Lecture 8: Ordered Dictionaries

Pure Consciousness is the field of perfect order, balance, and efficiency

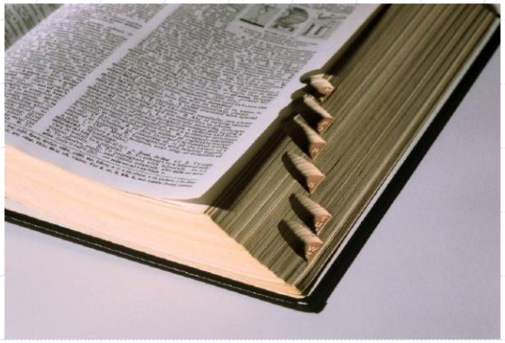
Hash Tables

- Hash tables are a highly efficient implementation of the Dictionary ADT
  - What are its disadvantages?

Wholeness Statement

A dictionary allows users to assign keys to elements then to access or remove those elements by key. An ordered dictionary maintains an order relation among keys allowing access to adjacent keys while supporting efficient implementation of the dictionary ADT. *Science of Consciousness*: Each of us has access to the source of thought which is a field of perfect order, balance, and efficiency.

Ordered Dictionaries



- Keys are assumed to come from a total order, i.e., the keys can be sorted.
- Specification of iterator operations:
  - keys()

    Returns an iterator of the keys in sorted order
  - values()

    Returns the element of the items in key-sorted order
  - items()

    Returns the (k, e) items in sorted order by key (k) of the item
- What would the running time be for creating these iterators for a Lookup Table from yesterday?
  - Constraint is that iteration through the items must take O(n) time where n is the number of items in the dictionary

Ordered Dictionaries

- What are the disadvantages of a sorted sequence (Lookup Table) as the basis of an ordered Dictionary?

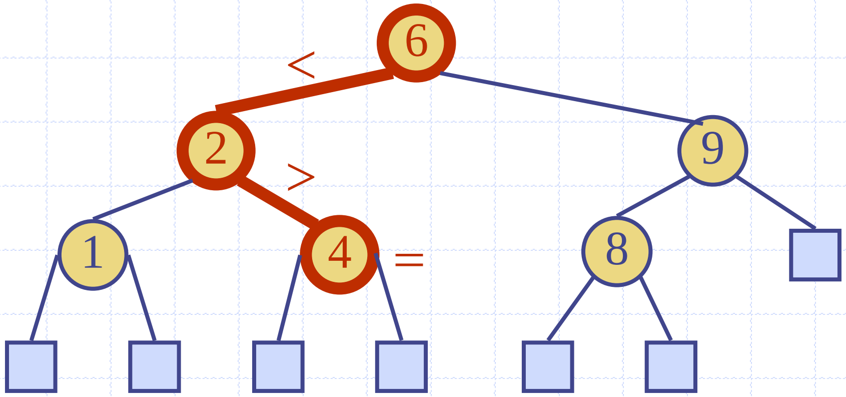
Overview

- Multi-Way Search Trees (§3.3.1)
- (2,4) Trees (§3.3.2)

- AVL Trees (§3.2)

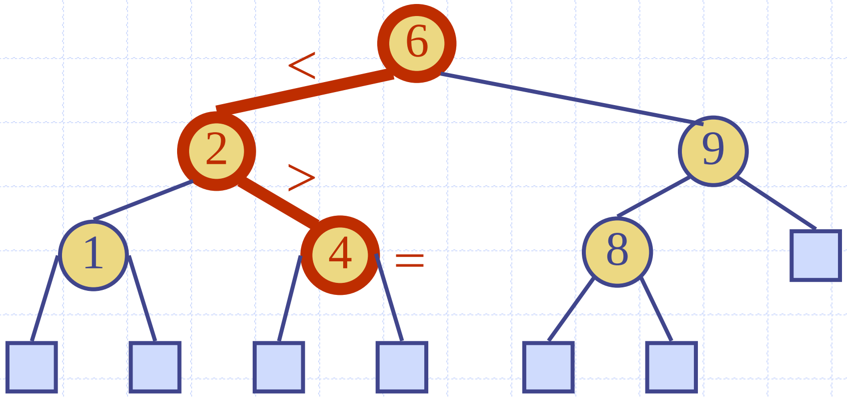
- Red-Black Trees (§3.3.3) (Tomorrow)

Binary Search Trees



Binary Search Tree (§3.1.2)

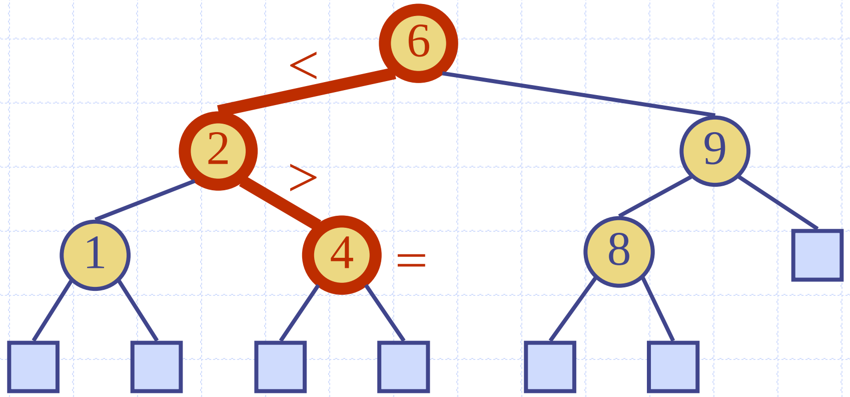
- A binary search tree is a binary tree storing keys (or key-element items) at its internal nodes and satisfying the following property:
  - Let ***u***, ***v***, and ***w*** be three nodes such that ***u*** is in the left subtree of ***v*** and ***w*** is in the right subtree of ***v***. We have $key(u) \leq key(v) \leq key(w)$
- External nodes do not store items
- An in-order traversal of a binary search tree visits the keys in increasing order



Search (§3.1.3)

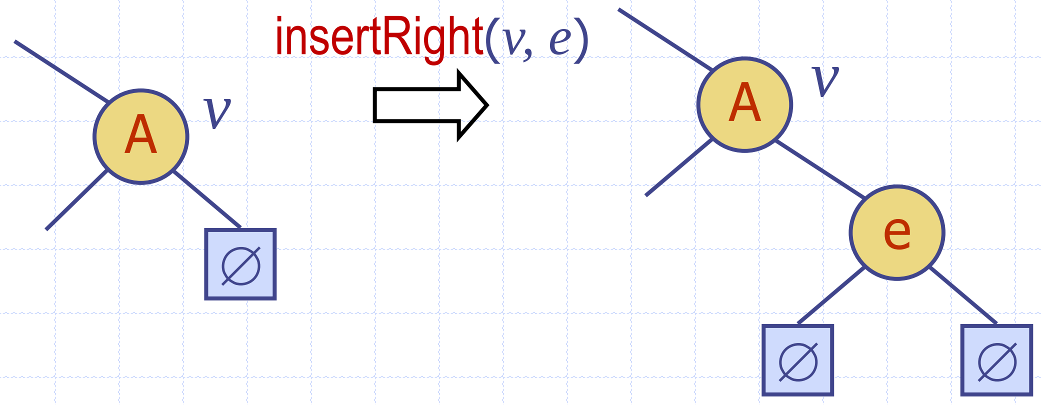
- To search for a key ***k***, we trace a downward path starting at the root
- The next node visited depends on the outcome of the comparison of ***k*** with the key of the current node
- If we reach a leaf, the key is not found and we return NO_SUCH_KEY
- Example: findValue of key 4

```
Algorithm findValue(T, k, v)
    if T.isExternal (v)
        return NO_SUCH_KEY
    if $k < key(v)$
        return findValue(T, k, T.leftChild(v))
    else if $k = key(v)$
        return element(v)
    else { $k > key(v)$ }
        return findValue(T, k, T.rightChild(v))
```

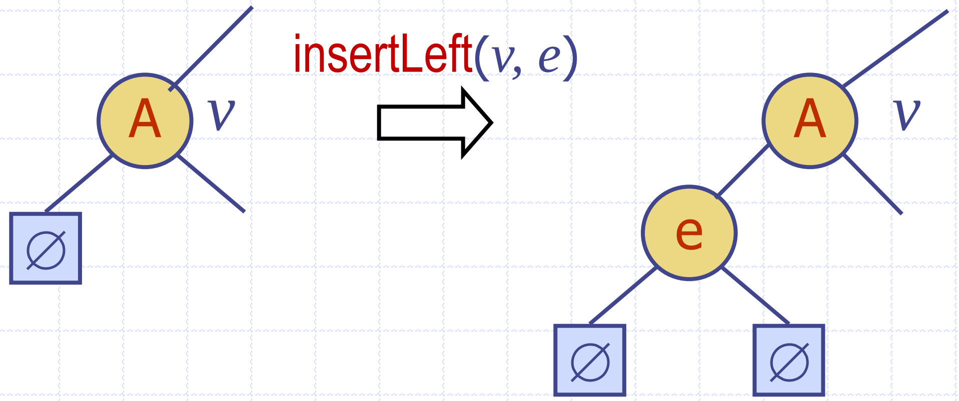


Insert Methods

insertRight(***v*, *e***)
Right child must be external.

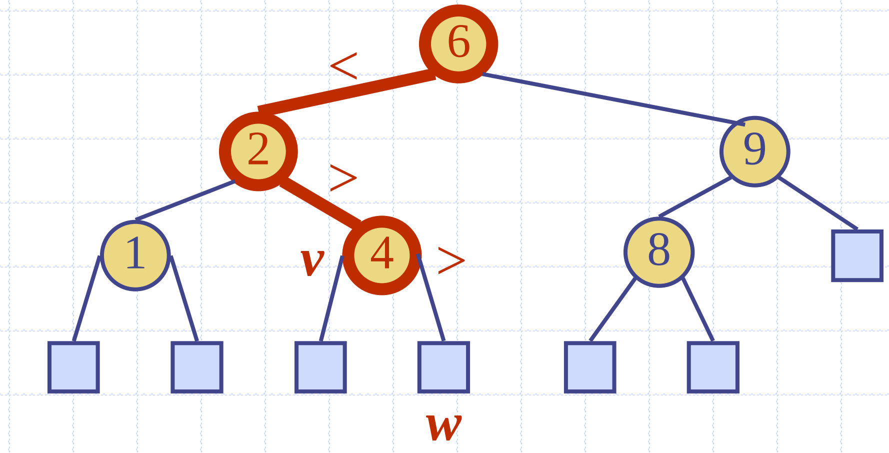


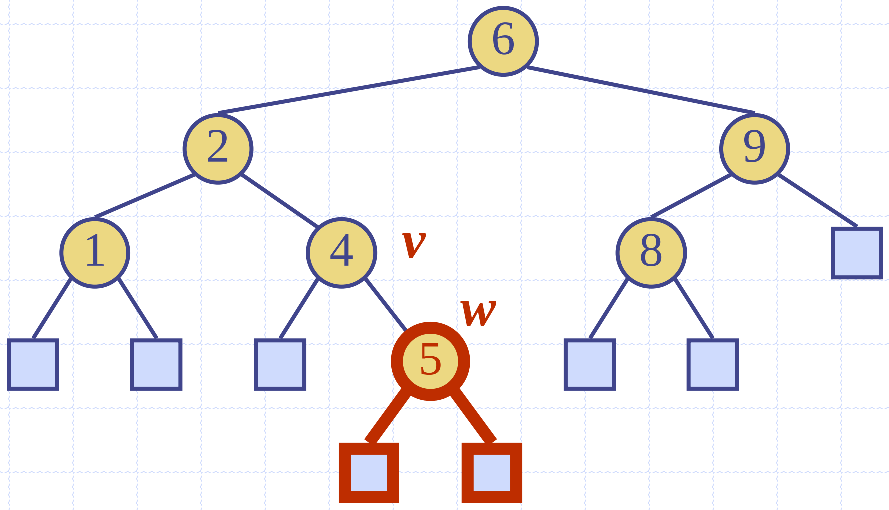
insertLeft(***v*, *e***)
Left child must be external.



