Типовой расчет

М3135

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Вариант 11

№1

а) $\int x\cdot e^{2x^2}dx = \frac{1}{2}\int e^{2x^2}dx^2 = \frac{1}{4}\int e^{2x^2}d(2x^2) =$
$= \frac{1}{4}e^{2x^2} + C$

б) $\int \frac{1-x}{\sqrt{1-x^2}}dx = \int\frac{dx}{\sqrt{1-x^2}} - \int\frac{x\,dx}{\sqrt{1-x^2}} = \arcsin x -$
$-\frac{1}{2}\int\frac{dx^2}{\sqrt{1-x^2}} = \arcsin x + \frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{(1-x^2)}} = \arcsin x +$
$+\frac{1}{2}\int t^{-\frac{1}{2}}dt = \arcsin x + \sqrt{1-x^2} + C$

в) $\int tg^2x\,dx = \int\frac{\sin^2x}{\cos^2x}dx = \int\frac{1-\cos^2x}{\cos^2x}dx =$
$= \int\frac{dx}{\cos^2x} - \int dx = tg\,x - x + C$

№2

а) $\int \cos 2x\cdot e^{-x}dx =$

$u = \cos 2x \qquad v = -e^{-x}$
$du = -2\sin 2x\,dx \quad dv = e^{-x}dx$

$= -e^{-x}\cos 2x - \int 2e^{-x}\sin 2x\,dx = -e^{-x}\cos 2x - 2\int e^{-x}\sin 2x\,dx =$

$u = \sin 2x \qquad v = -e^{-x}$
$du = 2\cos 2x\,dx \quad dv = e^{-x}dx$

$= -e^{-x}\cos 2x - 2\left(-e^{-x}\sin 2x + \int 2e^{-x}\cos 2x\,dx\right) =$
$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x - 4\int e^{-x}\cos 2x\,dx$

$5\int\cos 2x\cdot e^{-x}dx = 2e^{-x}\sin 2x - e^{-x}\cos 2x$

$\int\cos 2x\cdot e^{-x}dx = \frac{1}{5}e^{-x}(2\sin 2x - \cos 2x) + C$

б) $\int \sqrt{x^2-2x-1}\cdot dx = \int \sqrt{(x^2-2x+1)-2}\, dx =$

$= \int \sqrt{(x-1)^2-2}\, dx = \int \sqrt{t^2-2}\, dt =$

$u = \sqrt{t^2-2} \qquad v = t$

$du = \frac{t}{\sqrt{t^2-2}} dt \quad dv = dt$

$= t\sqrt{t^2-2} - \int \frac{t^2}{\sqrt{t^2-2}} \cdot dt = t\sqrt{t^2-2} -$

$- \int \frac{(t^2-2)+2}{\sqrt{t^2-2}} dt = t\sqrt{t^2-2} - \int\sqrt{t^2-2}\,dt -$

$- 2\int \frac{dt}{\sqrt{t^2-2}}$

$2\int \sqrt{t^2-2}\, dt = t\sqrt{t^2-2} - 2\ln(t+\sqrt{t^2-2})$

$\int \sqrt{x^2-2x-1}\, dx = \frac{1}{2}(x-1)\sqrt{x^2-2x-1} - \ln(x-1+\sqrt{x^2-2x-1}) + C$

№3

$\int \frac{-3x^3+13x^2-13x+1}{(x^2-x+1)(x-2)^2} dx =$

$= \int \frac{A}{x-2} dx + \int \frac{B}{(x-2)^2} dx + \int \frac{Cx+D}{x^2-x+1} dx =$

$A(x-2)(x^2-x+1) + B(x^2-x+1) + Cx\cdot(x-2)^2 + D(x-2)^2 =$
$= -3x^3+13x^2-13x+1$

$x=2: \quad 3B = 3 \to B = 1$

$x=0: \quad -2A + 1 + 4D = 1 \Rightarrow A = 2D \to A = 0$

$x=1: \quad -1(2D) + 1 + C + D = -2 \Rightarrow C = D - 3 \Rightarrow C = -3$

$x=3: \quad 7(2D) + 7 + 3(-3+D) + D = -2$

$14D + 3D + D = -2 + 9 - 7 = 0 \Rightarrow D = 0$

$= \int \frac{dx}{(x-2)^2} - 3\int \frac{x\,dx}{(x^2-x+1)} = \frac{-1}{x-2} - 3\int \frac{(x-1/2)+1/2}{(x-1/2)^2+3/4} dx$

