Exercises and Assignment 3: ISYE 4133.
Due October 2

Instructions. Exercises are for your practice. You need not submit them. The problems in the Section 2 are to be submitted by the due date. You are supposed to write your own solutions but you can discuss your solution with at most *one* other person. If you discuss with somebody, mention their name on your submission.

1 Exercises

1. Consider the linear program.

$$\min\ (0, -3, -1, 0)x$$
$$\begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$
$$x \geq 0$$

   (a) Starting from the basis $B = \{1, 4\}$, solve the problem with the simplex method. At each step, choose the entering and leaving variable using the Bland's rule.

   (b) Give a certificate of optimality or unboundedness for the problem and verify it.

2. Consider the linear program

$$\min\ (-2, -1, 1, 0, 0, 0)x$$
$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & -3 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$
$$x \geq 0$$

   Starting from the basis $B = \{4, 5, 6\}$ and applying the Bland's rule solve the linear program. At termination, display the proof of optimality or unboundedness.

3. Consider the following linear programs of form $\min\{c^T x : Ax = b, x \geq 0\}$. Use the two phase simplex method using Bland's rule to the solve the linear programs.

   (a)

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

   (b)

$$A = \begin{pmatrix} 2 & -1 & 4 & -2 & 1 \\ -1 & 0 & -3 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 3 \\ 1 \\ -1 \\ -4 \\ -7 \end{pmatrix}$$

4. Consider the following table indicating the nutritional value of different food types.

Food	Price($) per serving	Calories per serving	Fat(g) per serving	Protein(g) per serving	Carbohydrate(g) per serving
Raw Carrots	0.14	23	0.1	0.6	6
Baked Potatoes	0.12	171	0.2	3.7	30
Wheat Bread	0.2	65	0	2.2	13
Cheddar Cheese	0.75	112	9.3	7	0
Peanut Butter	0.15	188	16	7.7	2

   You need to decide how many servings of each food to buy each day to minimize the total food cost while satisfying the daily nutritional value.

   - Calories must be at least 2000.
   - fat must be at least 50g.
   - protein must be at least 100 g.
   - carbohydrates must be at least 250 g.

   Write an LP that will help you decide how many servings of each of the aforementioned foods are needed to meet all the nutritional requirements, while minimizing the total cost of the food (you may buy fractional number of servings). Solve this in Gurobi using Simplex Algorithm. Observe the number of iterations used.

5. Consider a school district with I neighborhoods, J schools, and G grades at each school. Each school j has a capacity C_{jg} for grade g. In each neighborhood i, the student population of grade g is S_{ig}. Finally the distance of school j from neighborhood i is d_{ij}. Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance travelled by all students. (You may ignore the fact the number of students must be an integer).

6. Write down the dual linear program for the following linear programs.

   (a)

$$
\begin{aligned}
\min(5,6)x \\
s.t. \\
3x_1 - x_2 \geq 8 \\
2x_1 + 4x_2 = -2 \\
3x_1 + 2x_2 \leq 2 \\
-x_1 + x_2 = -3 \\
x_1 \geq 0, x_2 \text{ free}
\end{aligned}
$$

   (b)

$$
\begin{aligned}
\min c^T x \\
s.t. \\
Ax \geq b, \\
Dx - Iu \leq d, \\
x \leq 0, u \geq 0
\end{aligned}
$$

   Where A and D are matrices, I is the identity matrix, b, c and d are vectors and x, u are vectors of variables.

7. Let $a_1, \ldots, a_n$ be n distinct numbers. Consider the linear program.

$$\max\{x \in \mathbb{R} : x \leq a_i, \text{ for } i = 1, \ldots n\}.$$

   (a) Show that the optimal value of (P) is the minimum of $a_1, \ldots, a_n$.
   (b) Find the dual (D) of (P).
   (c) Explain in words what (D) is doing?

8. For each of these programs write their dual. Solve both primal and dual using Gurobi and record whether they have an optimal solution or are infeasible or unbounded.

   (a)

$$
\begin{aligned}
\max\ 2x_2 + x_3 \\
s.t. \\
x_1 - x_2 + x_3 \leq 5 \\
-2x_1 + x_2 \leq 3 \\
x_2 - 2x_3 \leq 5 \\
x_1, x_2, x_3 \geq 0
\end{aligned}
$$

(b)

$$\begin{aligned} \min\ -x_1 - x_2 \\ s.t. \\ -x_1 \geq 1 \\ x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{aligned}$$

(c)

$$\begin{aligned} \min\ -2x_1 + x_2 \\ s.t. \\ x_1 - x_2 \leq 1 \\ -x_1 + x_2 \leq -2 \\ x_1, x_2 \geq 0 \end{aligned}$$

9. An analytical control problem. We are given a system whose state at time $t = 0, \ldots, N$ is determined by vector $x(t) \in \mathbb{R}^n$. We assume that $x(0) = 0$. At every time $t = 1, \ldots, N-1$, we can *control* the system by choosing an input $u(t) \in \mathbb{R}$ where x and u are related via a recursion

$$x(t+1) = Ax(t) + b \cdot u(t)$$
$$x(0) = 0$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given data.

We consider the problem of maximizing a linear function of the final state of a linear system, subject to bounds on the inputs:

$$\begin{aligned} \text{maximize } d^T x(N) \\ \text{subject to} \\ |u(t)| \leq U, \quad t = 0, \ldots, N \\ \sum_{t=1}^{N-1} |u(t)| \leq \alpha, \end{aligned}$$

where $d \in \mathbb{R}^n$, $U, \alpha \in \mathbb{R}$ are also input data.

