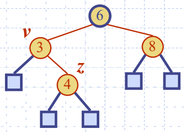
Lecture 9: Red-Black Trees

Perfect Balance and Efficiency



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Wholeness Statement

A red-black tree is an implementation of a (2, 4) tree that is optimized for space utilization. The insert and delete operations are also optimized to avoid backtracking; the operations are performed locally yet maintain balance and order in the whole. *Science of Consciousness*: Nature operates in accord with the law of least action while maintaining balance and order in the whole.

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Outline and Reading

- From (2,4) trees to red-black trees (§3.3.3)
- Red-black tree (§ 3.3.3)
  - Definition
  - Height
  - Insertion
    - restructuring
    - recoloring
  - Deletion
    - restructuring
    - recoloring
    - adjustment

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Balanced Search Trees

- History and development of balanced search trees
  - Started with AVL trees,1962
    - Two Soviet mathematicians (Adel'son-Vel'skiĭ and Landis)
    - Could have as many as O(log n) rotations
  - 2-3 trees,1970
  - B-trees, 1972
    - Generalization of 2-3 trees to any number keys per node
    - Variations (e.g., B*-tree, B+-tree) became popular file structures
  - Symmetric binary B-trees, 1972
  - Red-Black coloring introduced, 1978
    - Became popular implementation of balance binary trees in the late 1980's and early 1990's

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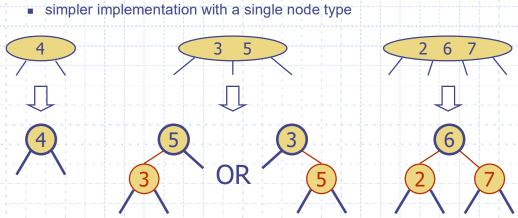
From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored red or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type



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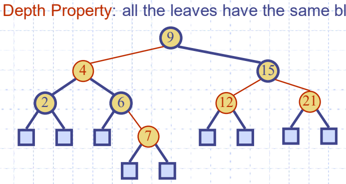
Red-Black Tree

- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black
  - Depth Property: all the leaves have the same black depth



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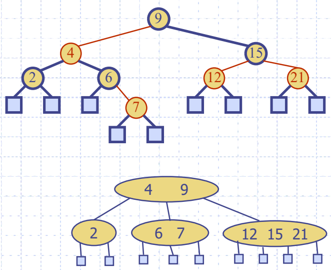
Red-Black Tree to (2,4) Tree



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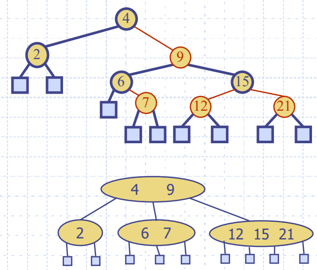
Different Red-Black Tree to Same (2,4) Tree



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Height of a Red-Black Tree

- Theorem: A red-black tree storing n items has height $O(\log n)$
  Proof:
  - The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $O(\log n)$
- The search algorithm for a red-black tree is the same as that for a binary search tree
- By the above theorem, searching in a red-black tree takes $O(\log n)$ time

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Red-Black Tree Search

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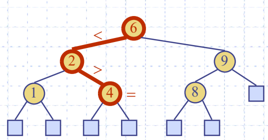
Search

- To search for a key k, we trace a downward path starting at the root using our helper ***findPosition***
- If we reach a leaf, the key is not found, so we return NO_SUCH_KEY
- Otherwise we return the element associated with the key k
- Example: findElement(4)

```
Algorithm findElement(k)
// The tree T is a field of the receiver this
    if isEmpty() then
        return NO_SUCH_KEY
    v ← findPosition(k, T.root()) // Helper method
    if k ≠ key(v) then
        return NO_SUCH_KEY
    else
        return v.element()
```



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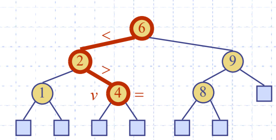
Helper *findPosition*

- To search for a key k, we trace a downward path starting at the root
- The next node visited is based on the comparison of k with the key of the current node
- If we reach a leaf, the key is not found and we return the parent of the external node
- If we find the key, then we return the node v containing k
- Example: findPosition of key 4 or 5 both return node v

```
Algorithm findPosition(k, v)
Output: the node containing key k or the parent of the node where k would be inserted into tree T
    if k = key(v) then
        return v        // node containing k
    else if k < key(v) then
        if isExternal (T.leftChild(v)) then
            return v   // node where k would be inserted
        else return findPosition(k, T.leftChild(v))
    else if isExternal (rightChild(v)) then
            return v   // node where k would be inserted
        else return findPosition(k, T.rightChild(v))
```



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Main Point

1. A red-black tree is an efficient way to implement an ordered dictionary ADT because it achieves logarithmic worst-case running times for both searching and updating (inserting and removing). *Science of Consciousness:* The TM technique is a very simple, effortless way to facilitate contact with the field of total knowledge, where the fulfillment of intellectual study is achieved, i.e., one feels at home with everything and everyone.