Insertion

- To perform operation insertItem(k, o), we search for key k using our helper
- If k is not already in the tree, then let v be the parent of the leaf w reached by the search
- We insert k as either the left or right child of the parent node v by expanding $w = item(k, e)$ into an internal node using either insertLeft(v, w) or insertRight(v, w) depending on k
- Example: insert 5

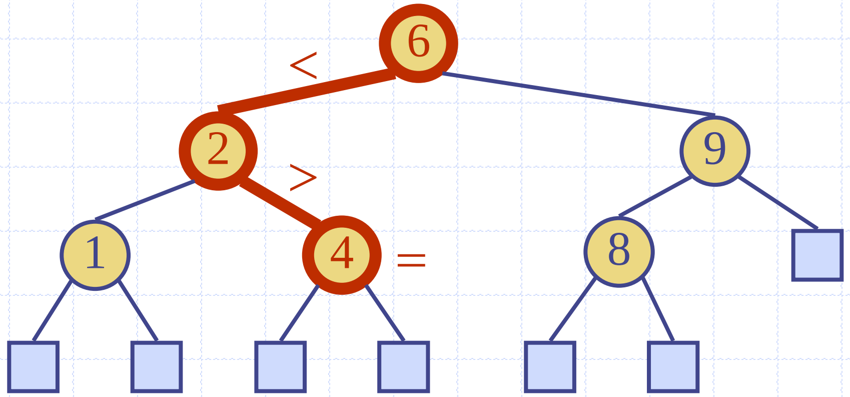




Insertion

- To search for a key ***k***, we trace a downward path starting at the root using our helper ***findPosition***
- If we reach a leaf, the key is not found, so we return NO_SUCH_KEY
- Otherwise we return the element associated with the key ***k***
- Example: findValue(4)

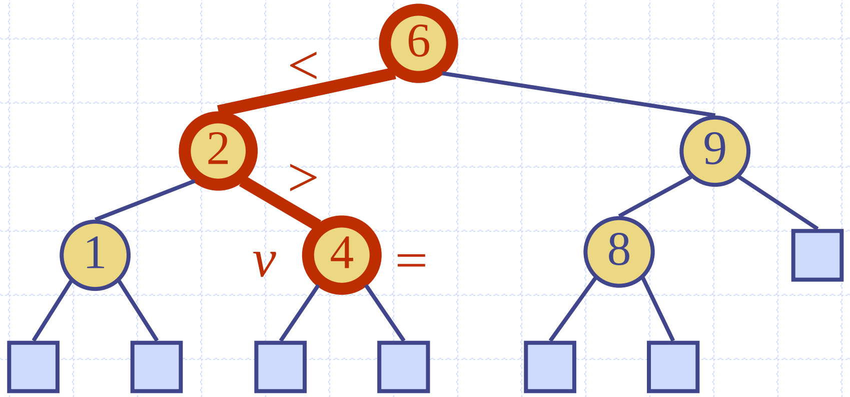
```
Algorithm insertItem(T, k, e)
    if T.isEmpty() then   // T is a field of this
        return T.insertRoot(k, e)
    v ← findPosition(T, k, T.root())
    if k = key(v) then   // k is already in Tree T
        return T.replaceElement(v, e) // return old elem
    else { v is the parent of the new item }
        w ← new Node(k, e)   // w is the new item
        if key(v) < k then
            return T.insertRight(v, w)
        else { k < key(v) }
            return T.insertLeft(v, w)
```



Helper *findPosition*

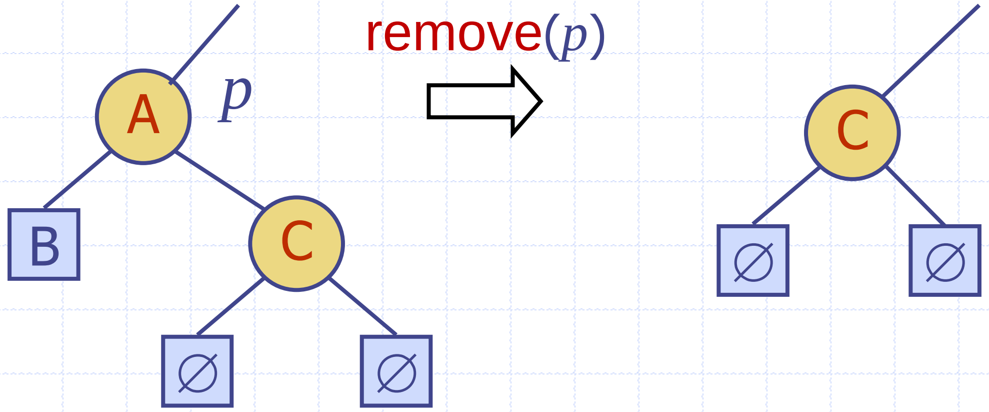
- To search for a key ***k***, we trace a downward path starting at the root
- The next node visited is based on the comparison of ***k*** with the key of the current node
- If we reach a leaf, the key is not found and we return the parent of the external node
- If we find the key, then we return the node ***v*** containing ***k***
- Example: findPosition of key 4 or 5 both return node ***v***

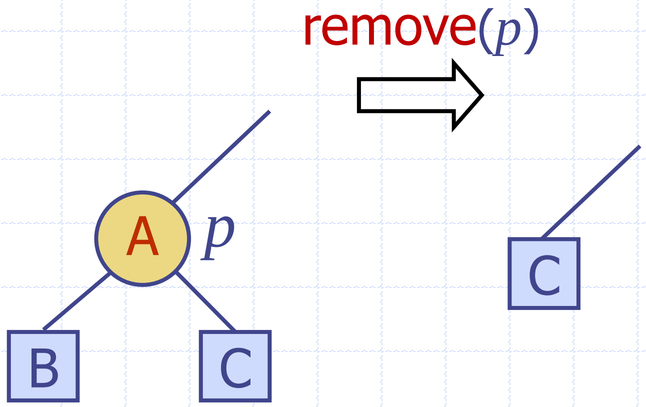
```
Algorithm findPosition(k, v)
Output: the node containing key k or the parent of the
node where k would be inserted into tree T
    if $k = key(v)$ then
        return v        // node containing k
    else if $k < key(v)$ then
        if isExternal (T.leftChild(v)) then
            return v   // node where k would be inserted
        else return findPosition(k, T.leftChild(v))
    else if isExternal (rightChild(v)) then
            return v   // node where k would be inserted
        else return findPosition(k, T.rightChild(v))
```

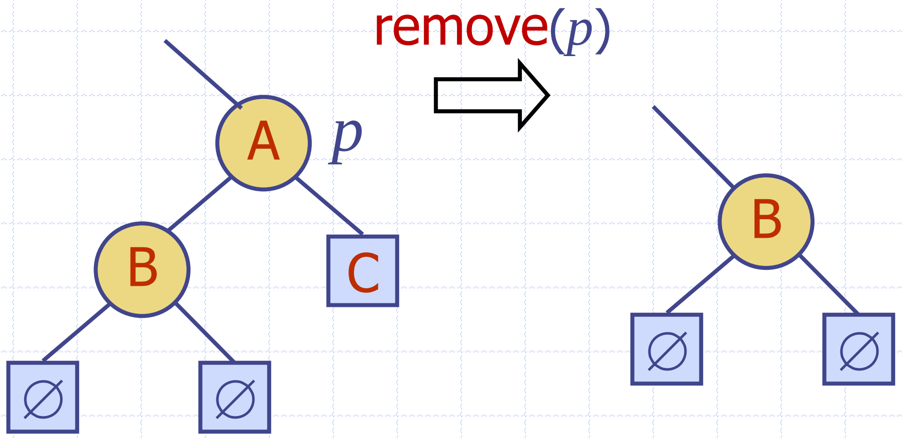


Remove Method

remove(***p***)
Either the left or right child must be external! We can only remove the node above an external node.

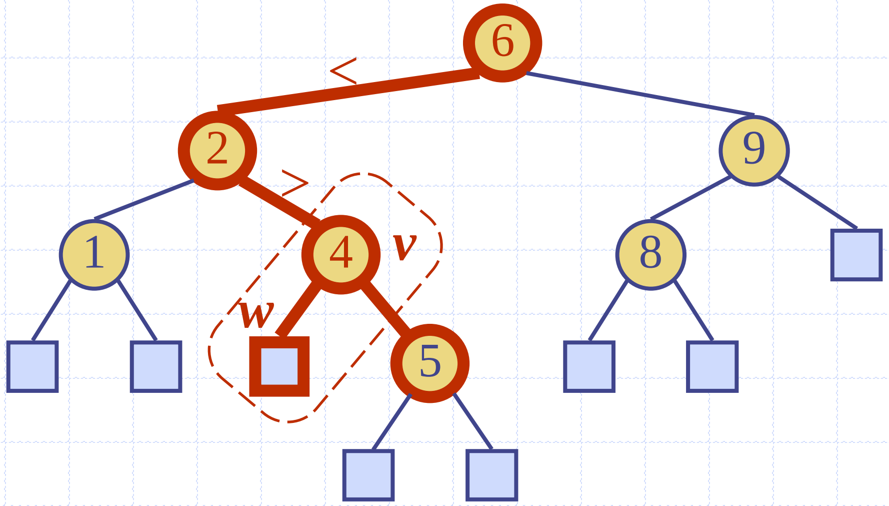


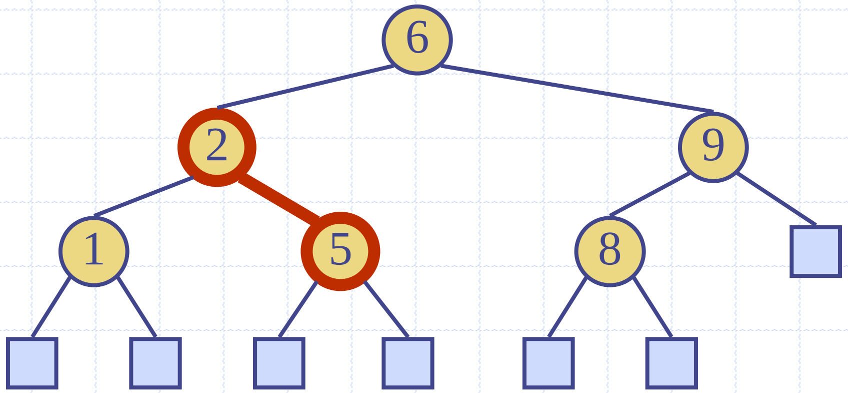




Deletion (Case 1)

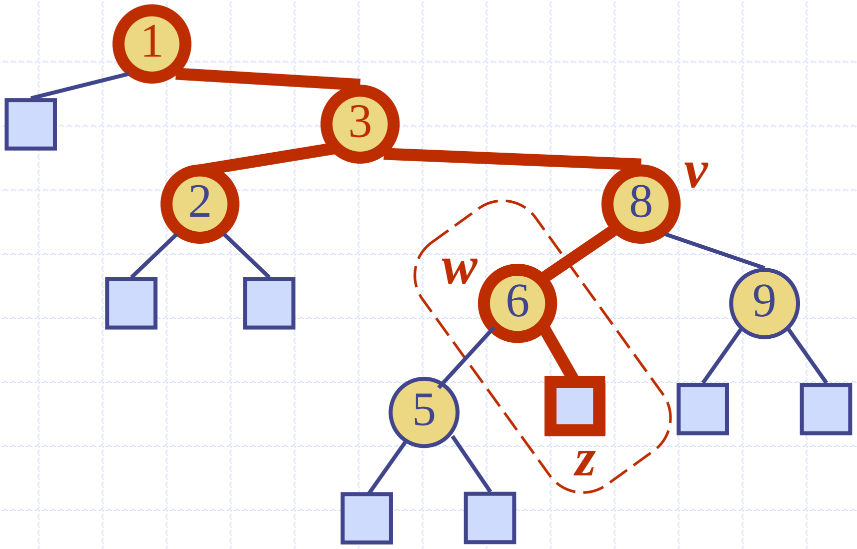
- To perform operation removeItem(*k*), we search for key *k*
- Assume key *k* is in the tree, and let *v* be the node storing *k*
- **Two cases**:
  - Node *v* has a leaf child *w*
  - Node *v* has no leaf child
- If node *v* has a leaf child *w*, we remove *v* and *w* from the tree with operation remove(*v*)
- Example: remove 4





Deletion (Case 2)

- We consider the case where the key ***k*** to be removed is stored at a node ***v*** whose children are both internal
  - we find the internal node ***w*** that precedes ***v*** in an in-order traversal
  - we copy ***key***(***w***) into node ***v***
  - we remove node ***w*** and its right child ***z*** (which must be a leaf) by means of operation remove(***w***)
- Example: remove 8



Deletion (Case 2)

