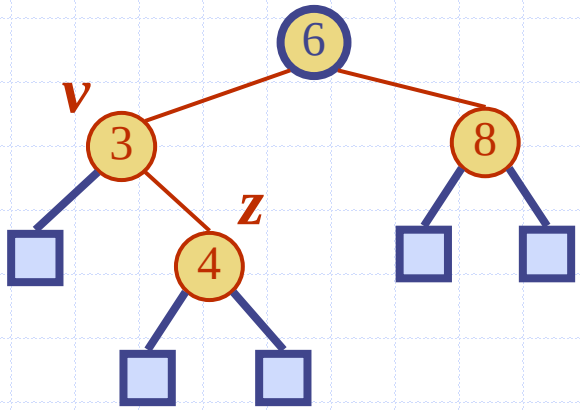
# Lecture 9: Red-Black Trees

Perfect Balance and Efficiency



# Wholeness Statement

A red-black tree is an implementation of a (2, 4) tree that is optimized for space utilization. The insert and delete operations are also optimized to avoid backtracking; the operations are performed locally yet maintain balance and order in the whole. *Science of Consciousness*: Nature operates in accord with the law of least action while maintaining balance and order in the whole.

# Outline and Reading

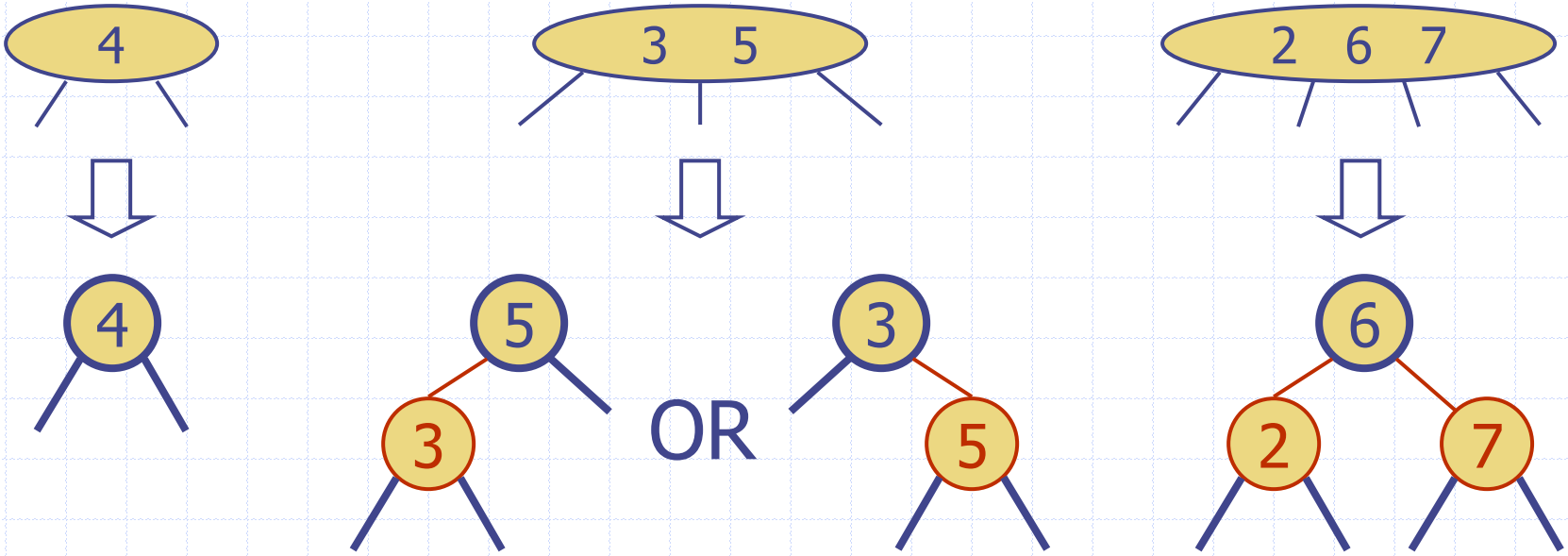
- From (2,4) trees to red-black trees (§3.3.3)
- Red-black tree (§ 3.3.3)
  - Definition
  - Height
  - Insertion
    - restructuring
    - recoloring
  - Deletion
    - restructuring
    - recoloring
    - adjustment

# Balanced Search Trees

- History and development of balanced search trees
  - Started with AVL trees,1962
    - Two Soviet mathematicians (Adel'son-Vel'skiĭ and Landis)
    - Could have as many as O(log n) rotations
  - 2-3 trees,1970
  - B-trees, 1972
    - Generalization of 2-3 trees to any number keys per node
    - Variations (e.g., B*-tree, B+-tree) became popular file structures
  - Symmetric binary B-trees, 1972
  - Red-Black coloring introduced, 1978
    - Became popular implementation of balance binary trees in the late 1980's and early 1990's

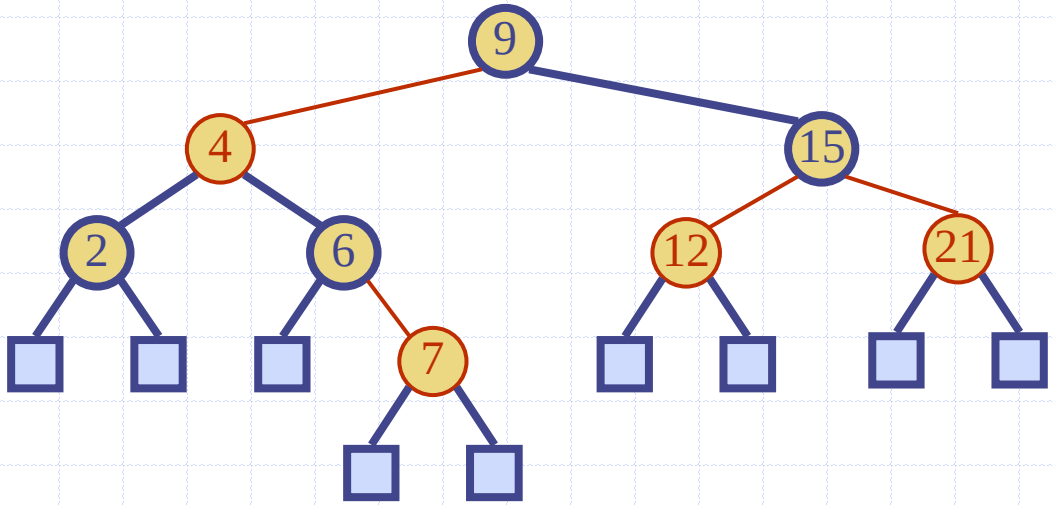
# From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored red or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type

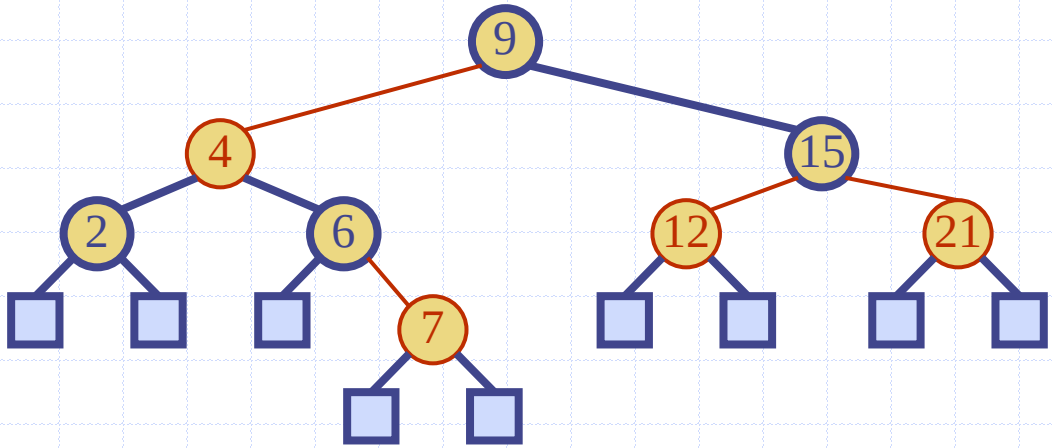


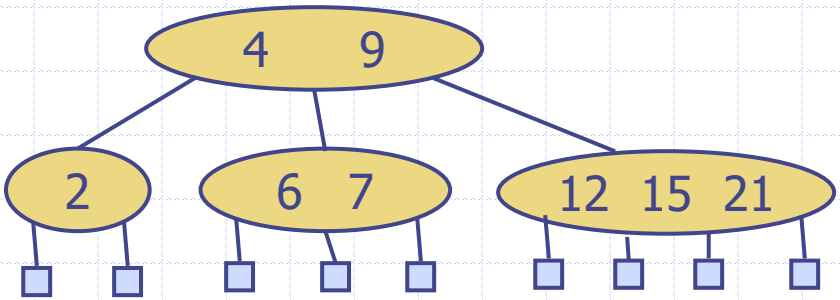
# Red-Black Tree

- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black
  - Depth Property: all the leaves have the same black depth