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Red-Black Tree Insertion

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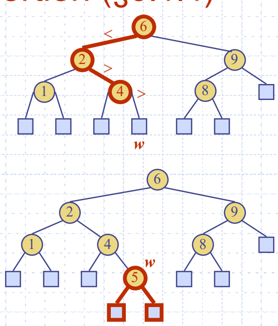
Recall
Binary Tree Insertion (§3.1.4)

- To perform operation insertItem(k, o), search for key k (k should not be in the tree)
- Let w be the leaf reached by the search
- Insert k at node w and expand w into an internal node
- Example: insert 5



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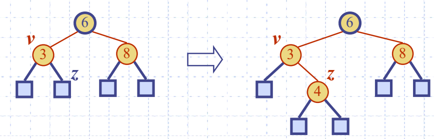
Insertion

- To perform operation insertItem(k, o), we execute the insertion algorithm for binary search trees and color the newly inserted node z red unless it is the root. We preserve the root, external, and depth properties.
  - If the parent v of z is black, we preserve the internal property and we are done…
  - …else (v is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree
- For example, the insertion of 4 causes a double red:



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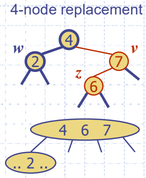
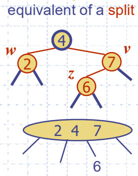
Remedying a Double Red

- Consider a double red with child z and parent v, and let w be the sibling of v

Case 1: z's uncle w is black
- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement



Case 2: z's uncle w is red
- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split



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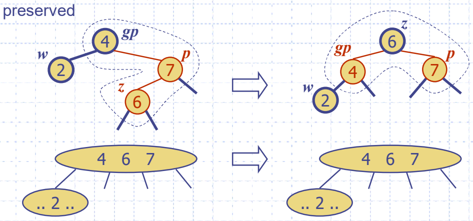
Case 1: Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved



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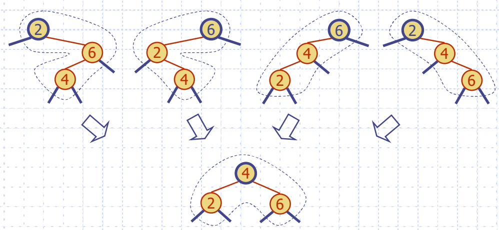
Restructuring (cont.)

- There are four restructuring configurations depending on whether the double red nodes are left or right children



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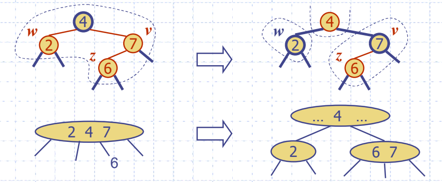
Case 2: Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent v and its sibling w become black and the grandparent u becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent u



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Example:

- Insert the following into an initially empty red-black tree in this order:
  (22, 5, 16, 45, 2, 10, 18, 30, 50, 12, 13, 33)

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Analysis of Insertion

Algorithm *insertItem*(k, o)

1. Search for key k to locate the insertion node z
2. Add the new item (k, o) at node z and color z red
3. **while** *doubleRed*(z)
   if *isBlack*(*sibling*(*parent*(z)))
   $z \leftarrow$ *restructure*(z)
   return
   else { *sibling*(*parent*(z)) is red }
   $z \leftarrow$ *splitRecolor*(z)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O($? $)$ time
- Step 2 takes $O($? $)$ time
- Step 3 takes $O($? $)$ time
- Thus, an insertion in a red-black tree takes $O($? $)$ time

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Analysis of Insertion

Algorithm *insertItem*(k, o)

