Math 3310 Homework 2

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Formal Systems Prompt 1. Understand the $\{M, U, I\}$-system well enough to state and prove a metatheorem about what strings are in S.

Axiom 0: $MI \in S$

Axiom 1: $xI \in S \Longrightarrow xIU \in S$

Axiom 2: $Mx \in S \Longrightarrow Mxx \in S$

Axiom 3: $xIIIy \in S \Longrightarrow xUy \in S$

Axiom 4: $xUUy \in S \Longrightarrow xy \in S$

Proof. Take a string (as defined in Dr. Brown's *An introduction to formal systems, and therefore to Mathematics*) to have the form $w_1 w_2 \dots w_i$ where w_i represents the ith character in the string. The metatheorem I will prove is that the character M must be present in a string *exclusively* as w_1 in order for that string to be a member of S. That is, the character M *must* be present as the first character of a string, and be present *nowhere else* in that string in order for that string to possibly be a member of S. This can be proven by the fact that axiom 0 is the only axiom to guarantee that a specific string is in S, that string being MI. Through the other axioms, the characters of MI may be manipulated. In order for the character M to appear anywhere other than w_1, an axiom must allow for one of two things:

1) for an M to be substituted for another character or group of characters

or

2) for the character in the w_1 position (which we've established will always be M) to change to a different position

None of the axioms allow for either of these manipulations. Therefore, the character M *must* be present as the first character of a string, and be present *nowhere else* in that string in order for that string to possibly be a member of S. $\square$

Set Theory Exercise 6. Let $P = \{(a, b, c) : a, b, c \in \mathbb{Z}, \text{ and } a^2 + b^2 = c^2\}$, and $T = \{(p, q, r) : p = x^2 - y^2, q = 2xy, \text{ and } r = x^2 + y^2, \text{ where } x, y \in \mathbb{Z}\}$. Show that $T \subseteq P$.

Proof. If it can be shown that p, q, and r have the same form as a, b, and c, it would then clearly follow that $T \subseteq P$. That is, if $p^2 + q^2 = r^2$ (the same form as $a^2 + b^2 = c^2$), then $p, q, r \in P$, and therefore $T \subseteq P$. So
$p^2 + q^2 = r^2$
$= (x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$
$= x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = x^4 + 2x^2y^2 + y^4$
$= x^4 + 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4$
shows that $p^2 + q^2 = r^2$, which is analogous to $a^2 + b^2 = c^2$ because $x, y, a, b, c \in \mathbb{Z}$. Therefore $p, q, r \in P$, and therefore $T \subseteq P$. $\square$

Set Theory Exercise 9. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and $A_1 \cap A_2 \cap A_3 = \varnothing$. Express the answer in the form $2^a 3^b 5^c 7^d$ where a, b, c, d are non-negative integers.

Proof. In order to determine the number of ordered triples (A_1, A_2, A_3) of sets with the stated properties, the number of unique ways each element may appear in the sets must be determined. Take x to represent some element of $A_1 \cup A_2 \cup A_3$, that is $x \in (A_1 \cup A_2 \cup A_3)$. In order to also meet the criteria that $A_1 \cap A_2 \cap A_3 = \varnothing$, x can be present in no more than two of the sets, but must be present in as few as one, and therefore there are 6 possible configurations by which x can be present in A_1, A_2, A_3, as shown in the table below:

Table 1: Legal possibilities for the presence of x in sets A_1, A_2, A_3

Configuration	A_1	A_2	A_3
1	T	F	F
2	F	T	F
3	F	F	T
4	T	T	F
5	F	T	T
6	T	F	T

T denotes the presence of x in the corresponding set, while F denotes its absence

Any other configurations of x in the three sets would violate the properties set forth in the problem statement. Therefore there are 6 possible configurations by which x can be present in A_1, A_2, A_3, which is analogous to stating that for each element in the desired outcome of the union of A_1, A_2, A_3, there are 6 possible configurations by which that element may be present in the ordered triple of sets. Because our desired union of sets is the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (a set of 10 elements), and because each element has 6 possible configurations, there are therefore $2^{10} \cdot 3^{10} \cdot 5^0 \cdot 7^0 = 6^{10}$ possible ordered triples of sets. $\square$

Logic Exercise 6. Prove or disprove that $\{M, \Phi, \neg, \vee, \wedge, \implies\}$ can be reduced to $\{M, \Phi, \triangledown\}$, where $x \triangledown y$ is equivalent to $\neg(x \vee y)$.

Proof. By definition, two operations, two operators, or an operator and operation are equal if they have the same truth table (see Dr. Brown's *A Brief Treatment of Logic*). The following four truth tables show that $\neg, \vee, \wedge$, and $\implies$ can all be expressed using $\triangledown$, and therefore that $\{M, \Phi, \neg, \vee, \wedge, \Rightarrow\}$ can be reduced to $\{M, \Phi, \triangledown\}$. The last two columns of each table show the final result of the truth table.

P	Q	$P \triangledown Q$	$(P \triangledown Q) \triangledown (P \triangledown Q)$	$P \vee Q$
T	T	F	T	T
T	F	F	T	T
F	T	F	T	T
F	F	T	F	F

P	Q	$P \triangledown P$	$Q \triangledown Q$	$(P \triangledown P) \triangledown (Q \triangledown Q)$	$P \wedge Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	F	F

P	Q	$P \triangledown P$	$(P \triangledown P) \triangledown Q$	$((P \triangledown P) \triangledown Q) \triangledown ((P \triangledown P) \triangledown Q)$	$P \implies Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	T	T

P	$P \triangledown P$	$\neg P$
T	F	F
F	T	T

$\square$