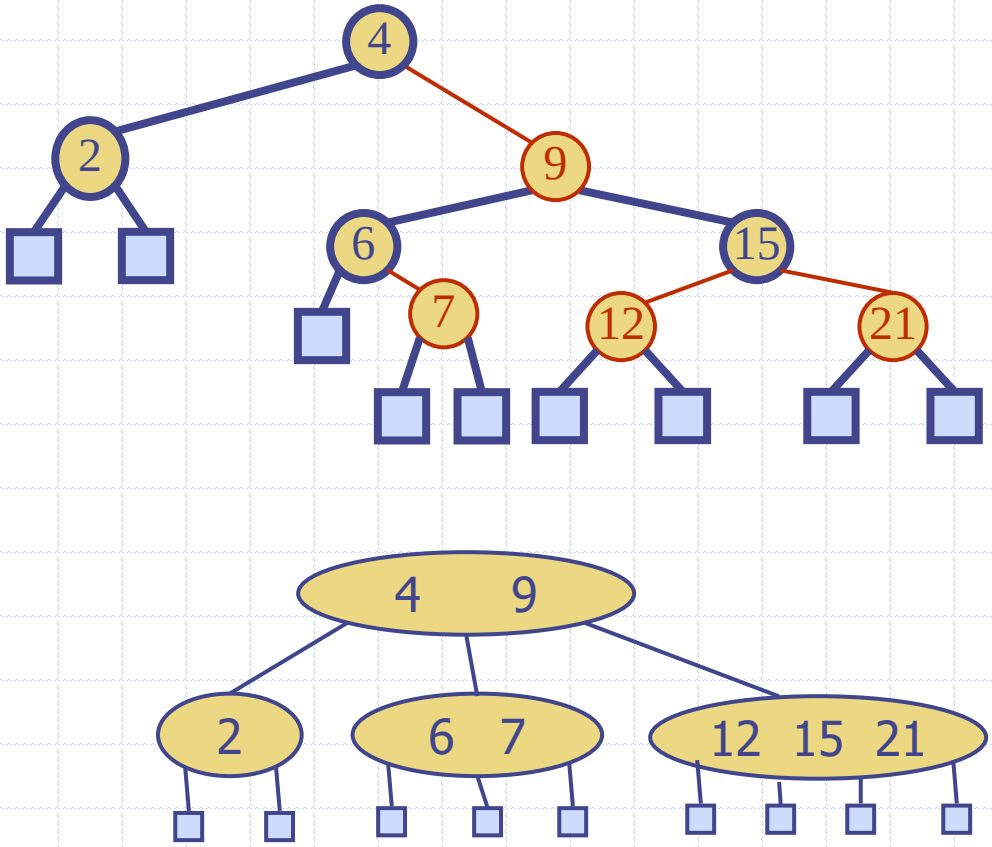


# Red-Black Tree to (2,4) Tree



# Different Red-Black Tree to Same (2,4) Tree



# Height of a Red-Black Tree

- Theorem: A red-black tree storing $n$ items has height $O(\log n)$

  Proof:

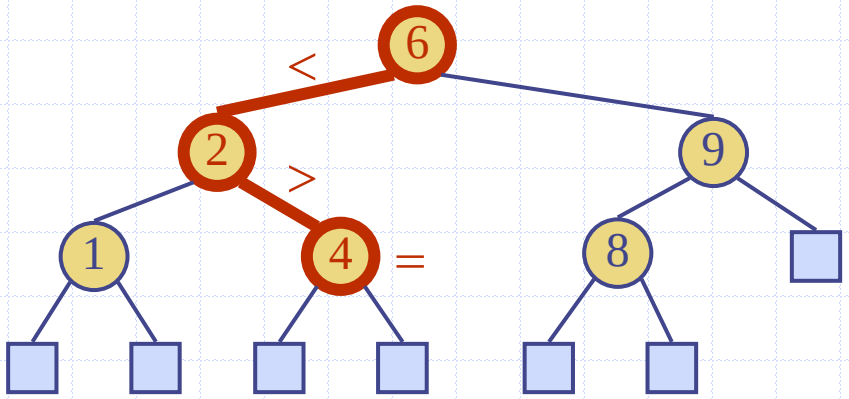
  - The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $O(\log n)$
- The search algorithm for a red-black tree is the same as that for a binary search tree
- By the above theorem, searching in a red-black tree takes $O(\log n)$ time

# Red-Black Tree Search

# Search

- To search for a key ***k***, we trace a downward path starting at the root using our helper ***findPosition***
- If we reach a leaf, the key is not found, so we return NO_SUCH_KEY
- Otherwise we return the element associated with the key ***k***
- Example: findElement(4)

```
Algorithm findElement(k)
// The tree T is a field of the receiver this
    if isEmpty() then
        return NO_SUCH_KEY
    v ← findPosition(k, T.root()) // Helper method
    if k ≠ key(v) then
        return NO_SUCH_KEY
    else
        return v.element()
```



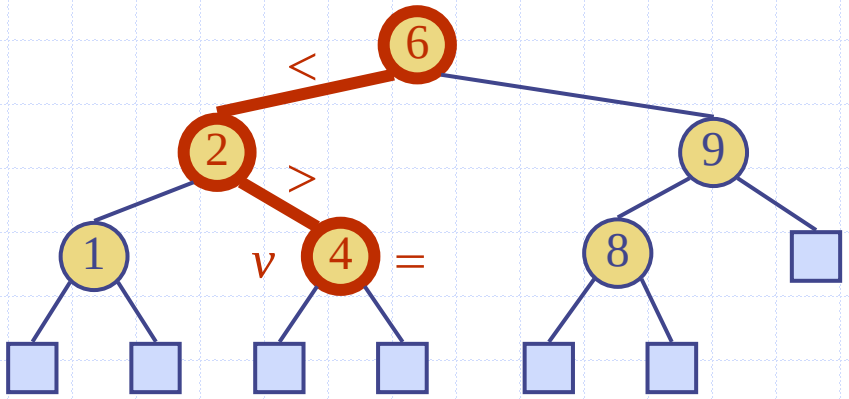
# Helper *findPosition*

- To search for a key ***k***, we trace a downward path starting at the root
- The next node visited is based on the comparison of ***k*** with the key of the current node
- If we reach a leaf, the key is not found and we return the parent of the external node
- If we find the key, then we return the node ***v*** containing ***k***
- Example: findPosition of key 4 or 5 both return node ***v***

**Algorithm** ***findPosition***(***k***, ***v***)

**Output**: the node containing key k or the parent of the node where k would be inserted into tree *T*

```
if k = key(v) then
    return v        // node containing k
else if k < key(v) then
    if isExternal (T.leftChild(v)) then
        return v    // node where k would be inserted
    else return findPosition(k, T.leftChild(v))
else if isExternal (rightChild(v)) then
        return v    // node where k would be inserted
    else return findPosition(k, T.rightChild(v))
```



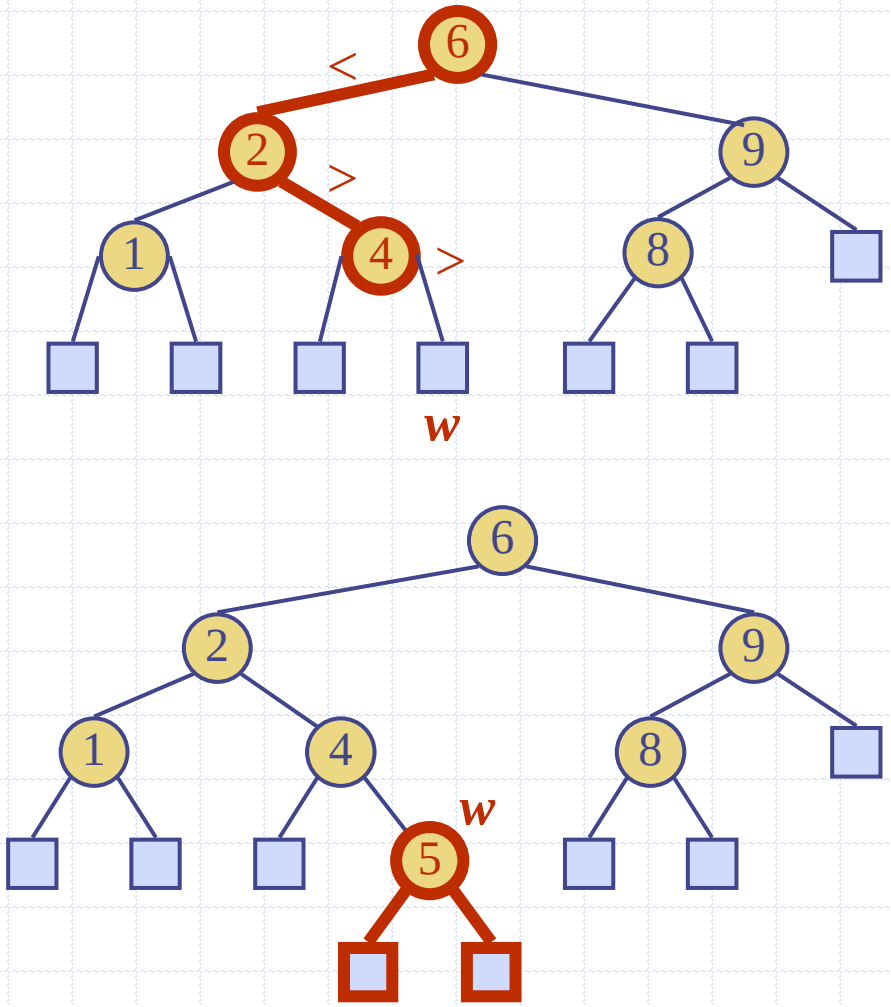
# Main Point

1. A red-black tree is an efficient way to implement an ordered dictionary ADT because it achieves logarithmic worst-case running times for both searching and updating (inserting and removing). *Science of Consciousness:* The TM technique is a very simple, effortless way to facilitate contact with the field of total knowledge, where the fulfillment of intellectual study is achieved, i.e., one feels at home with everything and everyone.

