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ISYE 4133 Assignment 2

1

For inventory pile $i = 1, \ldots, 8$ and customers $k = 1, \ldots, 25$
x_{ik} : tons of phosphate rock from inventory i for customer k
b_i : average BPL content per ton for pile i
a_i : asset value per ton for pile i
r_{ik} : contract net profit per ton for pile i and customer k
h_i : starting inventory for pile i
q_i : expected quantity to arrive for pile i

$$\text{Maximize } \sum_i \left(\sum_k \left[r_{ik} x_{ik} + a_i (h_i - x_{ik} + q_i) \right] \right)$$

$$\text{Subject To} \quad \underline{s_k} \leq \sum_i x_{ik} \leq \overline{s_k}$$

$$\text{If } b_i \geq \overline{p_k}, \text{ then } x_{ik} = 0$$

$$\text{If } b_i \leq \underline{p_k}, \text{ then } x_{ik} = 0$$

$$\sum_k x_{ik} \leq h_i + q_i$$

2

(a)

For $i = 1, \ldots, 7$ and $j = 1, \ldots, 5$
x_{ij} : number of hearts produced of size i from supplier j
c_j : unit cost of hearts produced by supplier j
a_j : maximum quantity availiable from supplier j
r_j : number of required hearts of size i
f_{ij} : fraction of hearts from supplier j and size i

$$\text{Minimize } \sum_j \left(c_j \sum_i x_{ij} \right)$$

$$\text{Subject To } \quad x_{ij} \le f_{ij} a_j$$

$$\sum_j x_{ij} \ge r_i$$

(b)

In [16]:
```
from gurobipy import *

def printOptimal(m):
    if m.Status == GRB.OPTIMAL:
        print('Variable Values:')
        for v in m.getVars():
            print(v.VarName, v.X)

        print('\nObjective Value: {}\n'.format(str(m.objVal)))
        print('\nRuntime: {}\n'.format(m.Runtime))
```

In [132]:
```
import numpy as np
f = np.array([[.4,.1,0,0,0],[.4,.2,0,0,0],[.2,.3,.4,.2,0],[0,.2,.3,.2,
0],[0,.2,.3,.2,.2],[0,0,0,.2,.3],[0,0,0,.2,.5]])
r = np.array([20,30,120,200,150,60,45])
c = np.array([2.5,3.2,3.0,2.1,3.9])
a = np.array([500,330,150,650,300])
```

In [140]:
```
m = Model()

X = m.addVars(7,5, name = 'X', lb = 0)

m.setObjective(np.sum(c[j]*X.sum('*',j) for j in range(5)), GRB.MINIMI
ZE)

m.addConstrs(X[i,j] <= f[i,j]*a[j] for i in range(7) for j in range(5)
)

m.addConstrs(X.sum(i,'*') >= r[i] for i in range(7))

m.optimize()
printOptimal(m)
```

```
Optimize a model with 42 rows, 35 columns and 70 nonzeros
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [2e+00, 4e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [2e+01, 2e+02]
Presolve removed 36 rows and 19 columns
Presolve time: 0.01s
Presolved: 6 rows, 16 columns, 16 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    4.8800000e+02   5.312500e+01   0.000000e+00      0s
       6    1.4185000e+03   0.000000e+00   0.000000e+00      0s

Solved in 6 iterations and 0.01 seconds
Optimal objective  1.418500000e+03
Variable Values:
X[0,0] 20.0
X[0,1] 0.0
X[0,2] 0.0
X[0,3] 0.0
X[0,4] 0.0
X[1,0] 30.0
X[1,1] 0.0
X[1,2] 0.0
X[1,3] 0.0
X[1,4] 0.0
X[2,0] 0.0
X[2,1] 0.0
X[2,2] 0.0
X[2,3] 120.0
X[2,4] 0.0
X[3,0] 0.0
X[3,1] 25.0
```

```
X[3,2] 45.0
X[3,3] 130.0
X[3,4] 0.0
X[4,0] 0.0
X[4,1] 0.0
X[4,2] 20.0
X[4,3] 130.0
X[4,4] 0.0
X[5,0] 0.0
X[5,1] 0.0
X[5,2] 0.0
X[5,3] 60.0
X[5,4] 0.0
X[6,0] 0.0
X[6,1] 0.0
X[6,2] 0.0
X[6,3] 45.0
X[6,4] 0.0

Objective Value: 1418.5


Runtime: 0.013051033020019531


/Users/Karen_Loscocco/miniconda3/lib/python3.7/site-packages/ipykern
el_launcher.py:5: DeprecationWarning: Calling np.sum(generator) is d
eprecated, and in the future will give a different result. Use np.su
m(np.fromiter(generator)) or the python sum builtin instead.
  """
```

