Disjoint Sets

COMP 2210 – Dr. Hendrix

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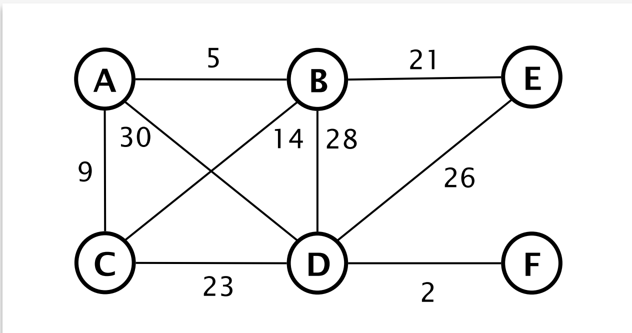
Applications – equivalence classes, connectedness

Kruskall's algorithm



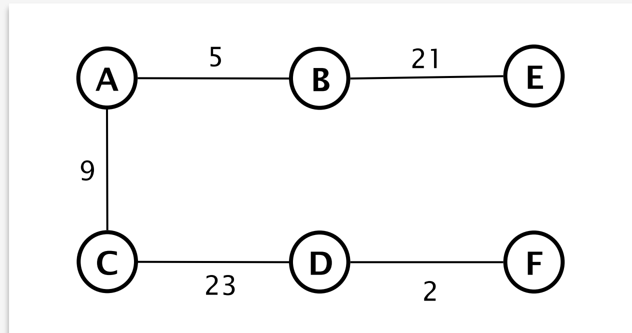
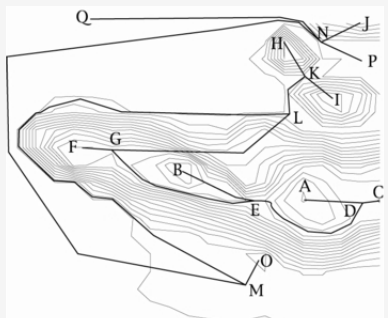
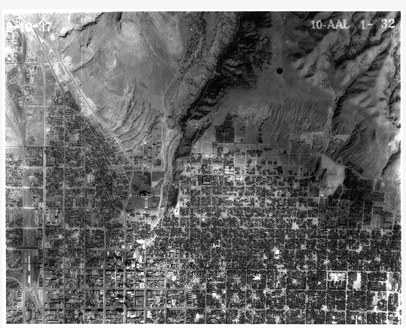
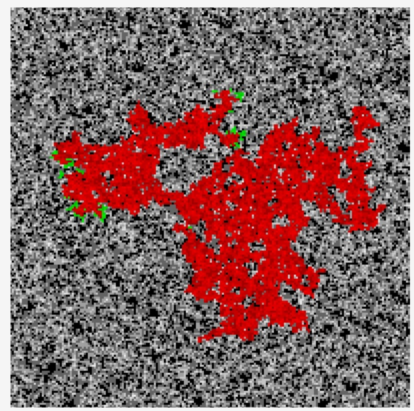


Image processing





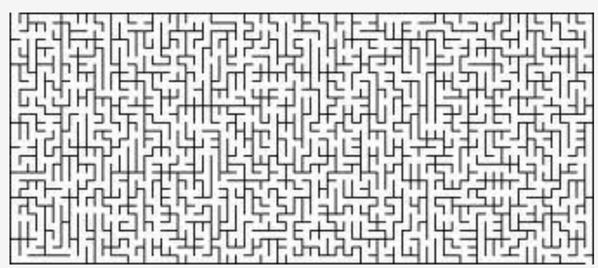
Percolation system modeling



water, oil, etc. through ground

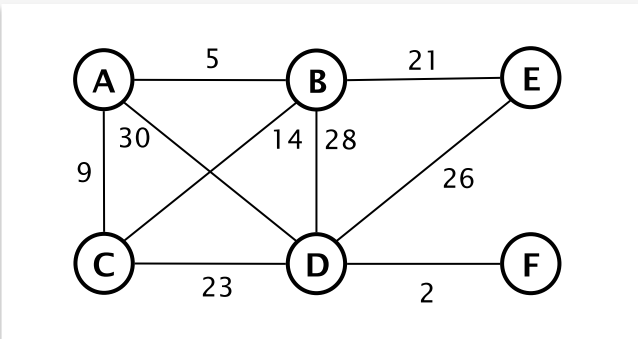
disease through populations

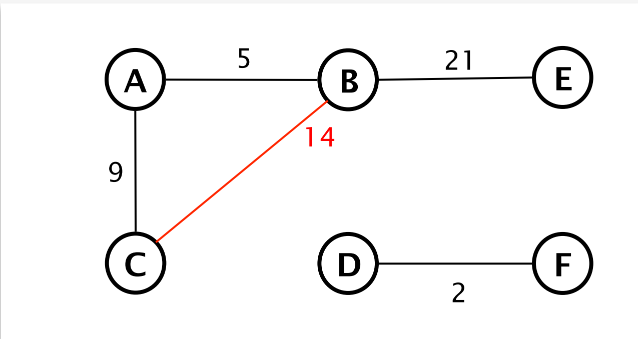
Maze layout, generation



Context – Kruskall's MST algorithm

```
Initialize MST with all vertices but no edges
Add all the edges to a collection E
while (still more edges in E) && (have not added n-1 edges to the MST) {
     Remove the edge with minimum cost from E.
     Add it to the MST if it does not create a cycle.
}
```





Maintain a set of the connected components in the MST.

```
if (find(u) != find(v)) {
   // add edge (u,v) to MST
   union(u, v);
}
```

```
if (!connected(u, v)) {
   // add edge (u,v) to MST
   union(u, v);
}
```