# Red-Black Tree Insertion

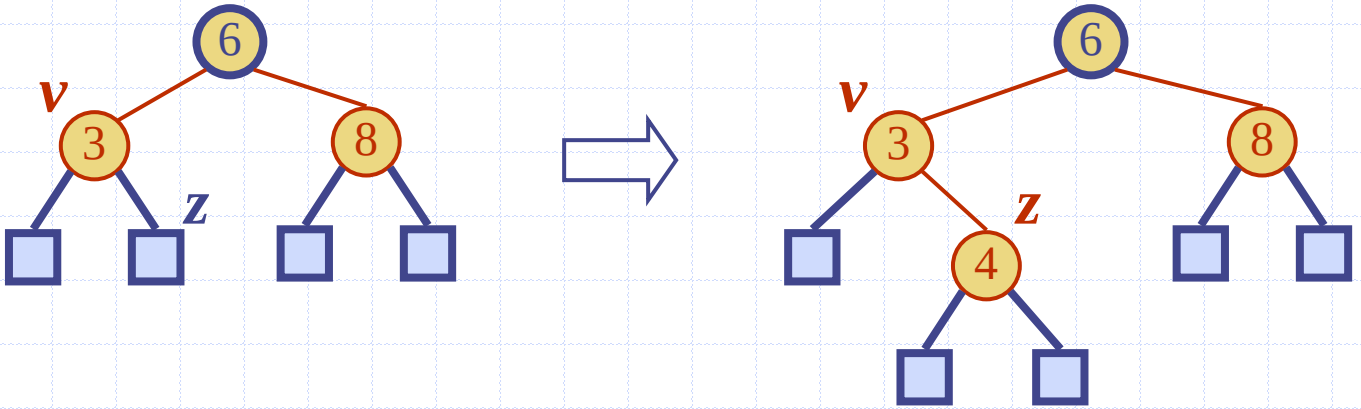
# Recall
# Binary Tree Insertion (§3.1.4)

- To perform operation insertItem(k, o), search for key k (k should not be in the tree)
- Let w be the leaf reached by the search
- Insert k at node w and expand w into an internal node
- Example: insert 5



# Insertion

- To perform operation insertItem($k$, $o$), we execute the insertion algorithm for binary search trees and color the newly inserted node $z$ red unless it is the root. We preserve the root, external, and depth properties.
  - If the parent $v$ of $z$ is black, we preserve the internal property and we are done…
  - …else ($v$ is red ) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree
- For example, the insertion of 4 causes a double red:

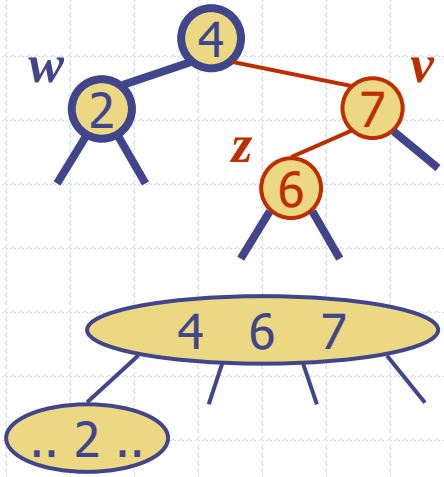


# Remedying a Double Red

- Consider a double red with child $z$ and parent $v$, and let $w$ be the sibling of $v$

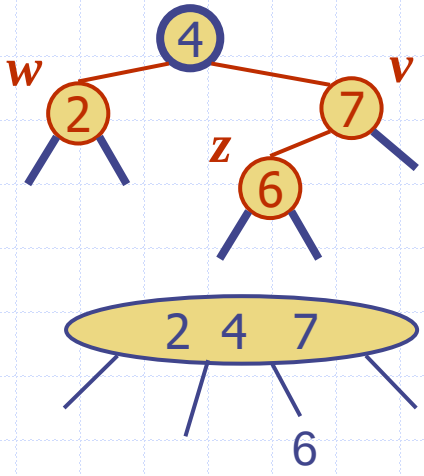
Case 1: $z$'s uncle $w$ is black

- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement



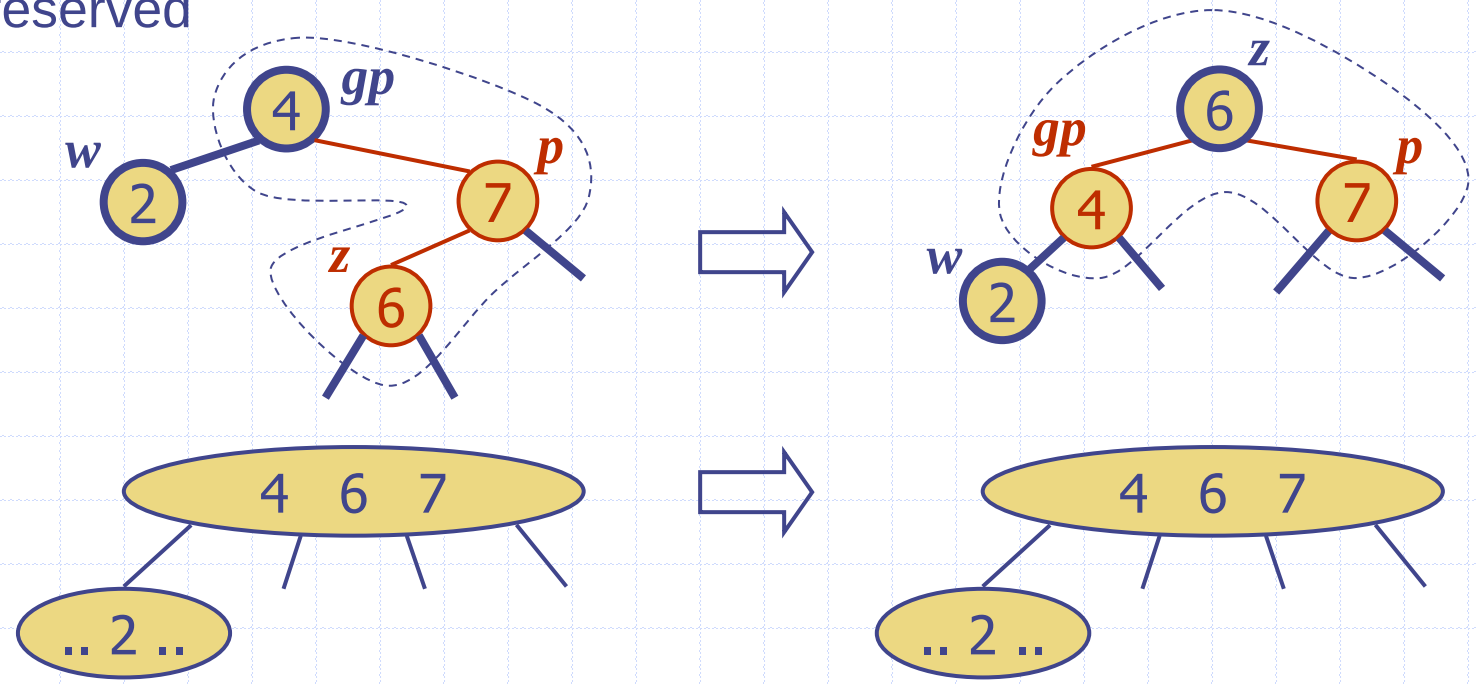

Case 2: $z$'s uncle $w$ is red

- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split



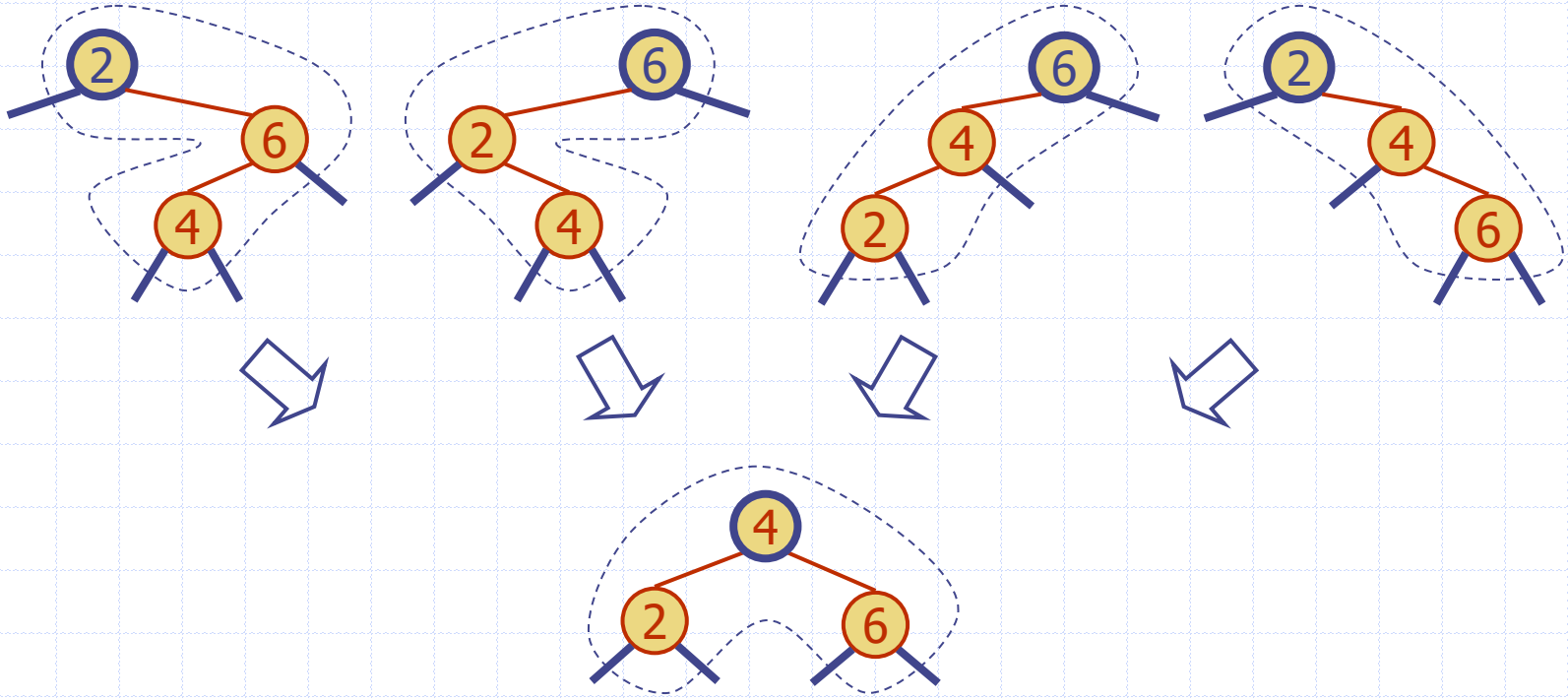
# Case 1: Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved



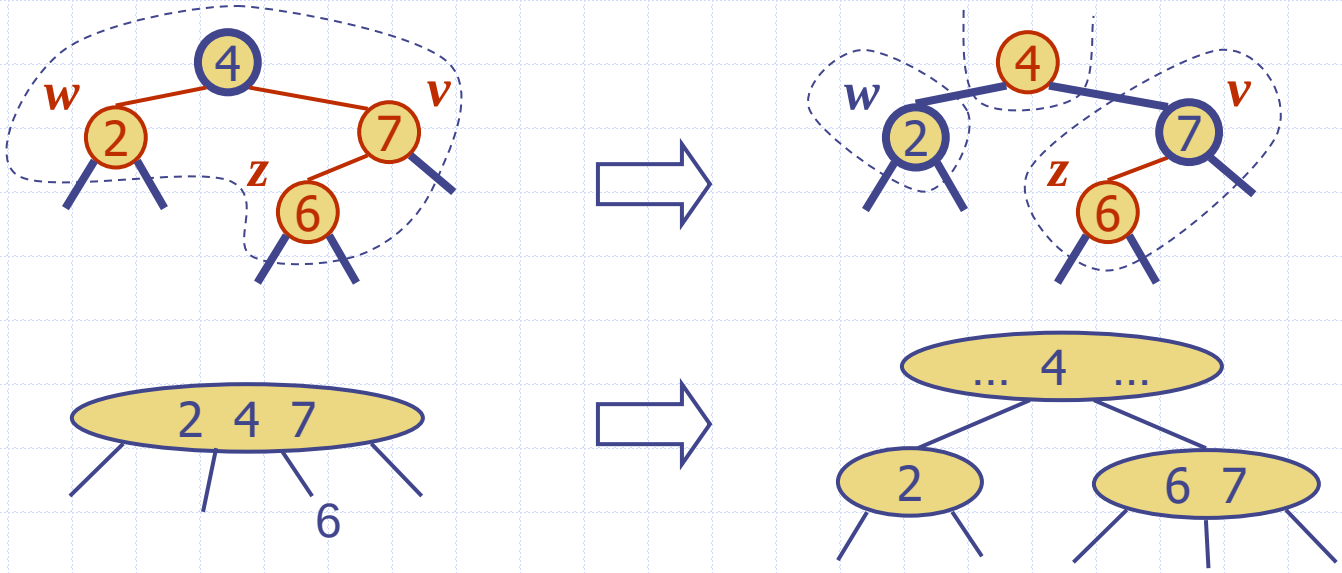
# Restructuring (cont.)

- There are four restructuring configurations depending on whether the double red nodes are left or right children



# Case 2: Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent $v$ and its sibling $w$ become black and the grandparent $u$ becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent $u$



# Example:

- Insert the following into an initially empty red-black tree in this order:

  (22, 5, 16, 45, 2, 10, 18, 30, 50, 12, 13, 33)

# Analysis of Insertion

**Algorithm** ***insertItem***(***k***, ***o***)

1. Search for key ***k*** to locate the insertion node ***z***

2. Add the new item (***k***, ***o***) at node ***z*** and color ***z*** red

3. **while** ***doubleRed***(***z***)
   **if** ***isBlack***(***sibling***(***parent***(***z***)))
      ***z*** ← ***restructure***(***z***)
      **return**
   **else** { ***sibling***(***parent***(***z***)) is red }
      ***z*** ← ***splitRecolor***(***z***)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(?)$ time
- Step 2 takes $O(?)$ time
- Step 3 takes $O(?)$ time
- Thus, an insertion in a red-black tree takes $O(?)$ time

# Analysis of Insertion

**Algorithm** ***insertItem***(***k***, ***o***)

1. Search for key ***k*** to locate the insertion node ***z*** using ***findPosition***
2. Add the new item (***k***, ***o***) at node ***z*** and color ***z*** red
3. **while** ***doubleRed***(***z***)
   **if** ***isBlack***(***sibling***(***parent(z)***))
      ***gpz*** ← ***parent***(***parent(z)***)
      ***restructure***(***z***)
      ***setColor(gpz, RED)***
      ***setColor(parent(gpz), BLK)***
      **return**
   **else** { ***sibling***(***parent***(***z***)) is red }
      ***z*** ← ***splitRecolor***(***z***)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

# Analysis of Insertion

```
Algorithm isDoubleRed(z)
    if isRoot(z) then
        setColor(z, BLK)
        return False
    else
        return isRed(parent(z))
```

```
Algorithm splitRecolor(z)
    pz ← parent(z)
    setColor(pz, BLK)
    setColor(sibling(pz), BLK)
    gpz ← parent(pz)
    setColor(gpz, RED)
    return gpz
```

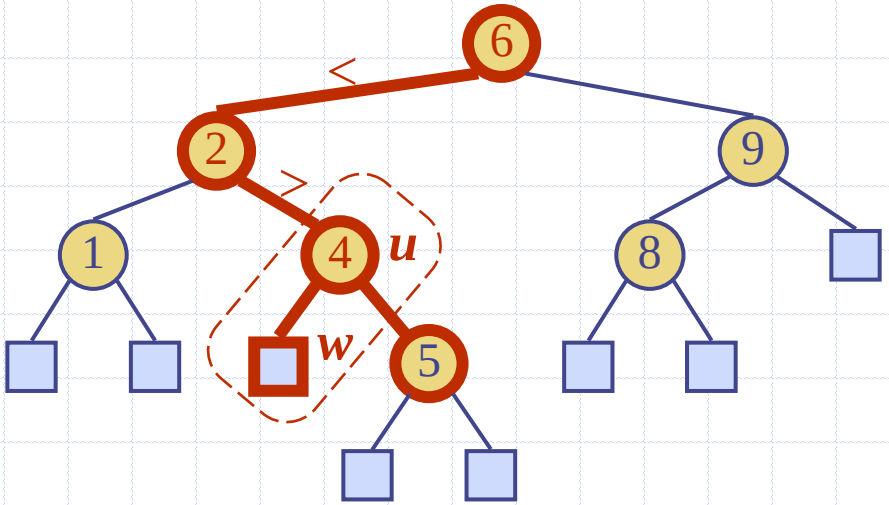
```
Algorithm restructure(z)
    pz ← parent(z)
    if isLeft(z) then
        if isLeft(pz) then
            rotateRight(pz)
        else
            rotateRight(z)
            rotateLeft(z)
    else { z is a right child }
        if isLeft(pz) then
            rotateLeft(z)
            rotateRight(z)
        else
            rotateLeft(pz)
```