$= \frac{-1}{x-2} - 3\int \frac{t+1/2}{t^2+3/4} dt = \frac{-1}{x-2} - 3\int \frac{t\,dt}{t^2+3/4} - \frac{3}{2}\int \frac{dt}{t^2+3/4}$

$= \frac{-1}{x-2} - \frac{3}{2}\int \frac{d(t^2+3/4)}{t^2+3/4} - \frac{3}{2}\cdot\frac{2}{\sqrt{3}} \operatorname{arctg}\frac{2t}{\sqrt{3}} =$

$$= \frac{-1}{x-2} - \sqrt{3}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt{3}} - \frac{3}{2}\ln(x^2-x+1)+C$$

N4

$$\int \frac{\sin x\,dx}{(1+\sin x)^2} \overset{[t = tg x}{=} \int \frac{\frac{2t}{1+t^2}\cdot\frac{2dt}{1+t^2}}{\left(1+\frac{2t}{1+t^2}\right)^2} = \int \frac{\frac{4t\,dt}{(t^2+1)^2}}{\left(\frac{t^2+2t+1}{t^2+1}\right)^2} =$$

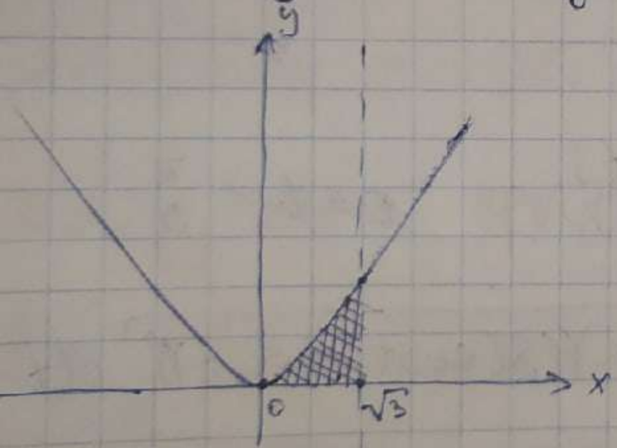
$$= \int \frac{\frac{4t\,dt}{(t^2+1)^2}}{\frac{(t+1)^4}{(t^2+1)^2}} = \int \frac{4t\,dt}{(t+1)^4} = 4\int\frac{(t+1)-1}{(t+1)^4}dt =$$

$$= 4\int\frac{dt}{(t+1)^3} - 4\int\frac{dt}{(t+1)^4} = \frac{4}{3(t+1)^3} - \frac{2}{(t+1)^2} + C =$$

$$= \frac{4}{3(tg\,x/2+1)^3} - \frac{2}{(tg\,x/2+1)^2} + C$$

N5

$$y = x\cdot \mathrm{arctg}\,x, \quad x=\sqrt{3}, \quad y=0$$



$$\int x\cdot\mathrm{arctg}\,x\,dx =$$

$$u = \mathrm{arctg}\,x \qquad v = \frac{x^2}{2}$$

$$du = \frac{dx}{x^2+1} \qquad dv = x\cdot dx$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int\frac{x^2dx}{x^2+1} =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int\frac{(x^2+1)-1}{x^2+1}dx =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{x^2+1} = \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}x +$$

$$+ \frac{1}{2}\,\mathrm{arctg}\,x + C$$

$$S = \int_0^{\sqrt{3}} x\cdot\mathrm{arctg}\,x\cdot dx = \left(\frac{3}{2}\cdot\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\pi}{3}\right) =$$

$$= \frac{\pi}{2} + \frac{\pi}{6} - \frac{\sqrt{3}}{2} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