1. Search for key k to locate the insertion node z using *findPosition*
2. Add the new item (k, o) at node z and color z red
3. **while** *doubleRed*(z)
   if *isBlack*(*sibling*(*parent*(z)))
   gpz $\leftarrow$ *parent*(*parent*(z))
   restructure(z)
   setColor(*gpz*, *RED*)
   setColor(*parent*(*gpz*), *BLK*)
   return
   else { *sibling*(*parent*(z)) is red }
   $z \leftarrow$ *splitRecolor*(z)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

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Analysis of Insertion

Algorithm *isDoubleRed*(z)
if *isRoot*(z) **then**
setColor(z, *BLK*)
return False
else
return isRed(parent(z))

Algorithm *splitRecolor*(z)
pz $\leftarrow$ *parent*(z)
setColor(*pz*, *BLK*)
setColor(*sibling*(*pz*), *BLK*)
gpz $\leftarrow$ *parent*(*pz*)
setColor(*gpz*, *RED*)
return *gpz*

Algorithm *restructure*(z)
pz $\leftarrow$ *parent*(z)
if *isLeft*(z) **then**
if *isLeft*(*pz*) **then**
rotateRight(*pz*)
else
rotateRight(z)
rotateLeft(z)
else { z is a right child }
if *isLeft*(*pz*) **then**
rotateLeft(z)
rotateRight(z)
else
rotateLeft(*pz*)

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Red-Black Tree Deletion

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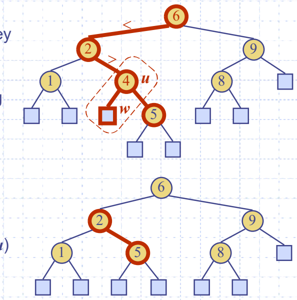
Recall
Binary Tree Deletion (Case 1)

- To perform operation remove(k), first search for key k
- Assume key k is in the tree, and let u be the node storing k
- Two cases:
  1. Node u has a leaf child w
  2. Node u has no leaf child
- If node u has a leaf child w, we remove u and w from the tree with operation remove(u)
- Example: remove 4



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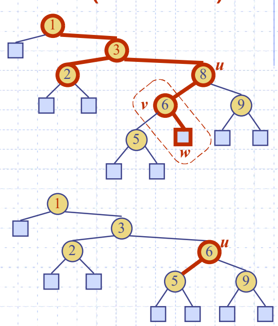
Recall
Binary Tree Deletion (Case 2)

- We consider the case where the key k to be removed is stored at a node u whose children are both internal
  - we find the internal node v that precedes u in an in-order traversal (v in the example)
  - we copy $key(v)$ into node u
  - we remove node v and its external child w by means of operation remove(v)
- Example: remove 8



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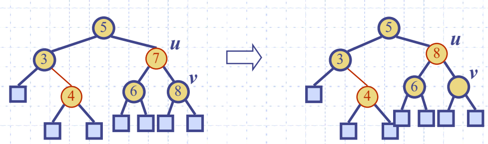
Deletion

- To perform operation remove(k), first execute the deletion algorithm for binary search trees
  - If node to be removed, u, does not have an external child, find next internal node by inorder traversal, called v, and move key at v to u, then remove v.
  - Thus, every removal occurs at an internal node with an external child.



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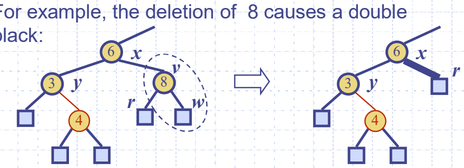
Deletion

- Let v be the internal node removed, w the external node removed, and r the sibling of w
  - If either v or r was red, we color r black and we are done
  - Else (v and r were both black), so we color r ***double black*** (a fictitious color), which is a violation of the internal property requiring a reorganization of the tree (denotes underflow)
- For example, the deletion of 8 causes a double black:



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Remedying a Double Black

- The algorithm for remedying a double black node r with sibling y considers three cases

Case 1: sibling y is black and has a red child z

Case 2: sibling y is black and its children are both black

Case 3: sibling y is red

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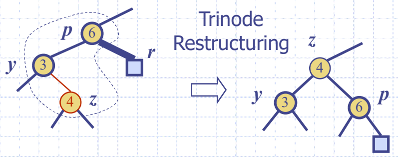
Case 1a – sibling y is black and has a red child z



Color y and p black, give z the former color of p, and set r to black

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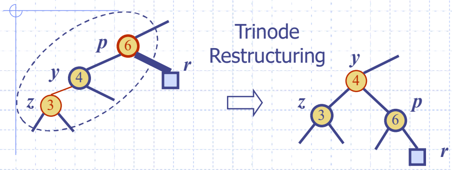
Case 1b – sibling y is black and has a red child z



