Lecture 10c:
Reasoning About Correctness

Spontaneous Right Action

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So far in the course

- Important basic data structures
  - Lists, Vectors, Sequences, Trees, Priority Queues, Heaps, Dictionaries, Hash Tables, and Binary Search Trees
  - Search Trees
- Important algorithms
  - Sorting (insertion, heap, PQ, merge, Quick, bucket, radix)
  - Searching (Dictionary: binary search, hash table, BST)
  - Selection (Quick, deterministic)
- Design strategies
  - Exhaustive Search, Divide-and-Conquer, Prune-and-Search, and randomization
- Solution to recurrences
- Amortized analysis

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What is the loop invariant?

- An assertion that is necessarily true immediately before and immediately after each iteration of a loop
- Could be false part way through the loop, but must be re-established before the end of the loop body
- **The invariant at termination of the loop should imply the goal of the loop!!!!**

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Reasoning About Loops

- Identify the loop invariant
- Make sure the invariant holds every time through the loop
- Make sure the loop is making progress toward termination
- Make sure the loop terminates (i.e., check boundary conditions)

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Binary Search Algorithm (What's wrong)

Algorithm BinarySearch(*S, k*):
Input: *Ordered vector S storing n items, accessed by key(), and key k*
Output: *An element of S with key k.*
low ← 0
high ← S.size() - 1
while low < high **do**
    mid ← (low + high)/2
    if k = key(S.elemAtRank(mid)) **then**
        return elem(S.elemAtRank(mid))
    if k < key(S.elemAtRank(mid)) **then**
        high ← mid - 1
    else
        low ← mid + 1
return NO_SUCH_KEY

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Error in binary search

- Does not handle the case when low equals high (boundary condition)
  - When the segment is size 1, the key may not be found because we do not enter the loop

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Binary Search Algorithm (Corrected)

```
Algorithm BinarySearch(S, k):
  Input: Ordered vector S storing n items, accessed by key(), and key k
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low ≤ high do
     mid ← (low + high)/2
     if k = key(S.elemAtRank(mid))  then
        return  elem(S.elemAtRank(mid))
     if k < key(S.elemAtRank(mid))  then
        high ← mid - 1
     else
        low ← mid + 1
  return NO_SUCH_KEY
```

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Introducing Errors through Copy-Paste

- We wish to have only one key comparison during each iteration of the loop
- So we copy from above version then modify as described
  - Move the check for equality after the loop
  - Now we do not exit the loop early and thus do half the key comparisons during each iteration

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Binary Search Algorithm (what's wrong? Look at red)

```
Algorithm BinarySearch(S, k):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low ≤ high do
     mid ← (low + high)/2
     if  k < key(S.elemAtRank(mid))  then // one key comparison per iteration
        high ← mid - 1
      else
        low ← mid + 1
  if  k = key(S.elemAtRank(mid))   then  // done once outside the loop now
     return elem(S.elemAtRank(mid))
  else
     return NO_SUCH_KEY
```

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Errors

- Does not handle a Vector with 0 elements
  - mid is not initialized since loop is not entered and, further, it cannot be initialized to handle an empty Vector
- Does not handle a Vector with 1 element that matches k
  - The else eliminates mid when it hasn't yet been eliminated, so delete the + 1 from the else branch

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Binary Search Algorithm (better, but what else?)

```
Algorithm BinarySearch(S, k):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  mid ← 0
  while low < high do
     mid ← (low + high)/2
     if  k < key(S.elemAtRank(mid))   then // one key comparison per iteration
        high ← mid - 1
     else
        low ← mid    // + 1 because mid has not been eliminated yet
  if S.size() > 0  /\  k = key(S.elemAtRank(low))   then   // handles empty S
     return elem(S.elemAtRank(low))
  else
     return NO_SUCH_KEY
```

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Error: loop does not terminate

- Does not handle a Vector with 2 items (or a segment with 2 items) when the key of first item is less than or equal to key k
  - The loop does not always terminate
    - mid needs to be the ceiling of the expression otherwise mid and low do not/cannot change

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Binary Search Algorithm (even better, but … ?)

```
Algorithm BinarySearch(S, k):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  mid ← 0
  while low ≤ high do
     mid ← (low + high + 1)/2          // needs to be the ceiling to terminate
     if  k < key(S.elemAtRank(mid))   then // one key comparison per iteration
        high ← mid - 1
     else
        low ← mid    // + 1 because mid has not been eliminated here
  if  S.size() > 0  ∧  k = key(S.elemAtRank(mid))   then // handles empty S
     return elem(S.elemAtRank(mid))
  else
     return NO_SUCH_KEY
```

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Error: loop does not terminate

- Does not handle a Vector with 1 item (or a segment with 1 item) when its key matches k
  - The loop does not terminate
    - Modify the loop condition from ≤ to < so the loop terminates when high = low since low does not change when the key of the item equals k
- The rank mid may not contain the item with the key after fixing the loop's terminating condition
  - Either low or high will contain the key if it is in the Vector
  - Fixing this eliminates the need to initialize mid before the loop since mid will only used inside the loop now

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Binary Search Algorithm (green shows corrections)

```
Algorithm BinarySearch(S, k):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low < high do                   // needs to be < to terminate
     mid ← (low + high + 1)/2          // needs to be the ceiling to terminate
     if  k < key(S.elemAtRank(mid))   then // one key comparison per iteration
        high ← mid - 1
     else
        low ← mid    // + 1 because mid has not been eliminated yet
  if  S.size() > 0  ∧  k = key(S.elemAtRank(high))   then // handles empty S
     return elem(S.elemAtRank(high))    // high or low contain matching key
  else
     return NO_SUCH_KEY
```