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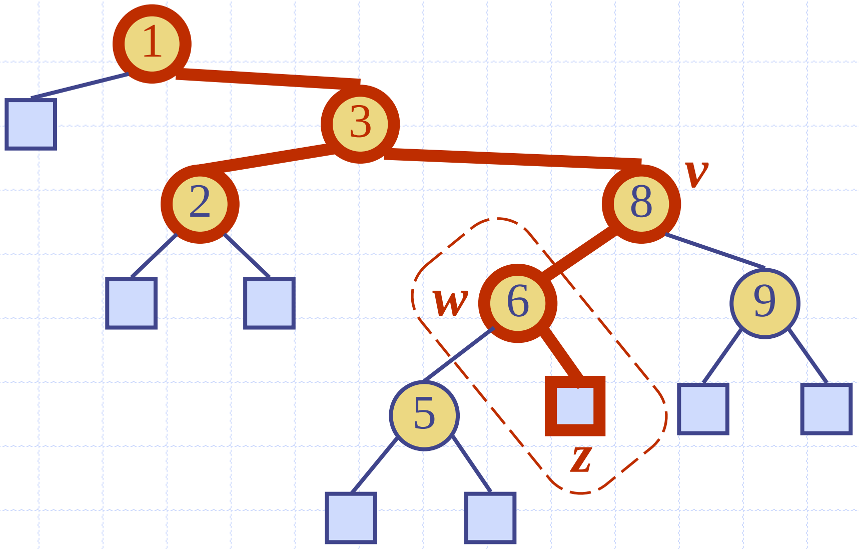
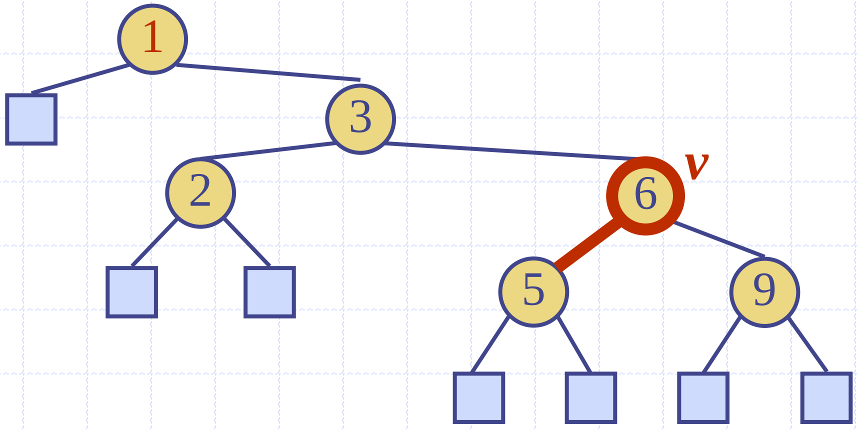


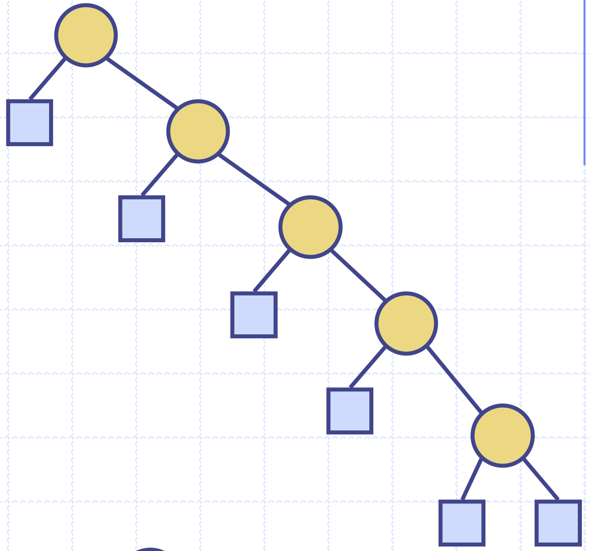


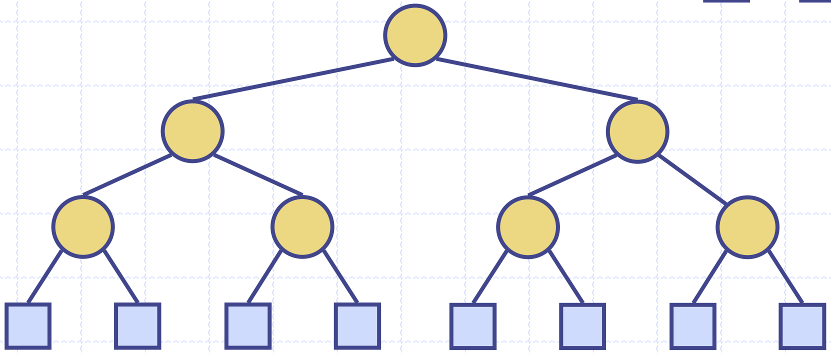
Binary Search Trees

- What are the problems with binary search trees as the basis of a Dictionary?

Performance (§3.1.6)

- Consider a dictionary with n items implemented by means of a binary search tree of height h
  - the space used is $O(n)$
  - methods findValue , insertItem and removeItem take $O(h)$ time
- The height h is $O(n)$ in the worst case and $O(\log n)$ in the best case

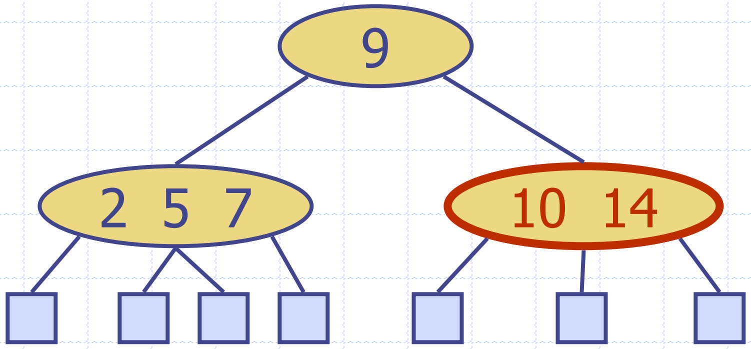




Main Point

1. A binary search tree is a binary tree with the property that the value at each node is greater than the values in the nodes of its left subtree (child) and less than the values in the nodes of its right subtree. When implemented properly, the operations (search, insert, and remove) can be efficiently accomplished in O(log n).
   Such data structures reflect the following SCI principles: law of least action, principle of diving, perfect order.

(2,4) Trees

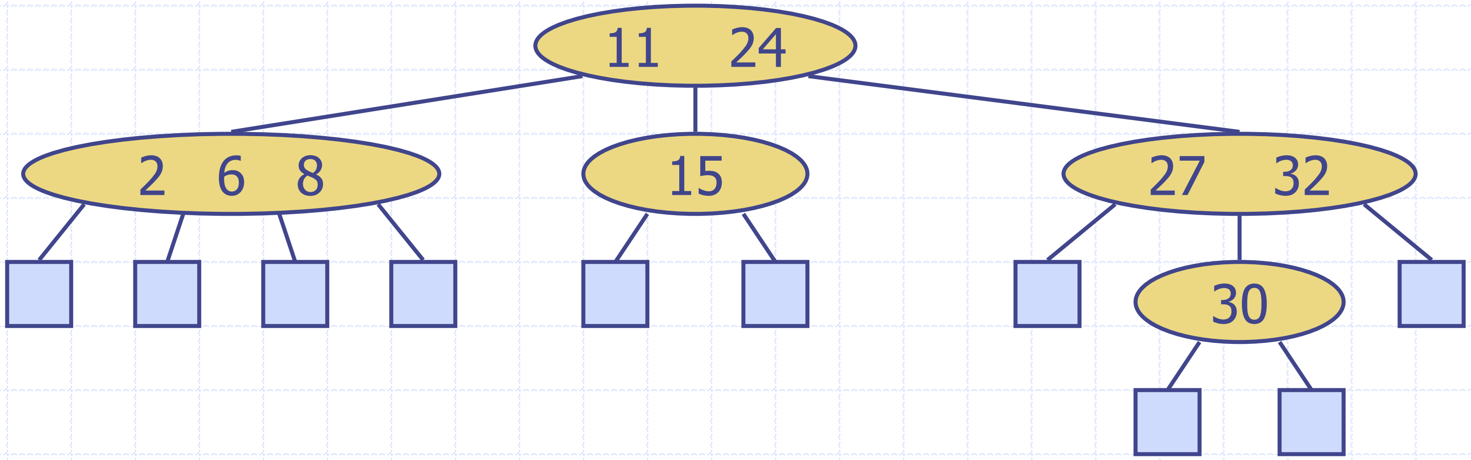


Outline and Reading

- Multi-way search tree (§3.3.1)
  - Definition
  - Search
- (2,4) tree (§3.3.2)
  - Definition
  - Search
  - Insertion
  - Deletion
- Comparison of dictionary implementations

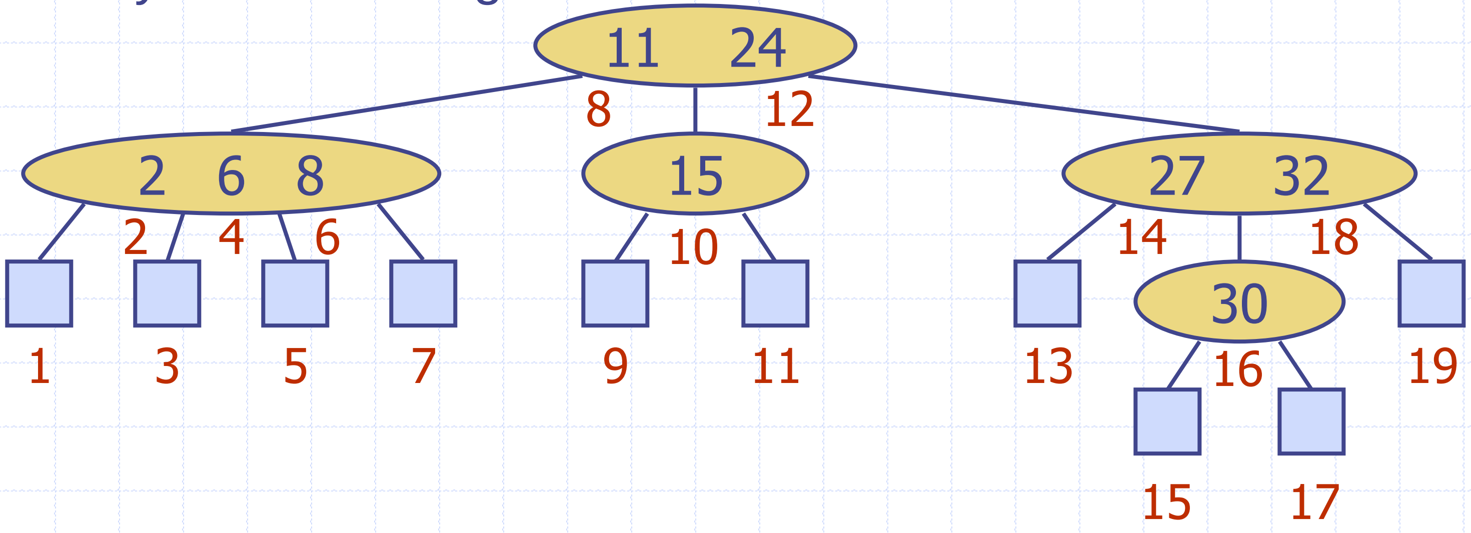
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\ v_2\ \dots\ v_d$ storing keys $k_1\ k_2\ \dots\ k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \dots, d-1)$
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders



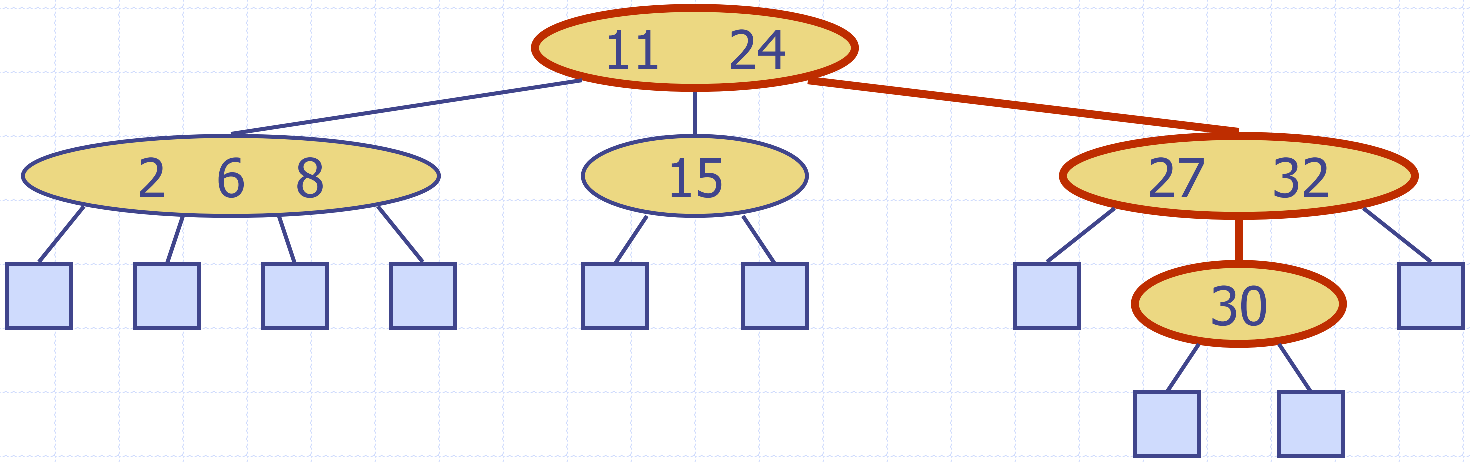
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order