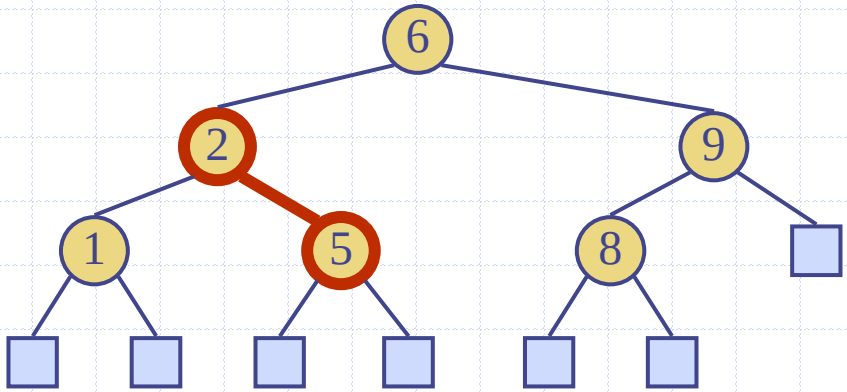
# Red-Black Tree Deletion

# Recall

# Binary Tree Deletion (Case 1)

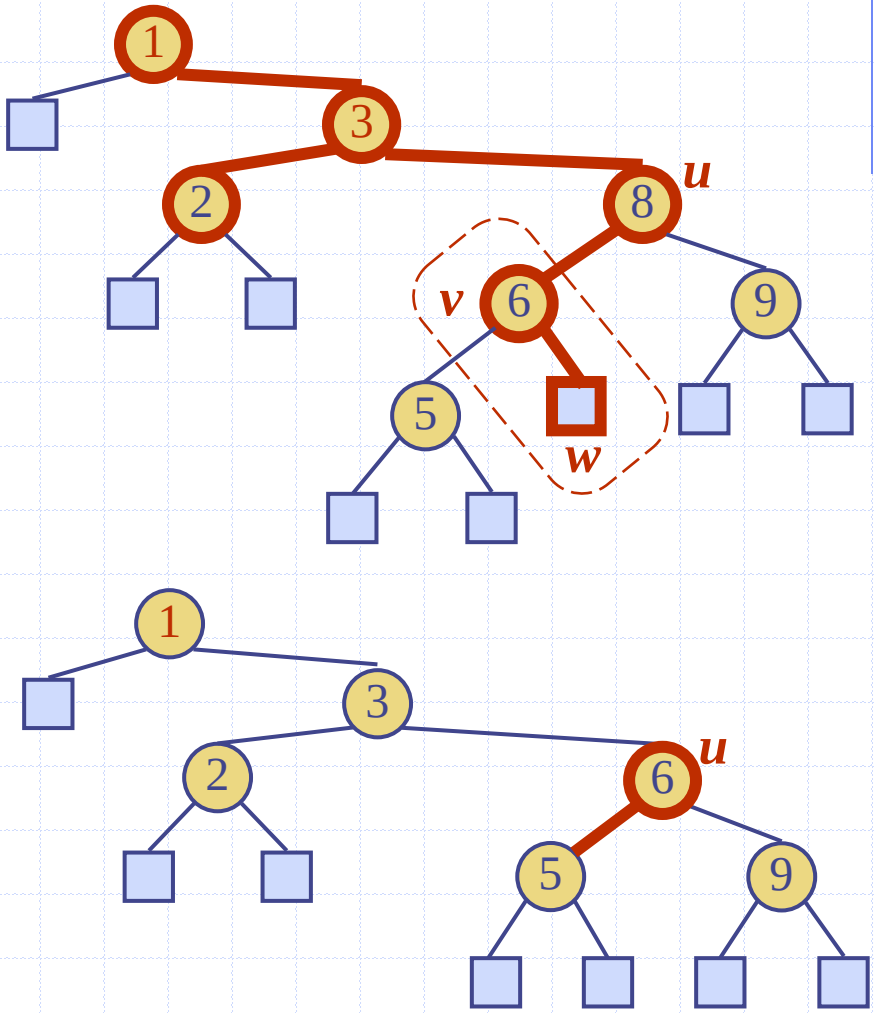
- To perform operation remove($k$), first search for key $k$
- Assume key $k$ is in the tree, and let $u$ be the node storing $k$
- Two cases:
  1. Node $u$ has a leaf child $w$
  2. Node $u$ has no leaf child
- If node $u$ has a leaf child $w$, we remove $u$ and $w$ from the tree with operation remove($u$)
- Example: remove 4



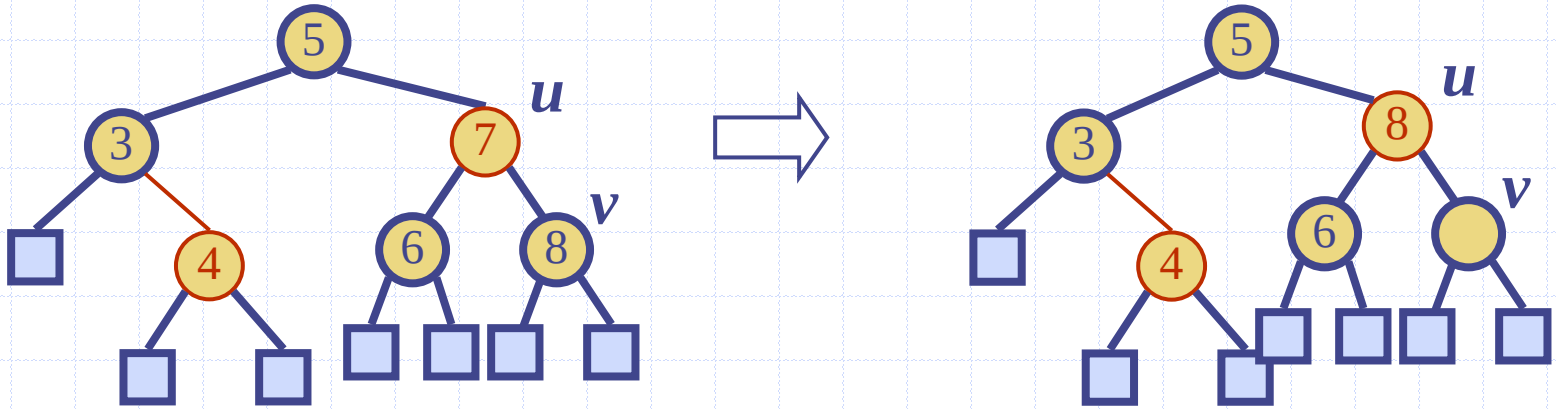
# Recall
# Binary Tree Deletion (Case 2)

- We consider the case where the key $k$ to be removed is stored at a node $u$ whose children are both internal
  - we find the internal node $v$ that precedes $u$ in an in-order traversal ($v$ in the example)
  - we copy $key(v)$ into node $u$
  - we remove node $v$ and its external child $w$ by means of operation remove($v$)
- Example: remove 8



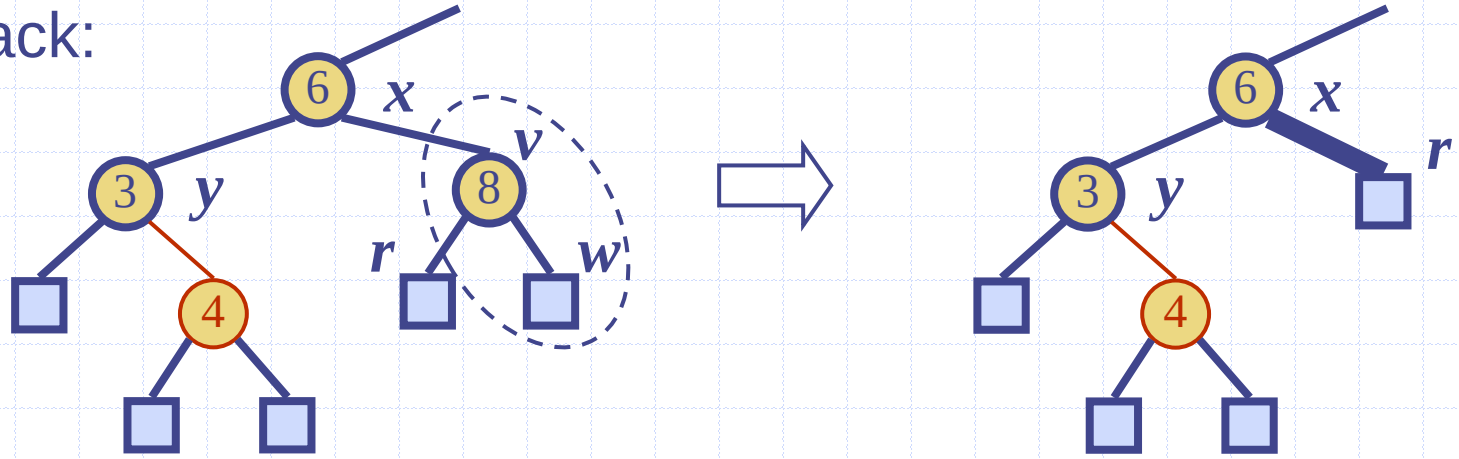
# Deletion

- To perform operation remove($k$), first execute the deletion algorithm for binary search trees
  - If node to be removed, *u,* does not have an external child, find next internal node by inorder traversal, called *v,* and move key at *v* to *u,* then remove *v.*
  - Thus, every removal occurs at an internal node with an external child.



# Deletion

- Let $v$ be the internal node removed, $w$ the external node removed, and $r$ the sibling of $w$
  - If either $v$ or $r$ was red, we color $r$ black and we are done
  - Else ($v$ and $r$ were both black), so we color $r$ ***double black*** (a fictitious color), which is a violation of the internal property requiring a reorganization of the tree (denotes underflow)
- For example, the deletion of 8 causes a double black:



# Remedying a Double Black

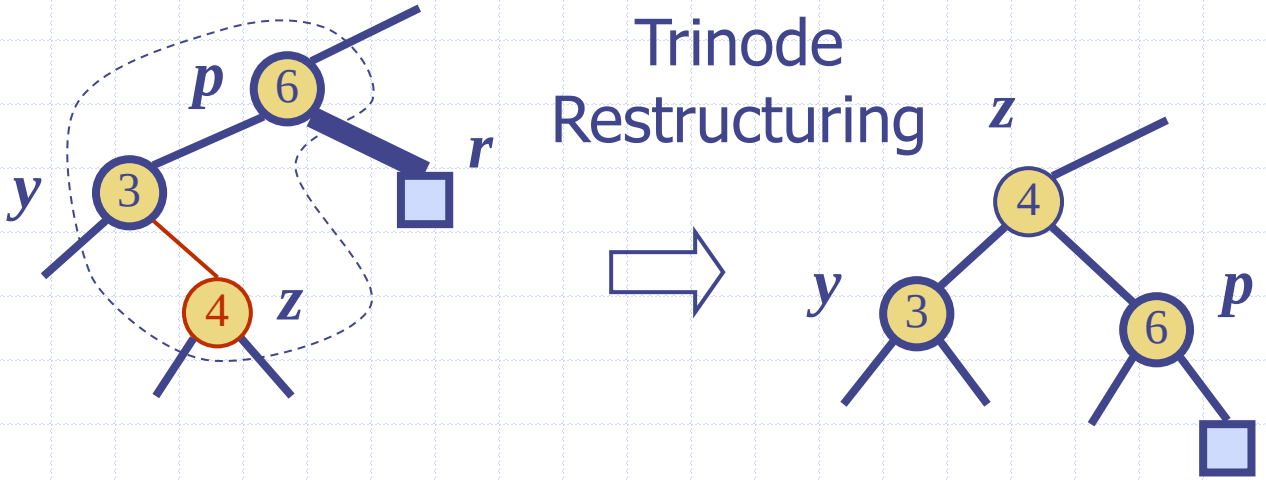
- The algorithm for remedying a double black node $r$ with sibling $y$ considers three cases

