

Definition 1.1.12 implies that a loop cannot belong to any basis of M, while a co-loop will be included in every basis of M.

To apply the definition of circuits, consider the following example.

Example 1.1.13. Let M be the matroid over the ground set $E = \{1,2,3,4\}$ and bases, $\mathcal{B} = \{\{1,2\},\{2,3\}\}$.

Here, the $\mathcal{C}(\mathsf{M})$ = { $\{4\}$, $\{1,3\}$ }, 2 is a co-loop and 4 is a loop.

1.2 Polytopes

This section defines Polytopes, which is another important topic for this paper. The main reference for this section is the book Lectures on Polytopes by Ziegler [Zie95].

The space we are going to define our polytope in is called the *affine space*. To understand affine spaces you must understand what an *affine combination* is.

Definition 1.2.1. Suppose $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^d$. Then a vector $\mathbf{b}$ is called an *affine combination* of $\mathbf{a}_1, \ldots, \mathbf{a}_k$ if and only if

$$\sum_{i=1}^{k} \lambda_i = 1 \text{ and } \sum_{i=1}^{k} \lambda_i \mathbf{a}_i = \mathbf{b} \tag{1.2.1}$$

where $\lambda_i \in \mathbb{R}$.

We have defined affine combinations for any dimension d. However, in the context of this paper, you should expect d to be a natural number. For example, consider the following affine combination.

Example 1.2.2. Let $\mathbf{a}_1 = (0,1)^T$ and $\mathbf{a}_2 = (0,1)^T$. Then consider $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_2 = 1 - \lambda_1$. Therefore, an arbitrary affine combination of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ can be denoted by the vector $\mathbf{b} = (\lambda_1, 1 - \lambda_1)$. As you might have noticed, the line $y = 1 - x$ represents all possible values for the coordinates of the vector $\mathbf{b}$. Refer to the image below for a visual representation.

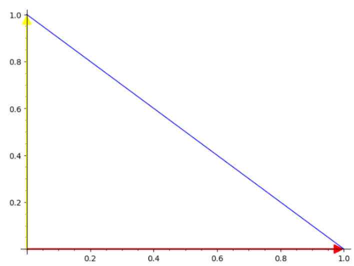


Figure 1.2.1: Affine combinations (blue) of $\mathbf{a_1}$ (red) and $\mathbf{a_2}$ (yellow)

Now we can define *affine space*.

Definition 1.2.3. An *affine space* is a space that is closed under affine combinations. In other words, $\mathbb{L}$ is an affine space if and only if:

$$\forall \mathbf{a}_1, ..., \mathbf{a}_k \in \mathbb{L}, \forall \lambda_1, \ldots, \lambda_m \in \mathbb{R}, s.t. \sum_{i=1}^{m} \lambda_i = 1 \Rightarrow \sum_{i=1}^{m} \lambda_i \mathbf{a}_i \in \mathbb{L} \tag{1.2.2}$$

In fact, we have already seen our first affine space in the form of a line $y = x - 1$ *spanning* the affine combinations of the vectors $(1,0)^T$ and $(0,1)^T$ in Example 1.2.2.

For the sake of clarity, it is important to define the term *span*. In the context of affine spaces, the word span is a shorthand for *affine span*.

Definition 1.2.4. The *affine span* of $A \subseteq \mathbb{R}^n$ is the set of all affine combinations of vectors in A, denoted by $L(A)$.

Definition 1.2.4 implies that $L(\emptyset)$ is $\emptyset$ itself. Additionally, the affine span of a single vector is the same vector itself.

The concept of *dependence* and *independence* in affine spaces is defined as the following:

Definition 1.2.5. The set X is *affinely dependent* if and only if $\exists \mathbf{a}_i \in \mathbb{L}$ where $\mathbf{a}_i$ is an affine combination of elements from the set $X - \{\mathbf{a}_i\}$. Otherwise, if and only if no such $\mathbf{a}_i$ exists then X is *affinely independent*.

To consolidate our understanding of affine independence and dependence, consider the next example.

Example 1.2.6. First, consider the set X within $\mathbb{R}^2$. Let $X = \{(0,1),(0,1),(\frac{1}{2},\frac{1}{2})\}$. Clearly,

$$(\frac{1}{2},\frac{1}{2}) = \frac{1}{2} * (0,1) + \frac{1}{2} * (1,0) \tag{1.2.3}$$

Therefore by Definition 1.2.1 $(\frac{1}{2},\frac{1}{2})$ is an affine combination of (1,0) and (0,1). Thus implying that the set X is affinely dependent by Definition 1.2.5.

Next, consider the set $Y = \{(1,0)\}$. Note that $Y - (1,0) = \emptyset$. As noted earlier, the span of $\emptyset$ is $\emptyset$. Thus Y is affinely independent by Definition 1.2.5.

Now, we have finally finished defining the concept of affine spaces and can begin with the creation of objects within this space. For starters, consider the simple object called a *convex*.

Definition 1.2.7. Let $\mathbb{L}$ be an affine space. A set of points $K \subseteq \mathbb{L}$ is *convex* if and only if given any two points $\mathbf{x}, \mathbf{y} \in K$, K also contains the straight line segment $[\mathbf{x}, \mathbf{y}] = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} : 0 \leq \lambda \leq 1\}$ between them.

Let us go back to the affine space that we created in Example 1.2.2, which is the line $y = 1 - x$, and create an example of a set of points K that is a convex and a set of points W that is not a convex.

Example 1.2.8. Here is the image of the affine space (blue line) from Definition 1.2.1. On top of the image look at the point sets K and W.By Definition 1.2 the set K is a convex because it contains the green line as shown in the image below. Similarly, the set W is not a convex because it does not contain the red line as shown in the image below.

Every intersection of convex is convex. Thus, a convex can contain infinitely many convex within it. The most minimal convex of a point set is called the *convex hull*.

Definition 1.2.9. Let K be a set of points in an arbitrary affine space $\mathbb{L}$. The *convex hull* of K is denoted by $\mathrm{conv}(K)$ and is the intersection of all convex sets K' in $\mathbb{L}$ that contain K :

$$\mathrm{conv}(K) \stackrel{\text{def}}{=} \bigcap \{K' \subseteq \mathbb{L} : K \subseteq K', K' \text{ convex }\} \tag{1.2.4}$$

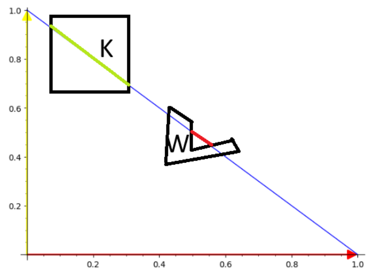


Figure 1.2.2: Image of convex K and point set W which is not a convex

If $K = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathbb{L}$ is itself finite, then its convex hull is

$$\operatorname{conv}(K) \stackrel{\text{def}}{=} \left\{ \lambda_1 \mathbf{x}_1 + \ldots + \lambda_n \mathbf{x}_n : n \geq 1, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\} \tag{1.2.5}$$

In the example ahead, we familiarize ourselves with both versions of Definition 1.2.9.

Example 1.2.10. Let the set {(0,1), (1,0)} be denoted by K. By Definition 1.2.9 the convex hull of {(0,1), (1,0)} is

$$\operatorname{conv}(K) = \{\lambda_1(0,1) + \lambda_2(1,0) : \lambda_1, \lambda_2 \geq 0, \lambda_2 = 1 - \lambda_1\} \tag{1.2.6}$$

Notice that this is the first quadrant of the affine span from Example 1.2.2.

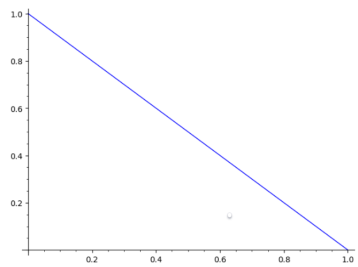


Figure 1.2.3: The blue line is a visual representation of $\operatorname{conv}(K)$

The convex hull is a well-defined operation on affine spaces that allows us to create objects. The object that arises from a convex hull is named the *$\mathcal{V}$-polytope*.

Definition 1.2.11. A *$\mathcal{V}$-polytope* is the convex hull of a finite set of points in some affine space.

The geometrical objects such as *hyper-cubes* can be generalized as polytopes.

Example 1.2.12. To create a hypercube in 3 dimensions, consider the point set, {(1, -1, -1), (1, 1, -1), (1, 1, 1), (1, -1, 1), (-1, -1, 1), (-1, -1, -1), (-1, 1, -1), (-1, 1, 1)}

By taking the convex hull of these points we derive a cube as a $\mathcal{V}$-polytope in $\mathbb{R}^3$

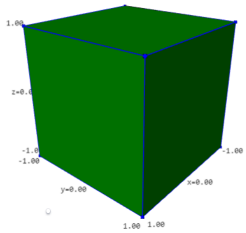


Figure 1.2.4: Visual representation cube as a $\mathcal{V}$-polytope in $\mathbb{R}^3$

There are situations where $\mathcal{V}$-polytope is not the most ideal because the convex hull might be tricky or expensive to compute. As an alternative, we can take the intersection of the half-spaces to describe our object. This representation of polytopes is called the *$\mathcal{H}$-polytope*.

Definition 1.2.13. Let $\mathbb{R}^d$ be an arbitrary affine space in dimension d. A *$\mathcal{H}$-polytope* is an intersection of closed half-spaces: a set $P \subseteq \mathbb{R}^d$ presented in the form:

$$P = P(A, \mathbf{z}) = \left\{ \mathbf{x} \in \mathbb{R}^d : A\mathbf{x}^T \leq \mathbf{z} \right\} \text{ for some } A \in \mathbb{R}^{m \times d}, \mathbf{z} \in \mathbb{R}^m \tag{1.2.7}$$

Here $A\mathbf{x} \leq \mathbf{z}$ is the notation for a system of inequalities $\mathbf{a}_1\mathbf{x}^T \leq z_1, \cdots, \mathbf{a}_m\mathbf{x}^T \leq z_m$ where $\mathbf{a}_1, \cdots, \mathbf{a}_m$ are rows of A and $z_1, \cdots, z_m$ are components of the vector $\mathbf{z}$.

