Exercises and Assignment 2: ISYE 4133.
Due September 18

Instructions. Exercises are for your practice. You need not submit them. The problems in the Section 2 are to be submitted by the due date. You are supposed to write your own solutions but you can discuss your solution with at most *one* other person. If you discuss with somebody, mention their name on your submission.

1 Exercises

1. Show that the inverse of a non-singular matrix is unique.
2. Revise how to compute the inverse of a matrix. Compute the inverse of the following matrices.

   (a)
   $$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

3. Compute solutions to the following system of equations.

   (a)
   $$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} x = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

4. Are these vectors linearly independent?

   (a)
   $$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}$$

5. Is the vector

$$\begin{pmatrix}2\\3\\-5\end{pmatrix}$$

in the span of the following vectors?

$$\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

6. Find the rank of the following matrix.

$$\begin{pmatrix}1 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 1 & 0\\1 & 0 & 0 & 0 & 1\\0 & 1 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\end{pmatrix}$$

7. Consider a system of linear equations $Ax = b$ where A is a $m \times n$ matrix, $b \in \mathbb{R}^m$. Let $a_1, \ldots, a_m$ denote the rows of A. Suppose the rank of the matix is strictly less than m, i.e., the rows are linearly dependent. Show one of the following.
   (a) The system $Ax = b$ has no solution.
   (b) The system $Ax = b$ has a redundant constraint, i.e., we can remove that constraint without changing the set of solutions.

8. Using the previous exercise, show that for any linear program in the standard form $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ which is non-empty, we can assume that the rows of A are linearly dependent.

9. Suppose there are N available currencies and assume that one unit of currency i can be exchanged for r_{ij} units of currency j. (Naturally $r_{ij} > 0$).
   (a) *Currency arbitrage* is the phenomenon where one can start from a unit of certain currency and exchange it for another and then for another and so on till we end up with more than one unit of first currency. Thus we have made money by just exchanging currencies. Write a linear program that checks for currency arbitrage.
   (b) Assume that there is no currency arbitrage (as is likely. Surprisingly, such arbitrage are not uncommon in Crypto currency exchanges!). Suppose that we start with B units of currency 1 and that we would like to maximize the number of units of currency N that we end up after all exchanges on a single day. Moreover, there is a limit u_i on the total amount of currency i that can be exchanged on a given day. Provide a linear programming formulation to find the sequence of exchanges to do.

(c) Consider the following data given in the table. Implement your model in Gurobi to check if there is any arbitrage. Also, can you find the best way to exchange 1000 dollars to Euros when we have a total limit of 500 GBP and 10000 JPY for exchanging. Entry in i^{th} row and column j denotes the value of r_{ij}. Thus, if sell 1 dollar, we can 98.5408 JPY and so on.

	USD	EUR	JPY	GBP
USD	-	0.7796	98.5408	0.6994
EUR	1.11	-	123.73	0.86
JPY	0.008793	0.0075267	-	0.006520
GBP	1.260	1.084	136.178	-

(d) Can your code solve if there are 10 currencies? What about 20 currencies or say 100? Your answer should discuss the issues if your solution does not scale. (You can use random values for r_{ij} between 0 and 1 to test your code.)

10. A company produces two kinds of products. A product of the first type requires $\frac{1}{4}$ hours of assembly labor, $\frac{1}{8}$ hours of testing and 1.2 dollars worth of raw materials. A product of the second type requires $\frac{1}{3}$ hours of labor, $\frac{1}{3}$ hours of testing and 0.9 dollars worth of raw materials. Given the current personnel of the company, there can be at most 90 hours of assembly labor and 80 hours of testing, each day. Products of the first and second type have a market value of 9 and 8 dollars, respectively.

    (a) Formulate a linear programming problem that can be used to maximize the daily profit of the company.

    (b) Consider the following two modifications of the problem.

        i. Suppose that up to 50 hours of overtime associated assembly labor can be scheduled, at a cost of 7 dollars per hour.

        ii. Suppose that the raw material supplier provides a discount of 10 per cent if the daily bill is over 300 dollars.

    Which of the above two elements can be easily incorporated into the linear programming formulation and how?