Color z and p black, give y the former color of p, and set r to black

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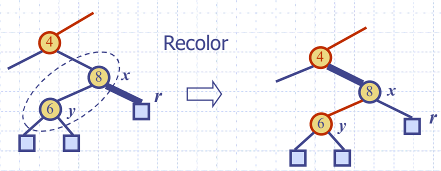
Case 2 – sibling y is black and its children are both black



Color r black and y red;
if x is red, color x black else color x double black

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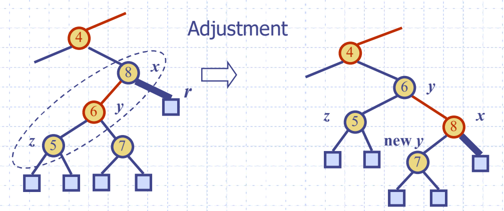
Case 3 – sibling y is red



Execute the trinode restructure operation that makes y the parent of x;
Color y black and x red;

Then apply Case 1 or 2, with new y.

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Remedying a Double Black

- The algorithm for remedying a double black node r with sibling y considers three cases

  Case 1: sibling y is black and has a red child
  - Perform a restructuring, equivalent to a transfer, and we are done

  Case 2: sibling y is black and its children are both black
  - Perform a recoloring, equivalent to a fusion, which may propagate the double black violation up to parent

  Case 3: sibling y is red
  - We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies

- Deletion in a red-black tree takes $O(\log n)$ time

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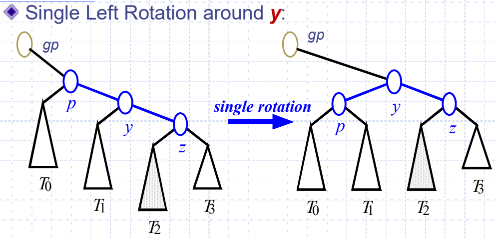
Restructuring (as Single Rotations)

- Single Left Rotation around ***y***:



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Left Rotation

```
Algorithm rotateLeft(T, y)
    Input Binary Tree T and node y in T
    Output a left rotation around node y is performed

    if T.isRoot(y) then throw InvalidRightRotation
    p ← T.parent(y)
    gp ← T.parent(p)

    T.setRightChild(p, T.leftChild(y))
    if T.isInternal(T.rightChild(y)) then {external node is a null reference}
        T.setParent(T.rightChild(y), p)

    T.setLeftChild(y, p)
    T.setParent(p, y)

    if T.isRoot(p)
        then T.setRoot(y)
        else
            if T.rightChild(gp) = p
                then T.setRightChild(gp, y)
                else T.setLeftChild(gp, y)
            T.setParent(y, gp)
```

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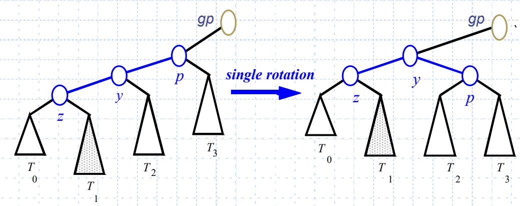
Exercise: Do Single Right Rotation



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Right Rotation

```
Algorithm rotateRight(T, y)
    Input Binary Tree T and node y in T
    Output a left rotation around node y is performed

    if T.isRoot(y) then throw InvalidLeftRotation
    p ← T.parent(y)
    gp ← T.parent(p)

    T.setLeftChild(p, T.rightChild(y))
    if T.isInternal(T.leftChild(y)) then {external node is a null reference}
        T.setParent(T.leftChild(y), p)

    T.setRightChild(y, p)
    T.setParent(p, y)

    if T.isRoot(p)
        then T.setRoot(y)
        else
            if T.rightChild(gp) = p
                then T.setRightChild(gp, y)
                else T.setLeftChild(gp, y)
            T.setParent(y, gp)
```

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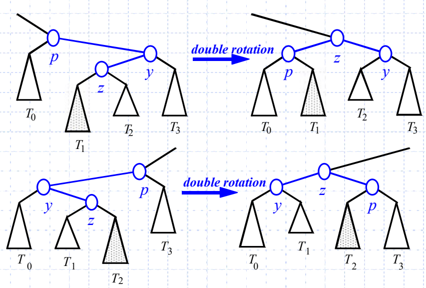
Restructuring (as Double Rotations)

