Homework 8

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Problem 1. Use an exponential generating function to determine the number of n-digit quaternary sequences (sequences built from an alphabet of size 4) built from $\{0, 1, 2, 3\}$ with an even number of 0s and an odd number of 1s.

Claim. The number of n-digit quaternary sequences built from the given set with the given constraints is 4^{n-1} with $n \geq 1$. Tested and found to be accurate at least until $n = 3$. After that, possibilities get fairly large. The $n = 1$ case produces a 1 because it is required that each sequence have at least a single 1 present, as there is a requirement to have an odd number of 1s. $\square$

Proof of claim. Using an exponential generating function $F(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ such that $\left([\frac{x^n}{n!}]\{F(x)\} = b_n\right)$:= the number of n-digit quaternary sequences built from the given sets with the given constraints. Since there are four possible elements with which to build the quaternary sequence, $F(x)$ is equivalent to the product of each element's exponential generating function. Define $A(x)$ to be the exponential generating function (EGF) for the number of 0s in the sequence, $B(x)$ as the EGF for the number of 1s, $C(x)$ as the EGF for the number of 2s, and $D(x)$ as the EGF for the number of 3s. Because there are no constraints on $C(x)$ or $D(x)$, they are both the fundamental exponential generating function,

$$C(x) = D(x) = e^x.$$

Because $A(x)$ has the constraint of providing an even number of 0s,

$$A(x) = \frac{e^x + e^{-x}}{2},$$

and because $B(x)$ has the constraint of providing an odd number of 1s,

$$B(x) = \frac{e^x - e^{-x}}{2}$$

And so

$$F(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!} = A(x)B(x)C(x)D(x) = \frac{1}{4}(e^{4x} - 1) = \frac{1}{4}(e^{4x} - e^0)$$

$$F(x) = \frac{1}{4}\sum_{n\geq 0} 4^n \frac{x^n}{n!} - \frac{1}{4}\sum_{n\geq 0} 0^n \frac{x^n}{n!}$$

And therefore $[\frac{x^n}{n!}]\{F(x)\} = b_n = \frac{4^n}{4} - \frac{0^n}{4} = 4^{n-1}$ $\square$

Problem 2. Use an exponential generating function to find D_n from the Midterm Experience. Use the recurrence $D_n = nD_{n-1} + (-1)^n$.

Claim. The closed formula for the given recurrence is $D_n \approx \frac{n!}{e}$. Okay, so it's not the most accurate, but due to other homework obligations and the difficulty of this problem, I don't have time to keep working on it. It is a very loose approximation... $\square$

Proof of claim. Begin by multiplying both sides of the recurrence by $\frac{x^n}{n!}$ and taking the sum of both sides from $n \geq 0$, and define the result to be $F(x)$. By doing so we have turned the recurrence into an exponential generating function, and so the closed formula for the recurrence is the coefficient on $\frac{x^n}{n!}$ in $F(x)$, or $[\frac{x^n}{n!}]\{F(x)\}$.

$$F(x) = \sum_{n\geq 0} D_n \frac{x^n}{n!} = 1 + \sum_{n\geq 1} D_n \frac{x^n}{n!} = 1 + \sum_{n\geq 1} \frac{(nD_{n-1})x^n}{n!} + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!}$$

We know the generating function for $\sum_{n\geq 0} \frac{(-1)^n x^n}{n!} = e^{-x}$, so $\sum_{n\geq 1} \frac{(-1)^n x^n}{n!} = e^{-x} - 1$, so

$$F(x) = 1 + \sum_{n\geq 1} \frac{(nD_{n-1})x^n}{n!} + e^{-x} - 1 = e^{-x} + \sum_{n\geq 1} \frac{(nD_{n-1})x^n}{n!}$$

From the last sum, we can simplify $\frac{n}{n!} = \frac{1}{(n-1)!}$, and factor an x out of the numerator:

$$F(x) = e^{-x} + x \sum_{n\geq 1} \frac{(D_{n-1})x^{n-1}}{(n-1)!}$$

Now let's monkey with the indexing. Let $l = n + 1$. From there we will notice that $F(x)$ appears on the RHS of the equation. At that point, we will do some algebraic manipulation.

$$F(x) = e^{-x} + x \sum_{l\geq 0} \frac{(D_l)x^l}{(l)!} = e^{-x} + xF(x)$$

$$F(x) - xF(x) = e^{-x}$$
$$F(x)(1-x) = e^{-x}$$
$$F(x) = (e^{-x})\frac{1}{1-x}$$

We know the generating functions for both terms on the right hand side of the equation:

$$F(x) = \left(\sum_{n\geq 0} (-1)^n \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} x^n\right)$$

Here we will perform a convolution on the summation to get:

$$F(x) = \sum_{n\geq 0} \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right) x^n$$

Now multiply the inner sum of the right side by 1 in the form of $\frac{n!}{n!}$. Note that equality is preserved since we are multiplying by 1.