Before moving on to an example to fully explain Definition 1.2.13 note that $\mathcal{V}$-polytope and $\mathcal{H}$-polytope are interchangeable as per the following remark.

Remark 1.2.14. A subset $P \subseteq \mathbb{R}^d$ is the convex hull of a point set ($\mathcal{V}$-polytope)

$$P = \operatorname{conv}(V) \text{ for some } V \in \mathbb{R}^{d \times n} \tag{1.2.8}$$

if and only if it is bounded intersection of half-spaces ($\mathcal{H}$-polytope)

$$P = P(A, \mathbf{z}) \text{ for some } A \in \mathbb{R}^{m \times d}, \mathbf{z} \in \mathbb{R}^m \tag{1.2.9}$$

We will call this theorem the *Main Theorem of Polytopes*.

Example 1.2.15. Define a d-simplex as the convex hull of any $d+1$ affinely independent points in $\mathbb{R}^n (n \geqslant d)$. Thus a d-simplex is a polytope of dimension of d with $d+1$ vertices. We define the standard d-simplex to be denoted by Δ_d, which consists of $d+1$ vertices in $\mathbb{R}^{d+1}$.

$$\Delta_d \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \mathbb{1}\mathbf{x}^T = (\mathbf{1}), x_i \geqslant 0 \right\} = \operatorname{conv}\left\{ \mathbf{e}_1, \cdots, \mathbf{e}_{d+1} \right\} \tag{1.2.10}$$

Note that $\mathbb{1}$ here represents the matrix of size $d+1 \times 1$ completely filled with 1s.

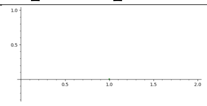
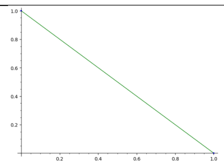
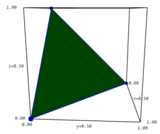
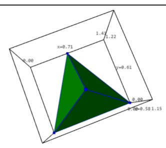
Example	Polytope Representation	H-Representation	V-Representation
Δ_0		$\begin{pmatrix}1\end{pmatrix}\begin{pmatrix}x_0\end{pmatrix} = \begin{pmatrix}1\end{pmatrix}$	$\mathrm{conv}\{e_1\}$
Δ_1		$\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}x_0\\x_1\end{pmatrix}^T = \begin{pmatrix}1\end{pmatrix}$	$\mathrm{conv}\{e_1, e_2\}$
Δ_2		$\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}^T = \begin{pmatrix}1\end{pmatrix}$	$\mathrm{conv}\{e_1, e_2, e_3\}$
Δ_3		$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix}^T = \begin{pmatrix}1\end{pmatrix}$	$\mathrm{conv}\{e_1, e_2, e_3, e_4\}$

Table 1.2.1: Representation of Simplex Polytopes Δ_0 to Δ_3

Definition 1.2.16. A *cone* is a nonempty set of vectors $C \subseteq \mathbb{R}^d$ that with any finite set of vectors also contains all their linear combinations with non-negative coefficients.

Similar to convex hull, for cones we define a *conical hull*. For an arbitrary subset $Y \subseteq \mathbb{R}^d$, conical hull, $\mathrm{cone}(Y)$ is the intersection of all cones in $\mathbb{R}^d$ that contain Y.

$$\mathrm{cone}(Y) = \{\lambda_1 \mathbf{y}_1 + \ldots + \lambda_k \mathbf{y}_k : \{\mathbf{y}_1, \ldots, \mathbf{y}_k\} \subseteq Y, \lambda_i \geq 0\} \tag{1.2.11}$$

In the case where $Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_n\} \subseteq \mathbb{R}^d$ is a finite set this reduces to

$$\mathrm{cone}(Y) \stackrel{\mathrm{def}}{=} \{t_1 \mathbf{y}_1 + \ldots + t_n \mathbf{y}_n : t_i \geq 0\} \tag{1.2.12}$$

Let us do a simple example to familiarize ourselves with cones.

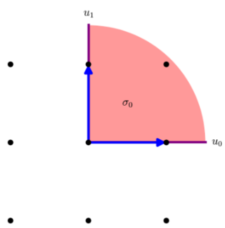


Figure 1.2.5: $\mathrm{cone}(\{(1,0),(0,1)\})$

Example 1.2.17. Consider $\text{cone}(\{(1,0),(0,1)\})$ in $\mathbb{R}^2$. By Definition 1.2.16, $\text{cone}(\{(1,0),(0,1)\}) = \{t_1(0,1) + t_2(1,0) : t_1, t_2 \geq 0\}$. This should cover the entire first quadrant as shown in Figure 1.2.5.

1.3 Matroid polytopes

The polytopes that arise from matroids are called *matroid polytopes*.

Definition 1.3.1. Let M be a matroid with an arbitrary labeling of the ground set by $0, \ldots, n-1$. A *matroid polytope* is defined as the convex hull of the vectors $\mathbf{e}_B$ over all bases $\mathbf{B}$ of the matroid.

$$\mathbf{e}_B = \sum_{i \in B} \mathbf{e}_i \tag{1.3.1}$$

Here $\mathbf{e}_i$ are the standard basis vectors of $\mathbb{R}^n$.

Given n number of elements in the ground set and choosing all combinations of subsets of size r as bases results in a matroid as per Definition 1.1.1. These types of matroids are called *uniform matroids*.

Definition 1.3.2. To illustrate the previous definition (Definition 1.3.1), examine a class of matroids known as *uniform matroids*. In a uniform matroid $\mathsf{U}_{r,n}$, all subsets of size r or less are independent, while all larger subsets are dependent. For further information on uniform matroids and their properties, refer to [Oxl2011], page 660.

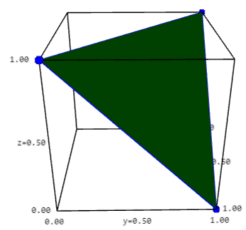


Figure 1.3.1: Visual of $P(U_{2,3})$

Example 1.3.3. Consider the uniform matroid $\mathsf{U}_{2,3}$. Then

$$P(\mathsf{U}_{2,3}) = \text{conv}\{(1,1,0),(1,0,1),(0,1,1)\} \tag{1.3.2}$$

This is shown in Figure 1.3.1

Similar to the notion of duality in matroids, there is a notion of *polarity* in polytopes that we are going to leverage in this paper.

Definition 1.3.4. Let P be a polytope in $\mathbb{R}^d$. The *polar dual*, P° of P is defined as the set of all points $\mathbf{y}$ in $\mathbb{R}^d$ such that for all points $\mathbf{x}$ in P, the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ is less than or equal to 1:

$$P^\circ = \left\{ \mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{x} \in P \right\} \tag{1.3.3}$$

Here are some important remarks about P°:

Remark 1.3.5. It is worth noting that the polar dual P° is a polytope.

Remark 1.3.6. If P is bounded, then P° is also bounded.

Remark 1.3.7. Similar to duality in matroids, the polar dual of the polar dual of a polytope P is P itself, symbolically, $(P^\circ)^\circ = P$.

Moreover, in the next example, we obtain the polar dual of the polytope from Example 1.3.2, $P^\circ(U_{2,3})$.

Example 1.3.8. For this, recall that our vertices are the following:

$$\mathbf{e_{0,1}} = (1,1,0)\ ,\ \mathbf{e_{0,2}} = (1,0,1)\ ,\ \mathbf{e_{1,2}} = (0,1,1) \tag{1.3.4}$$

We therefore have the following inequalities for the x, y and z components of point (x, y, z) in the polar dual $P^\circ(U_{2,3})$: $\langle \mathbf{e_{0,1}}, (x,y,z)\rangle \leq 1 \Rightarrow x + y \leq 1$;$\langle \mathbf{e_{0,2}}, (x,y,z)\rangle \leq 1 \Rightarrow x + z \leq 1$; $\langle \mathbf{e_{1,2}}, (x,y,z)\rangle \leq 1 \Rightarrow y + z \leq 1$.

By taking the intersection of the inequalities we get the following visual of $P^\circ(U_{2,3})$:

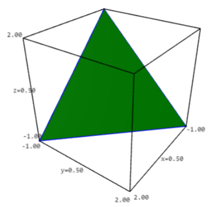


Figure 1.3.2: Visual of $P^\circ(U_{2,3})$

CHAPTER 2

INTRODUCTION

This chapter aims to gain insight into more powerful tools to finally combinatorically compute the *volume* of the polar dual of the matroid polytope.

2.1 HEPP BOUND FOR GRAPHS

Hepp Bound is introduced by Panzer [Pan22], it has many interesting properties that are not the focus of this paper. However, learning about the Hepp Bound enables us to understand the nature of the volume of a polar matroid polytope.

Before computing Hepp bound for Matroids for simplicity first consider computing Hepp bound for undirected graphs. Like Panzer, we first define the *co-rank* of a graph.

Definition 2.1.1. For an arbitrary undirected graph $G = (V, E)$, the co-rank $\mathrm{crk}(G)$ is defined in terms of the number of vertices $|V_G|$, number of edges $|E|$ and the number of connected components $\kappa(G)$ as follows:

$$\mathrm{crk}(G) \stackrel{\text{def}}{=} |E| - |V_G| + \kappa(G) \tag{2.1.1}$$

Notice that when a graph G has no loops $\mathrm{crk}(G) = 0$. Thus, co-rank is the number of edges that need to be removed to make the input graph a forest. A forest is a graph with no cycles.

Consider the undirected graph $G_\triangle$, which consists of three vertices labelled $1, 2$, and 3. The edges of $G_\triangle$ are defined by the set $\{(1,2), (2,3), (3,1)\}$, where each tuple (u, v) denotes an edge connecting vertex u to vertex v.

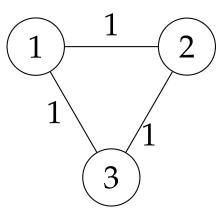


Figure 2.1.1: Visual representation of the graph $G_\triangle$.