  Case 1: sibling $y$ is black and has a red child $z$

  Case 2: sibling $y$ is black and its children are both black

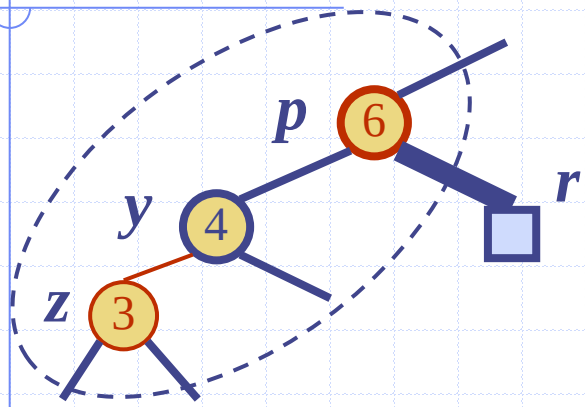
  Case 3: sibling $y$ is red

# Case 1a – sibling y is black and has a red child z

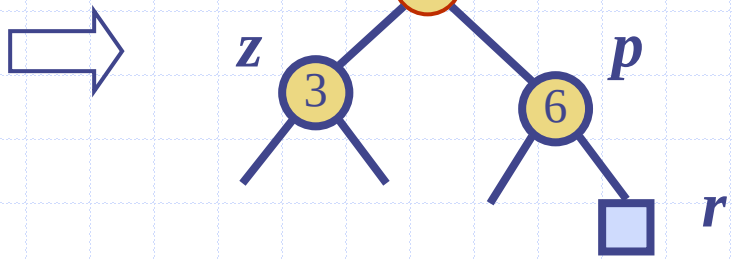



Color y and p black, give z the former color of p, and set r to black

# Case 1b – sibling y is black and has a red child z



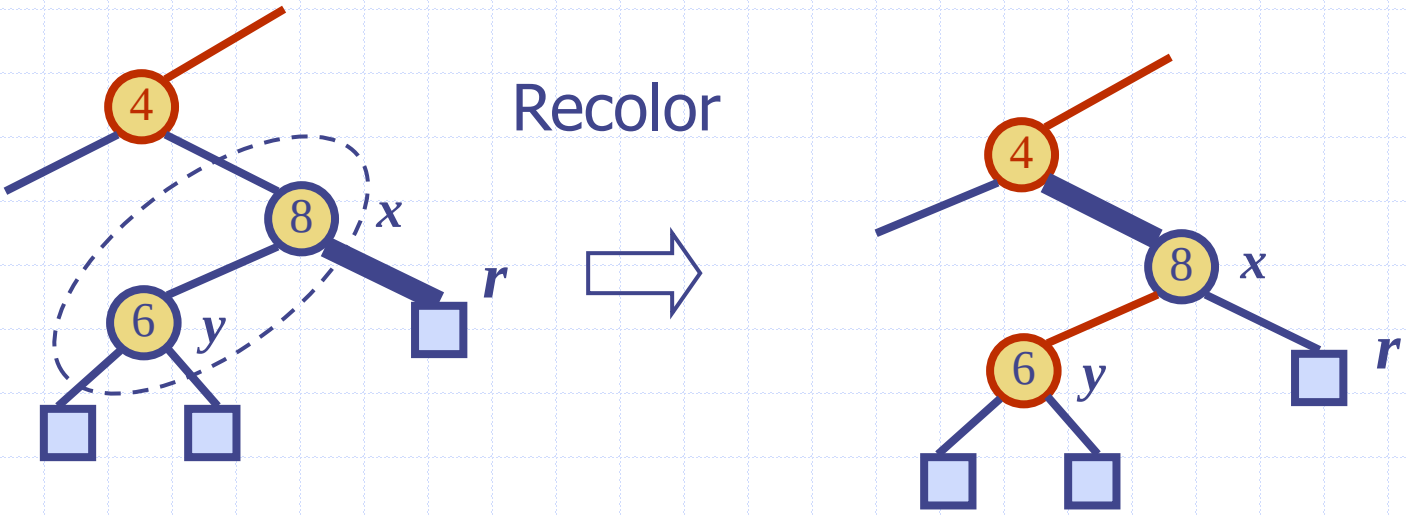

Trinode Restructuring





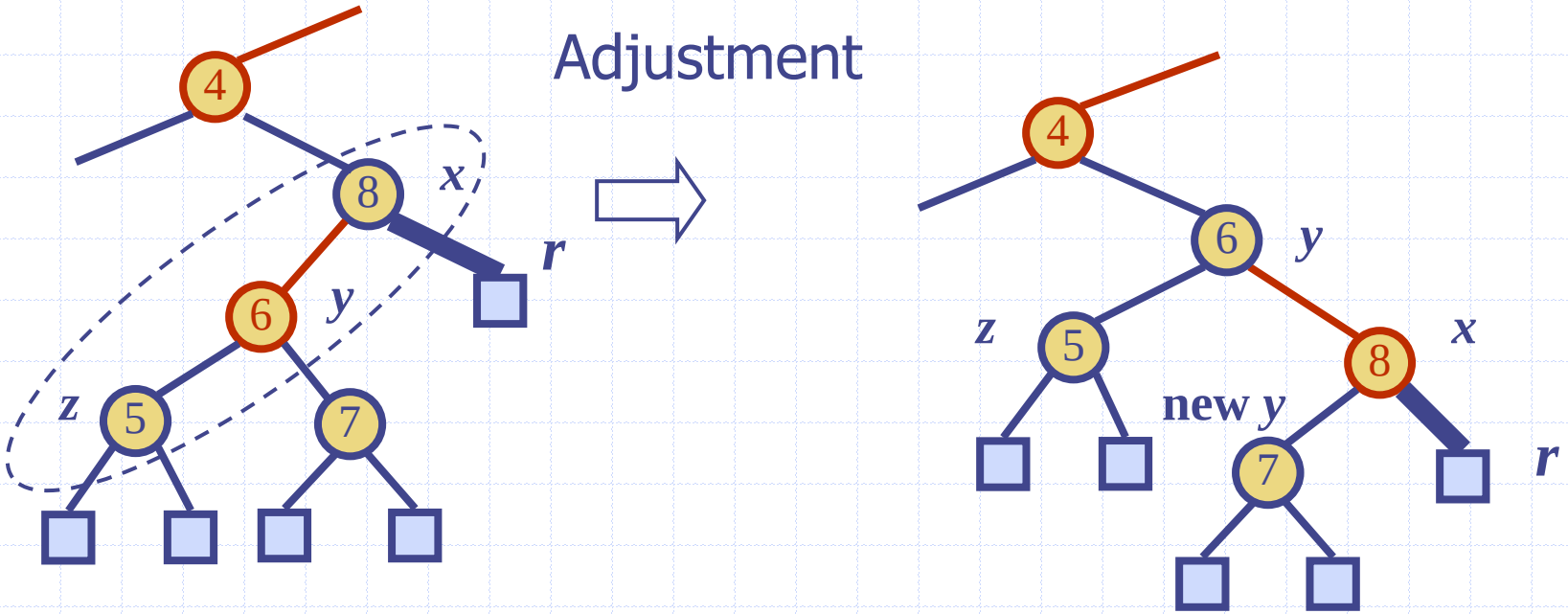

Color z and p black, give y the former color of p, and set r to black

# Case 2 – sibling y is black and its children are both black




Color r black and y red;
if x is red, color x black else
color x double black

# Case 3 – sibling y is red




Execute the trinode restructure operation that makes y the parent of x; Color y black and x red;

Then apply Case 1 or 2, with new y.