(c)

In [138]:

```
m_integer = Model()

X = m_integer.addVars(7,5, name = 'X',vtype = GRB.INTEGER, lb = 0)

m_integer.setObjective(np.sum(c[j]*X.sum('*',j) for j in range(5)), GR
B.MINIMIZE)

m_integer.addConstrs(X[i,j] <= f[i,j]*a[j] for i in range(7) for j in
range(5))

m_integer.addConstrs(X.sum(i,'*') >= r[i] for i in range(7))

m_integer.optimize()
printOptimal(m_integer)
```

```
Optimize a model with 42 rows, 35 columns and 70 nonzeros
Variable types: 0 continuous, 35 integer (0 binary)
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [2e+00, 4e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [2e+01, 2e+02]
Found heuristic solution: objective 1598.7000000
Presolve removed 41 rows and 31 columns
Presolve time: 0.00s
Presolved: 1 rows, 4 columns, 4 nonzeros
Found heuristic solution: objective 1511.6000000
Variable types: 0 continuous, 4 integer (0 binary)

Root relaxation: objective 1.418500e+03, 1 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |
Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/N
ode Time

*    0     0               0    1418.5000000 1418.50000  0.00%     -
0s

Explored 0 nodes (1 simplex iterations) in 0.01 seconds
Thread count was 8 (of 8 available processors)

Solution count 3: 1418.5 1511.6 1513.6

Optimal solution found (tolerance 1.00e-04)
Best objective 1.418500000000e+03, best bound 1.418500000000e+03, ga
p 0.0000%
Variable Values:
X[0,0] 20.0
X[0,1] -0.0
X[0,2] 0.0
X[0,3] 0.0
X[0,4] 0.0
X[1,0] 30.0
X[1,1] -0.0
X[1,2] 0.0
X[1,3] 0.0
X[1,4] 0.0
X[2,0] -0.0
X[2,1] -0.0
X[2,2] -0.0
X[2,3] 120.0
X[2,4] 0.0
X[3,0] 0.0
X[3,1] 25.0
```

```
X[3,2] 45.0
X[3,3] 130.0
X[3,4] 0.0
X[4,0] 0.0
X[4,1] -0.0
X[4,2] 20.0
X[4,3] 130.0
X[4,4] -0.0
X[5,0] 0.0
X[5,1] 0.0
X[5,2] 0.0
X[5,3] 60.0
X[5,4] -0.0
X[6,0] 0.0
X[6,1] 0.0
X[6,2] 0.0
X[6,3] 45.0
X[6,4] -0.0

Objective Value: 1418.5


Runtime: 0.01876211166381836


/Users/Karen_Loscocco/miniconda3/lib/python3.7/site-packages/ipykern
el_launcher.py:5: DeprecationWarning: Calling np.sum(generator) is d
eprecated, and in the future will give a different result. Use np.su
m(np.fromiter(generator)) or the python sum builtin instead.
  """
```

The two take about the same amount of time and give the same objective value.

3

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

$$x \geq 0$$

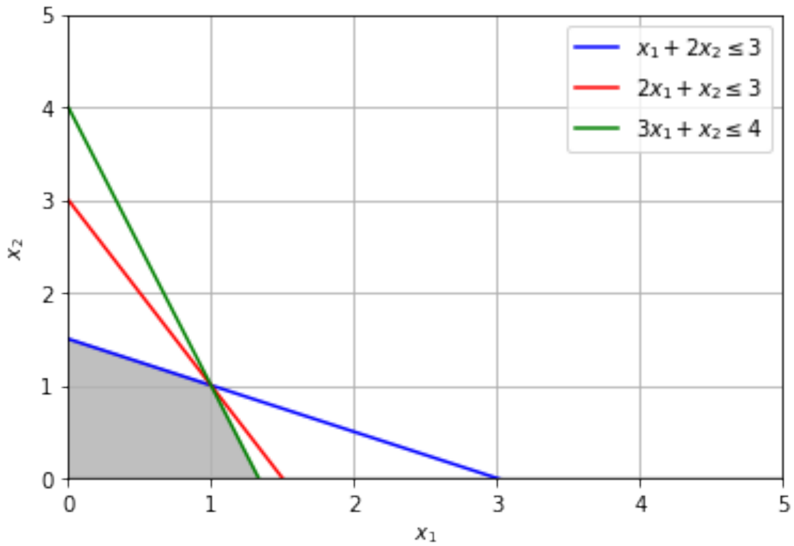
In [37]:
```
import matplotlib.pyplot as plt

x = np.linspace(0,5,100)

plt.plot(x, (3/2)-.5*x, '-b', label=r'$x_1+2x_2\leq 3$')
plt.plot(x, 3- 2*x , '-r', label=r'$2x_1+x_2\leq 3$')
plt.plot(x, 4- 3*x, '-g', label=r'$3x_1+x_2\leq 4$')

plt.xlabel(r'$x_1$')
plt.ylabel(r'$x_2$')
plt.legend()
plt.grid()
plt.xlim(0,5)
plt.ylim(0,5)

plt.fill_between(x ,np.minimum((3/2)-.5*x,4- 3*x), color='grey', alpha
='0.5')
plt.show()
```