Here is the co-rank computation for $G_\triangle$.

Example 2.1.2. The Graph $G_{\triangle}$ has 3 vertices, 3 edges, and forms a single connected component.

Thus by Definition 2.1.1 the co-rank of $G_{\triangle}$ is 1.

The *superficial degree of convergence* of a sub-graph of an arbitrary undirected graph is a linear function defined by Panzer.

Definition 2.1.3. For a sub-graph $\gamma \subseteq G$, the *superficial degree of convergence* $\omega(\gamma)$ is defined as follows:

$$\omega(\gamma) \stackrel{\text{def}}{=} \sum_{e \in \gamma} a_e - \frac{d}{2} \cdot \text{crk}(\gamma) \tag{2.1.2}$$

where a_e represents the weight of edge e, d is the dimension, and $\text{crk}(\gamma)$ is the co-rank of the sub-graph γ.

Like Panzer, we impose the condition $\omega(G) = 0$ for all undirected graphs G. We will call this constraint *logarithmic divergence*. Notice that the dimension d is irrelevant for graphs without loops as the co-rank evaluates to 0. For graphs with loops, the dimension d is defined as follows:

Definition 2.1.4. For an undirected graph G with n edges with weights $a_1, \cdots, a_n$ the dimension, d is defined as:

$$d \stackrel{\text{def}}{=} 2\frac{a_1 + \cdots + a_N}{\text{crk}(G)} \tag{2.1.3}$$

Here is the dimension computation for $G_{\triangle}$:

Example 2.1.5. The Graph $G_{\triangle}$ has 3 all with the weight of 1. Thus by Definition 2.1.4 the dimension of $G_{\triangle}$ is $2\frac{3}{1} = 6$.

Before giving an example of computation of the superficial degree of convergence it is important to make the creation of sub-graphs well-defined.

Definition 2.1.6. Let G be a graph with $n \geq 1$ edges and let σ be a permutation of its edges. We denote the sub-graphs formed by the first k edges in the order σ by the symbol G_k^{σ} which is defined as:

$$G_k^{\sigma} \stackrel{\text{def}}{=} \{\sigma(1), \ldots, \sigma(k)\} \tag{2.1.4}$$

As an example refer to the table below for all the sub-graphs of $G_{\triangle}$ and their superficial degree of convergence.

Example 2.1.7. The Graph $G_{\triangle}$ has 3 edges $(1,2)$, $(2,3)$, and $(1,3)$ as illustrated in Figure 2.1.1. Label these edges as 1, 2 and 3 respectively. All the possible permutations in lexicographical order are 123, 132, 213, 231, 312, and 321. Label all of these permutations from 1-6 respectively. Refer to Table 2.1.1 for all the sub-graphs of $G_{\triangle}$ and their superficial degree of convergence.

Set of Edges	Label	Superficial Degree of Convergence
$\emptyset$	$G_0^1, G_0^2, G_0^3, G_0^4, G_0^5, G_0^6$	0
$\{(1,2)\}$	G_1^1, G_1^2	1
$\{(1,3)\}$	G_1^3, G_1^4	1
$\{(2,3)\}$	G_1^5, G_1^6	1
$\{(1,2),(2,3)\}$	G_2^1, G_2^3	2
$\{(1,2),(1,3)\}$	G_2^2, G_2^5	2
$\{(2,3),(1,3)\}$	G_2^4, G_2^6	2
$\{(1,2),(2,3),(1,3)\}$	$G_3^1, G_3^2, G_3^3, G_3^4, G_3^5, G_3^6$	-6

Table 2.1.1: Superficial Degree of Convergence for sub-graphs of $G_\triangle$.

The Hepp bound for graphs is defined as follows:

Definition 2.1.8. Let G be a graph with n edges. The Hepp bound of Graph G, with the vector of weights of its edges $\mathbf{a}$ is the homogeneous rational function:

$$\mathcal{H}(G,\mathbf{a}) \stackrel{\text{def}}{=} \sum_{\sigma\in\mathfrak{S}_n} \frac{1}{\omega\left(G_1^\sigma\right)\cdots\omega\left(G_{n-1}^\sigma\right)} \tag{2.1.5}$$

Using all the previous examples in this section we can compute the Hepp bound for $G_\triangle$.

Example 2.1.9. Using Definition 2.1.8, $\mathcal{H}(G_\triangle,(1,1,1))$ evaluates to 3. Here is the computation:

$$\begin{aligned}
\mathcal{H}\left(G_\Delta,(1,1,1)\right) &= \sum_{\sigma\in\mathfrak{S}_3} \frac{1}{\omega\left(G_1^\sigma\right)\cdot\omega\left(G_2^\sigma\right)} \\
&= \frac{1}{\omega\left(G_1^1\right)\cdot\omega\left(G_2^1\right)} + \frac{1}{\omega\left(G_1^2\right)\cdot\omega\left(G_2^2\right)} + \\
&\frac{1}{\omega\left(G_1^3\right)\cdot\omega\left(G_2^3\right)} + \frac{1}{\omega\left(G_1^4\right)\cdot\omega\left(G_2^4\right)} + \frac{1}{\omega\left(G_1^5\right)\cdot\omega\left(G_2^5\right)} + \\
&\frac{1}{\omega\left(G_1^6\right)\cdot\omega\left(G_2^6\right)} \\
&= \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot 2} + \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot(2)} \\
&= 3
\end{aligned}$$

Next, we create another graph defined as $G_\square$.

The Graph $G_\square$ consists of four vertices labelled as $1,2,3$, and 4. The edges of $G_\square$ are defined by the following tuple:

$$((1,2),(1,3),(2,3),(2,4),(3,4))$$

Each tuple (u,v) denotes an edge connecting vertex u to vertex v. For each of the edges, the weight vector is defined as:

$$(1,2,3,4,5)$$

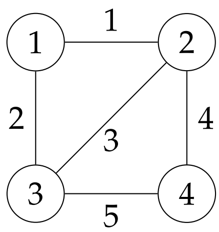


Figure 2.1.2: Illustration of $G_{\square}$

Notice that to compute the Hepp bound for $G_{\square}$ we need to consider 5! * 4 sub-graphs. That is why, next we present the simple sage script to calculate the Hepp bound of any given graph.

Example 2.1.10. By using the script above the Hepp bound of $G_{\square}$ evaluates to $\frac{-117415}{2387616} \approx -0.049$.

In the next example set all the weights of edges of $G_{\square}$ to 1 and evaluate the Hepp bound.

Example 2.1.11. When all weights of $G_{\square}$ are set to 1 the Hepp bound evaluates to $\frac{-20}{9} \approx -2.22$

For the next example consider any undirected graph G with 2 isolated edges.

Figure 2.1.3: Illustration of an arbitrary undirected graph with 2 isolated edges.

Example 2.1.12. Let $a_1, a_2 \in \mathbb{R}$ be weights of the edges $(1,2)$ and $(3,4)$ respectively. The Hepp bound of G evaluates to $\frac{1}{a_1} + \frac{1}{a_2}$. However, due to the logarithmic divergence, $a_1 + a_2 = 0$ is a constraint imposed on G. If $a_1 + a_2 = 0$, then we can rearrange the equation to express one variable in terms of the other. For example, we can express a_1 in terms of a_2 as $a_1 = -a_2$.

Now, we can substitute $a_1 = -a_2$ into the expression $\frac{1}{a_1} + \frac{1}{a_2}$ and simplify:

$$\frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{a_1} + \frac{1}{-a_1} = \frac{1}{a_1} - \frac{1}{a_1} = 0$$

Therefore the Hepp Bound of G evaluates to 0 for any given vector of weights.

Remark 2.1.13. The logarithmic divergence causes the Hepp-Bound of all forests consisting of 2 or more edges to be 0 as shown by Panzer.

2.2 Hepp Bound for Matroids

All of the concepts that we discussed in 2.1 can be extended to matroids with some modification.

First, we need to track the number of surplus edges, also known as the *co-rank*.

Definition 2.2.1. Let E be the ground set of a matroid. For the subset of E, γ the number of dependent elements concerning γ is the *co-rank*:

$$\mathrm{crk}(\gamma) \stackrel{\text{def}}{=} |\gamma| - \mathrm{rk}(\gamma) \tag{2.2.1}$$

Consider the following example of the computation of co-rank.

Example 2.2.2. Consider a uniform matroid $\mathsf{U}_{2,3}$, where every subset of the ground set $\{1,2,3\}$ of size 2 is a basis. The co-rank of the set $\{1,2,3\}$ as per Definition 2.2.1 is 1.

Another way of thinking about $\mathrm{crk}(\gamma) = k$ is that we would need to remove k number of elements from the set γ to make it independent.

Similar to graphs we assign some weight to each element of the ground set. The superficial degree of convergence, which is defined in 2.1.3 extends to matroid using the matroid version of co-rank. Therefore, the Hepp bound for matroids is identical to the one defined in 2.1.8.

Definition 2.2.3. Let M be a matroid with E as the ground set and β as the collection of basis. For the set $\gamma \subseteq E$, the *superficial degree of convergence* $\omega(\gamma)$ is defined as follows:

$$\omega(\gamma) \stackrel{\text{def}}{=} \sum_{e\in\gamma} a_e - \frac{d}{2} \cdot \mathrm{crk}(\gamma) \tag{2.2.2}$$

where a_e represents the weight of element e from the set E, d is the dimension, and $\mathrm{crk}(\gamma)$ is the co-rank of the sub-graph γ.

Like Panzer, we impose the condition $\omega(\mathsf{M}) = 0$ for all matroids M. We will call this constraint *logarithmic divergence*. This means that d is irrelevant for matroids that have $\mathrm{crk}(E) = 0$. For matroids with loops, the dimension d is defined as follows:

Definition 2.2.4. Let M be a matroid with E as the ground set and β as the collection of basis. Let $|E| = n$ with weights $a_1, \cdots, a_n$ the dimension, d is defined as:

$$d \stackrel{\text{def}}{=} 2\frac{a_1 + \cdots + a_N}{\mathrm{crk}(E)} \tag{2.2.3}$$

Here is the dimension computation for $\mathsf{U}_{2,3}$ with all the edges having weight equal to 1:

Example 2.2.5. The matroid $\mathsf{U}_{2,3}$ has 3 elements in the ground set, all with weight equal to 1. Thus by Definition 2.2.4 the dimension of $\mathsf{U}_{2,3}$ is $2\frac{3}{1} = 6$.