- double rotations:



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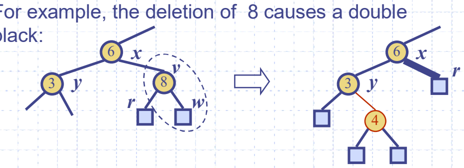
Deletion

- Let v be the internal node removed, w the external node removed, and r the sibling of w
  - If either v or r was red, we color r black and we are done
  - Else (v and r were both black), so we color r ***double black*** (a fictitious color), which is a violation of the internal property requiring a reorganization of the tree (denotes underflow)
- For example, the deletion of 8 causes a double black:



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Analysis of Deletion

```
Algorithm removeElement(k)
1.  v ← findNode2Remove(k) Search
    for key k, then find deletion node v
    with external child w. Let
    y=sibling(v) and r=sibling(w).
2.  Remove node v (removes v and w
    and returns r=sibling(w)). If v and r
    are both black then color r double
    black else r was red so color r black
    and we're done
3. while isDoubleBlack(r)
     y ← sibling(r)
     if isRed(y) then
         y ← adjustment(y)
     if hasRedChild(y) then //transfer
         r ← restructure(r)
         return
     else {sibling(r) has no red child}
         r ← fusionRecolor(r)
```

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most two restructurings taking $O(1)$ time each
- Thus, a deletion in a red-black tree takes $O(\log n)$ time

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Analysis of Deletion (details)

```
Algorithm findNode2Remove(k)
  Input returns node r containing key k, node r has at least one external node
     v ← findPosition(k, root() )
     r ← v
     if isInternal(leftChild(v)) ∧ isInternal(rightChild(v)) then // one child must be external
        r ← findPosition(k, leftChild(v))   // finds node containing predecessor of k
        swapElements(v, r)   // swaps items so r is node containing key being deleted
     return r   // r is the node to be deleted and contains key k unless k is not in tree

Algorithm fusionRecolor(y, p, r)
  Input r and y are siblings, p is their parent
     setColor(y, RED)
     if isRed(p) then
        setColor(p, BLK)
     else
        setColor(p, DOUBLE_BLACK)
     if isInternal(r) then
        setColor(r, BLK)
     return p
```

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Analysis of Deletion (details)

```
Algorithm removeDoubleBlack(y, r)
  Input r is the double black node and y is sibling(r)
  if isDoubleBlack(r) then
     if isRed(y) then  {Case 3: when y, the sibling of r, is red}
        y ← adjustment(y)
     p ← parent(y)
     z ← redChildOf(y)
     if isBlack(z) then {Case 1: when y has no red child}
        r ← fusionRecolor(y, p, r)
        if isRoot(r) then
           setColor(r, BLK)
        else
           removeDoubleBlack(sibling(r), r)  // recursive call
     else  {Case 2: y has a red child z, so we do a transfer}
        restructure(z)
        setColor(parent(p), color(p))
        setColor(p, BLK)
        setColor(z, BLK)
        if isInternal(r) then
           setColor(r, BLK) // make sure r is not external/null
```

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Main Point

2. Restoring balance after insertion or deletion in a red-black tree only requires a constant number of trinode restructurings (0, 1, or 2) and at most O(log n) recolorings. The red-black tree is slightly more complicated than a (2,4) tree because of restructuring, but has a major advantage in space requirements and simplifies splitting and fusion of nodes. *Science of Consciousness:* The TM technique is a simple, effortless technique that restructures the physiology to a more balanced state.

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Red-Black Tree Reorganization

Insertion	remedy double red	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	correcting of 4-node representation	double red removed
recoloring	split	double red removed or propagated up

Deletion	remedy double black	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	transfer	double black removed
recoloring	fusion	double black removed or propagated up
adjustment	change of 3-node representation	restructuring or recoloring follows

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Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. A (2, 4) tree offers a simple and effective way of maintaining balance in a dynamic tree structure.
2. A red-black tree offers a refinement of the (2, 4) tree by eliminating data slots and optimizing operations.

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3. **Transcendental Consciousness** is the unbounded field of pure order and balance and is the basis of order and balance in creation.
4. **Impulses within Transcendental Consciousness**: The dynamic natural laws within this unbounded field create and maintain the order and balance in creation.
5. **Wholeness moving within itself:** In Unity Consciousness, the diversity of creation is experienced as waves of intelligence, perfectly orderly fluctuations of one's own self-referral consciousness.

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