# Remedying a Double Black

- The algorithm for remedying a double black node $r$ with sibling $y$ considers three cases

  Case 1: sibling $y$ is black and has a red child
  - Perform a restructuring, equivalent to a transfer, and we are done

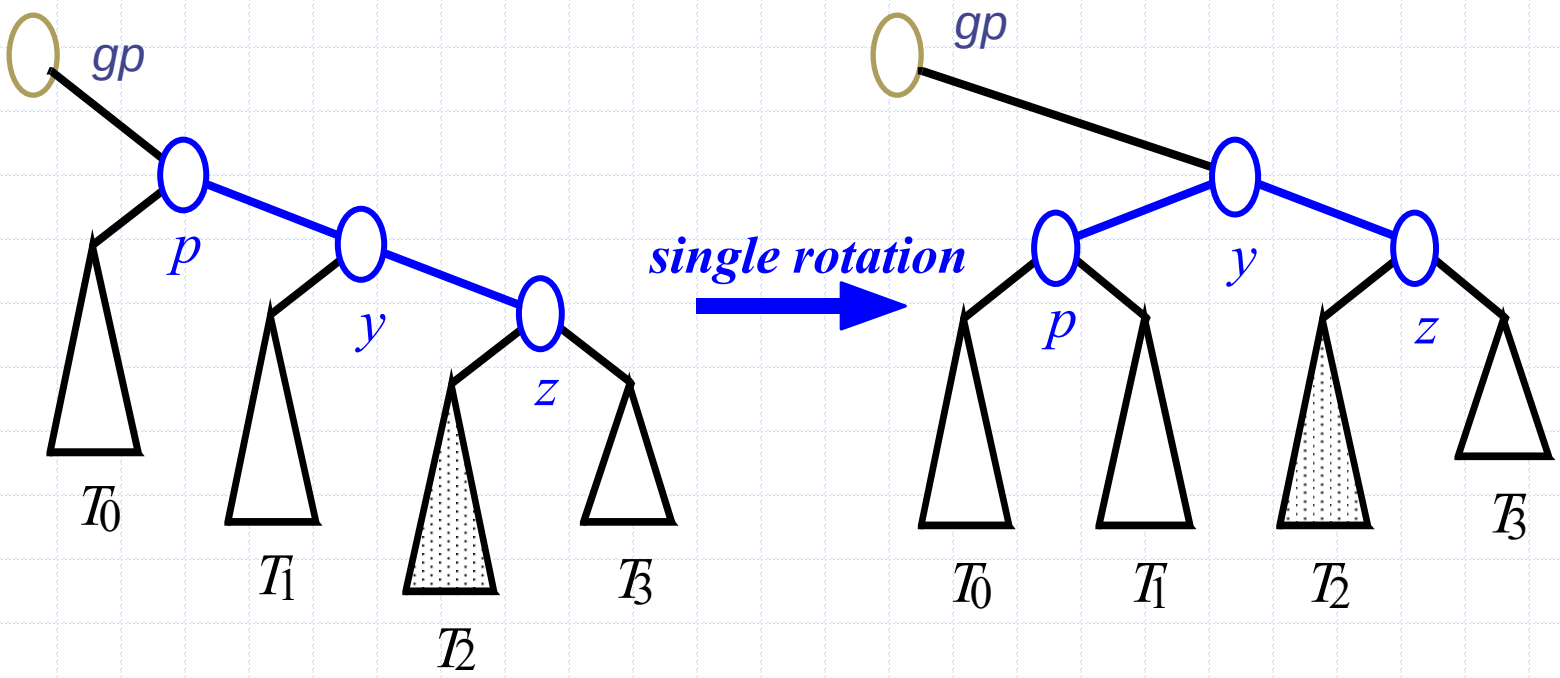
  Case 2: sibling $y$ is black and its children are both black
  - Perform a recoloring, equivalent to a fusion, which may propagate the double black violation up to parent

  Case 3: sibling $y$ is red
  - We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies

- Deletion in a red-black tree takes $O(\log n)$ time

# Restructuring (as Single Rotations)

- Single Left Rotation around ***y***:

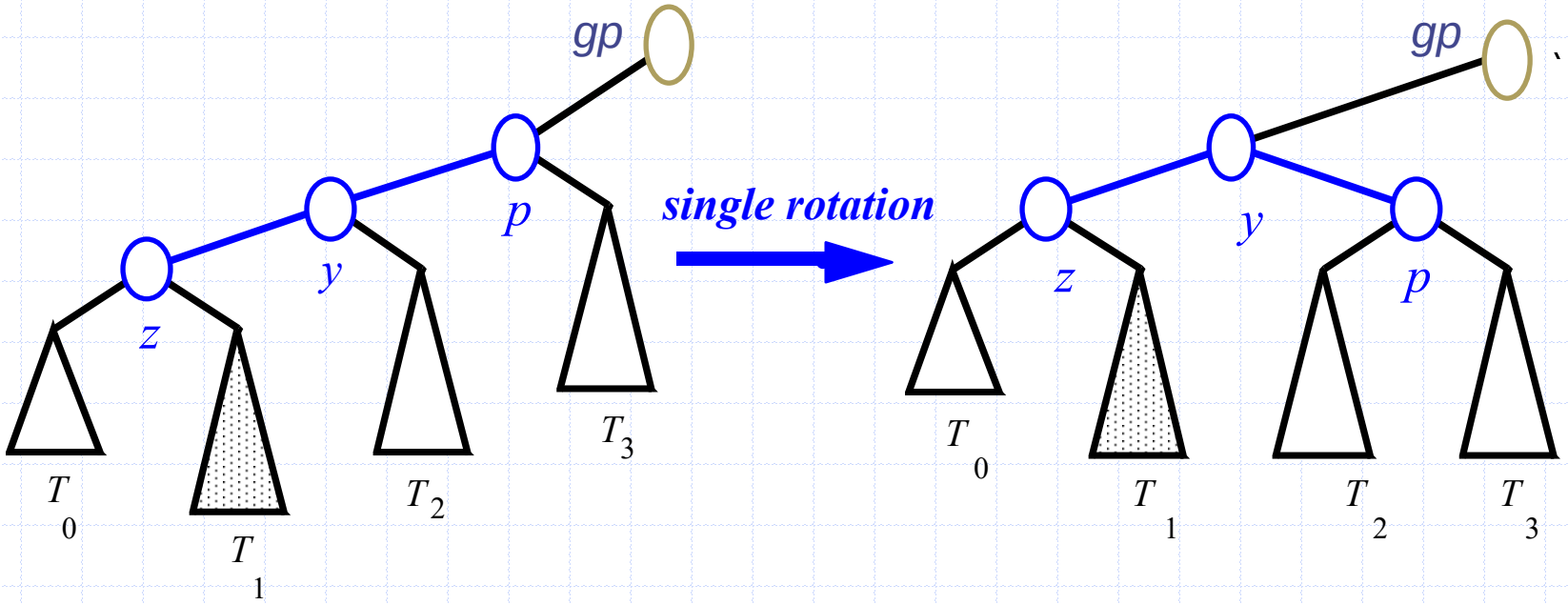


# Left Rotation

**Algorithm** *rotateLeft*(T, *y*)
**Input** Binary Tree T and node *y* in T
**Output** a left rotation around node *y* is performed

```
if T.isRoot(y) then throw InvalidRightRotation
p ← T.parent(y)
gp ← T.parent(p)

T.setRightChild(p, T.leftChild(y))
if T.isInternal(T.rightChild(y)) then {external node is a null reference}
    T.setParent(T.rightChild(y), p)

T.setLeftChild(y, p)
T.setParent(p, y)

if T.isRoot(p)
    then T.setRoot(y)
    else
        if T.rightChild(gp) = p
            then T.setRightChild(gp, y)
            else T.setLeftChild(gp, y)
        T.setParent(y, gp)
```