Before giving an example of computation of the superficial degree of convergence it is important to make the creation of a subset of the ground set well-defined. For convenience from now on we will refer to the elements of the ground set of the matroid as edges.

Definition 2.2.6. Let M be a graph with $n \geq 1$ edges. Assign an arbitrary order to the edges. and let σ be a permutation of the edges. We denote the subset formed by the first k edges in the order σ by the symbol M_k^σ which is defined as:

$$\mathsf{M}_k^\sigma \stackrel{\text{def}}{=} \{\sigma(1), \dots, \sigma(k)\} \tag{2.2.4}$$

Example 2.2.7. The Matroid $\mathsf{U}_{2,3}$ has 3 edges. Assign the edges an arbitrary labeling of 1, 2, and 3. All the possible permutations of edges in lexicographical order are 123, 132, 213, 231, 312, and 321. Label all of these permutations from 1-6 respectively. Refer to Table 2.2.1 for all the permutations of edges and their superficial degree of convergence.

To be clear, M is the symbol used to denote $\mathsf{U}_{2,3}$ in this table.

Set of Edges	Label	Superficial Degree of Convergence
$\emptyset$	$\mathsf{M}_0^1, \mathsf{M}_0^2, \mathsf{M}_0^3, \mathsf{M}_0^4, \mathsf{M}_0^5, \mathsf{M}_0^6$	0
$\{1\}$	$\mathsf{M}_1^1, \mathsf{M}_1^2$	1
$\{2\}$	$\mathsf{M}^3, \mathsf{M}_1^4$	1
$\{3\}$	$\mathsf{M}_1^5, \mathsf{M}_1^6$	1
$\{1,3\}$	$\mathsf{M}_2^1, \mathsf{M}_2^3$	2
$\{1,2\}$	$\mathsf{M}_2^2, \mathsf{M}_2^5$	2
$\{3,2\}$	$\mathsf{M}_2^4, \mathsf{M}_2^6$	2
$\{1,2,3\}$	$\mathsf{M}_3^1, \mathsf{M}_3^2, \mathsf{M}_3^3, \mathsf{M}_3^4, \mathsf{M}_3^5, \mathsf{M}_3^6$	-6

Table 2.2.1: Superficial Degree of Convergence for subsets of the ground set of $\mathsf{U}_{2,3}$.

The Hepp bound for matroids is defined as follows:

Definition 2.2.8. indexMatroid Hepp Bound

Let M be a matroid with n edges. The Hepp bound of the matroid M, with the vector of weights of its edges $\mathbf{a}$ is the homogeneous rational function:

$$\mathcal{H}(\mathsf{M}, \mathbf{a}) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{\omega\left(\mathsf{M}_1^\sigma\right) \cdots \omega\left(\mathsf{M}_{n-1}^\sigma\right)} \tag{2.2.5}$$

Using all the previous examples in this section we can compute the Hepp bound for $\mathsf{U}_{2,3}$.

Example 2.2.9. Using Definition 2.2.8, $\mathcal{H}(\mathsf{U}_{2,3}, (1,1,1))$ evaluates to 3. Here is the computation:

$$\mathcal{H}(\mathsf{U}_{2,3},(1,1,1)) = \sum_{\sigma\in \mathsf{M}_3} \frac{1}{\omega\left(\mathsf{M}_1^{\sigma}\right)\cdot\omega\left(\mathsf{M}_2^{\sigma}\right)}$$
$$= \frac{1}{\omega\left(\mathsf{M}_1^{1}\right)\cdot\omega\left(\mathsf{M}_2^{1}\right)} + \frac{1}{\omega\left(\mathsf{M}_1^{2}\right)\cdot\omega\left(\mathsf{M}_2^{2}\right)} +$$
$$\frac{1}{\omega\left(\mathsf{M}_1^{3}\right)\cdot\omega\left(\mathsf{M}_2^{3}\right)} + \frac{1}{\omega\left(\mathsf{M}_1^{4}\right)\cdot\omega\left(\mathsf{M}_2^{4}\right)} + \frac{1}{\omega\left(\mathsf{M}_1^{5}\right)\cdot\omega\left(\mathsf{M}_2^{5}\right)} +$$
$$\frac{1}{\omega\left(\mathsf{M}_1^{6}\right)\cdot\omega\left(\mathsf{M}_2^{6}\right)}$$
$$= \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot 2} + \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot(2)}$$
$$= 3$$

Consider the matroid $\mathsf{M}_{\square}$ from the Example 1.1.2. In the following example, we will compute the $\mathcal{H}(\mathsf{M}_{\square})$. Let the weights of the elements of the ground set of $(\mathsf{M}_{\square})$ be equal to 1.

Example 2.2.10. Recall that $\mathsf{M}_{\square}$ has the ground set $E = \{1,2,3\}$ and the collection of bases $\mathcal{B} = \{\{1,2\}.\{2,3\}\}$.

Subset of Ground Set	crk	Label	Superficial Degree of Convergence
$\{1\}$	0	$\mathsf{M}_1^1, \mathsf{M}_1^2$	1
$\{2\}$	0	$\mathsf{M}_1^3, \mathsf{M}_1^4$	1
$\{3\}$	0	$\mathsf{M}_1^5, \mathsf{M}_1^6$	1
$\{1,2\}$	0	$\mathsf{M}_2^1, \mathsf{M}_2^3$	2
$\{1,3\}$	1	$\mathsf{M}_2^2, \mathsf{M}_2^5$	-4
$\{2,3\}$	0	$\mathsf{M}_2^4, \mathsf{M}_2^6$	2

Table 2.2.2: Superficial Degree of Convergence for Subsets of Ground Set of $\mathsf{M}_{\square}$.

By using this table, $\mathcal{H}(\mathsf{M}_{\square},(1,1,1)) = 1$. Here is the computation:

$$\mathcal{H}(\mathsf{M}_{\square},(1,1,1)) = \sum_{\sigma\in \mathsf{M}_3} \frac{1}{\omega\left(\mathsf{M}_1^{\sigma}\right)\cdot\omega\left(\mathsf{M}_2^{\sigma}\right)}$$
$$= \frac{1}{\omega\left(\mathsf{M}_1^{1}\right)\cdot\omega\left(\mathsf{M}_2^{1}\right)} + \frac{1}{\omega\left(\mathsf{M}_1^{2}\right)\cdot\omega\left(\mathsf{M}_2^{2}\right)} +$$
$$\frac{1}{\omega\left(\mathsf{M}_1^{3}\right)\cdot\omega\left(\mathsf{M}_2^{3}\right)} + \frac{1}{\omega\left(\mathsf{M}_1^{4}\right)\cdot\omega\left(\mathsf{M}_2^{4}\right)} + \frac{1}{\omega\left(\mathsf{M}_1^{5}\right)\cdot\omega\left(\mathsf{M}_2^{5}\right)} +$$
$$\frac{1}{\omega\left(\mathsf{M}_1^{6}\right)\cdot\omega\left(\mathsf{M}_2^{6}\right)}$$
$$= \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot(-4)} + \frac{1}{(1)\cdot(2)} + \frac{1}{(1)\cdot 2} + \frac{1}{(1)\cdot(-4)} + \frac{1}{(1)\cdot(2)}$$
$$= \frac{3}{2}$$

In the case of uniform matroids, where all elements of the ground are weighed 1, the Hepp bound simplifies.

Example 2.2.11. Consider the uniform matroid $\mathsf{M} = U^r_n$. This matroid has co-rank $\mathrm{crk} = n - r$. Since all the elements of the ground have weight 1, the dimension d is n.

The superficial degrees of convergence for each permutation,

$$\omega\left(M^\sigma_k\right) = \begin{cases} k & \text{for } 1 \leq k \leq r \text{ and} \\ (n-k)\frac{r}{\mathrm{crk}} & \text{for } r < k < n. \end{cases} \tag{2.2.6}$$

Therefore the Hepp bound simplifies to:

$$\mathcal{H}\left(U^r_n\right) = \frac{n!}{(r-1)!\mathrm{crk}!}\left(\frac{\mathrm{crk}}{r}\right)^{\mathrm{crk}}. \tag{2.2.7}$$

Remark 2.2.12. According to Theorem 2.19 [Pan22] the Hepp bound of a matroid is zero if and only if the matroid is disconnected. A matroid is disconnected if and only if it can be partitioned into 2 or more disjoint collections of bases.

2.3 MOULDS

This section covers the introduction to *moulds*. Our main reference for this section is the paper by Chapoton and colleagues [Cha10]. A *mould* is a function of a variable number of variables.

Definition 2.3.1. A *mould* is a sequence $(f_n)_{n\geq 1}$, where f_n is a function of the variables $\{u_1, \ldots, u_n\}$. The degree of a mould is n if and only if the only non-zero component in the mould is f_n. Note, that in some cases a mould may not have a degree.

Here are some examples of moulds:

Example 2.3.2. Let u_i be a variable for each $i \in \mathbb{N}$. Define the function

$$\mathrm{sum}_n\left(u_1, \ldots, u_n\right) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n} u_i \tag{2.3.1}$$

where $n \in \mathbb{N}$.