{A, B, C}   {D, F}   {E}

{A, B, C, E}   {D, F}

Disjoint Set

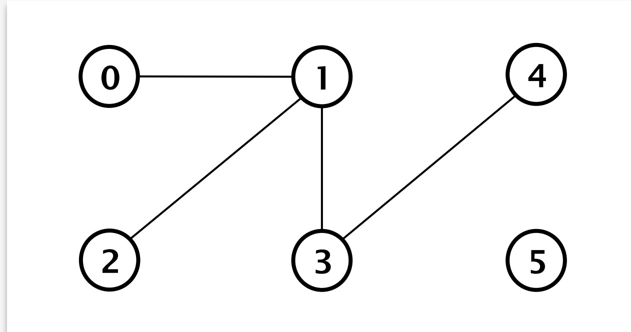
A **disjoint set** is a collection that contains a set of elements that are partitioned into disjoint (non-overlapping) subsets.

```
public interface DisjointSet {

   /**
    * combine components containing p and q
    */
   void union(int p, int q);

   /**
    * return component id for p
    */
   int find(int p);

   /**
    * are p and q in the same component?
    */
   boolean connected(int p, int q);

   /**
    * return number of connected components
    */
   int count();
}
```

Typical application problems involve N elements that begin as individual disjoint sets, and the problem solution involves a sequence of union and find operations.

Disjoint Set operations

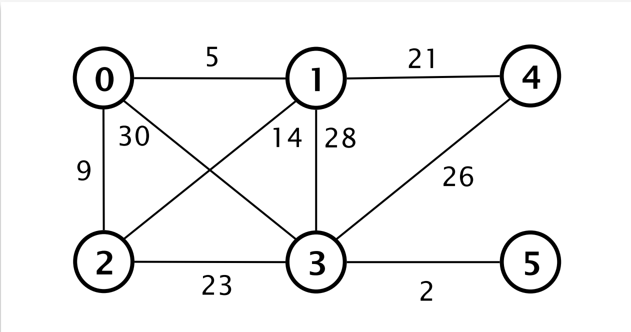
`count()`	6
`connected(3, 5)`	false
`connected(0, 4)`	false
`union(0, 1)`	
`union(3, 4)`	
`count()`	4
`connected(0, 4)`	false
`union(1, 3)`	
`connected(0, 4)`	true
`union(2, 1)`	
`count()`	2



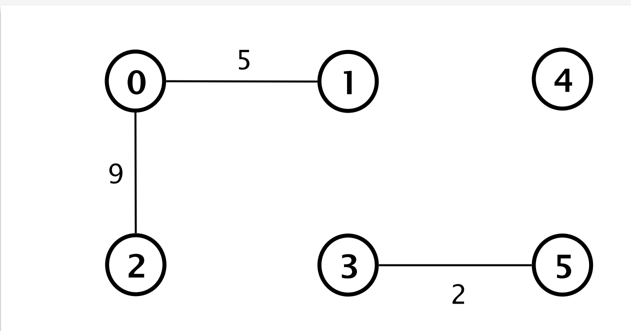

```
{0}   {1}   {2}   {3}   {4}   {5}
{0, 1}   {2}   {3}   {4}   {5}
{0, 1}   {2}   {3, 4}   {5}
{0, 1, 3, 4}   {2}   {5}
{0, 1, 2, 3, 4}   {5}
```

Kruskall's MST algorithm with disjoint sets

Original graph



Minimum Spanning Tree



Disjoint Set of MST components

`{0} {1} {2} {3} {4} {5}`

`connected(3, 5)` *false, ok to add*

`add (3, 5) to MST`

`union(3, 5)`

`{0} {1} {2} {3, 5} {4}`

`connected(0, 1)` *false, ok to add*

`add (0, 1) to MST`

`union(0, 1)`

`{0, 1} {2} {3, 5} {4}`

`connected(0, 2)` *false, ok to add*

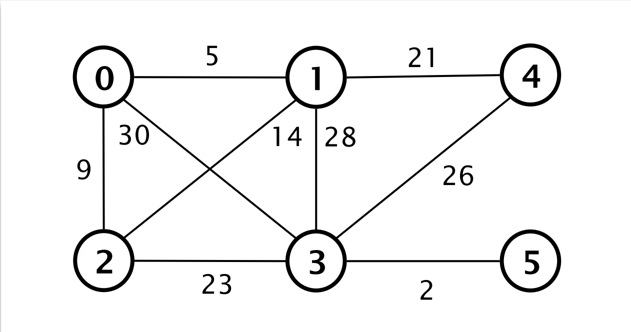
`add (0, 2) to MST`

`union(0, 2)`

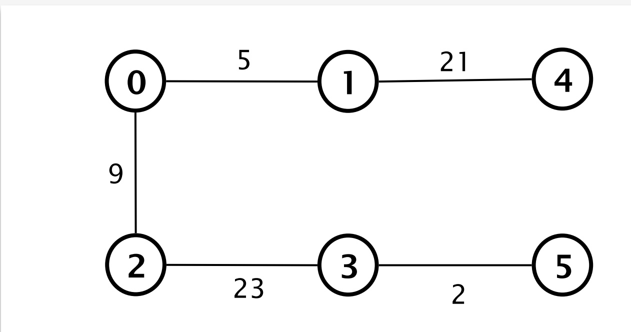
`{0, 1, 2} {3, 5} {4}`

Kruskall's MST algorithm with disjoint sets (cont.)