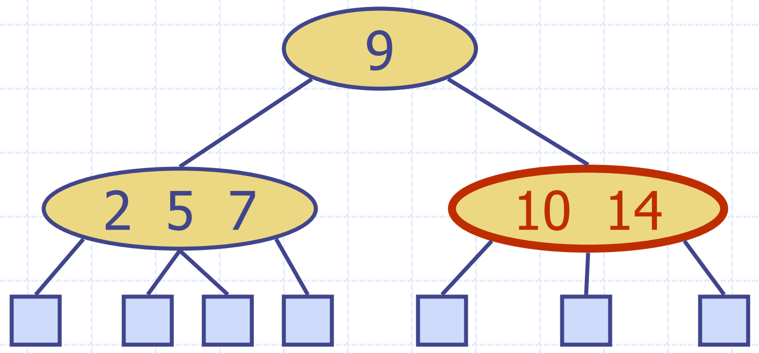


Multi-Way Searching

- Similar to search in a binary search tree
- A each internal node with children $v_1\ v_2\ \ldots\ v_d$ and keys $k_1\ k_2\ \ldots\ k_{d-1}$
  - $k = k_i$ $(i = 1, \ldots, d-1)$: the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ $(i = 2, \ldots, d-1)$: we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



(2,4) Tree



B-Trees

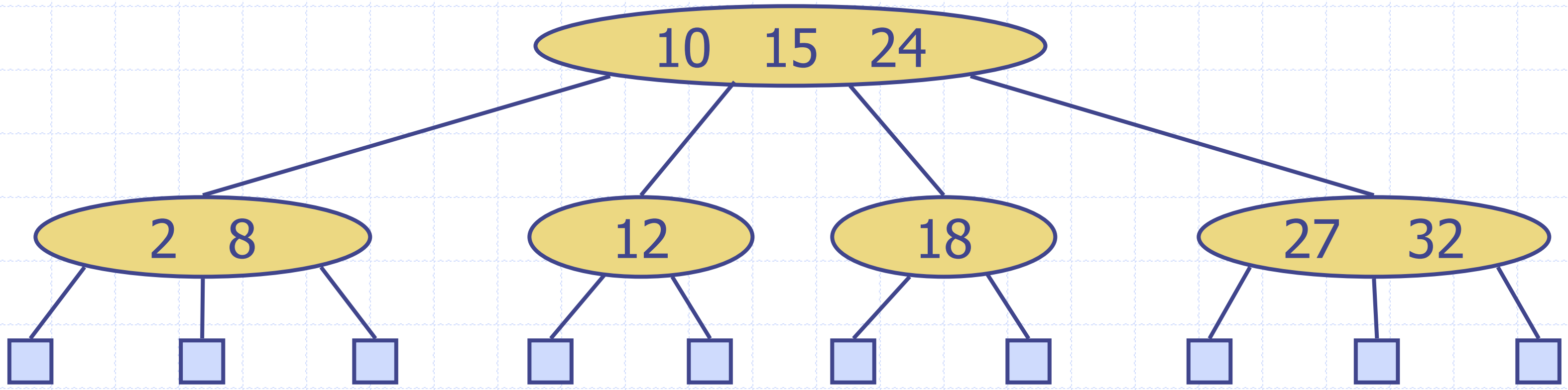
- A B-Tree is a balanced multi-way search tree, i.e., all leaves are at the same depth
- B-Trees are used to implement a file structure that allows random access by key as well as sequential access of keys in sorted order
- The size (maximum number of keys) of a node in a B-Tree file structure is determined by the size of a sector of a track of a disk file (also called a physical block)
- A (2,4) Tree is a special case of a B-Tree in which each node can contain 1, 2, or 3 keys

Why (2-4) Trees?

- A Red-Black tree is an implementation of a (2-4) Tree in a binary tree data structure
- If you understand the (2-4) Tree implementation, you will more easily understand what is done and why in a Red-Black Tree to keep it balanced
- You will appreciate that it's easier and more efficient in space and time to implement a Red-Black Tree than a (2-4) Tree

(2,4) Tree

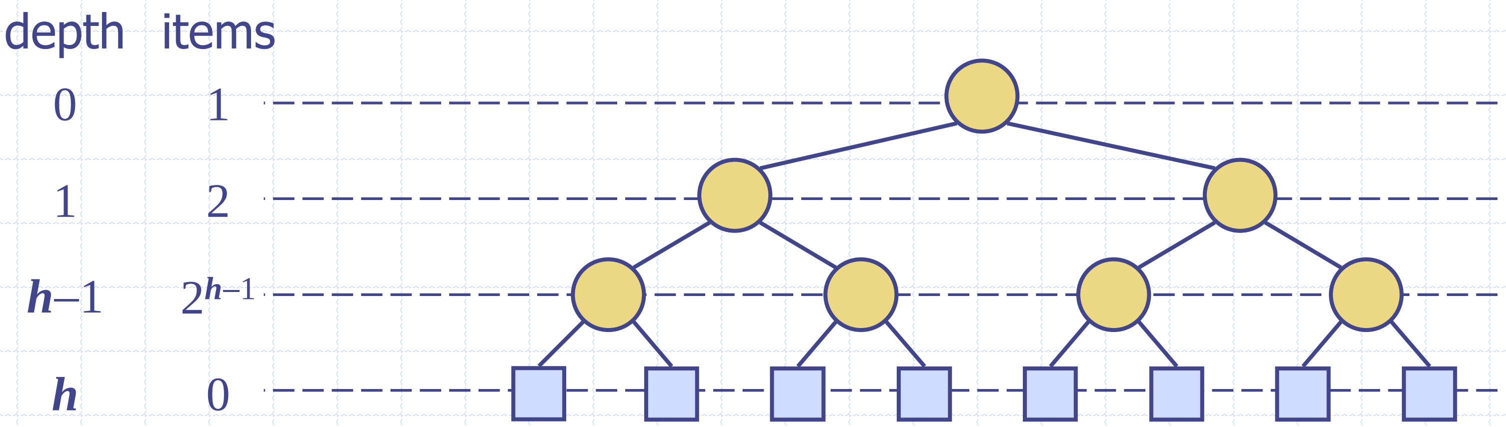
- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search tree with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



Height of a (2,4) Tree

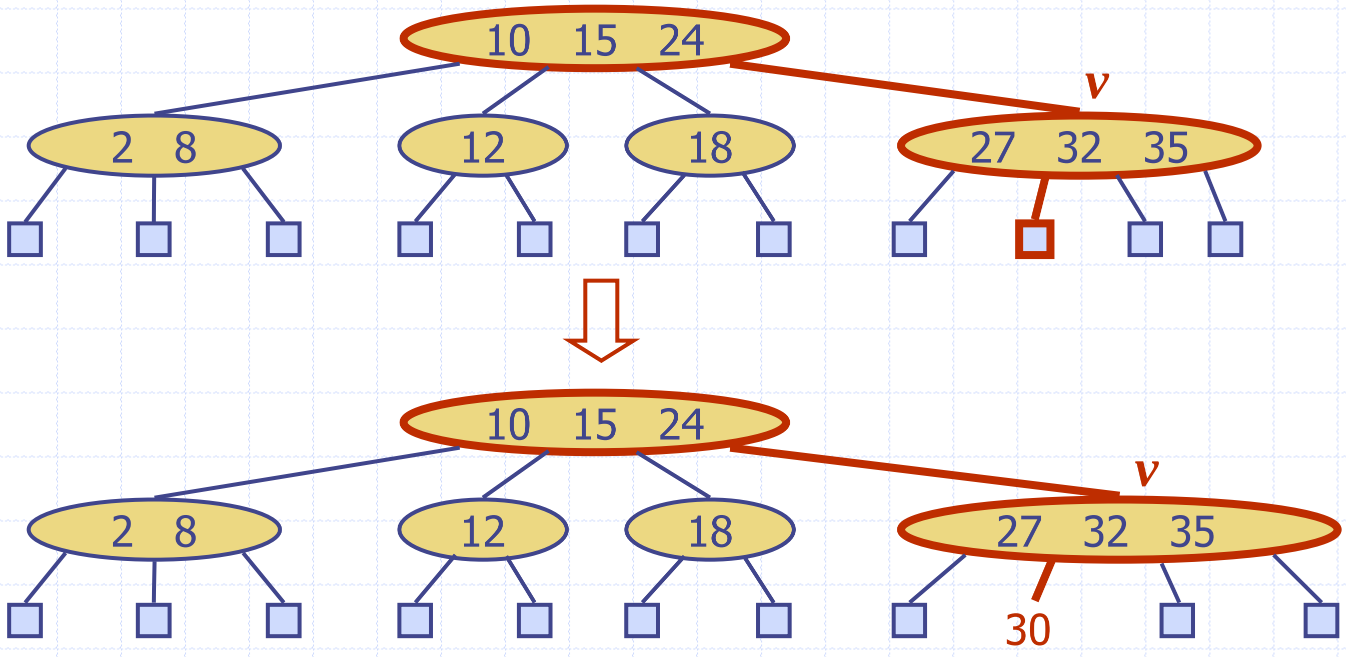
- Theorem: A (2,4) tree storing n items has height $O(\log n)$

  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h-1$ and no items at depth h, we have
  $$n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log (n+1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



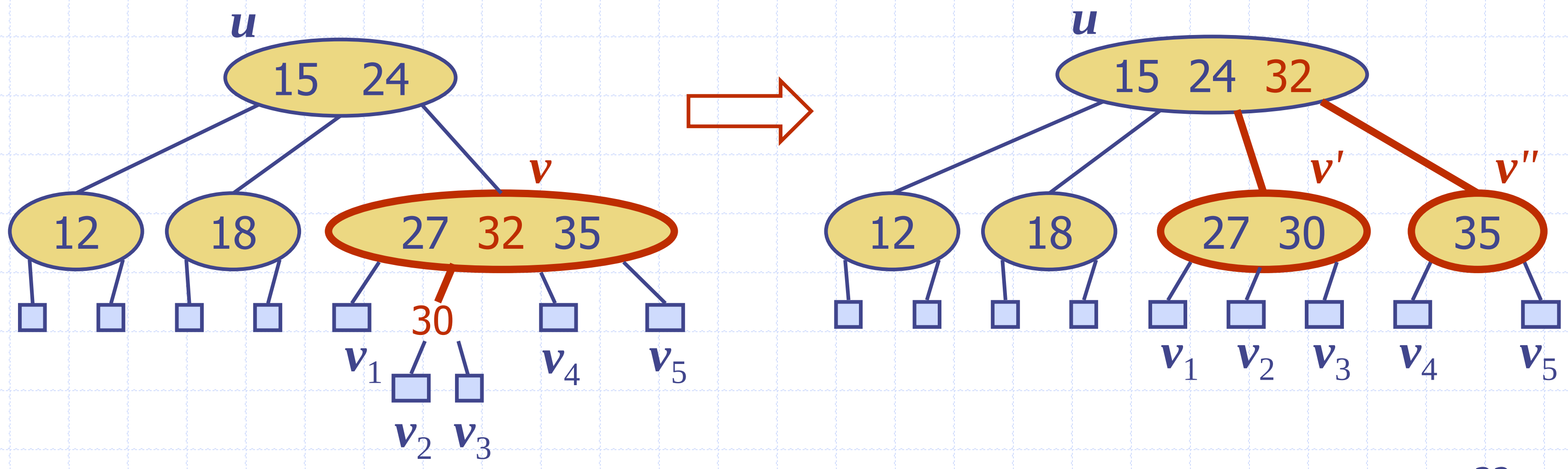
Insertion

- We insert a new item (***k***, ***o***) at the parent ***v*** of the leaf reached by searching for ***k***
  - We preserve the depth property but
  - We may cause an overflow (i.e., node ***v*** may become a 5-node)
- Example: inserting key 30 causes an overflow



Overflow and Split

- We handle an overflow at a 4-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced by nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - the middle key 32 of node v is inserted into the parent u of v (a new root may be created); the new key 30 is inserted into either v' or v''
- The overflow may propagate to the parent node u



Analysis of Insertion

```
Algorithm insertItem(k, o)

1.  We search for key k to locate the
    insertion node v

2.  We add the new item (k, o) at node v

3. while overflow(v)
       if isRoot(v)
           create a new empty root above v
       v ← split(v)
```

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(1)$ time
  - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