№6

а) $\ln(x^2-1)$ $\quad 2 \le x \le 3$

$$L = \int_2^3 \sqrt{1+\left(\frac{2x}{x^2-1}\right)^2}\, dx$$

$$\int \sqrt{1+\left(\frac{2x}{x^2-1}\right)^2}\, dx = \int \sqrt{1+\frac{4x^2}{(x^2-1)^2}}\, dx =$$

$$= \int \sqrt{\frac{x^4-2x^2+1+4x^2}{(x^2-1)^2}}\, dx = \int \sqrt{\left(\frac{(x^2+1)}{(x^2-1)}\right)^2}\, dx =$$

$$= \int \frac{x^2+1}{x^2-1}\, dx = \int \frac{(x^2-1)+2}{x^2-1}\, dx =$$

$$= \int dx + 2\int \frac{dx}{x^2-1} = x - 2\int \frac{dx}{1-x^2} =$$

$$= x - 2\cdot\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + C = x - \ln\left(\frac{x+1}{x-1}\right) + C$$

$$L = \int_2^3 \sqrt{1+\left(\frac{2x}{x^2-1}\right)^2}\, dx = 3 - \ln 2 - 2 + \ln 3 =$$

$$= 1 + \ln 3/2$$

б) $x = 8(\cos t + t\sin t)$
$y = 8(\sin t - t\cos t)$ $\quad 0 \le t \le \frac{\pi}{3}$

$$L = \int_0^{\pi/3} \sqrt{(8(\sin t - t\cos t)')^2 + ((8(\cos t + t\sin t))')^2}\, dt$$

$$\int \sqrt{64(\cos t - \cos t + t\sin t)^2 + 64(-\sin t + \sin t + t\cos t)^2}\, dt$$

$$= \int \sqrt{64t^2\sin^2 t + 64t^2\cos^2 t}\, dt = \int 8t\, dt =$$

$$= 8\int t\, dt = 4t^2 + C$$

$$L = \int_0^{\pi/3} \sqrt{(8(\sin t - t\cos t)')^2 + (8(\cos t + t\sin t)')^2}\, dt = 4\cdot\frac{\pi^2}{9}$$

в) $\rho = 2\varphi$, $0 \le \varphi \le 5/12$

$$L = \int_0^{5/12} \sqrt{(2\varphi)^2 + ((2\varphi)')^2}\, d\varphi$$

$$\int \sqrt{(2\varphi)^2 + 2^2}\, d\varphi = \int \sqrt{4\varphi^2 + 4}\, d\varphi = 2\int \sqrt{\varphi^2+1}\, d\varphi$$

$u = \sqrt{\varphi^2+1}$ $\qquad v = \varphi$

$du = \frac{\varphi}{\sqrt{\varphi^2+1}}\, d\varphi$ $\qquad dv = d\varphi$

$$= 2\varphi\sqrt{\varphi^2+1} - 2\int \frac{\varphi^2}{\sqrt{\varphi^2+1}}\, d\varphi = 2\varphi\sqrt{\varphi^2+1} - 2\int \frac{(\varphi^2+1)-1}{\sqrt{\varphi^2+1}}\, d\varphi$$

$$= 2\varphi\sqrt{\varphi^2+1} - 2\int\sqrt{\varphi^2+1}\, d\varphi + 2\int \frac{d\varphi}{\sqrt{\varphi^2+1}} =$$

$$2\int\sqrt{\varphi^2+1}\, d\varphi = 2\varphi\sqrt{\varphi^2+1} + 2\ln(\varphi + \sqrt{\varphi^2+1}) - 2\int\sqrt{\varphi^2+1}\, d\varphi$$

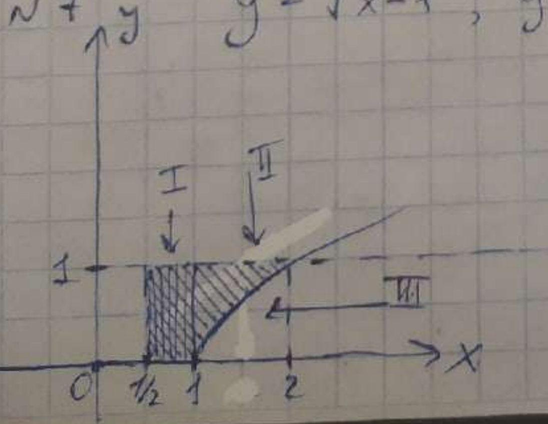
$$2\int\sqrt{\varphi^2+1}\, d\varphi = \varphi\sqrt{\varphi^2+1} + \ln(\varphi + \sqrt{\varphi^2+1}) + C$$