Original graph



Minimum Spanning Tree



Disjoint Set of MST components

`{0, 1, 2}  {3, 5}  {4}`

`connected(1, 2)` *true, adding would create cycle*

`connected(1, 4)` *false, ok to add*

`add (1, 4) to MST`

`union(1, 4)`

`{0, 1, 2, 4}  {3, 5}`

`connected(2, 3)` *false, ok to add*

`add (2, 3) to MST`

`union(2, 3)`

`{0, 1, 2, 3, 4, 5}`

Disjoint Set

A **disjoint set** is a collection that contains a set of elements that are partitioned into disjoint (non-overlapping) subsets.

```
public interface DisjointSet {

   /**
    * combine components containing p and q
    */
   void union(int p, int q);

   /**
    * return component id for p
    */
   int find(int p);

   /**
    * are p and q in the same component?
    */
   boolean connected(int p, int q);

   /**
    * return number of connected components
    */
   int count();
}
```

Typical application problems involve N elements that begin as individual disjoint sets, and the problem solution involves a sequence of union and find operations.

Implementation strategies must consider the cost of a *sequence* of operations, not just the individual operations themselves.

Disjoint Set – fast find/connected

Fast find strategy: Let each component be identified by the label of one vertex in that component. Store these “component ids” in an int array such that id[i] == component id of element i.

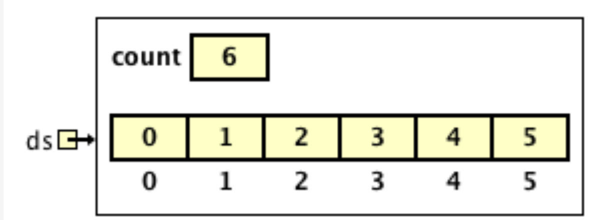
Initially:

{0} {1} {2} {3} {4} {5}

0	1	2	3	4	5
0	1	2	3	4	5

```
public class FastFindDS implements DisjointSet {

   private int[] id;  // component ids
   private int count; // number of components

   public FastFindDS(int N) {
      count = N;
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }
```

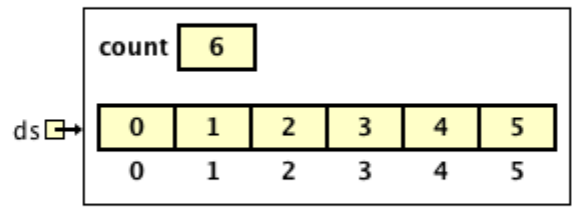
```
FastFindDS ds;

ds = new FastFindDS(6);
```



Disjoint Set – fast find/connected

```
public class FastFindDS implements DisjointSet {

   private int[] id;  // component ids
   private int count; // number of components

   /**
    * return component for p
    */
   public int find(int p) {
      return id[p];
   }

   /**
    * are p and q in the same component?
    */
   public boolean connected(int p, int q) {
      return find(p) == find(q);
   }

   /**
    * return number of connected components
    */
   public int count() {
      return count;
   }
```

OR: `return id[p] == id[q];`



Disjoint Set – fast find/connected

Fast find strategy: Let each component be identified by the label of one vertex in that component. Store these “component ids” in an int array such that id[i] == component id of element i.

`{0} {1} {2} {3} {4} {5}`

0	1	2	3	4	5
0	1	2	3	4	5

`union(3, 5)` `{0} {1} {2} {3, 5} {4}`

0	1	2	5	4	5
0	1	2	3	4	5

`union(0, 2)` `{0, 2} {1} {3, 5} {4}`

2	1	2	5	4	5
0	1	2	3	4	5

`union(1, 0)` `{0, 1, 2} {3, 5} {4}`

2	2	2	5	4	5
0	1	2	3	4	5

Disjoint Set – fast find/connected

Fast find strategy: Let each component be identified by the label of one vertex in that component. Store these “component ids” in an int array such that id[i] == component id of element i.

```
              {0}  {1}  {2}  {3}  {4}  {5}
union(3, 5)   {0}  {1}  {2}  {3, 5}  {4}
union(0, 2)   {0, 2}  {1}  {3, 5}  {4}
union(1, 0)   {0, 1, 2}  {3, 5}  {4}
```

0	1	2	3	4	5
0	1	2	3	4	5

0	1	2	5	4	5
0	1	2	3	4	5

2	1	2	5	4	5
0	1	2	3	4	5

2	2	2	5	4	5
0	1	2	3	4	5

Parameter order can’t matter, so union isn’t as efficient as the last example might have implied.

Disjoint Set – fast find/connected

Fast find strategy: Let each component be identified by the label of one vertex in that component. Store these "component ids" in an int array such that id[i] == component id of element i.

`{0}  {1}  {2}  {3}  {4}  {5}`

0	1	2	3	4	5
0	1	2	3	4	5

`union(3, 5)`

`{0}  {1}  {2}  {3, 5}  {4}`

0	1	2	5	4	5
0	1	2	3	4	5

`union(0, 2)`

`{0, 2}  {1}  {3, 5}  {4}`

2	1	2	5	4	5
0	1	2	3	4	5

`union(0, 1)`

`{0, 1, 2}  {3, 5}  {4}`

1	1	1	5	4	5
0	1	2	3	4	5

2	2	2	5	4	5
0	1	2	3	4	5

1	1	1	5	4	5
0	1	2	3	4	5

→ `{0, 1, 2}  {3, 5}  {4}`

Disjoint Set – fast find/connected

```
public class FastFindDS implements DisjointSet {

   private int[] id;  // component ids
   private int count; // number of components

   /**
    * combine components containing p and q
    */
   public void union(int p, int q) {
      int pid = find(p);
      int qid = find(q);
      if (pid == qid)    return;

      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
      count--;
}
```

Could be shortened with connected(p, q). Expressed this way for consistency among different implementations.

Iterates over the entire array and changes every pid to qid.