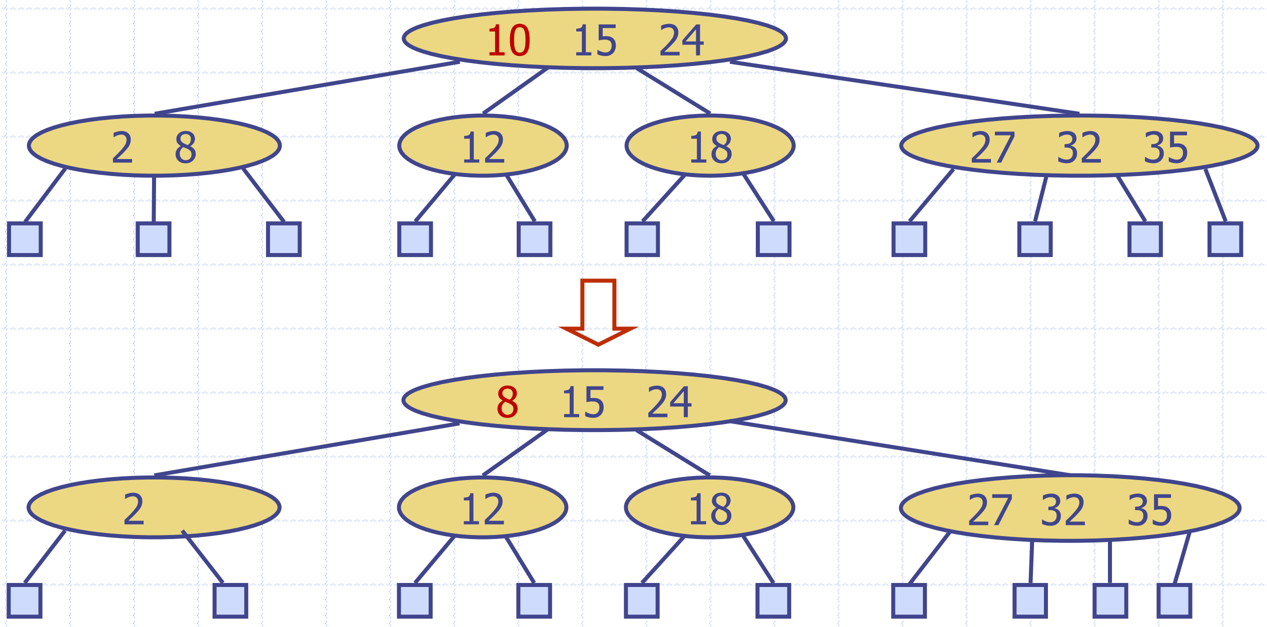
Example:

- Insert the following into an initially empty 2-4 tree in this order:

  (16, 5, 22, 45, 2, 10, 18, 30, 50, 12, 1, 33)

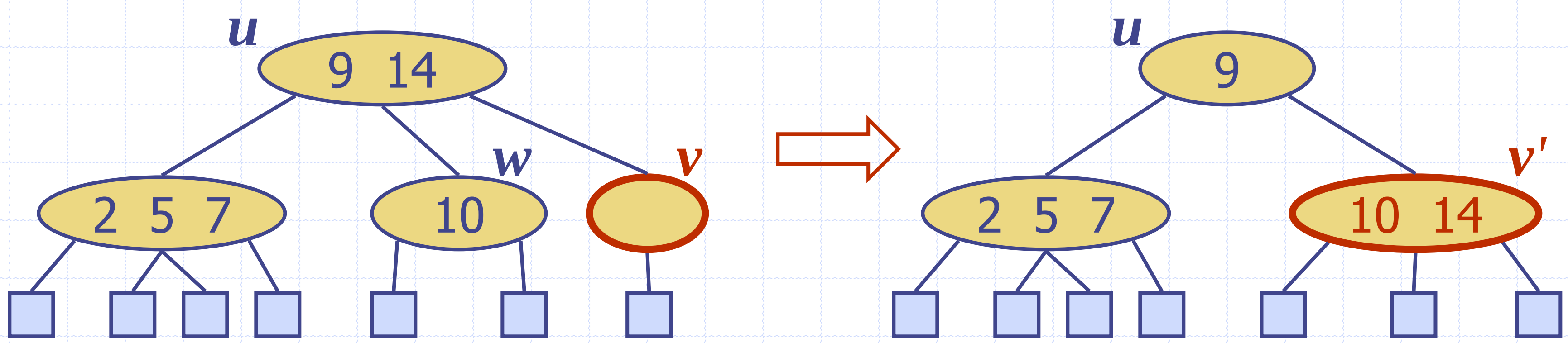
Deletion

- We reduce deletion of an item to the case where the item is at the node with leaf children
- Otherwise, we replace the item with its inorder predecessor (or, equivalently, with its inorder successor) and delete the latter item
- Example: to delete key 10, we replace it with 8 (inorder predecessor)



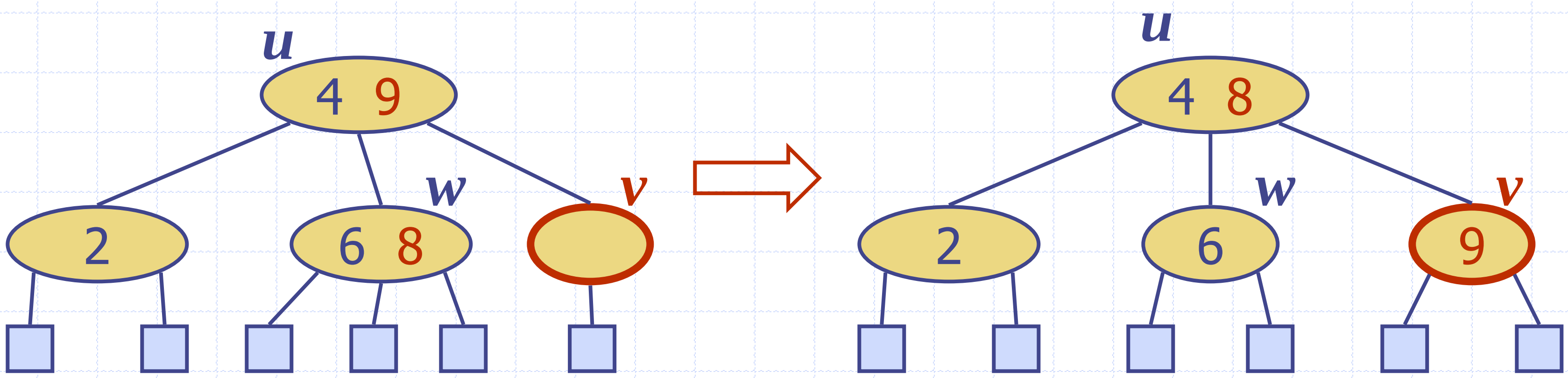
Underflow and Fusion

- Deleting an item from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an item from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Underflow and Transfer

- To handle an underflow at node v with parent u, we also consider a second case
- Case 2: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move an item from u to v
    2. we move an item from w to u
  - After a transfer, no underflow occurs



Analysis of Deletion

```
Algorithm deleteItem(k)

1. We search for key k and locate the
   deletion node v

2. while underflow(v) do
       if isRoot(v)
           change the root to child of v; return
       if a sibling(v) = u is a 3- or 4-node
           transfer(u, v); return
       else {both siblings are 2-nodes}
           fusion(u, v) {merge v with sibling u}
           v ← parent(v)
```

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(\log n)$ time because each fusion takes $O(1)$ time and we perform $O(\log n)$ fusions
- Thus, a deletion in a (2,4) tree takes $O(\log n)$ time

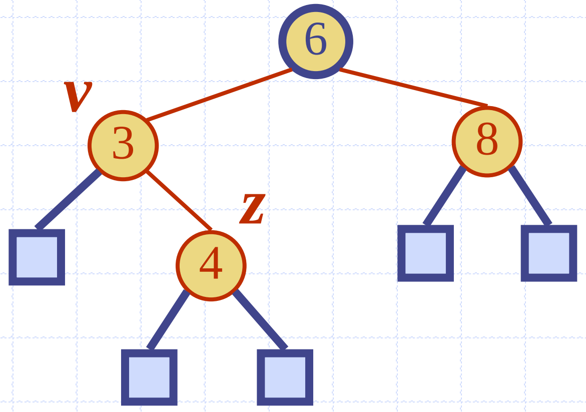
Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the item
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

Main Point

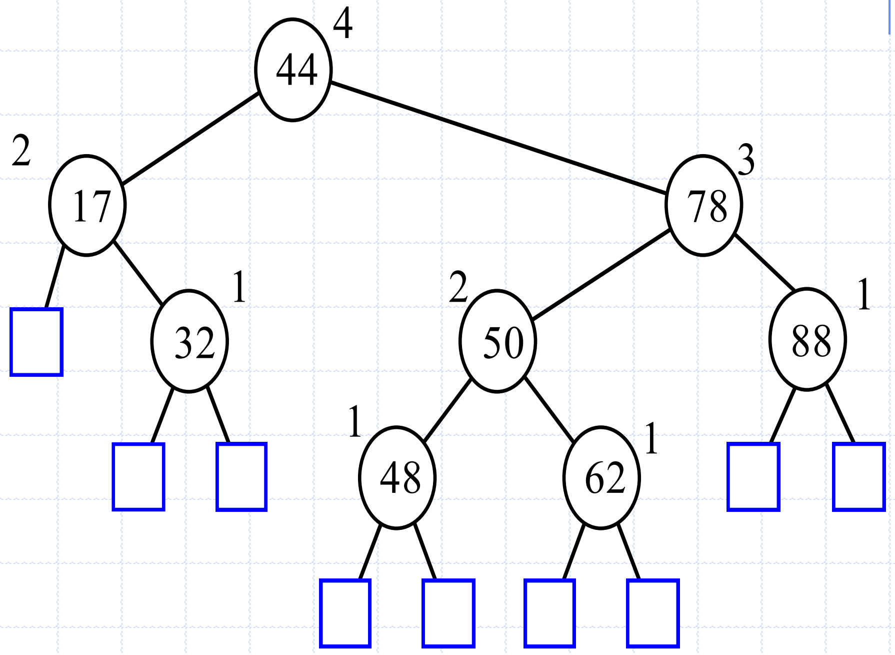
3. By introducing some flexibility in the data content of each node, all leaf nodes of a (2,4) Tree can be kept at the same depth.
   Stability and adaptability are fundamentals of progress and evolution in nature.

AVL Trees



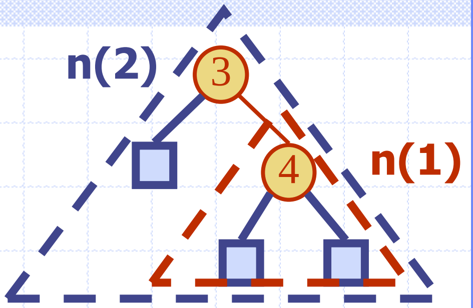
AVL Tree Definition

- **AVL trees are balanced.**
- An AVL Tree is a ***binary search tree*** such that for every internal node v of T, the *heights of the children of v can differ by at most 1.*



An example of an AVL tree where the heights are shown next to the nodes:

Height of an AVL Tree



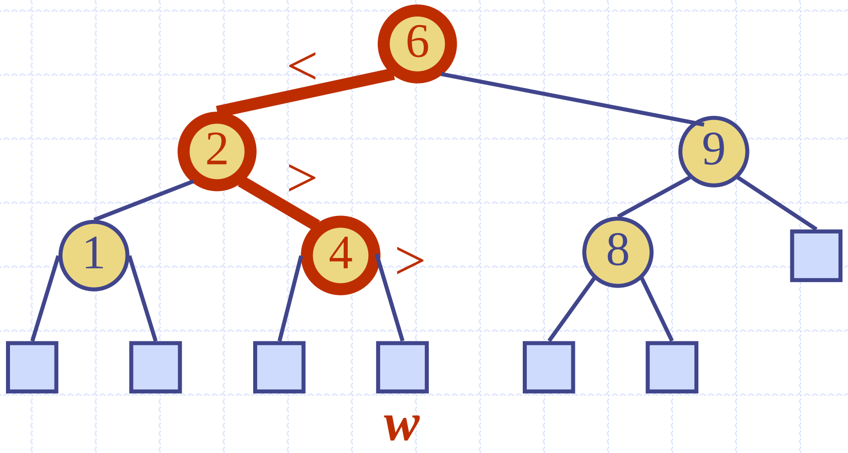
- **Fact**: The ***height*** of an AVL tree storing n keys is O(log n).
- **Proof**: Let us bound **n(h):** the minimum number of internal nodes of an AVL tree of height h.
- We easily see that n(1) = 1 and n(2) = 2
- For n > 2, an AVL tree of height h contains the root node, one AVL subtree of height h-1 and another of height h-2.
- That is, n(h) = 1 + n(h-1) + n(h-2)
- Knowing n(h-1) > n(h-2), we get n(h) > 2n(h-2). So

