Assignment 1: ISYE 4133. September 4.

August 24, 2019

Instructions. Exercises are for your practice. You need not submit them. The problems in the Section 2 are to be submitted by the due date before class starts. You are supposed to write your own solutions but you can discuss your solution with at most *one* other person. If you discuss with somebody, mention their name on your submission. Any code must be written by you. We may check the code for plagiarism using software for such purpose.

1 Exercises

1. Convert the following linear program in standard form.

   (a)

$$\begin{aligned} &\text{minimize } 2x_1 + x_2 - x_3 \\ &\text{subject to} \\ x_1 + x_2 - 4x_3 &\geq 10 \\ 2x_1 - 5x_2 + 7x_3 &\leq 40 \\ x_1 &\geq 0 \\ x_2 &\leq 0 \end{aligned}$$

   (b)

$$\begin{aligned} &\text{maximize } x_1 - x_2 + x_4 \\ &\text{subject to} \\ x_1 + x_2 + x_3 &\leq 4 \\ 2x_1 + 5x_2 + 4x_4 &= 30 \\ x_1 &\leq 0 \\ x_2 \geq 0 x_4 &\leq 0 \end{aligned}$$

2. Solve the above linear program in Gurobi in both the form that is given and standard form. Note the different solutions and see how the optimal solution to LP in original form corresponds to the optimal solution in standard form.

3. Suppose we have m food items and our goal is to decide the quantity of each of them to buy to make sure we obtain enough nutrition as well as we spend the minimum amount of money.

We are given a collection of n different nutrients and for each nutrient $1 \leq i \leq n$ and each food item $1 \leq j \leq m$, we are given $\alpha_{ij} \in \mathbb{R}_+$ that denotes the amount of this nutrient in a unit amount of j^{th} food item. We are given for each nutrient i, a requirement r_i of the amount of nutrient needed. Each food item j also a cost of c_j per unit amount of food. Formulate a linear program that finds out how much of each food item to buy to satisfy the requirements and is of minimum cost.

4. Suppose there is an oil supplier that aims to minimize its operating cost while ensuring it can meet all the demand that it will receive for oil in the next four months. It has estimated the demand for heating oil as follows.

Month	1	2	3	4
Demand (litres)	5000	8000	9000	6000

At the beginning of each month, it can purchase oil from a regional supplier at the current market rate. The market rate at the beginning of each month has been estimated using historical trends as follows.

Month	1	2	3	4
Price (dollars per litre)	0.75	0.72	0.92	0.90

The company also has a small storage tank of capacity 4000 and currently (beginning of month 1) contains 2000 litres. The company wants to know how much oil it should buy at the beginning of each month so it can satisfy the demand at the minimum possible cost. Note, that the oil not used can be stored in the tank (up to its capacity) that can be used later in future. Write a linear program for the problem and formulate a model in Gurobi to solve it.

5. Meat Free Burger's (MFB) new meat-free burgers are so popular that company cannot keep up with the demand. Regional demand shown in the following table total 2000 burgers per week, but MFB can only produce 60 percent of that number.

	NE	SE	MW	W
Demand	620	490	510	380
Profit	1.60	1.40	1.90	1.20

The table also shows the different profit levels per burger experienced in the region due to competition and regional demand. MFB wants to find a maximum profit plan that fulfills between 50 percent and 70 percent of the demand in each region.

   (a) Formulate a LP to choose an optimal distribution plan.

   (b) Solve your model using Gurobi.

6. A construction contractor has undertaken a job with 7 major tasks. Some of the tasks can begin at any time, but others have predecessors that must be completed first. The following table shows that predecessor task numbers, together with the minimum and maximum time (in days) allowed for each task, and the total cost that would be associated with the accomplishing each task in its minimum and maximum times (more time usually saves expense).

j	Min Time	Max. Time	Cost Min.	Cost Max.	Predecessor Tasks
1	6	12	1600	1000	None
2	8	16	2400	1800	None
3	16	24	2900	2000	2
4	14	20	1900	1300	1,2
5	4	16	3800	2000	3
6	12	16	2900	2200	3
7	2	12	1300	800	3

The contractor seeks a way to complete all work in 40 days at least total cost, assuming that cost of each task is linearly interpolated for times between the minimum and maximum.