`FastFindDS ds = new FastFindDS(6);`

index	0	1	2	3	4	5
id	0	1	2	3	4	5

`ds.union(0, 1);` id[0] == 0, id[1] == 1

index	0	1	2	3	4	5
before	0	1	2	3	4	5
after	1	1	2	3	4	5

`ds.union(3, 4);` id[3] == 3, id[4] == 4

index	0	1	2	3	4	5
before	1	1	2	3	4	5
after	1	1	2	4	4	5

`ds.union(1, 3);` id[1] == 1, id[3] == 4

index	0	1	2	3	4	5
before	1	1	2	4	4	5
after	4	4	2	4	4	5

`ds.union(2, 1);` id[2] == 2, id[1] == 4

index	0	1	2	3	4	5
before	4	4	2	4	4	5
after	4	4	4	4	4	5

Disjoint Set – fast find/connected

Before / **After**

union(5, 6)

Before:

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

After:

0	1	2	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

union(2, 9)

Before:

0	1	2	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

After:

0	1	9	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

union(4, 9)

Before:

0	1	9	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

After:

0	1	9	3	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

union(3, 4)

Before:

0	1	9	3	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

After:

0	1	9	9	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

union(9, 6)

Before:

0	1	9	9	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

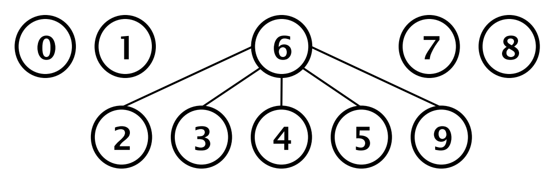
After:

0	1	6	6	6	6	6	7	8	6
0	1	2	3	4	5	6	7	8	9

Final

0	1	6	6	6	6	6	7	8	6
0	1	2	3	4	5	6	7	8	9

{0} {1} {2, 3, 4, 5, 6, 9} {7} {8}



Disjoint Set – fast find/connected

Fast find strategy: Let each component be identified by the label of one vertex in that component. Store these "component ids" in an int array such that id[i] == component id of element i.

Advantage: find, connected, and count are fast (and trivial to write)

Disadvantage: union will access every array element each time it's called.

Cost of find

O(1) per find

O(N) for a sequence of N finds

Cost of union

O(N) per union

O(N^2) for a sequence of N unions

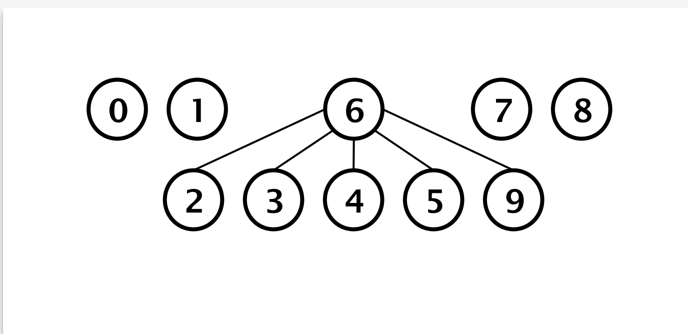
Disjoint Set – fast union

Fast union strategy: Let each component be identified by the label of one vertex in that component, its “root”. Store these “component ids” in an int array such that id[i] == the “parent” of element i. The i for which id[i] == i is the component root.

```
{0} {1} {2, 3, 4, 5, 6, 9} {7} {8}
```

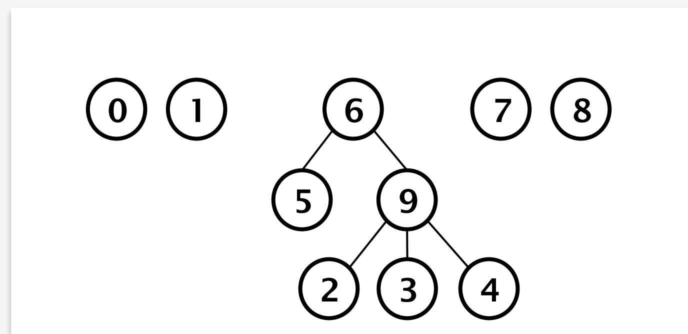
Fast find

0	1	6	6	6	6	6	7	8	6
0	1	2	3	4	5	6	7	8	9



Fast union

0	1	9	9	9	6	6	7	8	6
0	1	2	3	4	5	6	7	8	9



Disjoint Set – fast union

Fast union strategy: Let each component be identified by the label of one vertex in that component, its "root". Store these "component ids" in an int array such that id[i] == the "parent" of element i. The i for which id[i] == i is the component root.

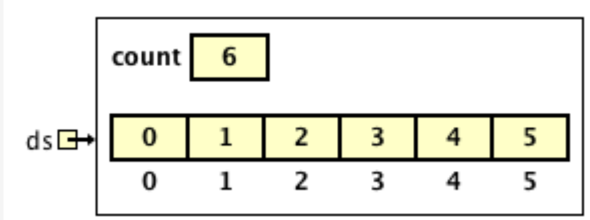
Initially:

```
{0}  {1}  {2}  {3}  {4}  {5}
```

0	1	2	3	4	5
0	1	2	3	4	5

```
public class FastUnionDS implements DisjointSet {

   private int[] id;  // component ids
   private int count; // number of components

   public FastUnionDS(int N) {
      count = N;
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }
```

```
FastUnionDS ds;

ds = new FastUnionDS(6);
```



Disjoint Set – fast union

Fast union strategy: Let each component be identified by the label of one vertex in that component, its "root". Store these "component ids" in an int array such that id[i] == the "parent" of element i. The i for which id[i] == i is the component root.

`{0} {1} {2} {3} {4} {5}`

0	1	2	3	4	5
0	1	2	3	4	5

For each union(x, y), set id[root(x)] to root(y).