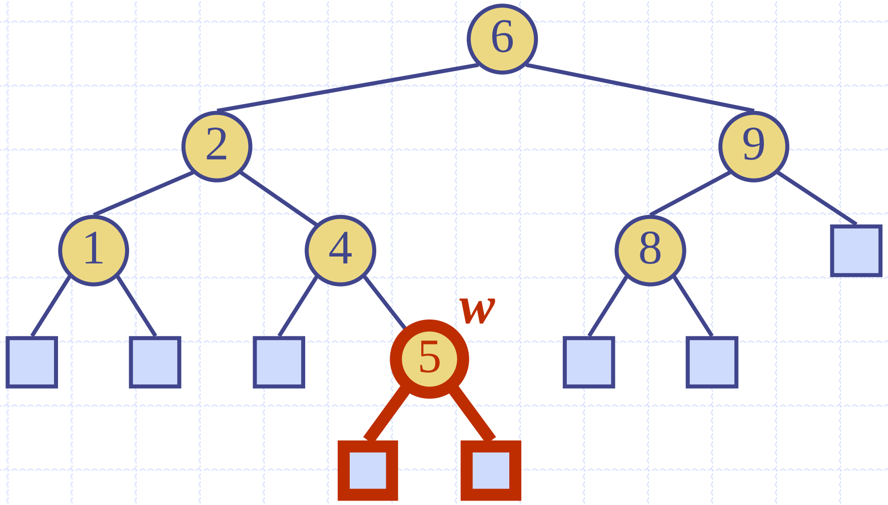
  n(h) > 2n(h-2), n(h) > 4n(h-4), n(h) > 8n(h-6), … (by induction),

  $n(h) > 2^i n(h-2i)$
- Solving the base case
  - Pick $i = \lceil h/2 \rceil - 1$ since value of the base cases are n(1) = 1 and n(2) = 2
    - i.e., pick i such that $1 \leq h-2i \leq 2$
  - Thus we get: $n(h) > 2^{h/2-1}$
- Taking logarithms: h < 2log n(h) +2
- Thus the height of an AVL tree is O(log n)

Recall
Insertion into a BST (§3.1.4)

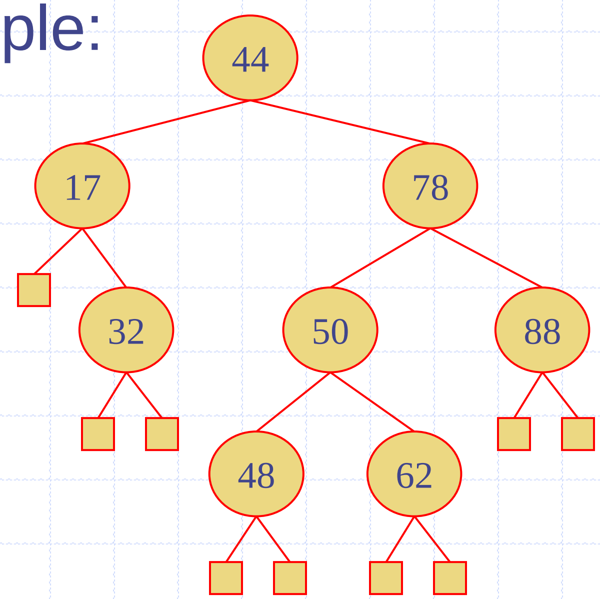
- To perform operation insertItem(k, o), we search for key k
- Assume k is not already in the tree, and let let w be the leaf reached by the search
- We insert k at node w and expand w into an internal node
- Example: insert 5



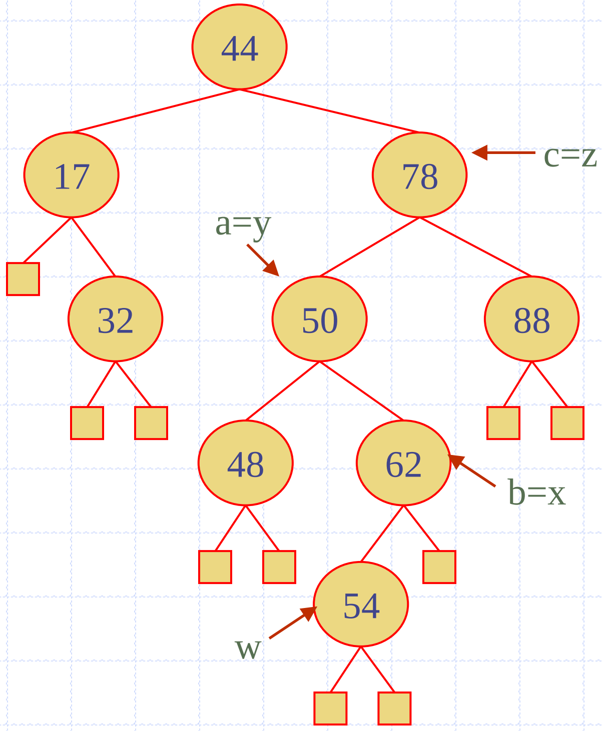


Insertion in an AVL Tree

- Insertion is as in a binary search tree
- Always done by expanding an external node.
- Example:



before insertion

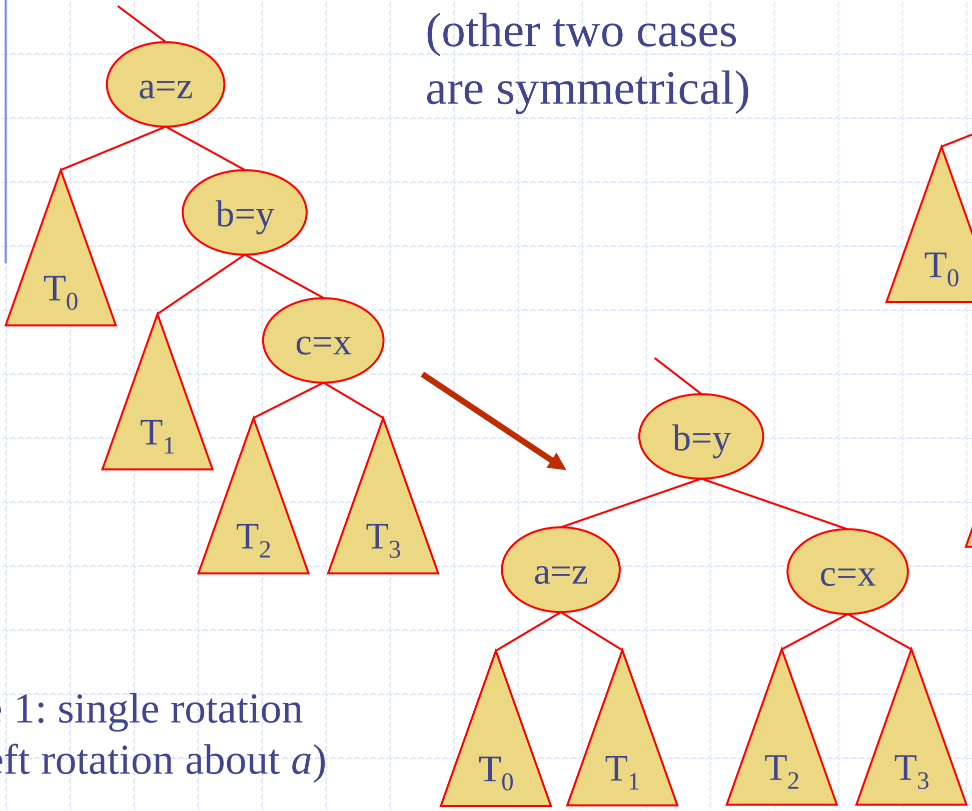


after insertion

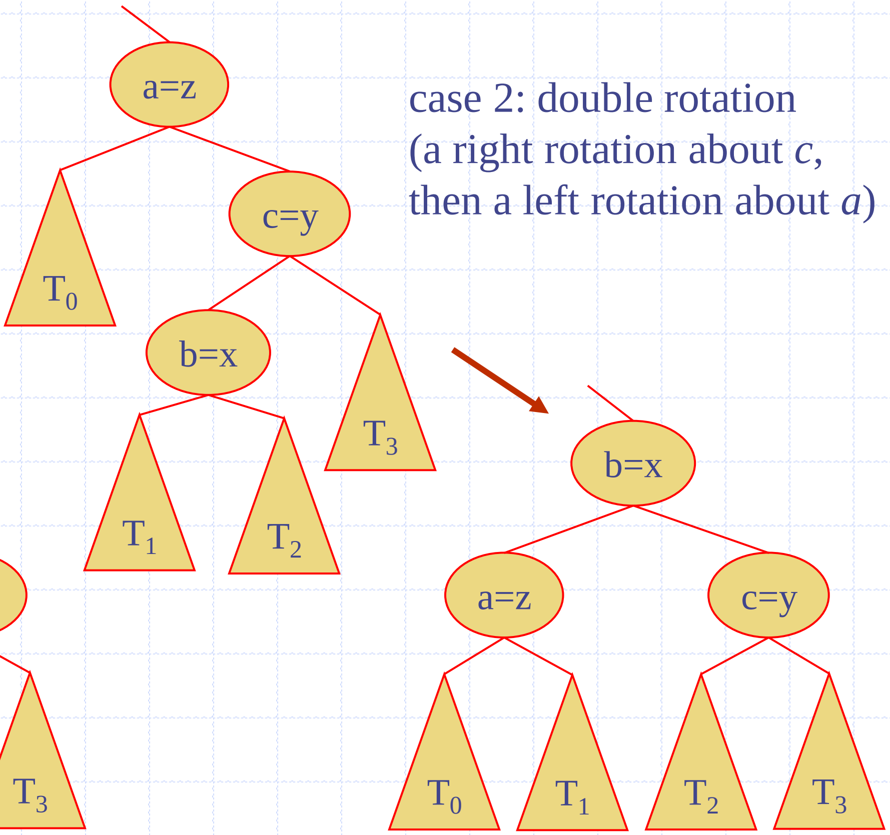
Trinode Restructuring

- let (*a*,*b*,*c*) be an inorder listing of *x*, *y*, *z*
- perform the rotations needed to make *b* the topmost node of the three

(other two cases are symmetrical)



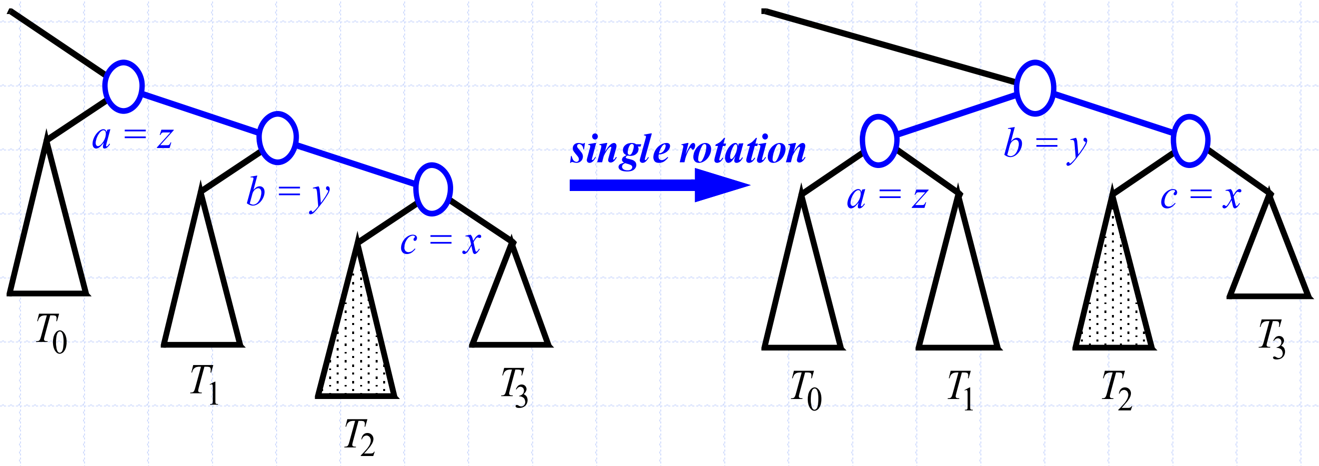
case 1: single rotation (a left rotation about *a*)

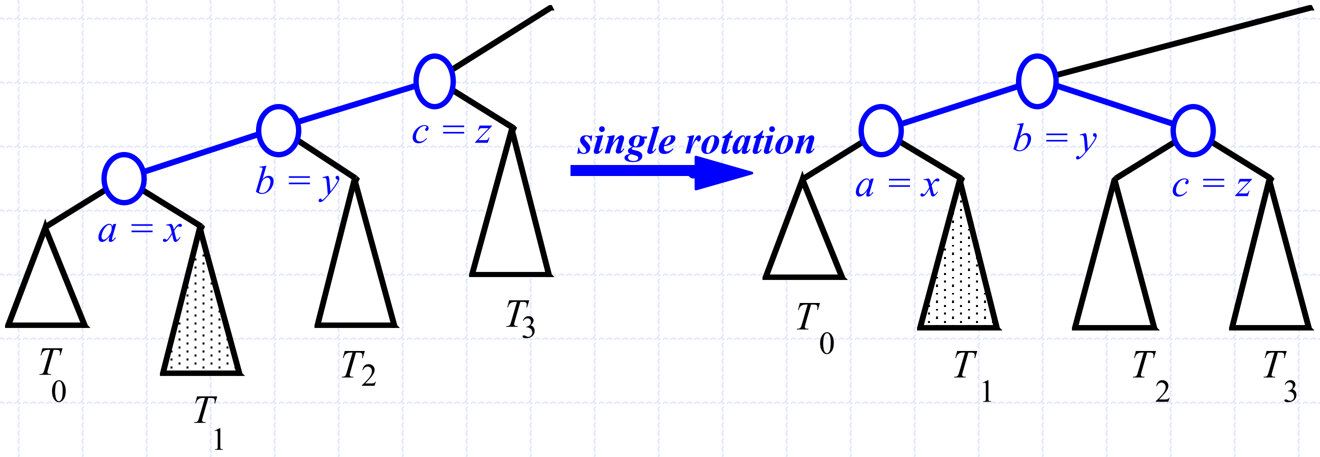


case 2: double rotation (a right rotation about *c*, then a left rotation about *a*)