$$L = \int_0^{5/12} \sqrt{(2\varphi)^2 + ((2\varphi)')^2}\, d\varphi = \frac{5}{12}\sqrt{\frac{25}{144}+1} + \ln\left(\frac{5}{12} + \sqrt{\frac{25}{144}+1}\right) =$$

$$= \frac{5\cdot 13}{12\cdot 12} + \ln\left(\frac{5}{12} + \frac{13}{12}\right) = \frac{65}{144} + \ln 3/2$$

N 7 $\quad y = \sqrt{x-1}$; $y = 0$; $y = 1$; $x = 0.5$

Вращение вокруг оси оу



$$V_I = h\cdot(\pi R^2 - \pi r^2) =$$

$$= 1\cdot(\pi 1^2 - \pi (1/2)^2) = \frac{3\pi}{4}$$

$$V_{II} = 1\cdot(\pi 2^2 - \pi 1^2) - V_{III} = 3\pi - V_{III}$$

$$V_{III} = 2\pi \int_1^2 x\cdot\sqrt{x-1}\,dx$$

$$\int x\cdot\sqrt{x-1}\,dx = \int((x-1)+1)\sqrt{x-1}\,dx =$$

$$= \int (x-1)^{3/2} d(x-1) + \int \sqrt{x-1}\,d(x-1) = \frac{2}{5}(x-1)^{5/2} +$$

$$+ \frac{2}{3}(x-1)^{3/2} + C$$

$$\int_1^2 x\cdot\sqrt{x-1}\,dx = \frac{2}{5}\cdot 1^{5/2} + \frac{2}{3}\cdot 1^{3/2} - 0 - 0 = \frac{16}{15}$$

$$V_{III} = \frac{32\pi}{15} \Rightarrow V_{II} = 3\pi - \frac{32\pi}{15} = \frac{13\pi}{15}$$

$$V = V_I + V_{II} = \frac{3\pi}{4} + \frac{13\pi}{15} = \frac{97\pi}{60}$$

Часть Вторая

I Найти площадь, ограниченную заданными кривыми

а) $r = \dfrac{4-\sqrt{3}}{2-\cos\varphi}$ $\quad r = \dfrac{1}{\cos 2\varphi}$

Найдём точки пересечения

$$\frac{4-\sqrt{3}}{2-\cos\varphi} = \frac{1}{\cos 2\varphi}$$

$$(4-\sqrt{3})\cos 2\varphi = 2-\cos\varphi$$

$$(4-\sqrt{3})(2\cos^2\varphi - 1) = 2-\cos\varphi$$

$$(8-2\sqrt{3})\cos^2\varphi + \cos\varphi - (6-\sqrt{3}) = 0$$

$$D = 1 + 4\cdot(8-2\sqrt{3})(6-\sqrt{3}) = 1 + 8(27-10\sqrt{3}) = (5-8\sqrt{3})^2$$

$$\cos\varphi = \frac{-1 \pm \sqrt{(5-8\sqrt{3})^2}}{4(4-\sqrt{3})} = \frac{-1 \pm (5-8\sqrt{3})}{4(4-\sqrt{3})}$$

$$\cos\varphi_1 = \frac{4-8\sqrt{3}}{4(4-\sqrt{3})} = \frac{1-2\sqrt{3}}{4-\sqrt{3}} = \frac{(1-2\sqrt{3})(4+\sqrt{3})}{13} = \frac{-2-7\sqrt{3}}{13}$$

$$\cos\varphi_2 = \frac{-6+8\sqrt{3}}{4(4-\sqrt{3})} = \frac{4\sqrt{3}-3}{2(4-\sqrt{3})} = \frac{(4\sqrt{3}-3)(4+\sqrt{3})}{13\cdot 2} = \frac{\sqrt{3}}{2}$$

$$\cos\varphi = \frac{\sqrt{3}}{2}. \qquad \varphi_1 = -\frac{\pi}{6} \qquad \varphi_2 = \frac{\pi}{6}$$

