Information and Signal Theory

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3.2 Discrete Time Convolution

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Summary

Introduction

Convolution, one of the most important concepts in electrical engineering, can be used to determine the output a system produces for a given input signal. It can be shown that a linear time invariant system is completely characterized by its impulse response. The sifting property of the discrete time impulse function tells us that the input signal to a system can be represented as a sum of scaled and shifted unit impulses. Thus, by linearity, it would seem reasonable to compute of the output signal as the sum of scaled and shifted unit impulse responses. That is exactly what the operation of convolution accomplishes. Hence, convolution can be used to determine a linear time invariant system's output from knowledge of the input and the impulse response.

Convolution and Circular Convolution

Convolution

Operation Definition

Discrete time convolution is an operation on two discrete time signals defined by the integral

$$(f * g)[n] = \sum_{k=-\infty}^{\infty} f[k]\, g[n-k]$$

for all signals f, g defined on $\mathbb{Z}$. It is important to note that the operation of convolution is commutative, meaning that

$$f * g = g * f$$

for all signals f, g defined on $\mathbb{Z}$. Thus, the convolution operation could have been just as easily stated using the equivalent definition

$$(f * g)[n] = \sum_{k=-\infty}^{\infty} f[n-k]\, g[k]$$

for all signals f, g defined on $\mathbb{Z}$. Convolution has several other important properties not listed here but explained and derived in a later module.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

by the sifting property of the unit impulse function. By linearity

$$H(x[n]) = \sum_{k=-\infty}^{\infty} x[k]\,H(\delta[n-k]).$$

Since $H(\delta[n-k])$ is the shifted unit impulse response $h[n-k]$, this gives the result

$$H(x[n]) = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = (x*h)[n].$$

Hence, convolution has been defined such that the output of a linear time invariant system is given by the convolution of the system input with the system unit impulse response.

Graphical Intuition

It is often helpful to be able to visualize the computation of a convolution in terms of graphical processes. Consider the convolution of two functions f, g given by

$$(f*g)[n] = \sum_{k=-\infty}^{\infty} f[k]\,g[n-k] = \sum_{k=-\infty}^{\infty} f[n-k]\,g[k].$$

The first step in graphically understanding the operation of convolution is to plot each of the functions. Next, one of the functions must be selected, and its plot reflected across the $k=0$ axis. For each real n, that same function must be shifted left by n. The point-wise product of the two resulting plots is then computed, and then all of the values are summed.

Recall that the impulse response for a discrete time echoing feedback system with gain a is

$$h[n] = a^n u[n],$$

and consider the response to an input signal that is another exponential

$$x[n] = b^n u[n].$$

We know that the output for this input is given by the convolution of the impulse response with the input signal