Restructuring (as Single Rotations)

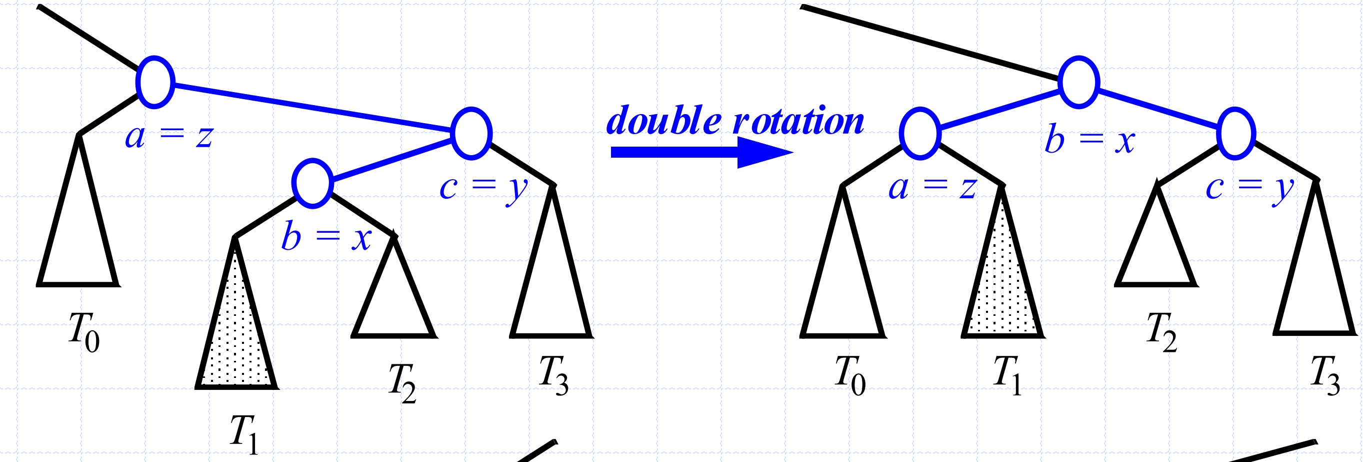
- Single Rotations:



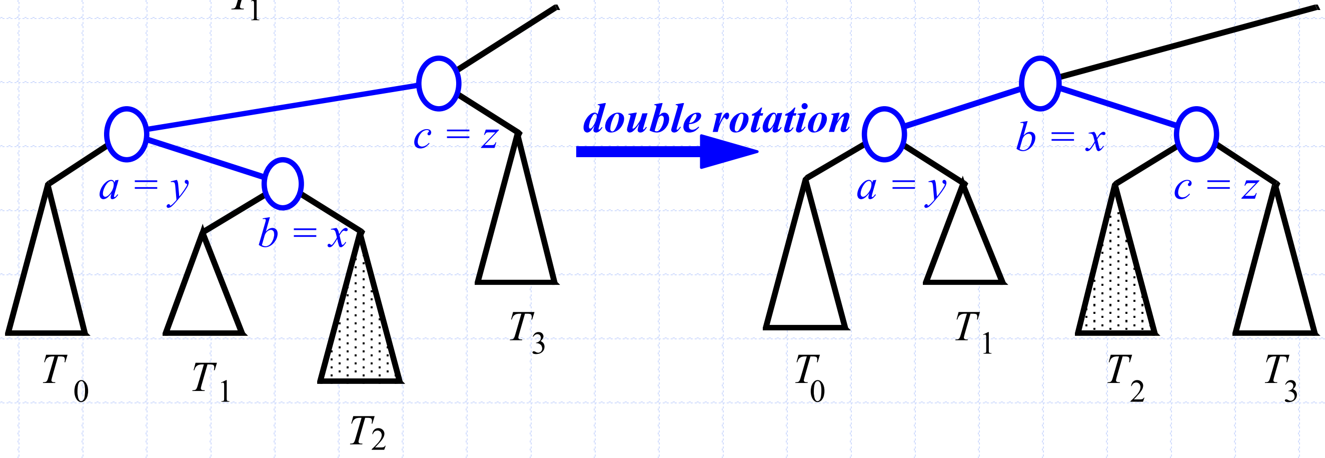


Restructuring (as Double Rotations)

- double rotations:

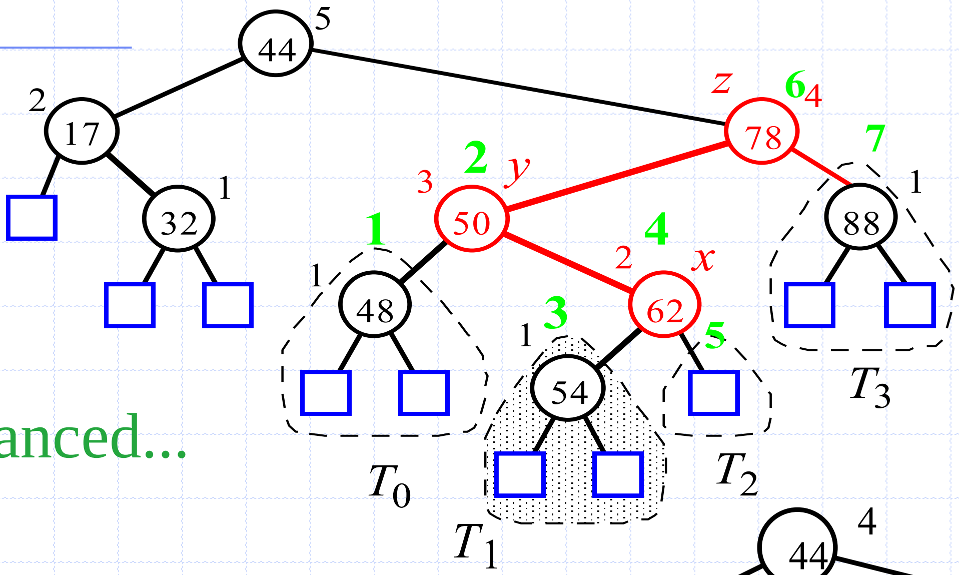


Right Left

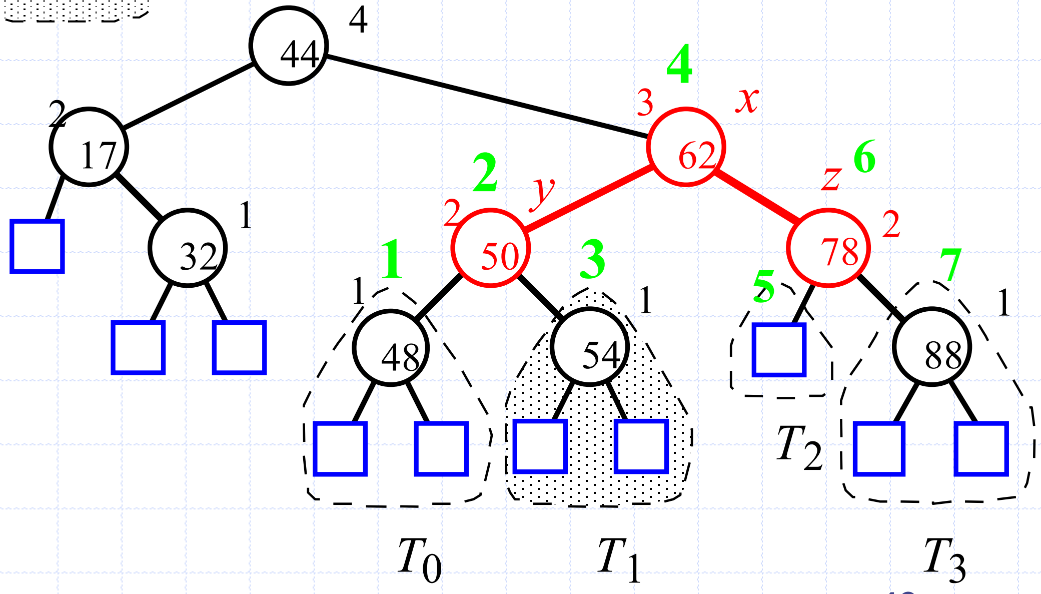


Left Right

Insertion Example, continued



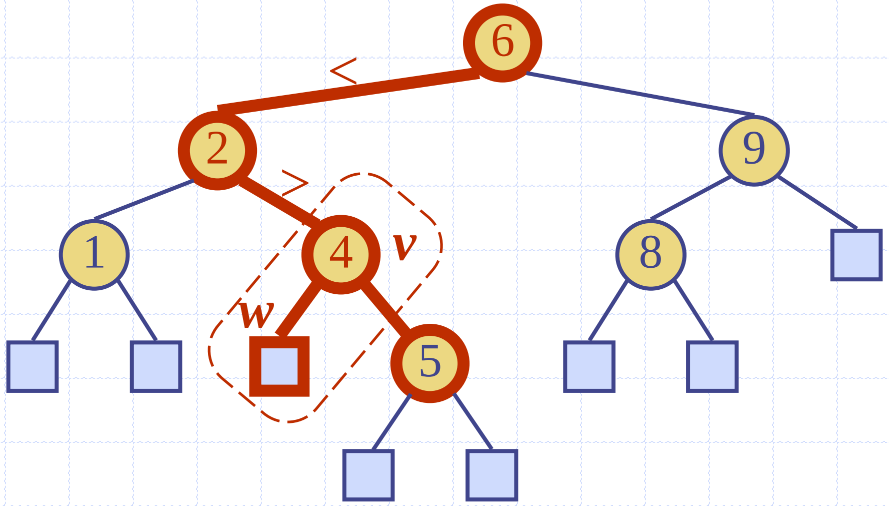
unbalanced...

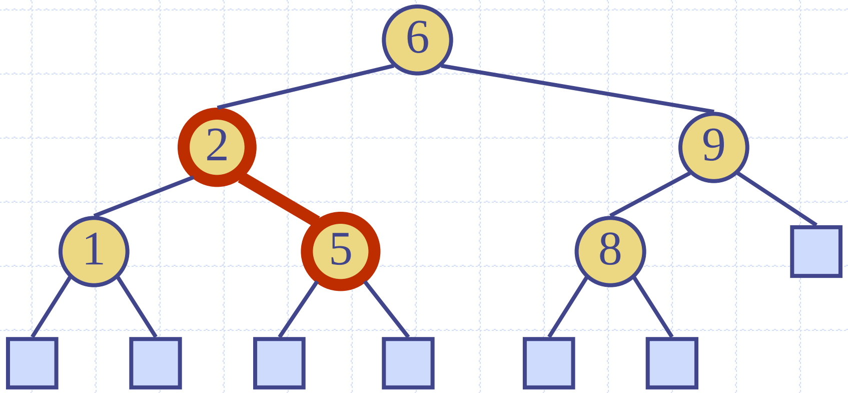


...balanced

Recall Deletion (Case 1)

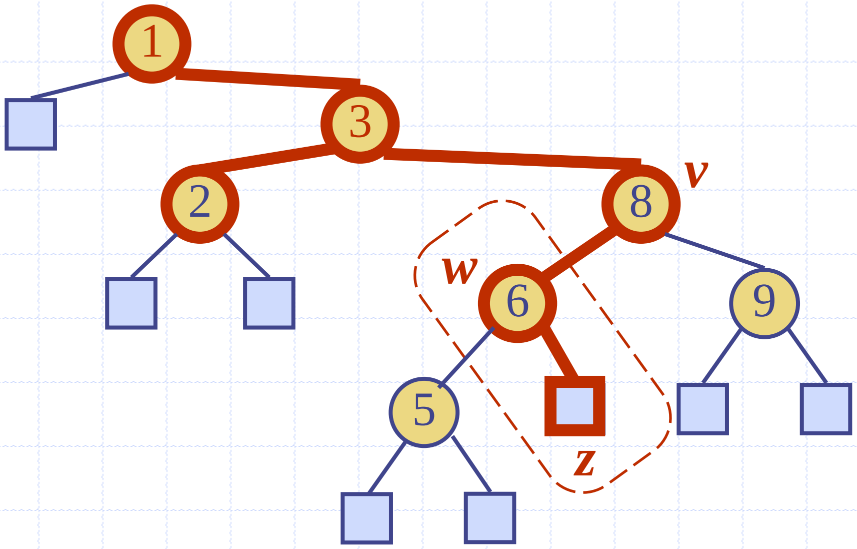
- To perform operation removeItem(***k***), we search for key ***k***
- Assume key ***k*** is in the tree, and let ***v*** be the node storing ***k***
- **Two cases**:
  - Node ***v*** has a leaf child ***w***
  - Node ***v*** has no leaf child
- If node ***v*** has a leaf child ***w***, we remove ***v*** and ***w*** from the tree with operation remove(***v***)
- Example: remove 4