`union(3, 5)` `{0} {1} {2} {3, 5} {4}`

0	1	2	5	4	5
0	1	2	3	4	5

`union(0, 2)` `{0, 2} {1} {3, 5} {4}`

2	1	2	5	4	5
0	1	2	3	4	5

`union(1, 0)` `{0, 1, 2} {3, 5} {4}`

2	2	2	5	4	5
0	1	2	3	4	5

Disjoint Set – fast union

Before | | **After**

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

`union(5, 6)`

0	1	2	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

0	1	2	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

`union(2, 9)`

0	1	9	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

0	1	9	3	4	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

`union(4, 9)`

0	1	9	3	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

0	1	9	3	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

`union(3, 4)`

0	1	9	9	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

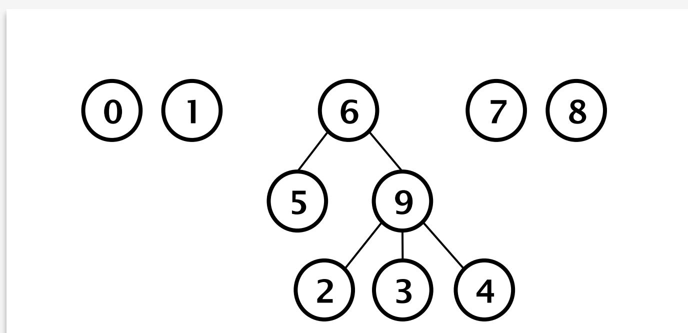
0	1	9	9	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9

`union(9, 6)`

0	1	9	9	9	6	6	7	8	6
0	1	2	3	4	5	6	7	8	9

Final

0	1	9	9	9	6	6	7	8	6
0	1	2	3	4	5	6	7	8	9



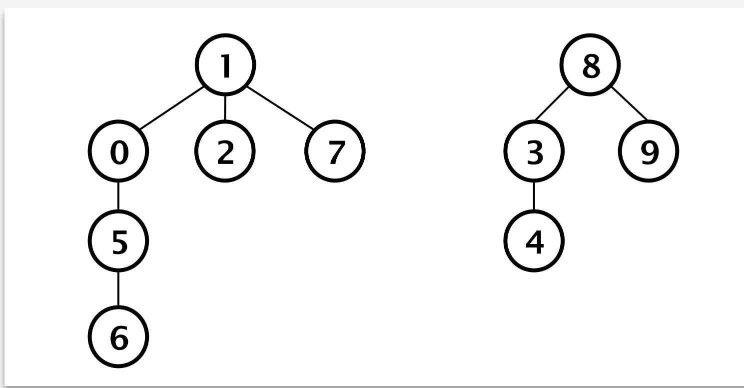
Disjoint Set – fast union

```java
public class FastUnionDS implements DisjointSet {

   private int[] id;  // component ids
   private int count; // number of components

   /**
    * return component for p
    */
   public int find(int p) {
      while (p != id[p])
         p = id[p];
      return p;
   }
```

Must traverse parents until the root is found

id[]	1	1	1	8	3	0	5	1	8	8
	0	1	2	3	4	5	6	7	8	9

`find(8) == 8`   `find(9) == 8`   `find(4) == 8`   `find(6) == 1`



Disjoint Set – fast union

```
public class FastUnionDS implements DisjointSet {

   private int[] id;  // component ids
   private int count; // number of components

   /**
    * combine components containing p and q
    */
   public void union(int p, int q) {
      int pr = find(p);
      int qr = find(q);
      if (pr == qr) return;

      id[pr] = qr;
      count--;
   }
```

Exactly the same as in the fast find version, **except** that find() returns the component root.

Updates only one id per call. The root of p is set to the root of q

`FastUnionDS ds = new FastUnionDS(6);`

0	1	2	3	4	5
0	1	2	3	4	5

`ds.union(0, 1);` r(0) == 0, r(1) == 1

0	1	2	3	4	5
0	1	2	3	4	5

0	1	2	3	4	5
1	1	2	3	4	5

`ds.union(3, 4);` r(3) == 3, r(4) == 4

0	1	2	3	4	5
1	1	2	3	4	5

0	1	2	3	4	5
1	1	2	4	4	5

`ds.union(1, 3);` r(1) == 1, r(3) == 4

0	1	2	3	4	5
1	1	2	4	4	5

0	1	2	3	4	5
1	4	2	4	4	5

`ds.union(2, 1);` r(2) == 2, r(1) == 4

0	1	2	3	4	5
1	4	2	4	4	5

0	1	2	3	4	5
1	4	4	4	4	5

Disjoint Set – fast union

```
FastUnionDS ds = new FastUnionDS(10);
```

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(4, 3)
```

0	1	2	3	3	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

0	1	2	3	3	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(3, 8)
```

0	1	2	8	3	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

0	1	2	8	3	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(6, 5)
```

0	1	2	8	3	5	5	7	8	9
0	1	2	3	4	5	6	7	8	9

0	1	2	8	3	5	5	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(9, 4)
```

0	1	2	8	3	5	5	7	8	8
0	1	2	3	4	5	6	7	8	9

0	1	2	8	3	5	5	7	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(2, 1)
```

0	1	1	8	3	5	5	7	8	8
0	1	2	3	4	5	6	7	8	9

0	1	1	8	3	5	5	7	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(5, 0)
```

0	1	1	8	3	0	5	7	8	8
0	1	2	3	4	5	6	7	8	9

0	1	1	8	3	0	5	7	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(7, 2)
```

0	1	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

0	1	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(6, 1)
```

1	1	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

1	1	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(7, 3)
```

1	8	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

Disjoint Set – fast union

```java
public class FastUnionDS implements DisjointSet {

   private int[] id;  // component ids
   private int count; // number of components