Example 2.3.3. Another example of a mould g_n can be defined as follows:

$$g_n\left(u_1, \ldots, u_n\right) = \prod_{i=1}^{n} u_i$$

Example 2.3.4. Another example of a mould h_n can be defined as follows:

$$h_n\left(u_1, \ldots, u_n\right) = \prod_{i=1}^{n} \frac{1}{u_1 \cdot (u_1 + u_2) \cdots (u_1 + u_2 + \cdots + u_n)}$$

The *shuffle product* of any two sequences is defined as follows:

Definition 2.3.5. The (n,m)-shuffles $\mathfrak{S}_{n,m}$ are permutations $\sigma \in \mathfrak{S}_{n+m}$ that maintain the order up-to the first n elements and also the last m elements. The *shuffle product* of two sequences is

$$\langle s_1, \ldots, s_n \rangle \sqcup\!\sqcup \langle s_{n+1}, \ldots, s_{n+m} \rangle \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_{n,m}} \left\langle s_{\sigma(1)}, \ldots, s_{\sigma(n+m)} \right\rangle \tag{2.3.2}$$

where $\mathfrak{S}_{n,m}$ is the set of all (n,m)-shuffles.

Example 2.3.6. Consider an example of shuffles with $n = 2$ and $m = 3$. Suppose we have two words $\langle a, b \rangle$ and $\langle c, d \rangle$. The $(2,2)$-shuffles would be all possible permutations of the letters a, b, c, d that maintain the order of the first two letters and the last three letters.

The shuffle product of these two words is the sum of all such permutations, representing different combinations of the letters a, b, c, d, e while preserving the order within each set.

$$\langle a, b \rangle \sqcup\!\sqcup \langle c, d \rangle = \langle a, b, c, d \rangle + \langle a, c, b, d \rangle + \langle c, a, b, d \rangle + \langle a, c, d, b \rangle + \langle c, a, d, b \rangle + \langle c, d, a, b \rangle$$

Proposition 2.3.7. *For any 2 words, a, b the following holds:*

$$h_n(a \sqcup\!\sqcup b) = h_n(a)\dot{h}_n(b)$$

Proof. We will prove Proposition 2.3.7 using structural induction on the structure of the words a and b.

Suppose a and b are the empty word, Proposition 2.3.7 holds trivially in this case.

Let a and b be words of arbitrary lengths. We now claim that adding the letters α, and β to the end of words a and b respectively, should still allow Proposition 2.3.7 to hold.

Recall the definition of the shuffle product, Definition 2.3.5.

$$a\alpha \sqcup\!\sqcup b\beta = (a\alpha \sqcup\!\sqcup b)\beta + (a \sqcup\!\sqcup b\beta)\alpha.$$

Observe that the last letter of the word in each term of the summation must either be α or β.

Therefore we can expand out $h(a \sqcup\!\sqcup b)$ in the following manner:

$$\begin{aligned} h_n(a \sqcup\!\sqcup b) &= \frac{h_n(a \sqcup\!\sqcup b\beta) + h_n(a\alpha \sqcup\!\sqcup b)}{|a| + \alpha + |b| + \beta} \\ &= \frac{h_n(a)h_n(b\beta) + h_n(a\alpha)h_n(b)}{|a| + \alpha + |b| + \beta} \\ &= \frac{h_n(a)(|b| + \beta)h_n(b) + (|a| + \alpha)h_n(a)h_n(b)}{|a| + \alpha + |b| + \beta} \\ &= h_n(a)h_n(b) \end{aligned}$$

Note that $|a| + \alpha + |b| + \beta$ is the last term in the product defined in Example 2.3.4. Here $|a|$ denotes the sum of all the letters in the word a. The last step of the simplification matches Proposition 2.3.7 as needed.

Therefore, by structural induction Proposition 2.3.7 for all words a and b.
$\square$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, 3, \ldots\}$.

Let $\Bbbk$ be a field of characteristic zero and let Σ be a nonempty set.

$$\Sigma^* = \{w = w_1 w_2 \cdots w_l \mid l \in \mathbb{N}, w_1, \ldots, w_l \in \Sigma\}$$

It follows that:

$$\Sigma^* \times \Sigma^* \to \Sigma^* \quad (u_1 \cdots u_k, v_1 \cdots v_l) \mapsto u_1 \cdots u_k v_1 \cdots v_l$$

So $(\Sigma^*, \cdot)$ is a mould with the empty word $\varnothing$ as unit.

A $\Bbbk$ *mould* on Σ^* is a function $\Sigma^* \to \Bbbk$. The set of all moulds is denoted by $\Bbbk\Sigma^*$.

Given a mould $M : \Sigma^* \to \Bbbk$, it is customary to denote by M^w the value it takes on a word w.

Mould multiplication is defined as follows:

Definition 2.3.8. Let M and N be moulds and w be a word. Then *mould multiplication* is defined as:

$$(M \times N)(w) = \sum_{w=uv} M(u)N(v)$$

$\Bbbk\Sigma^*$ is an associative $\Bbbk$-algebra, noncommutative if Σ has more than one element, whose unit is the mould.

Definition 2.3.9. We say that a mould has *order p* if $M(w) = 0$ for each word w of length $< p$. For any given mould, M $\operatorname{ord}(M)$ denotes the order of M.

If $\operatorname{ord}(M) \geq p$ and $\operatorname{ord}(N) \geq q$ then $\operatorname{ord}(M \times N) \geq p + q$.

In particular, if $M(\varnothing) = 0$ then $\operatorname{ord}(M^k) \geq k$ for each $k \in \mathbb{N}$.

Hence $\exp M$ and $\log(M)$ are well-defined.

Recall that the shuffling of two words $u = u_1 u_2 \cdots u_k$ and $v = v_1 v_2 \cdots v_l$ is the set of all words w which can be obtained by interchanging the letters of u and those of v while preserving their internal orders in u and v.

That is, the words which can be written $w = w_{\sigma(1)} \cdots w_{\sigma(k+l)}$ with a permutation σ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

We define the shuffling coefficient $\operatorname{sh}(u, v, w)$ to be the number of such permutations σ, and we set $\operatorname{sh}(u, v, w) = 0$ whenever w is not a shuffling of u and v.

Eg. If $u = (x, y)$ and $v = (z)$, then $\operatorname{sh}((x, y), (z), (x, y, z)) = 1$.

We also define, for arbitrary words u and w:

$$\operatorname{Sh}(u, w) = \begin{cases} 1 & \text{if } u = w \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.3.10. A mould $M \in \Bbbk\Sigma^*$ is said to be *Alternal* if $M(\varnothing) = 0$ and $\sum_w \mathrm{sh}(u, v, w)M(w) = 0$ for any two words u and v.

Example 2.3.11. Here is an example of a Alternal mould: $E_a(w) = a^{|w|}$ where $a \in \Bbbk$.

Definition 2.3.12. A mould $M \in \Bbbk\Sigma^*$ is said to be *Symmetral* if $M(\varnothing) = 1$ and $\sum_w \mathrm{sh}(u, v, w)M(w) = M(u)M(v)$ for any two words u and v.

Example 2.3.13. Here is an example of a Symmetral mould: $E(w) = \frac{1}{|w|!}$.

Lemma 2.3.14. *1. The product of two Symmetral moulds is Symmetral.*

2. The logarithm of a Symmetral mould is Alternal.

3. The exponential of an Alternal mould is Symmetral.

Let $\Bbbk\Sigma^*$ be the linear span of the set of words, i.e., the vector space consisting of all formal linear combinations $\sum_w c_w w$ with finitely many non-zero coefficients $c_w \in \Bbbk$.

The set of moulds can be identified with the set of linear forms on $\Bbbk\Sigma^*$ if we identify $M \in \Bbbk\Sigma^*$ with $\sum_w M(w)w$.

The associative algebra structure of $\Bbbk\Sigma^*$ is then dual to the coalgebra structure of $\Bbbk\Sigma^*$.

Alternal moulds appear as infinitesimal characters of $\Bbbk\Sigma^*$: linear forms M such that $M(u \cdot v) = M(u) + M(v)$.

Symmetral moulds are characters of $\Bbbk\Sigma^*$: linear forms M such that $M(\varnothing) = 1$ and $M(u \cdot v) = M(u)M(v)$.

2.4 $\mathcal{G}$-INVARIANT

The $\mathcal{G}$-invariant is an invariant of matroids that takes the product of all the permutations of the edges of the ground set of the matroid. We will be using the paper by Bonin and colleagues as our main source in this section [BK17].

Definition 2.4.1. Let M be a matroid with n elements in the ground set (edges). The G-invariant of M will be denoted by $\mathcal{G}(\mathsf{M})$ and defined as follows:

$$\mathcal{G}(\mathsf{M}) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_{\mathsf{M}}} \left\langle \mathrm{rk}\left(\mathsf{M}_1^\sigma\right), \mathrm{rk}\left(\mathsf{M}_2^\sigma\right) - \mathrm{rk}\left(\mathsf{M}_1^\sigma\right), \ldots, \mathrm{rk}\left(\mathsf{M}_n^\sigma\right) - \mathrm{rk}\left(\mathsf{M}_{n-1}^\sigma\right) \right\rangle \in \mathbb{Z}\langle 0, 1\rangle \tag{2.4.1}$$

Consider the matroid $\mathsf{M}_\square$ from Example 1.1.2. The G-invariant for $\mathsf{M}_\square$ is computed as follows:

Example 2.4.2. Recall that the groundset of $\mathsf{M}_\square$ is $\{1, 2, 3\}$. There are 6 permutations of $\{1, 2, 3\}$ as noted in 2.2.10. Here are the ranks of each permutation of edges.

Subset of Ground Set	rk	Label
$\{1\}$	1	$\mathsf{M}_1^1, \mathsf{M}_1^2$
$\{2\}$	1	$\mathsf{M}_1^3, \mathsf{M}_1^4$
$\{3\}$	1	$\mathsf{M}_1^5, \mathsf{M}_1^6$
$\{1,2\}$	2	$\mathsf{M}_2^1, \mathsf{M}_2^3$
$\{1,3\}$	1	$\mathsf{M}_2^2, \mathsf{M}_2^5$
$\{2,3\}$	2	$\mathsf{M}_2^4, \mathsf{M}_2^6$
$\{1,2,3\}$	2	$\mathsf{M}_3^1, \mathsf{M}_3^2, \mathsf{M}_3^3, \mathsf{M}_3^4, \mathsf{M}_3^5, \mathsf{M}_3^6$

Table 2.4.1: Rank of each permutation of the ground set of $\mathsf{M}_\square$.