Найдём интегралы

① $$\frac{1}{2}\int\left(\frac{4-\sqrt{3}}{2-\cos\varphi}\right)^2 d\varphi = \frac{1}{2}\int \frac{19-8\sqrt{3}}{(2-\cos\varphi)^2}\,d\varphi =$$

$$\frac{1}{2}(19-8\sqrt{3})\int\frac{d\varphi}{(2-\cos\varphi)^2} = \left[\begin{matrix}\varphi = 2\,\mathrm{arctg}\, t\\ t = \mathrm{tg}\,\varphi/2\end{matrix}\right] = \frac{(19-8\sqrt{3})}{2}\int\frac{\frac{2dt}{1+t^2}}{\left(2-\frac{1-t^2}{1+t^2}\right)^2} =$$

$$= \frac{(19-8\sqrt{3})}{2}\int\frac{2dt}{3t^2+1} = \frac{2(19-8\sqrt{3})}{3\cdot 2}\int\frac{dt}{t^2+1/3} = \frac{19\sqrt{3}-24}{3}\cdot \mathrm{arctg}\sqrt{3}t + C$$

$$= \frac{19\sqrt{3}-24}{3}\,\mathrm{arctg}\left(\sqrt{3}\,\mathrm{tg}\frac{\varphi}{2}\right) + C$$

$$\frac{1}{2}\int_{-\pi/6}^{\pi/6}\left(\frac{4-\sqrt{3}}{2-\cos\varphi}\right)^2 d\varphi = \frac{19\sqrt{3}-24}{3}\left(\mathrm{arctg}\left(\sqrt{3}\cdot\frac{\sin\varphi}{1+\cos\varphi}\right)\right)\Big|_{-\pi/6}^{\pi/6} =$$

$$= \frac{19\sqrt{3}-24}{3}\left(\mathrm{arctg}(2\sqrt{3}-3) - \mathrm{arctg}(3-2\sqrt{3})\right) =$$

$$= \frac{38\sqrt{3}-48}{3}\,\mathrm{arctg}(2\sqrt{3}-3)$$

② $$\frac{1}{2}\int\left(\frac{1}{\cos 2\varphi}\right)^2 d\varphi = \frac{1}{4}\int\frac{1}{\cos^2 t}\,dt = \frac{1}{4}\,\mathrm{tg}\,2\varphi + C$$

$$\frac{1}{2}\int_{-\pi/6}^{\pi/6}\left(\frac{1}{\cos 2\varphi}\right)^2 d\varphi = \frac{1}{4}\,\mathrm{tg}\,2\varphi\Big|_{-\pi/6}^{\pi/6} = 2\sqrt{3}$$

$S_{\text{искомой фигуры}} = 2\sqrt{3} - \frac{38\sqrt{3}-48}{3} \operatorname{arctg}(2\sqrt{3}-3)$

б) $x = a\cos t \quad y = a\sin t \cdot \cos^2 t$

Так как у данных функций период 2π

то возьмём $0 \le t \le 2\pi$

Найдём интеграл

$\int a\sin t \cdot \cos^2 t \cdot (a\cos t)' dt =$

$= -a^2 \int \sin t \cdot \cos^2 t \cdot \sin t \, dt = -a^2 \int \frac{1}{4} \cdot 4 \cos^2 t \cdot \sin^2 t \, dt$

$= \frac{-a^2}{4} \int \sin^2 2t \, dt = \frac{-a^2}{8} \int \sin^2 k \, dk = \frac{-a^2}{8} \int \frac{1}{2}(1-\cos 2k) dk$

$= \frac{-a^2}{16}\left(\int dk - \int \cos 2k \, dk\right) = \frac{-a^2}{16} k - \frac{-a^2}{32} \sin m + C =$

$= \frac{-a^2}{8} t + \frac{a^2}{32} \sin 4t + C$

$\int_0^{2\pi} a\sin t \cdot \cos^2 t \, (a\cos t)' dt = \left(\frac{-a^2}{8} t + \frac{a^2 \sin 4t}{32}\right)\Big|_0^{2\pi} =$

$= \frac{-a^2 \pi}{4}$