    ($\star$) If one or both are not easy to incorporate, indicate how you might nevertheless solve the problem.

11. Find all the basic solutions and basic feasible solutions to the following polyhedron.

$$\begin{pmatrix} 1 & -1 & 2 & -1 & 0 \\ 2 & 0 & 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$x \geq 0$$

12. Consider the polyhedron defined by the following system. Find all its extreme points. Graphically check it.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$x \geq 0$$

2 Assignment

1. (10 points) A chemical company mines phosphate rock, collects in inventory poles $i = 1, \ldots, 8$ and blends it to meet contracts with customers $k = 1, \ldots, 25$. The critical measure of phosphate content in rock is its BPL (bone phosphate of lime). Piles correspond to different average BPL content b_i per ton, asset value a_i per ton, contract net profit r_{ik} per ton, starting inventory h_i and expected quantity q_i to arrive from mines vary accordingly. Each contract includes a minimum of $\underline{s_k}$ and maximum $\overline{s_k}$ number of tons to be shipped, along with a minimum of $\underline{p_k}$ and a maximum of $\overline{p_k}$ average BPL content. Managers want to schedule blending and sales to maximize total profit plus total inventory asset value. Formulate an LP formulation for this planning problem.

2. (10 points) A medical devices company makes artificial heart valves from pig hearts. One thing that makes the process complicated is that pig hearts come in various sizes depending on breed, age etc. The following (fictitious) table shows he fraction of hearts from suppliers $j = 1, \ldots, 5$ yielding each of the value sizes $i = 1, \ldots, 7$ along with maximum quantity available from each supplier per week and unit cost of hearts obtained.

	Supplier j				
Size	1	2	3	4	5
1	0.4	0.1	-	-	-
2	0.4	0.2	-	-	-
3	0.2	0.3	0.4	0.2	-
4	-	0.2	0.3	0.2	-
5	-	0.2	0.3	0.2	0.2
6	-	-	-	0.2	0.3
7	-	-	-	0.2	0.5
Availability	500	330	150	650	300
Cost	2.5	3.2	3.0	2.1	3.9

The company wants to decide how to purchase hearts to meet weekly requirement of 20 size 1, 30 size 2, 120 size 3, 200 size 4, 150 size 5, 60 size 6 and 45 size 7 valves at minimum cost.

   (a) Formulate an LP model of this heart purchase problem where you do not impose integrality on the number of hearts purchased.

   (b) Solve the LP model in Gurobi.

   (c) Now insist the variables are integer and resolve the problem. Discuss the difference in solutions as well as time taken to solve the two instances.

3. (10 points) Consider the polyhedron defined by the following set of inequalities. List all its basic solutions and basic feasible solutions. Plot it graphically and check it.

$$
\begin{aligned}
x_1 + 2x_2 &\leq 3 \\
2x_1 + x_2 &\leq 3 \\
3x_1 + x_2 &\leq 4 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

4. (10 points) Consider the following polyhedron in $\mathbb{R}^7$.

$$
\begin{pmatrix} 1 & 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 0 & -2 & 1 \\ 1 & 2 & 1 & 5 & 4 & 3 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}
$$
$$
x \geq 0
$$

and the following vectors

(a) $(1, 1, 0, 0, 0, 0, 0)^T$,
(b) $(2, -1, 2, 0, 0, 0, 0)^T$,
(c) $(1, 0, 1, 0, 1, 0, 0)^T$,
(d) $(0, 0, 1, 1, 0, 0, 0)^T$,
(e) $(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 1)^T$.

Which of the following points are basic solutions. Also, which one are basic feasible solutions. You need to justify your answer.

5. (10 points) Consider the polyhedron defined by the following system.

$$
\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 7 \end{pmatrix}
$$
$$
x \geq 0
$$

(a) What is the rank of the matrix

$$
\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \end{pmatrix}
$$

(b) Can you give a different description of the form $\{x \in \mathbb{R}^3 : Ax = b, x \geq 0\}$ of the above polyhedron where rows of A are linearly independent. (Use Exercise 7 and 8).

(c) Find all basic solutions and basic feasible solutions of your formulation.