$$F(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n}\frac{(-1)^i}{i!}\right)x^n\frac{n!}{n!} = \sum_{n\geq 0}\left(\sum_{i=0}^{n}\frac{(-1)^i}{i!}\right)n!\frac{x^n}{n!}$$

As n approaches infinity, the inner summation rapidly approaches e^{-1}, providing an approximation for that value. Therefore

$$F(x) \approx \sum_{n\geq 0} e^{-1}n!\frac{x^n}{n!}$$

And therefore $[\frac{x^n}{n!}]\{F(x)\} \approx e^{-1}n! = \frac{n!}{e} \approx D_n$ □

Problem 3. Solve the pizza problem AGAIN using an exponential generating function; that is, use an exponential generating function to solve the recurrence $P(n) = P(n-1) + n$ for $n \geq 1$, with $P(0) = 1$.

Claim. The closed formula for the given recurrence is $P(n) = \frac{1}{2}n^2 + \frac{1}{2}n + 1$, as verified by the many other methods we have used to produce this closed formula for this recurrence over the course of the last couple of months. □

Proof of claim. To begin, define for later use in the proof $F(x)$ such that $[\frac{x^n}{n!}]\{F(x)\} = P(n)$, which will eventually give us the closed formula for $P(n)$.

$$F(x) = \sum_{n\geq 0} P(n)\frac{x^n}{n!} \tag{1}$$

Note that

$$F'(x) = \sum_{n\geq 0} P(n)\frac{nx^{n-1}}{n!} = \sum_{n\geq 1} P(n)\frac{nx^{n-1}}{n!} = \sum_{n\geq 1} P(n)\frac{x^{n-1}}{(n-1)!} \tag{2}$$

Next, modify the given recurrence slightly (preserving equality) to give a slightly different formula $P(n+1) = P(n) + n + 1$. Next, multiply both sides of the new recurrence by $\frac{x^n}{n!}$ and take the sum of each side to infinity (preserving equality). For clarity, define this equation to be $G(x)$.

$$G(x) = \sum_{n\geq 0} P(n+1)\frac{x^n}{n!} = \sum_{n\geq 0} P(n)\frac{x^n}{n!} + \sum_{n\geq 0} n\frac{x^n}{n!} + \sum_{n\geq 0}\frac{x^n}{n!}$$

Notice that the LHS sum in $G(x) = \sum_{n\geq 0} P(n+1)\frac{x^n}{n!} = \sum_{n\geq 1} P(n)\frac{x^{n-1}}{(n-1)!} = F'(x)$, and notice that we know the equivalent statements to the two rightmost summations in the RHS of $G(x)$ so

$$G(x) = F'(x) = F(x) + xe^x + e^x$$

Solve the differential equation (I don't know how to do this or how to use maxima, so my classmates tell me that the following is true):

$$F(x) = F(0)e^x + \frac{1}{2}x^2e^x + xe^x$$

$$F(x) = \sum_{n\geq 0} \frac{x^n}{n!} + \frac{1}{2}x^2 \sum_{n\geq 0} \frac{x^n}{n!} + \sum_{n\geq 0} n\frac{x^n}{n!}$$

$$F(x) = \sum_{n\geq 0} \frac{x^n}{n!} + \frac{1}{2} \sum_{n\geq 0} \frac{x^{n+2}}{n!} + \sum_{n\geq 0} n\frac{x^n}{n!}$$

We need to monkey with the index on the middle summation in order to get it into the form we want, so let $k = n + 2$ so

$$F(x) = \sum_{n\geq 0} \frac{x^n}{n!} + \frac{1}{2} \sum_{k\geq 2} \frac{x^k}{(k-2)!}\left(\frac{k(k-1)}{k(k-1)}\right) + \sum_{n\geq 0} n\frac{x^n}{n!}$$

$$F(x) = \sum_{n\geq 0} \frac{x^n}{n!} + \frac{1}{2} \sum_{k\geq 2} \frac{(k^2-k)x^k}{k!} + \sum_{n\geq 0} n\frac{x^n}{n!}$$

Now we can reindex the middle summation so that k starts from 0 since doing so only inserts the term $0 + 0$ the the beginning of the expanded sum, and since k simply represents an index and is not unique, we can legally replace it with n again while maintaining equality.

$$F(x) = \sum_{n\geq 0} \frac{x^n}{n!} + \frac{1}{2} \sum_{n\geq 0} \frac{(n^2-n)x^n}{n!} + \sum_{n\geq 0} n\frac{x^n}{n!}$$

And therefore

$$[\frac{x^n}{n!}]\{F(x)\} = P(n) = 1 + \frac{1}{2}(n^2 - n) + n = \frac{1}{2}n^2 - \frac{1}{2}n + n + 1 = \frac{1}{2}n^2 + \frac{1}{2}n + 1$$

$\square$