(a) Express the above problem as a linear program.

(b) Solve the linear program for the following data.

$$A = \begin{pmatrix} 9.9007 \cdot 10^{-1} & 9.934 \cdot 10^{-3} & -9.453 \cdot 10^{-3} & 9.453 \cdot 10^{-3} \\ 9.9340 \cdot 10^{-2} & 9.066 \cdot 10^{-1} & -9.453 \cdot 10^{-2} & 9.453 \cdot 10^{-2} \\ 9.9502 \cdot 10^{-3} & 4.9793 \cdot 10^{-4} & 9.9952 \cdot 10^{-1} & 4.8172 \cdot 10^{-4} \\ 4.9793 \cdot 10^{-3} & 9.5201 \cdot 10^{-2} & 4.8172 \cdot 10^{-3} & 9.9518 \cdot 10^{-1} \end{pmatrix}$$

$$b = \begin{pmatrix} 9.9502 \cdot 10^{-2} \\ 4.9793 \cdot 10^{-3} \\ 4.9834 \cdot 10^{-3} \\ 1.6617 \cdot 10^{-4} \end{pmatrix}$$

$$U = 2$$

$$d = (0, 0, 1, -1)^T$$

$$\alpha = 161$$

$$N = 100.$$

2 Assignment

1. (10 points) The following linear program is in standard form.

$$\min\ (-1, 2, 0, 1, 3)x$$

$$\begin{pmatrix} 1 & -1 & 2 & -1 & 0 \\ 2 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x \geq 0$$

   (a) Find an equivalent linear program in canonical form for basis $\{1, 4\}$ and $\{3, 5\}$.

   (b) In each case, state whether the basis is feasible and/or optimal.

   (c) Apply one iteration of the simplex method starting from each of the solution using the Bland's rule.

2. (10 points) Consider the following linear programs of form $\min\{c^T x : Ax = b, x \geq 0\}$. Use the two phase simplex method using Bland's rule to the solve the linear programs.

   (a)

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}$$

   (b)

$$A = \begin{pmatrix} 2 & 4 & 2 \\ -1 & -3 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} -27 \\ -2 \\ 6 \end{pmatrix}$$

3. (10 points) The princess wedding ring can be made from four types of gold $1, 2, 3$ or 4 with the following amounts of milligrams of impurity per gram.

Type	1	2	3	4
mg of lead	1	2	2	1
mg of cobalt	0	1	1	2
value	1	2	3	2

   (a) Set up a linear program that finds the most valuable rung that can be made containing at most $6mg$ of lead and $10mg$ of cobalt.

   (b) Put the linear program in standard form.

   (c) Solve the linear program using the simplex method by hand and by Gurobi.

4. (10 points) Write down the dual linear program for the following linear programs.

   (a)

$$\begin{aligned} \max(1, 3, 1)x \\ s.t. \\ x_1 + x_2 + 7x_3 \leq -3 \\ x_2 - x_3 = 9 \\ 9x_1 \leq 5 \\ x_1 - x_2 + x_3 = 0 \\ x_1, x_2 \geq 0, x_3 \text{ free} \end{aligned}$$

   (b)

$$\begin{aligned} \min c^T x + d^T u \\ s.t. \\ Ax + Du = b, \\ x \geq 0, u \text{ free} \end{aligned}$$

   Where A and D are matrices, b, c and d are vectors and x, u are vectors of variables.

5. (10 points) For each of (a) and (b), do the following:

   - Write the dual (D) of (P).
   - Write the complementary slackness condition for (P) and (D).
   - Use weak duality to prove $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D) or show they are not optimal.
   - Use complementary slackness to prove that $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D) or show they are not optimal.

(a)

$$\begin{aligned}
\max(-2,-1,0)x \\
s.t. \\
x_1+3x_2+2x_3 \geq 5 \\
-x_1+4x_2+2x_3 \leq 7 \\
x_1 \leq 0, x_2 \geq 0, x_3 \geq 0.
\end{aligned}$$

and

$$\bar{x}=(-1,0,3)^T,\ \bar{y}=(-1,1)^T.$$

(b)

$$\begin{aligned}
\min(-5,-7,-5)x \\
s.t. \\
x_1+2x_2+3x_3 \geq 3 \\
4x_1+5x_2+6x_3 \leq 9 \\
7x_1+8x_2+9x_3 \geq 2 \\
x_1,x_2,x_3 \geq 0.
\end{aligned}$$

and

$$\bar{x}=(1,1,0)^T,\ \bar{y}=(-1,-1,0)^T.$$

6. (10 points) Consider the following linear program

$$\begin{aligned}
\min -10x_1-3x_2 \\
x_1+x_2+x_3=4 \\
5x_1+2x_2+x_4=11 \\
x_2+x_5=4 \\
x_1,x_2,x_3,x_4,x_5 \geq 0
\end{aligned}$$

(a) Starting from the basis $B=\{2,3,4\}$, solve the linear program using the simplex method.

(b) Removing the slack variables, we have the equivalent formulation.

$$\begin{aligned}
\min -10x_1-3x_2 \\
x_1+x_2 \leq 4 \\
5x_1+2x_2 \leq 11 \\
x_2 \leq 4 \\
x_1,x_2 \geq 0
\end{aligned}$$

Plot the feasible region and mark the extreme points. As the simplex algorithm moves in the first formulation, plot its path in the two dimensional feasible region of the new linear program.