   /**
    * return component for p
    */
   public int find(int p) {
      while (p != id[p])
         p = id[p];
      return p;
   }

   /**
    * are p and q in the same component?
    */
   public boolean connected(int p, int q) {
      return find(p) == find(q);
   }

   /**
    * return number of connected components
    */
   public int count() {
      return count;
   }
```

Must traverse parents until the root is found

No change – exactly the same as before

Disjoint Set – fast union

Fast union strategy: Let each component be identified by the label of one vertex in that component, its “root”. Store these “component ids” in an int array such that id[i] == the “parent” of element i. The i for which id[i] == i is the component root.

Advantage: Union updates less on average

Disadvantage: Find and connected no longer constant; union has linear worst case

Cost of find

O(N) per find

O(N^2) for a sequence of N finds

Cost of union

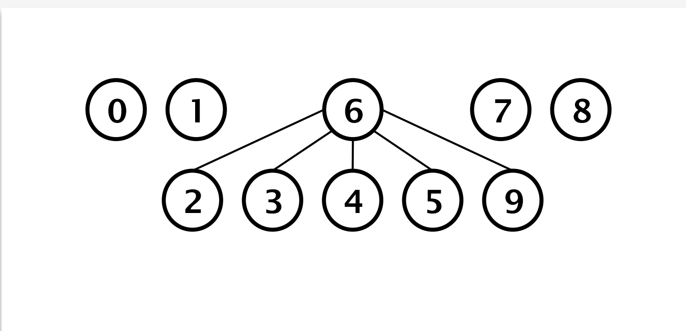
O(N) per union

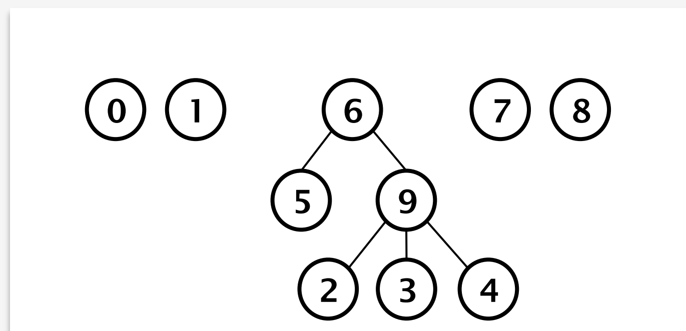
O(N^2) for a sequence of N unions

Comparison of unions

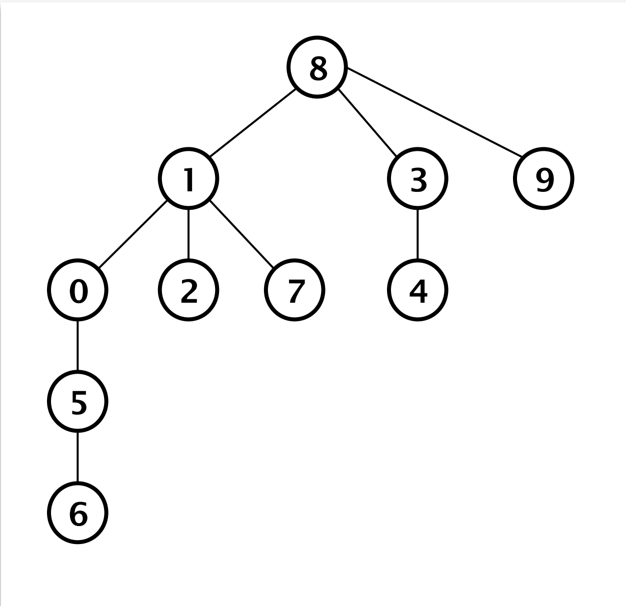
FastFindDS		FastUnionDS
0 1 2 3 4 5 6 7 8 9	`union(5, 6)`	0 1 2 3 4 5 6 7 8 9
0 1 2 3 4 6 6 7 8 9	`union(2, 9)`	0 1 2 3 4 6 6 7 8 9
0 1 9 3 4 6 6 7 8 9	`union(4, 9)`	0 1 9 3 4 6 6 7 8 9
0 1 9 3 9 6 6 7 8 9	`union(3, 4)`	0 1 9 3 9 6 6 7 8 9
0 1 9 9 9 6 6 7 8 9	`union(9, 6)`	0 1 9 9 9 6 6 7 8 9
0 1 6 6 6 6 6 7 8 6		0 1 9 9 9 6 6 7 8 6

(Array indices below each row: 0 1 2 3 4 5 6 7 8 9)





Fast union allows tall trees



1	8	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

```
FastUnionDS ds = new FastUnionDS(10);
```

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(4, 3)
```

0	1	2	3	3	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(3, 8)
```

0	1	2	8	3	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(6, 5)
```

0	1	2	8	3	5	5	7	8	9
0	1	2	3	4	5	6	7	8	9

```
ds.union(9, 4)
```

0	1	2	8	3	5	5	7	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(2, 1)
```

0	1	1	8	3	5	5	7	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(5, 0)
```

0	1	1	8	3	0	5	7	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(7, 2)
```

0	1	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