Basic solutions: $(0, 0), (\frac{4}{3}, 0), (1, 1), (0, \frac{3}{2})$
Basic feasible solutions: $(0, 0), (\frac{4}{3}, 0), (1, 1), (0, \frac{3}{2})$

4

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 0 & -2 & 1 \\ 1 & 2 & 1 & 5 & 4 & 3 & 3 \end{pmatrix}$$

To test for basic solution (BS), the solution of Ax must equal: $\begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$

To test for basic feasible solution (BFS), the solution must be a basic solution and also satisfy $x \geq 0$.

In [7]:
```
A = np.array([[1,1,0,2,1,1,1],[0,2,2,0,0,-2,1],[1,2,1,5,4,3,3]])
```

(a)

Neither BS or BFS

In [8]:
```
x = np.array([1,1,0,0,0,0,0])
np.dot(A,x)
```

Out[8]:
```
array([2, 2, 3])
```

(b)

Neither BS or BFS

In [9]:
```
x = np.array([2,-1,2,0,0,0,0])
np.dot(A,x)
```

Out[9]:
```
array([1, 2, 2])
```

(c)

BS and BFS

In [10]:
```
x = np.array([1,0,1,0,1,0,0])
np.dot(A,x)
```

Out[10]:
```
array([2, 2, 6])
```

(d)

BS and BFS

In [11]:
```
x = np.array([0,0,1,1,0,0,0])
np.dot(A,x)
```

Out[11]:
```
array([2, 2, 6])
```

(e)

BS and BFS

In [12]:
```
x = np.array([0,.5,0,0,.5,0,1])
np.dot(A,x)
```

Out[12]:
```
array([2., 2., 6.])
```

5

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \end{pmatrix}$$

(a)

The rank of A is 2.

(b)

The following shows a way to find the linearly independent rows of A:

$$A^T = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix}$$

$$RREF(A^T) = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

This means the third column of A^T is linearly dependent:

$$\frac{1}{2}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

The linearly independent columns of A^T are $\begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 1 & 3 \end{pmatrix}$ and therefore, the linearly independent rows of A are

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \end{pmatrix}.$$

Based on the above, I can describe the polyhedron in the following form:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

(c)

$n - m = 3 - 2 = 1$ and so there is one nonbasic variable. In other words, for each basic solution, there is one x_i which equals zero.

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & 3 & 3 & 9 \end{array}\right)$$

In [101]:

```
from sympy import *
Ab = Matrix([[2,1,1,5],[4,3,3,9]])
Ab
```

Out[101]:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & 3 & 3 & 9 \end{bmatrix}$$

Case 1: $x_1 = 0$

In [96]:
```
Ab[:,1:]
```
Out[96]:
$$\begin{bmatrix} 1 & 1 & 5 \\ 3 & 3 & 9 \end{bmatrix}$$

In [97]:
```
Ab[:,1:].rref()[0]
```
Out[97]:
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 = 0$
$x_2 + x_3 = 0$
$0 \neq 1$ so there is no solution to this system.

Case 2: $x_2 = 0$

In [100]:
```
Ab[:,[0,2,3]]
```
Out[100]:
$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 3 & 9 \end{bmatrix}$$

In [99]:
```
Ab[:,[0,2,3]].rref()[0]
```
Out[99]:
$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$x_1 = 3$
$x_2 = 0$
$x_3 = -1$

Case 2: $x_3 = 0$

In [93]:
```
Ab[:,[0,1,3]]
```
Out[93]:
$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 3 & 9 \end{bmatrix}$$

In [94]:
```
Ab[:,[0,1,3]].rref()[0]
```

Out[94]:
$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$x_1 = 3$
$x_2 = -1$
$x_3 = 0$

The basic solutions are the following:
$(x_1, x_2, x_3) = (3, 0, -1), (3, -1, 0)$

There are **no basic feasible solutions** because the basic solutions do not satisfy the condition that $x \geq 0$