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Binary Search Algorithm (change < to > in the loop)

```
Algorithm BinarySearch(S, k):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low < high do                   // needs to be < to terminate
     mid ← (low + high + 1)/2          // needs to be the ceiling to terminate
     if  k > key(S.elemAtRank(mid))   then // change to > instead of <
        low ← mid + 1       // changed due to change of condition
     else
        high ← mid           // changed due to change of condition
  if  S.size() > 0  ∧  k = key(S.elemAtRank(high))   then // handles empty S
     return elem(S.elemAtRank(high))    // high or low contain matching key
  else
     return NO_SUCH_KEY
```

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Errors

- The loop does not always terminate
  - mid needs to be the floor of the expression otherwise mid and high do not/cannot change which causes non-termination

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Binary Search Algorithm (green shows corrections)

```
Algorithm BinarySearch(S, k):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low < high do                   // needs to be < to terminate
     mid ← (low + high)/2              // needs to be the floor to terminate
     if  k > key(S.elemAtRank(mid))   then // changed to > instead of <
        low ← mid + 1       // changed
     else
        high ← mid           // changed
  if  S.size() > 0  ∧  k = key(S.elemAtRank(low))   then // handles empty S
     return elem(S.elemAtRank(low))    // high or low contain matching key
  else
     return NO_SUCH_KEY
```

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Why a third version?

- Depends on the purpose
- The third version is an improvement in the binary search used by the Lookup Table

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Errors (none)

- Handles a Vector with 0 elements
- Handles a Vector with 1 element that matches the key k
- We do not want the ceiling((high+low)/2) this time
- The loop terminates
  - mid is initialized correctly with the floor of the expression (does not add 1)
- Handles a Vector with 2 elements (or a segment with 2 elements) with one matching the key k
  - Two cases: first and second element
- Finds the key when it is in the vector by using rank low although could have left it as high

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The loop invariant of the loop in function BinarySearch

if the key k is in the Vector S, then
S.elemAtRank(low) $\leq$ k $\leq$ S.elemAtRank(high)

- Informally, if key k is in the Vector S, then k is the key of an item in S at a rank between low and high

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Is it worth exiting early from the loop?

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Binary Search Algorithm (Two comparisons per iteration)

```
Algorithm BinarySearch(S, k):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k.
  low ← 0
  high ← S.size() - 1
  while low ≤ high do
     mid ← (low + high)/2
     if k = key(S.elemAtRank(mid)) then {exit early from the loop}
        return elem(S.elemAtRank(mid))
     else  if k < key(S.elemAtRank(mid)) then
        high ← mid - 1
     else
        low ← mid + 1
  return NO_SUCH_KEY
```

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Binary Search Algorithm (One comparison per iteration)

```
Algorithm BinarySearch( S, k ):
  Input: An ordered vector S storing n items, accessed by keys()
  Output: An element of S with key k and rank between low & high.
  low ← 0
  high ← S.size() - 1

  while low < high do
     mid ← (low + high)/2
     if k > key(S.elemAtRank(mid)) then {always does log n comparisons}
        low ← mid+1
     else
        high ← mid  // - 1
  if S.size() > 0  Λ  k = key(S.elemAtRank(low)) then
     return elem(S.elemAtRank(low))
  else
     return NO_SUCH_KEY
```

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Homework

- Both algorithms make *O*(log n) key comparisons
- Which algorithm makes fewer actual key comparisons when the key is not in S?
- Which makes fewer comparisons, **on average**, when the key is in S, assuming the keys are equally probable?

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What's Wrong with this In Place Version of Partition

```
Algorithm inPlacePartition(S, lo, hi)
    Input Sequence S and ranks lo and hi, 0 ≤ lo ≤ hi < S.size()
    Output the pivot is now stored at its sorted rank

    p ← a random integer between lo and hi
    S.swapElements(S.atRank( lo ), S.atRank( p ))
    pivot ← S.elemAtRank(lo)
    j ← lo + 1
    k ← hi
    while  j ≤ k  do
        while  k > j  ∧ S.elemAtRank( k ) ≥ pivot  do
            k ← k – 1
        while  j < k  ∧ S.elemAtRank( j ) ≤ pivot  do
            j ← j + 1

        if j < k then
            S.swapElements(S.atRank( j ), S.atRank( k ))

    S.swapElements(S.atRank( lo ), S.atRank( k )) {move pivot to sorted rank}
    return k
```

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Error

- Does not terminate!
- Every other swap could incorrectly move to elements

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Corrected In Place Version of Partition

```
Algorithm inPlacePartition(S, lo, hi)
    Input Sequence S and ranks lo and hi, 0 ≤ lo ≤ hi < S.size()
    Output the pivot is now stored at its sorted rank

    p ← a random integer between lo and hi
    S.swapElements(S.atRank( lo ), S.atRank( p ))
    pivot ← S.elemAtRank(lo)
    j ← lo + 1
    k ← hi
    while  j ≤ k  do
        while  k ≥ j  ∧ S.elemAtRank( k ) ≥ pivot  do
            k ← k – 1
        while  j ≤ k  ∧ S.elemAtRank( j ) ≤ pivot  do
            j ← j + 1

        if j < k then
            S.swapElements(S.atRank( j ), S.atRank( k ))

    S.swapElements(S.atRank( lo ), S.atRank( k )) {move pivot to sorted rank}
    return k
```

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