```
ds.union(6, 1)
```

1	1	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

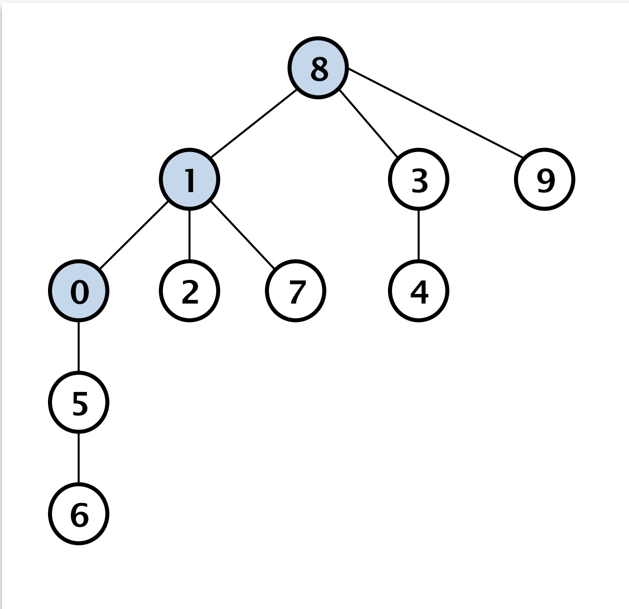
```
ds.union(7, 3)
```

1	8	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

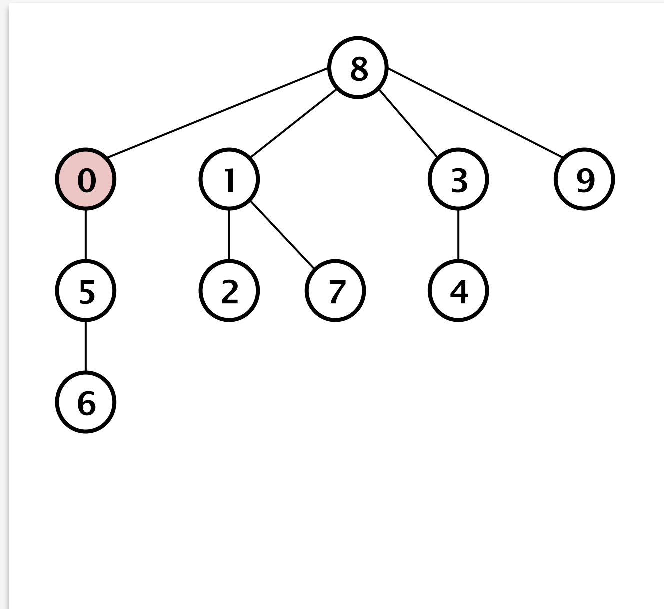
Disjoint Set – fast union with path compression

Each time we find the root of an element, replace the parent value of each examined element with the root value.

`find(0) == 8`



1	8	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

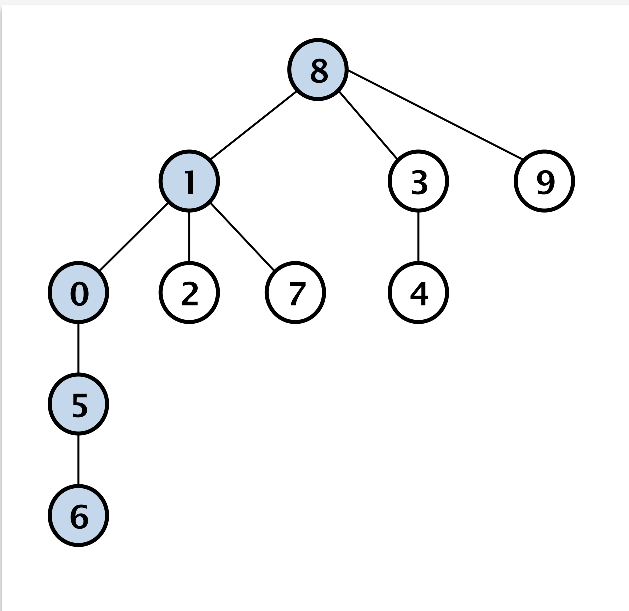


8	8	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9

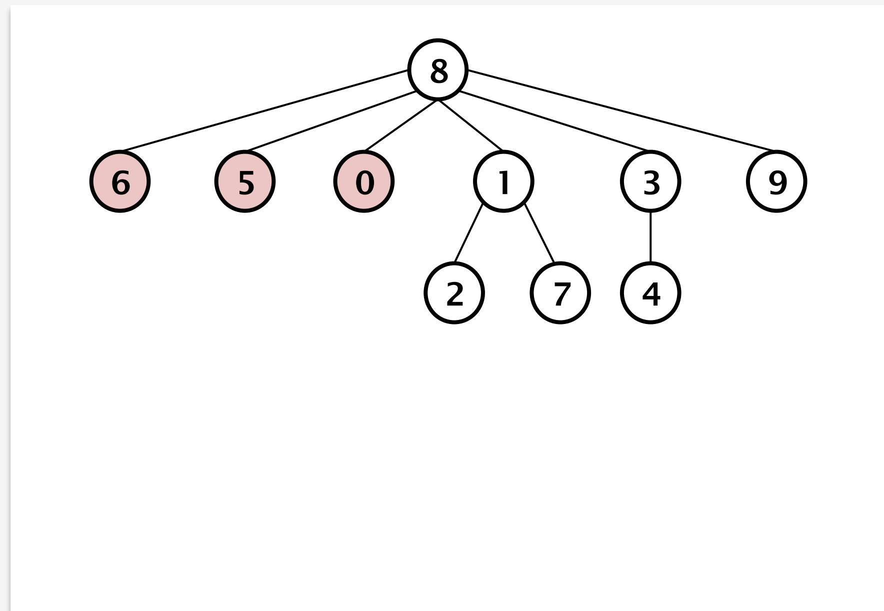
Disjoint Set – fast union with path compression

Each time we find the root of an element, replace the parent value of each examined element with the root value.