Using the table above, $\mathcal{G}(\mathsf{M}_\square)$ is computed as follows:

$$\begin{aligned}\mathcal{G}(\mathsf{M}_\square) &= \langle 1, 2-1, 2-2\rangle + \langle 1, 1-1, 2-1\rangle + \langle 1, 2-1, 2-2\rangle + \langle 1, 2-1, 2-2\rangle + \\ &\quad \langle 1, 1-1, 2-1\rangle + \langle 1, 2-1, 2-2\rangle \\ &= \langle 1,1,0\rangle + \langle 1,0,1\rangle + \langle 1,1,0\rangle + \langle 1,1,0\rangle + \langle 1,0,1\rangle + \langle 1,1,0\rangle \\ &= \langle 6,4,2\rangle\end{aligned}$$

For uniform matroid $\mathsf{U}_{2,3}$ $\mathcal{G}(\mathsf{U}_{2,3})$ is computed as follows:

Example 2.4.3.

$$\begin{aligned}\mathcal{G}(\mathsf{U}_{2,3}) &= \langle 1, 2-1, 2-2\rangle + \langle 1, 2-1, 2-2\rangle + \langle 1, 2-1, 2-2\rangle + \langle 1, 2-1, 2-2\rangle + \\ &\quad \langle 1, 2-1, 2-2\rangle + \langle 1, 2-1, 2-2\rangle \\ &= \langle 1,1,0\rangle + \langle 1,1,0\rangle + \langle 1,1,0\rangle + \langle 1,1,0\rangle + \langle 1,1,0\rangle + \langle 1,1,0\rangle \\ &= \langle 6,6,0\rangle\end{aligned}$$

2.5 MELLIN INTEGRAL

This section serves as an introduction to Mellin integral. The main source for this section is Chapter 12 from the book "The Transforms and Applications" [J00]. The Mellin integral is the operation mapping a given function into another function on the complex plane as defined below:

Definition 2.5.1. Let $f(x)$ be a function defined for $x > 0$. For a given complex number s, we define the Mellin transform of $f(x)$ as $F(s)$:

$$F(s) \stackrel{\text{def}}{=} \int_0^\infty x^{s-1} f(x)\, dx \tag{2.5.1}$$

Consider the Mellin transform of the function $f(x) = x^n$, where n is a real number.

Example 2.5.2. The Mellin integral for the function f is:

$$F(s) = \int_0^\infty x^{s-1} x^n \, dx \tag{2.5.2}$$

$$= \int_0^\infty x^{s+n-1} \, dx \tag{2.5.3}$$

Consider different values of the complex variable s:

Example 2.5.3. $s = 1$:

$$F(1) = \int_0^\infty x^{1+n-1}\,dx = \int_0^\infty x^n\,dx \tag{2.5.4}$$

This integral converges if $n > -1$.

CHAPTER 3

COMBINATORIAL VOLUMES

3.1 VOLUME COMPUTATION

Given a convex polytope with support function $h_P(x)$, we have:

$$h_P(x) = \sup\left(x^T y \mid y \in P\right) \tag{3.1.1}$$

where P is the convex polytope, and the support function $h_P(x)$ is defined as the supremum of the dot product $x^T y$ over all points y in the polytope P.

Definition 3.1.1. The polar of the polytope P, denoted P°, is defined as:

$$P^\circ = \{x \mid h_P(x) \leq 1\} \tag{3.1.2}$$

This set contains all points x such that the support function $h_P(x)$ is less than or equal to 1.

Definition 3.1.2. To calculate the *volume* of the polar dual polytope C°, we use the formula:

$$\mathrm{Vol}(C^\circ) = \frac{1}{n!}\int_{\mathbb{R}^n} e^{-h_P(x)}\,dx \tag{3.1.3}$$

3.2 COMPUTATION USING HEPP BOUND

We are going to begin this section by defining a new invariant, $\mathcal{R}(\mathsf{M})$.

Definition 3.2.1. Let M be a matroid on $E = \{1, 2, \ldots, n\}$ of rank d. Define the *reciprocal sum* $\mathcal{R}(\mathsf{M})$ by:

$$\mathcal{R}(\mathsf{M}) \overset{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \frac{n^{n-2}/(n-1)!}{\pi(\mathsf{M}_1)\cdot\pi(\mathsf{M}_2^\sigma)\cdots\pi(\mathsf{M}_{n-1}^\sigma)} \tag{3.2.1}$$

where,

$$\pi(\mathsf{M}_k^\sigma) = \mathrm{rk}_{\mathsf{M}}\left(\left\{\sigma(1), \sigma(2), \ldots, \sigma(k)\right\}\right) * \mathrm{rk}_{\mathsf{M}^*}\left(\left\{\sigma(1), \sigma(2), \ldots, \sigma(k)\right\}\right) \tag{3.2.2}$$

Definition 3.2.2. Let M be a loopless matroid on $E = \{1, 2, \dots, n\}$ of rank d. Define the T-*invariant* $\mathsf{T}(\mathsf{M})$ by:

$$\mathsf{T}(\mathsf{M}) \stackrel{\text{def}}{=} C \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{n} \frac{1}{\text{rk}(\{\sigma(1), \dots, \sigma(j)\})} \tag{3.2.3}$$

$$\stackrel{\text{def}}{=} C \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{\text{rk}(\{\sigma(1)\})\text{rk}(\{\sigma(1), \sigma(2)\}) \cdots \text{rk}(\{\sigma(1), \dots, \sigma(n)\})} \tag{3.2.4}$$

Proposition 3.2.3. *For uniform matroid $\mathsf{U}_{r,n}$, where r is the rank of the matroid and n is the size of the ground set, $\mathsf{T}(\mathsf{U}_{r,n})$ can be simplified as follows:*

$$\mathsf{T}(\mathsf{U}_{r,n}) = C \frac{n!}{r! r^{n-r}} \tag{3.2.5}$$

Proof. The subsets of the ground set of the matroid $\mathsf{U}_{r,n}$ are independent if and only if the size of the subset is less than or equal to r. Thus, for any given subset of the ground set of the matroid $\mathsf{U}_{r,n}$, γ, the rank of γ, $\text{rk}(\gamma) = \min(|\gamma|, r)$. Therefore, we can make the following simplification:

$$\mathsf{T}(\mathsf{U}_{r,n}) = C \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{r! \cdot \cdots \text{rk}(\{\sigma(1), \dots, \sigma(n)\})} \tag{3.2.6}$$

There are $n - r$ terms in the denominator product that are bounded by r. Therefore, we can make the following simplification:

$$\mathsf{T}(\mathsf{U}_{r,n}) = C \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{r! r^{n-r}} \tag{3.2.7}$$

There are $n!$ permutations of the ground set of the matroid $\mathsf{U}_{r,n}$ that have the same value for the term $\frac{1}{r! r^{n-r}}$. Therefore, we can make the following simplification:

$$\mathsf{T}(\mathsf{U}_{r,n}) = C \frac{n!}{r! r^{n-r}} \tag{3.2.8}$$

This matches proposition 3.2.3 as needed.

□

Proposition 3.2.4. *If* **s** *and* **t** *are two rank-jump sequence of length m and n then*

$$a\mathbf{s} \sqcup\!\sqcup b\mathbf{t} = ab\,(\mathbf{s} \sqcup\!\sqcup \mathbf{t}) \tag{3.2.9}$$

Example 3.2.5. Let $\mathbf{s} = (1, 0)$ and let $\mathbf{t} = (1)$, then

$$2\mathbf{s} \sqcup\!\sqcup 1\mathbf{t} = 2(1,0) \sqcup\!\sqcup 1(1) \tag{3.2.10}$$

$$= [(1,0) + (1,0)] \sqcup\!\sqcup (1) \tag{3.2.11}$$

$$= (1,1,0) + (1,1,0) + (1,0,1) + (1,1,0) + (1,1,0) + (1,0,1) \tag{3.2.12}$$

$$= 2(1,0) \sqcup\!\sqcup (1) \tag{3.2.13}$$

Lemma 3.2.6. *The number of total shuffles is given by the binomial coefficient. Say the size of the first word is p and the size of the other word is q then the total number of shuffles is defined as:*

$$\#Sh(p,q) = \binom{p+q}{p}.$$

Due to the preservation of order in the two words a shuffle $\sigma \in Sh(p,q) \subset Aut(\{1,\dots,p+q\})$ is fully determined by which p elements of $\{1,\dots,p+q\}$ are in the image of $\{1,\dots,p\}$, which is the number of choices of p out of $p+q$ elements. Here $Aut(S)$ denotes the automorphism group of a set S, which is the group of all bijective mappings from S to itself.

Lemma 3.2.7. *Let* $\mathbf{s} = (\underbrace{1,1,\dots,1}_{k},\underbrace{0,0,\dots,0}_{m-k})$ *and* $\mathbf{t} = (\underbrace{1,1,\dots,1}_{\ell},\underbrace{0,0,\dots,0}_{n-\ell})$. *The count of the specific sequence* $\mathbf{w} = (\underbrace{1,1,\dots,1}_{k+\ell},\underbrace{0,0,\dots,0}_{m+n-k-\ell})$ *generated from shuffling* $\mathbf{s}$ *and* $\mathbf{t}$ *is given by:*

$$[\mathbf{w}](\mathbf{s} \sqcup\!\sqcup \mathbf{t}) = \binom{k+\ell}{\ell} \times \binom{m+n-k-\ell}{n-\ell} \tag{3.2.14}$$

where $\binom{k+\ell}{\ell}$ is the number of ways to choose ℓ positions for ones out of $k+\ell$ possible positions, and $\binom{m+n-k-\ell}{n-\ell}$ is the number of ways to choose $n-\ell$ positions for zeros out of $m+n-k-\ell$ possible positions.