Recall Deletion (Case 2)

- We consider the case where the key *k* to be removed is stored at a node *v* whose children are both internal
  - we find the internal node *w* that precedes *v* in an in-order traversal
  - we copy ***key***(*w*) into node *v*
  - we remove node *w* and its right child *z* (which must be a leaf) by means of operation remove(*w*)
- Example: remove 8



Deletion (Case 2)

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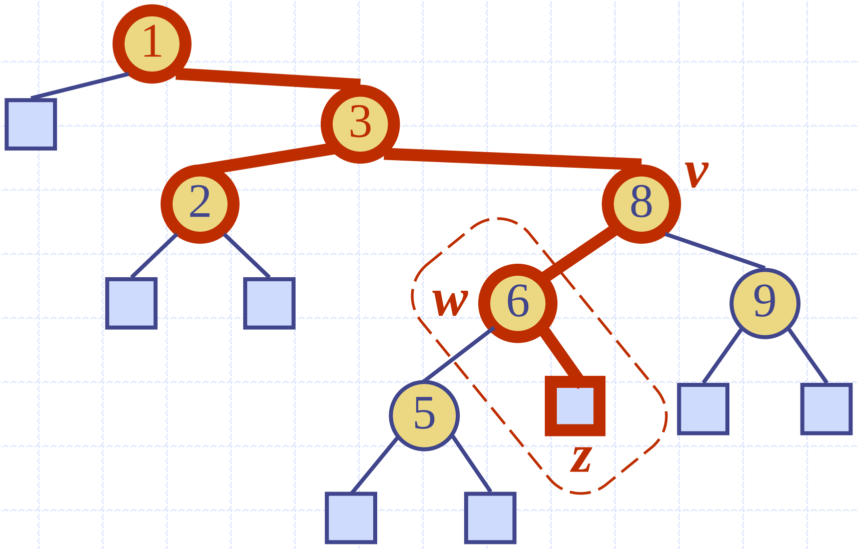
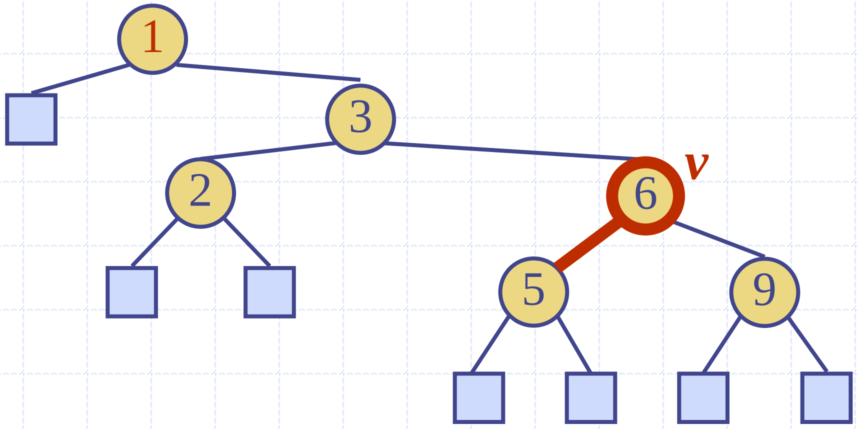
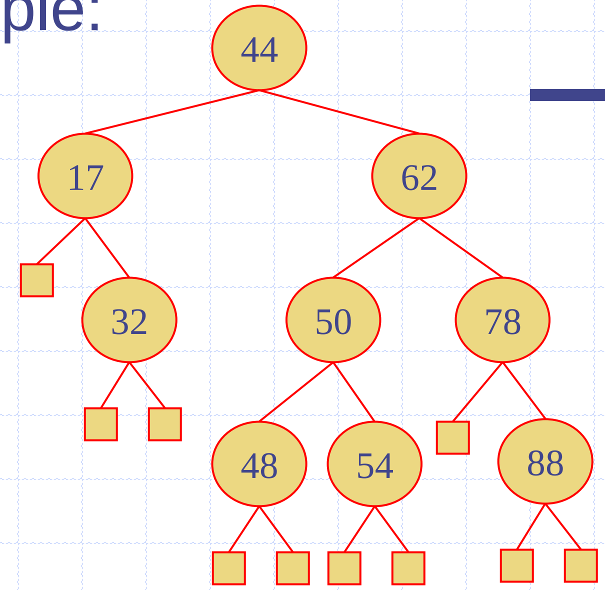
  - we find the internal node ***w*** that precedes ***v*** in an in-order traversal
  - we copy ***key***(***w***) into node ***v***
  - we remove node ***w*** and its right child ***z*** (which must be a leaf) by means of operation remove(***w***)
- Example: remove 8



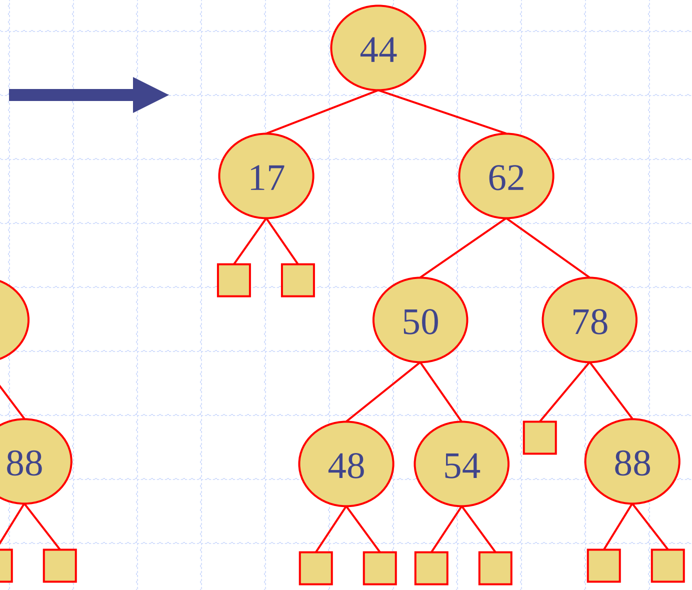


Removal in an AVL Tree

- Removal begins as in a binary search tree, which means the node removed will become an empty external node. Its parent, w, may cause an imbalance.
- Example:



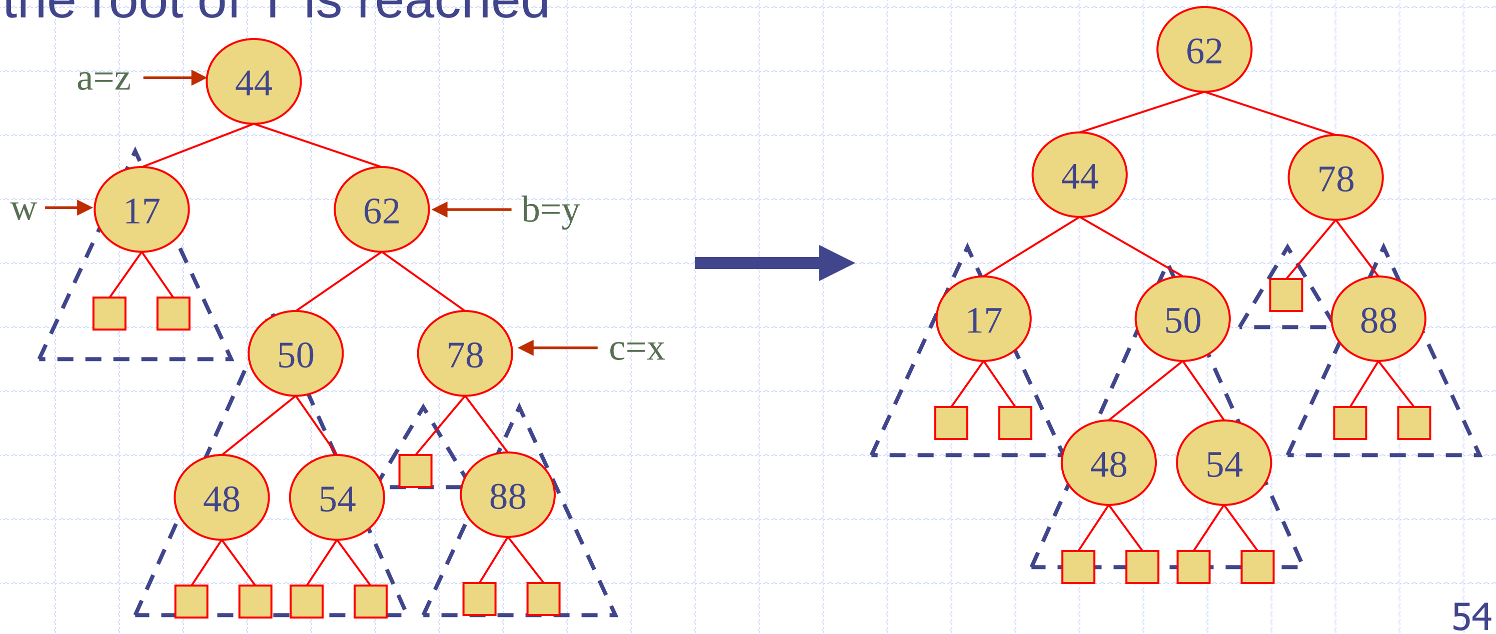
before deletion of 32



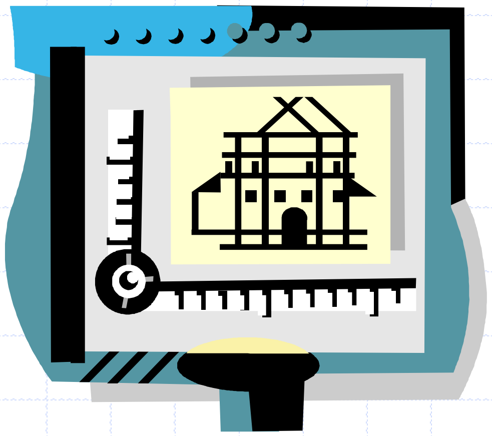
after deletion

Rebalancing after a Removal

- Let z be the first unbalanced node encountered while travelling up the tree from w. Also, let y be the child of z with the larger height, and let x be the child of y with the larger height.
- We perform restructure(x) to restore balance at z.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached



Running Times for AVL Trees



- a single restructure is O(1)
  - using a linked-structure binary tree
- find is O(log *n*)
  - height of tree is O(log *n*), no restructures needed
- insert is O(log *n*)
  - initial find is O(log *n*)
  - Restructuring up the tree, maintaining heights is O(log *n*)
- remove is O(log *n*)
  - initial find is O(log *n*)
  - Restructuring up the tree, maintaining heights is O(log *n*)

Advantages of Binary Search Trees

- When implemented properly, BST’s
  - perform insertions and deletions faster than can be done on Linked Lists
  - perform any find with the same efficiency as a binary search on a sorted array
  - keep all data in sorted order (eliminates the need to sort)

Main Point

2. The elimination of the worst case behavior of a binary search tree is accomplished by ensuring that the tree remains balanced, that is, the insert and delete operations do not allow any leaf to become significantly deeper than the other leaves of the tree.

   Science of Consciousness: Regular experience of pure consciousness during the TM technique reduces stress and restores balance in the physiology. The state of perfect balance, pure consciousness, is the basis for balance in activity.

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. In a (2,4) tree, each node has 2, 3, or 4 children and all leaf nodes are at the same depth so search, insertion, and deletion are efficient, O(log n).

2. The insert and delete operations in a (2,4) tree are carefully structured so that the activity at each node promotes balance in the tree as a whole. Each node contributes to the dynamic balance by giving and receiving keys during key transfer and the splitting and fusion of nodes.

3. **Transcendental Consciousness** is the state of perfect balance, the foundation for wholeness of life, the basis for balance in activity.
4. **Impulses within Transcendental Consciousness**: The dynamic natural laws within this unbounded field create and maintain the order and balance in creation.
5. **Wholeness moving within itself** : In Unity Consciousness, one experiences the dynamics of pure consciousness that gives rise to the laws of nature, the order and balance in creation, as nothing other than one’s own Self.