```
find(6) == 8
```



0	1	2	3	4	5	6	7	8	9
1	8	1	8	3	0	5	1	8	8



0	1	2	3	4	5	6	7	8	9
8	8	1	8	3	8	8	1	8	8

Disjoint Set – fast union with path compression

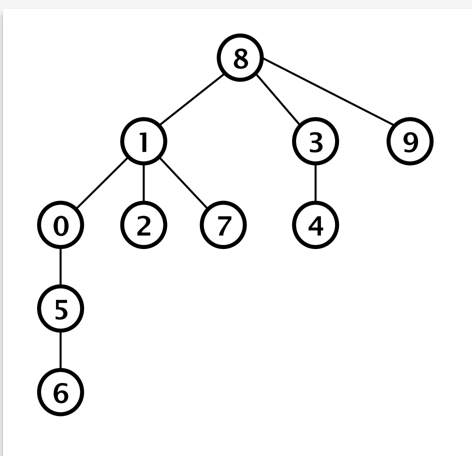
```
public class FastUnionDS implements DisjointSet {

   public int find(int p) {
      int root = p;
      while (root != id[root])
         root = id[root];

      while (p != root) {
         int newp = id[p];
         id[p] = root;
         p = newp;
      }
      return root;   }
```

Find the root of p.

Compress the path from p to root.

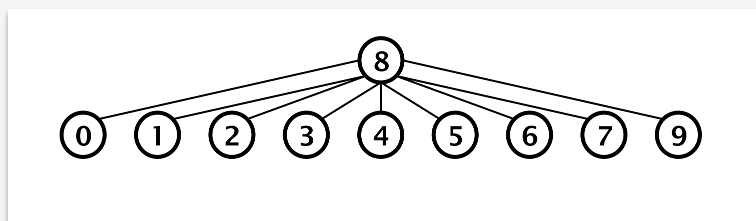
Without path compression:

1	8	1	8	3	0	5	1	8	8
0	1	2	3	4	5	6	7	8	9



With path compression:

8	8	8	8	8	8	8	8	8	8
0	1	2	3	4	5	6	7	8	9



Disjoint Set – fast union

Second implementation strategy: Let each component be identified by the label of one vertex in that component, its "root". Store these "component ids" in an int array such that id[i] == the "parent" of element i. The i for which id[i] == i is the component root.

Each time we find the root of an element, replace the parent value of each examined element with the root value.

Cost of find

O(log N) per find

O(N log N) for a sequence of N finds

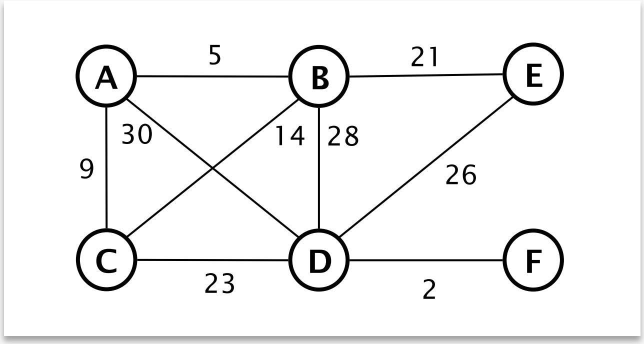
Cost of union

O(log N) per union

O(N log N) for a sequence of N unions

Applications

Kruskall's algorithm



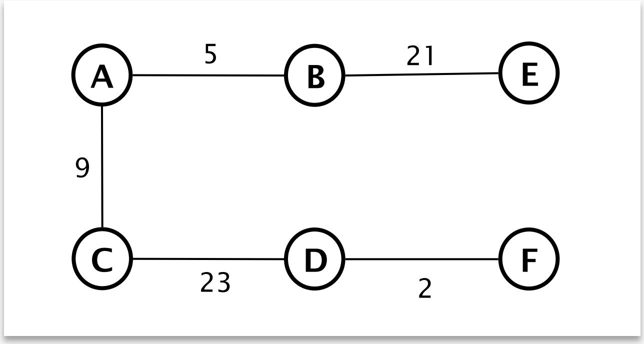
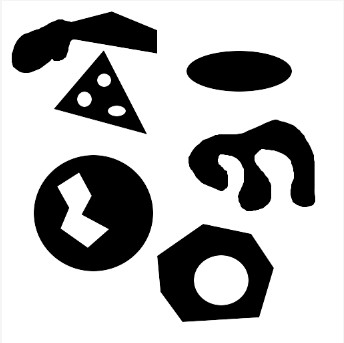
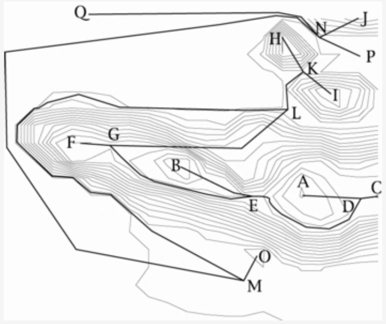
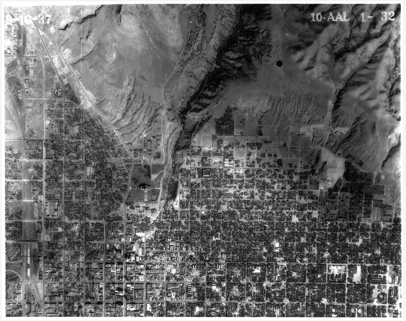


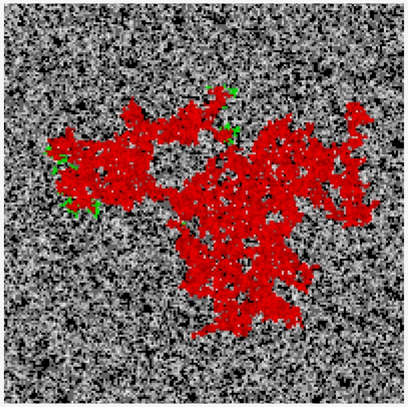
Image processing







Percolation system modeling



water, oil, etc. through ground

disease through populations