Example 3.2.8. If $\mathbf{s} = (1,0)$ and $\mathbf{t} = (1,1,0)$ then

$$[(1,1,1,0,0)](\mathbf{s} \sqcup\!\sqcup \mathbf{t}) = 6. \tag{3.2.15}$$

Example 3.2.9. If $\mathbf{s} = (1,0,0,0)$ and $\mathbf{t} = (1,0,0,0)$ then

$$[(1,1,0,0,0)](\mathbf{s} \sqcup\!\sqcup \mathbf{t}) = 40. \tag{3.2.16}$$

The either of brute force computation or Lemma 3.2.7 can be used to verify the examples above.

Definition 3.2.10. If M and R are matroids with ground sets E_{M} and E_{R}, their direct sum $\mathsf{N} = \mathsf{M} \oplus \mathsf{R}$ is a matroid where:

- The ground set of N is $E_{\mathsf{M}} \cup E_{\mathsf{R}}$.
- A set $\mathcal{S} \subseteq E_{\mathsf{M}} \cup E_{\mathsf{R}}$ is independent in N if and only if $\mathcal{S} \cap E_{\mathsf{M}}$ is independent in M and $\mathcal{S} \cap E_{\mathsf{R}}$ is independent in R.
- The rank function of N is additive:

$$\text{rank}_{\mathsf{N}}(\mathcal{S}) = \text{rank}_{\mathsf{M}}(\mathcal{S} \cap E_{\mathsf{M}}) + \text{rank}_{\mathsf{R}}(\mathcal{S} \cap E_{\mathsf{R}}).$$

Example 3.2.11. Consider the matroids $\mathsf{U}_{2,3}$ and $\mathsf{U}_{1,2}$.

The matroid $\mathsf{U}_{2,3}$ is defined on the ground set $\{1,2,3\}$, with rank 2. The independent sets are all subsets of size at most 2.

The matroid $\mathsf{U}_{1,2}$ is defined on the ground set $\{4,5\}$, with rank 1. The independent sets are all subsets of size at most 1.

Their direct sum $\mathsf{N} = \mathsf{U}_{2,3} \oplus \mathsf{U}_{1,2}$ is a matroid on the ground set $\{1,2,3,4,5\}$, with the collection of basis being: β = { {1,2,4}, {1,3,4}, {2,3,4}, {1,2,5}, {1,3,5}, {2,3,5}}.

The rank function of N is given by:

$$\text{rank}_{\mathsf{N}}(\mathcal{S}) = \text{rank}_{\mathsf{U}_{2,3}}(\mathcal{S} \cap \{1,2,3\}) + \text{rank}_{\mathsf{U}_{1,2}}(\mathcal{S} \cap \{4,5\}).$$

For instance, for the set $\{1,2,4\}$, the rank is:

$$\text{rank}_{\mathsf{N}}(\{1,2,4\}) = \text{rank}_{\mathsf{U}_{2,3}}(\{1,2\}) + \text{rank}_{\mathsf{U}_{1,2}}(\{4\}) = 2 + 1 = 3.$$

Proposition 3.2.12. *Let* M *be a matroid on* E_{M} *with* n_{M} *many elements and let* N *be a matroid on* E_{N} *with* n_{N} *many elements. Then*

$$f(\mathsf{M} \oplus \mathsf{N}) = \sum_{u \in \mathfrak{S}_{\mathsf{M}}} \sum_{v \in \mathfrak{S}_{\mathsf{N}}} (s_1, \ldots, s_{n_{\mathsf{M}}})_u \sqcup\!\sqcup (t_1, \ldots, t_{n_{\mathsf{N}}})_v \tag{3.2.17}$$

Proof. This proof aims to show that the combination of incremental ranks through all permutations in $f(\mathsf{M} \oplus \mathsf{N})$, matches $\sum_{u \in \mathfrak{S}_{\mathsf{M}}} \sum_{v \in \mathfrak{S}_{\mathsf{N}}} (s_1, \ldots, s_{n_{\mathsf{M}}})_u \sqcup \sqcup (t_1, \ldots, t_{n_{\mathsf{N}}})_v$.Specifically, for each permutation σ_{M} from M and σ_{N} from N, the RHS of Proposition 3.2.12 generates all shuffles, each representing a valid permutation of the direct sum $\mathsf{M} \oplus \mathsf{N}$. Each shuffle is treated as a new permutation of $\mathsf{M} \oplus \mathsf{N}$, and the incremental ranks are computed as if this permutation were directly drawn from $\mathsf{M} \oplus \mathsf{N}$. Summing across all shuffles effectively sums across all possible permutations of $\mathsf{M} \oplus \mathsf{N}$ since every shuffle corresponds uniquely to a permutation of the combined ground sets, respecting the internal order of elements from M and N.

□

Example 3.2.13. Note that

$$f(\mathsf{U}_{1,2}) = 2(1,0) \tag{3.2.18}$$
$$f(\mathsf{U}_{2,3}) = 6(1,1,0) \tag{3.2.19}$$

Now consider

$$\begin{aligned}
f(\mathsf{U}_{1,2} \oplus \mathsf{U}_{2,3}) &= 2(1,0) \sqcup\!\sqcup 6(1,1,0) \\
&= 12[(1,0) \sqcup\!\sqcup (1,1,0)] \\
&= 12\Big[(1,0,1,1,0) + (1,1,0,1,0) + (1,1,1,0,0) \\
&+ (1,1,1,0,0) + (1,1,0,1,0) + (1,1,1,0,0) + (1,1,1,0,0) + \\
&(1,1,1,0,0) + (1,1,1,0,0) + (1,1,0,1,0)\Big] \\
&= 12\Big[6(1,1,1,0,0) + 3(1,1,0,1,0) + (1,0,1,1,0)\Big]
\end{aligned}$$

Let us recall the mould h_n, that is given by

$$h_n(\mathbf{s}) = \frac{1}{s_1(s_1+s_2)\cdots(s_1+s_2+\cdots+s_n)} \tag{3.2.20}$$

then note that

$$\mathrm{T}(M) = h_n(f(M)) \tag{3.2.21}$$

Let us apply h_n on both sides of Proposition 3.2.12

$$T(M \oplus N) = \sum_{u\in\mathfrak{S}_M} \sum_{v\in\mathfrak{S}_N} h_{n_M}(s_1,\ldots,s_{n_M})_u h_{n_N}(t_1,\ldots,t_{n_N})_v \tag{3.2.22}$$

$$= \sum_{u\in\mathfrak{S}_M} h_{n_M}(s_1,\ldots,s_{n_M})_u \sum_{v\in\mathfrak{S}_N} h_{n_N}(s_1,\ldots,s_{n_N})_u \tag{3.2.23}$$

Proposition 3.2.14. *Let A and B be arbitrary matroids. The following is true for the direct sum of the matroids A and B:*

$$\mathrm{T}(A \oplus B) = \mathrm{T}(A)\mathrm{T}(B) \tag{3.2.24}$$

Proof. The proof for Proposition 3.2.14 follows from Proposition 2.3.7. First, observe that $T(M)$ is the mould h_n on the rank of the set of permutations of the ground set. □

Definition 3.2.15. For a connected matroid M on n edges with $\mathrm{crk}(M) \geq 1$ loops, define the set of flags of bridgeless submatroids of M as the following:

$$\mathcal{F}_M^{1\text{ pi}} := \{\varnothing = \gamma_0 \subsetneq \gamma_1 \subsetneq \cdots \subsetneq \gamma_\ell = M : \text{ each } \gamma_k \text{ is bridgeless with } \ell(\gamma_k) = k\}$$

Example 3.2.16. Consider the uniform matroid $\mathsf{U}_{2,3}$ on the ground set $\{1,2,3\}$. The set of flags of bridgeless submatroids $\mathcal{F}^{1\text{ pi}}_{\mathsf{U}_{2,3}}$ includes:

$$\mathcal{F}^{1\text{ pi}}_{\mathsf{U}_{2,3}} = \{\varnothing \subsetneq \{1,2,3\}\}$$

Each γ_k in the flag is bridgeless, and $\ell(\gamma_k) = k$.

Definition 3.2.17. Define a connected matroid M on n edges with $\mathrm{crk}(M) \geq 1$ loops. For any nested subsets $\delta \subseteq \gamma \subseteq M$, let $a_{\gamma/\delta} := \sum_{e\in\gamma\setminus\delta} a_e$ denote the sum of the indices of the additional edges in γ.

Proposition 3.2.18. *Define a connected matroid* M *on n edges with* $\mathrm{crk}(M) \geq 1$ *loops. The following holds for* $\mathcal{T}\mathsf{M}$*);*

$$\mathcal{T}(\mathsf{M}) = \sum_{\gamma\in\in\mathcal{F}_{\mathsf{M}}^{1pi}} \frac{a_{\gamma_1/\gamma_0}\cdots a_{\gamma_{\mathrm{crk}(\mathsf{M})}/\gamma_{\mathrm{crk}(\mathsf{M})-1}}}{\mathrm{rk}(\gamma_1)\cdots\mathrm{rk}\left(\gamma_{\mathrm{crk}(\mathsf{M})-1}\right)} \tag{3.2.25}$$

Example 3.2.19. Consider the Uniform Matroid $\mathsf{U}_{2,3}$. Here,

$$\mathcal{F}^{1\text{ pi}}_{\mathsf{U}_{2,3}} = \{\emptyset \subsetneq \{1,2,3\}\}$$

by Proposition 3.2.18,

$$\mathcal{T}(M) = \frac{3}{\text{rk}(\{1,2,3\})} = \frac{3}{2} \tag{3.2.26}$$

Example 3.2.20. Consider the matroid M which is the cycle matroid of the complete graph on four vertices, K_4. The ground set consists of the edges of the K_4 graph. The collection of basis, β of this matroid is given by:

$$\begin{aligned}\beta = \{&\{0,1,2\},\{0,1,4\},\{0,1,5\},\{0,2,3\},\{0,2,5\},\{0,3,4\},\{0,3,5\},\{0,4,5\},\\ &\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}.\end{aligned}$$

Each basis identifies a tree of the k_4 graph. Here is a visual of the k_4 graph.