# Exercise:
# Do Single Right Rotation

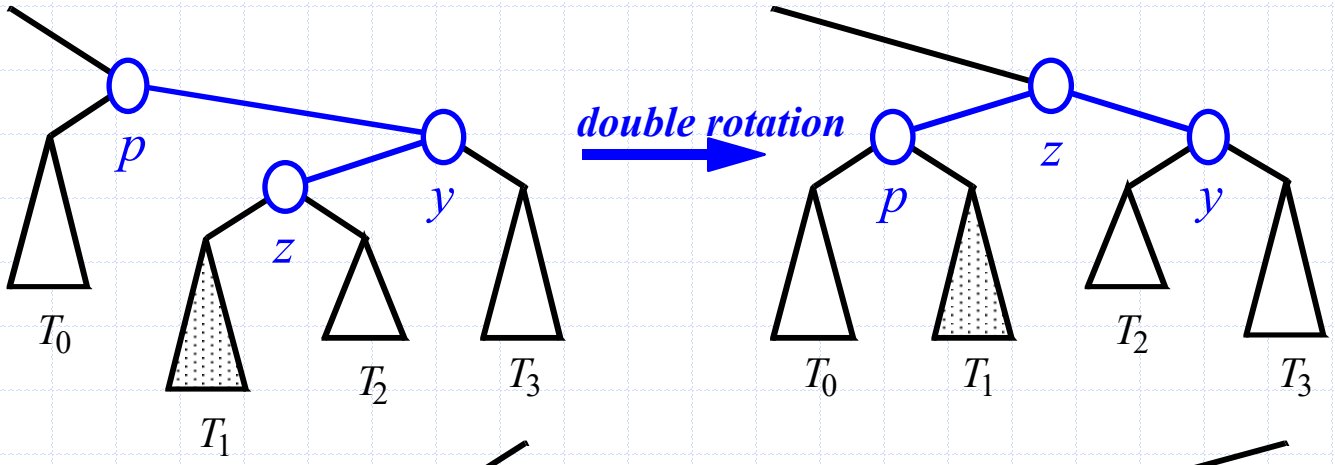


# Right Rotation

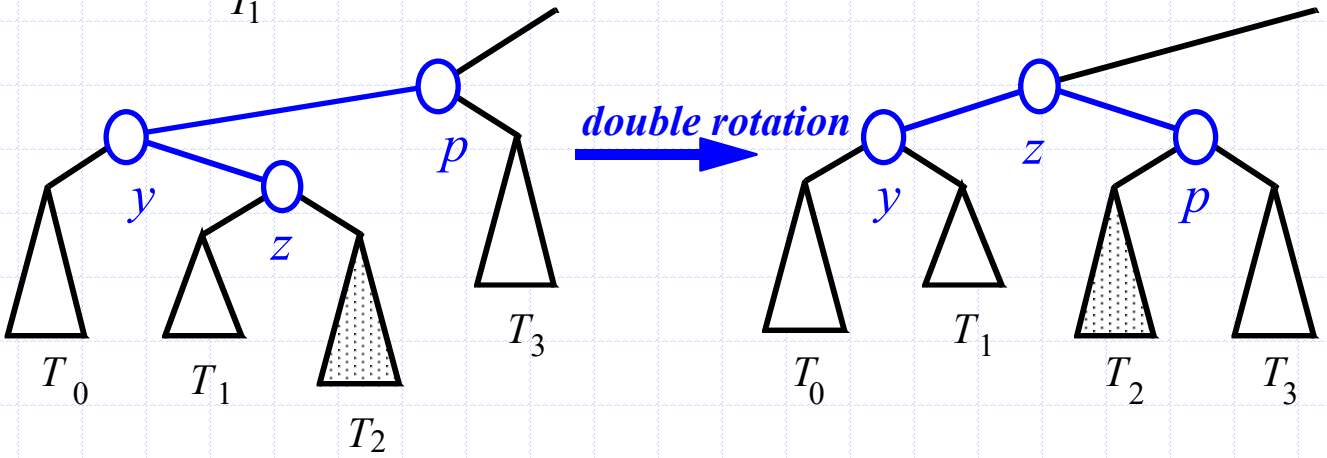
```
Algorithm rotateRight(T, y)
    Input Binary Tree T and node y in T
    Output a left rotation around node y is performed

     if  T.isRoot(y)  then   throw InvalidLeftRotation
     p ← T.parent(y)
     gp ← T.parent(p)

     T.setLeftChild(p, T.rightChild(y))
     if T.isInternal(T.leftChild(y))  then {external node is a null reference}
         T.setParent(T.leftChild(y), p)

     T.setRightChild(y, p)
     T.setParent(p, y)

     if  T.isRoot(p)
         then   T.setRoot(y)
         else
               if   T.rightChild(gp) =  p
                   then   T.setRightChild(gp, y)
                   else    T.setLeftChild(gp, y)
              T.setParent(y, gp)
```

# Restructuring (as Double Rotations)

- double rotations:



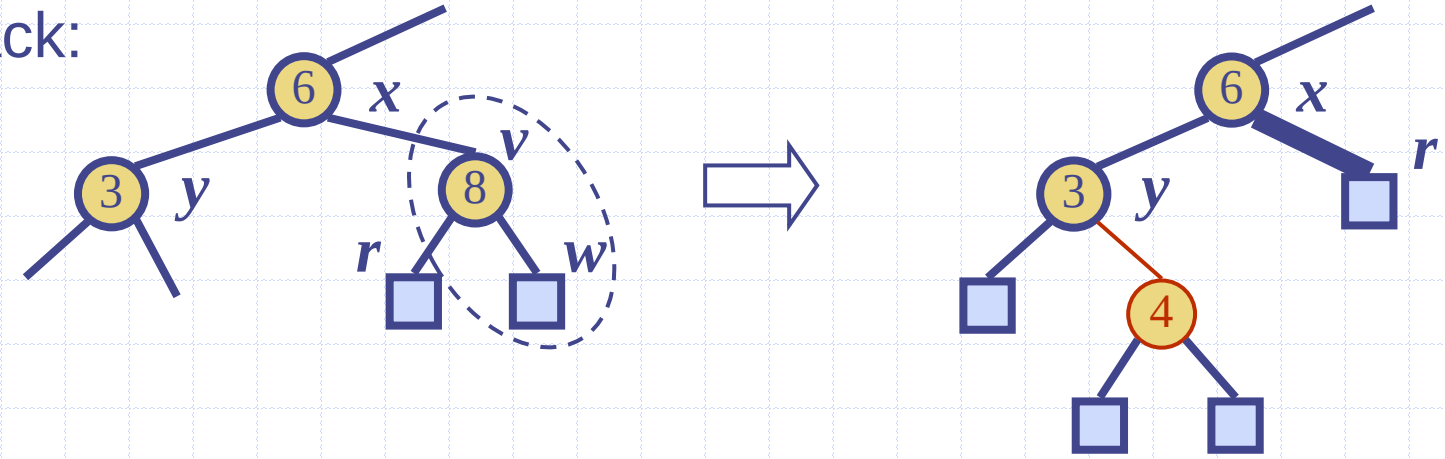

Right
Left




Left
Right

# Deletion

- Let $v$ be the internal node removed, $w$ the external node removed, and $r$ the sibling of $w$
  - If either $v$ or $r$ was red, we color $r$ black and we are done
  - Else ($v$ and $r$ were both black), so we color $r$ ***double black*** (a fictitious color), which is a violation of the internal property requiring a reorganization of the tree (denotes underflow)
- For example, the deletion of 8 causes a double black:



# Analysis of Deletion

**Algorithm** ***removeElement*(*k*)**

1. $v \leftarrow$ ***findNode2Remove*(*k*)** Search for key ***k***, then find deletion node ***v*** with external child *w*. Let ***y*=*sibling*(*v*)** and ***r*=*sibling***(w).
2. Remove node ***v*** (removes *v* and *w* and returns ***r***=sibling(w)). If ***v*** and *r* are both black then color ***r*** double black else *r* was red so color *r* black and we're done

3. **while** ***isDoubleBlack*(*r*)**
   - $y \leftarrow$ ***sibling***(*r*)
   - **if** ***isRed*(*y*) then**
     - $y \leftarrow$ ***adjustment***(*y*)
   - **if** ***hasRedChild*(*y*) then //transfer**
     - $r \leftarrow$ ***restructure***(r)
     - **return**
   - **else** {***sibling*(*r*)** has no red child}
     - $r \leftarrow$ ***fusionRecolor*(*r*)**

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most two restructurings taking $O(1)$ time each
- Thus, a deletion in a red-black tree takes $O(\log n)$ time

# Analysis of Deletion (details)

**Algorithm** *findNode2Remove*(*k*)
**Input** returns node *r* containing key k, node r has at least one external node
    *v* ← *findPosition*(*k*, *root*() )
    *r* ← *v*
    **if** *isInternal*(*leftChild*(*v*)) ∧ *isInternal*(*rightChild*(*v*)) **then** // one child must be external
        *r* ← *findPosition*(*k*, *leftChild*(v))  // finds node containing predecessor of k
        *swapElements*(*v*, *r*)  // swaps items so r is node containing key being deleted
    **return** *r*  // r is the node to be deleted and contains key k unless k is not in tree

**Algorithm** *fusionRecolor*(*y*, *p*, *r*)
**Input** *r* and *y* are siblings, *p* is their parent
    *setColor*(*y*, *RED*)
    **if** *isRed*(*p*) **then**
        *setColor*(*p*, *BLK*)
    **else**
        *setColor*(*p*, *DOUBLE_BLACK*)
    **if** *isInternal*(*r*) **then**
        *setColor*(*r*, *BLK*)
    **return** *p*

# Analysis of Deletion (details)

```
Algorithm removeDoubleBlack(y, r)
  Input r is the double black node and y is sibling(r)
  if isDoubleBlack(r) then
      if isRed(y) then  {Case 3: when y, the sibling of r, is red}
          y ← adjustment(y)
      p ← parent(y)
      z ← redChildOf(y)
      if isBlack(z) then {Case 1: when y has no red child}
          r ← fusionRecolor(y, p, r)
          if isRoot(r) then
              setColor(r, BLK)
          else
              removeDoubleBlack(sibling(r), r)  // recursive call
      else  {Case 2: y has a red child z, so we do a transfer}
          restructure(z)
          setColor(parent(p), color(p))
          setColor(p, BLK)
          setColor(z, BLK)
          if isInternal(r) then
              setColor(r, BLK) // make sure r is not external/null
```

# Main Point

2. Restoring balance after insertion or deletion in a red-black tree only requires a constant number of trinode restructurings (0, 1, or 2) and at most O(log n) recolorings. The red-black tree is slightly more complicated than a (2,4) tree because of restructuring, but has a major advantage in space requirements and simplifies splitting and fusion of nodes.

   *Science of Consciousness:* The TM technique is a simple, effortless technique that restructures the physiology to a more balanced state.

# Red-Black Tree Reorganization

| **Insertion** | remedy double red | |
|---|---|---|
| **Red-black tree action** | **(2,4) tree action** | **result** |
| restructuring | correcting of 4-node representation | double red removed |
| recoloring | split | double red removed or propagated up |

| **Deletion** | remedy double black | |
|---|---|---|
| **Red-black tree action** | **(2,4) tree action** | **result** |
| restructuring | transfer | double black removed |
| recoloring | fusion | double black removed or propagated up |
| adjustment | change of 3-node representation | restructuring or recoloring follows |

# Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. A (2, 4) tree offers a simple and effective way of maintaining balance in a dynamic tree structure.

2. A red-black tree offers a refinement of the (2, 4) tree by eliminating data slots and optimizing operations.

3. **Transcendental Consciousness** is the unbounded field of pure order and balance and is the basis of order and balance in creation.
4. **Impulses within Transcendental Consciousness**: The dynamic natural laws within this unbounded field create and maintain the order and balance in creation.
5. **Wholeness moving within itself:** In Unity Consciousness, the diversity of creation is experienced as waves of intelligence, perfectly orderly fluctuations of one's own self-referral consciousness.