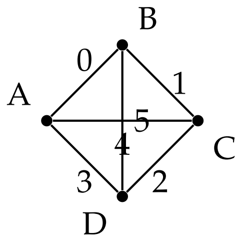


Figure 3.2.1: Image of the k_4 graph

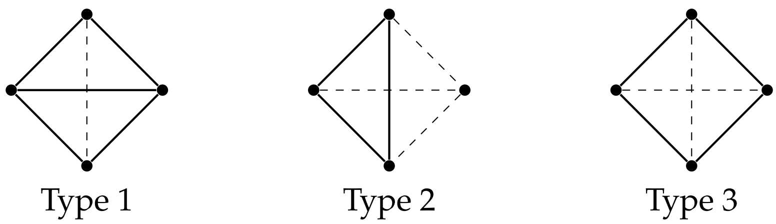


Figure 3.2.2: The three types of bridge-less subgraphs of K_4 are highlighted. To construct Type 1, only remove any edge of the k_4 graph. To construct Type 2, only remove all the edges connected to any of the vertex of the k_4 graph. To construct Type 3, only remove any 2 non adjacent edges of the k_4 graph.

$$\text{Total flag chains of } \emptyset \subset \text{Type 2} \subset \text{Type 1} \subset K_4 = 1 \cdot 4 \cdot \binom{3}{2} \cdot 1 = 12$$

. We will refer to the **set** of all possible chains of this type of chains as $\mathcal{F}^{1\text{ pi}}_{k_{4_a}}$.

$$\text{Total flag chains of } \emptyset \subset \text{Type } 3 \subset \text{Type } 1 \subset K_4 = 1 \cdot 3 \cdot \binom{2}{1} \cdot 1 = 6$$

. We will refer to the **set** of all possible chains of this type of chains as $\mathcal{F}_{k_{4b}}^{1\text{ pi}}$.

Note that $\mathcal{F}_{k_4}^{1\text{ pi}} = \mathcal{F}_{k_{4a}}^{1\text{ pi}} \cup \mathcal{F}_{k_{4b}}^{1\text{ pi}}$. Let $\mathcal{F}_1$ be an element of the set $\mathcal{F}_{k_{4a}}^{1\text{ pi}}$ and $\mathcal{F}_2$ be an element of the set $\mathcal{F}_{k_{4b}}^{1\text{ pi}}$.

According to Proposition 3.2.18:

$$\mathcal{T}(\mathsf{M}) = 12 \cdot \sum_{\gamma \in \in \mathcal{F}_1} \frac{a_{\gamma_1/\gamma_0} \cdots a_{\gamma_{\mathrm{crk}(\mathsf{M})}/\gamma_{\mathrm{crk}(\mathsf{M})-1}}}{\mathrm{rk}(\gamma_1) \cdots \mathrm{rk}\left(\gamma_{\mathrm{crk}(\mathsf{M})-1}\right)} + 6 \cdot \sum_{\gamma \in \in \mathcal{F}_2} \frac{a_{\gamma_1/\gamma_0} \cdots a_{\gamma_{\mathrm{crk}(\mathsf{M})}/\gamma_{\mathrm{crk}(\mathsf{M})-1}}}{\mathrm{rk}(\gamma_1) \cdots \mathrm{rk}\left(\gamma_{\mathrm{crk}(\mathsf{M})-1}\right)} \tag{3.2.27}$$

We can now evaluate both of the summations as follows:

$$\sum_{\gamma \in \in \mathcal{F}_1} \frac{a_{\gamma_1/\gamma_0} \cdots a_{\gamma_{\mathrm{crk}(\mathsf{M})}/\gamma_{\mathrm{crk}(\mathsf{M})-1}}}{\mathrm{rk}(\gamma_1) \cdots \mathrm{rk}\left(\gamma_{\mathrm{crk}(\mathsf{M})-1}\right)} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 3 \cdot 3} \tag{3.2.28}$$

$$\sum_{\gamma \in \in \mathcal{F}_2} \frac{a_{\gamma_1/\gamma_0} \cdots a_{\gamma_{\mathrm{crk}(\mathsf{M})}/\gamma_{\mathrm{crk}(\mathsf{M})-1}}}{\mathrm{rk}(\gamma_1) \cdots \mathrm{rk}\left(\gamma_{\mathrm{crk}(\mathsf{M})-1}\right)} = \frac{4 \cdot 1 \cdot 1}{3 \cdot 3 \cdot 3} \tag{3.2.29}$$

Thus, $\mathcal{T}(\mathsf{M}) = \frac{44}{9}$ which also matches the direct computation as defined in Definition 3.2.2.

Example 3.2.21. Consider the following graph G as shown in the image:

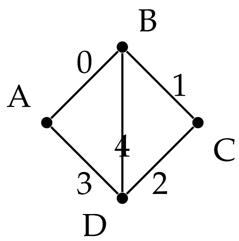


Figure 3.2.3: Image of the graph G

Let M be a matroid with the edges of the graph G as the ground-set, which is {0,1,2,3,4} and here the β, the collection of basis consisting the edges of all the trees of the graph G. For clarity the collection of basis is:

$$\beta = \{\{0,1,2\}, \{0,1,4\}, \{0,2,3\}, \{0,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

Besides G, there are only 3 possible sub-graphs of G that are bridge-less and non empty. These are:

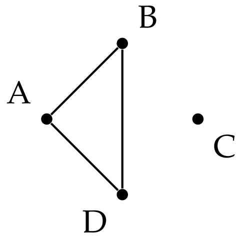


Figure 3.2.4: Bridgeless subgraph 1

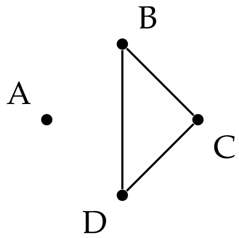


Figure 3.2.5: Bridgeless subgraph 2

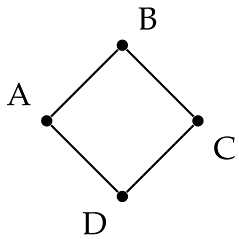


Figure 3.2.6: Bridgeless subgraph 3

Trivially, graph G with all of its edges removed is a sub graph of all of the graphs shown above and all of the graphs shown above are sub-graphs of the graph G. Thus, we get 4 chains of bridgeless flags which means according to the Propositon 3.2.18, $\mathcal{T}(\mathsf{M})$ evaluates to:

$$\mathcal{T}(\mathsf{M}) = \frac{3 \cdot 2}{2 \cdot 3} + \frac{3 \cdot 2}{2 \cdot 3} + \frac{4 \cdot 1}{3 \cdot 3} = \frac{22}{9} \tag{3.2.30}$$

Thus, $\mathcal{T}(\mathsf{M}) = \frac{22}{9}$ which also matches the direct computation as defined in Definition 3.2.2.

Proof. The proof for 3.2.18 is identical to the proof of Proposition 3.2 in Eric Panzer's paper [Pan22].

□

CHAPTER 4

TRANSLATED MATROID BASE POLYTOPE

In this section, we will create a translation that guarantees the origin is inside the translated matroid polytope. As a by-product, the origin is also inside the polar dual of the translated matroid polytope. It is important to note that translating the polar of the matroid polytope does not alter its volume.

4.1 INTRODUCTION

Definition 4.1.1. Let P be the matroid polytope associated with a matroid M, and let n denote the size of the ground set of M. The translation of the polytope P by a vector $\mathbf{v} \in \mathbb{R}^n$ results in a new polytope $P_{\mathbf{v}}$, where each vertex x of P is shifted by $\mathbf{v}$. Specifically, if P has vertex set $\{x_1, x_2, \ldots, x_k\}$, then the vertex set of the translated polytope Q are $\{x_1 + \mathbf{v}, x_2 + \mathbf{v}, \ldots, x_k + \mathbf{v}\}$. In this chapter, we study the translation of P by the vector $\mathbf{v} = -\dfrac{\mathrm{rk}(\mathsf{M})}{n}(1, 1, \ldots, 1)$, and we denote this by Q.

Example 4.1.2. Consider the graphical matroid M created from the minimum spanning trees of the graph shown in Example 2.1.9 with ground set $\{0, 1, 2, 3, 4\}$. To be clear, the collection of bases, $\mathcal{B}$, is shown below:

$$\mathcal{B} = \{\{0,1,3\}, \{0,1,4\}, \{0,2,3\}, \{0,2,4\}, \{0,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$$

As per Definition 1.3.1, the polytope of M, denoted as P, is the following convex polytope:

$$P = \mathrm{Conv}\begin{pmatrix} (1,1,0,1,0), & (1,1,0,0,1), & (1,0,1,1,0), & (1,0,1,0,1), \\ (1,0,0,1,1), & (0,1,1,1,0), & (0,1,1,0,1), & (1,0,1,0,1) \end{pmatrix}$$

The translated polytope Q is then:

$$Q = \mathrm{Conv}\left(\ (1,1,0,1,0) + (-\tfrac{3}{5}, -\tfrac{3}{5}, -\tfrac{3}{5}, -\tfrac{3}{5}, -\tfrac{3}{5}), \quad \ldots, \quad (1,0,1,0,1) + (-\tfrac{3}{5}, -\tfrac{3}{5}, -\tfrac{3}{5}, -\tfrac{3}{5}, -\tfrac{3}{5})\ \right)$$

Both Q and the polar dual of Q contain the origin $(0,0,0,0,0)$ as needed.

Lemma 4.1.3. *The origin is inside* Q_{M}.

Example 4.1.4. U15, U25, U35, U45, U55

Example 4.1.5.])

- Wherever you define a concept or a term, it must be followed by at least three examples.
- For definition 3.2.2 and definition/proposition 3.2.18 [already done], you must have atleast 3 concrete examples for both of them, and the number should agree.
- Correct the index [DONE]
- For Definition 4.1.1, include the examples of U15 to U55, as hyperplane explanation (H-representation) of polar duals [P1_translated.polar ¡- H-representation] of translated matroid polytopes and then their volume must be written as a table. Go up to U66.
- fix the inconsistent beta for collection of basis use curly B. M and M.

Call back to the T(M) after computing the volume https://github.com/Karanvir1729/SagePlo

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