(a) Formulate an LP model of this time/cost planning problem using decision variables ($j = 1, \ldots, 7$),

i. s_j := start time of the task j (in days).

ii. t_j := days to complete task j.

Both the above variables can take fractional values. Your model should have an objective function summing interpolated cost and main constraints to enforce precedence constraints and the time limit.

(b) Solve the above program in Gurobi.

2 Assignment

1. (10 points) Transforms the following linear programs in the standard form.

(a)

$$\begin{aligned} &\text{minimize } x_1 + 2x_2 \\ &\text{subject to} \\ &x_1 + x_2 \geq 10 \\ &2x_1 + 5x_2 \leq 40 \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} &\text{maximize } x_1 - x_2 \\ &\text{subject to} \\ &x_1 + x_2 \leq 4 \\ &2x_1 + 5x_2 = 30 \\ &x_1 \leq 0 \\ &x_2 \geq 0 \end{aligned}$$

2. (10 points) Implement and solve the linear programs in the above question in the given form as well as the standard form in Gurobi. Note the optimal solutions of the given LP and the LP in its standard form. Show how these solutions are related.

3. (10 points) A builder needs to build a house and has divided the process into a number of tasks, namely

   B. excavation and building the foundation.

   F. raising the wooden frame.

   E. electrical wiring.

   P. Plumbing.

   D. dry walls and flooring.

   L. Landscaping.

   The estimated time for each process (in weeks) is as follows.

Task	B	F	E	P	D	L
Duration	3	2	3	4	1	2

   Some of the tasks can only be started when some other tasks are completed. For instance, you can online build the frame once the foundation has been completed, i.e., F can only start after B is completed. All these precedence constraints are summarized as follows.

   - F can start only after B is completed.
   - L can start only after B is completed.
   - E can start only after F is completed.
   - P can start only after F is completed.
   - D can start only after E is completed.
   - D can start only after P is completed.

   The goal is to schedule the starting time of each task such that the entire project is completed as soon as possible.

   As an example here is a feasible schedule with a completion time of ten weeks.

Task	B	F	E	P	D	L
Starting Time	0	3	6	5	9	6
Ending Time	3	5	9	9	10	8

   Formulate a linear program that solves the problem. Explain your formulation. Note, there is no limit on the number of tasks that can be done in parallel. Solve the linear program using Gurobi and report the optimal solution.

4. (10 points) For $m \in \{10, 20, 50, 100, 500, 1000, 10000\}$ and $n \in \{10, 20, 50, 100, 1000, 10000\}$ generate matrices $A \in \mathbb{R}^{m \times n}$ whose entries are uniformly random between $[0, 1]$. Similarly generate $b \in \mathbb{R}^m$ randomly with entries randomly between $[0, 1000]$. Also generate a cost function $c \in \mathbb{R}^n$ with entries randomly between $[0, 1000]$.

(a) Formulate the linear program $\{\min c^T x : Ax \geq b, x \geq 0\}$ for the above random data. Solve 10 instances of the program for each pair of values of m and n with a time out of 2 minutes. Note the time taken and objective value for each run and average over the 10 runs for each pair of (m, n).

(b) Update the formulation to insist the variables are integers. Repeat the experiment. Note the time taken and objective value for each run.

(c) Plot the time and objective value as the y-axis and size $(m + n)$ as the x-axis.

5. (10 points) A furniture manufacturing company makes two models of tables for libraries and other university facilities. Both models use the same table tops but model A has 4 short (18-inch) legs and model B has 4 longer ones (30 inches). It takes 0.10 labor hour to cut and shape a short leg from stock, 0.15 labor hour to do the same for a long leg and 0.50 labor hour to produce a tabletop. An additional 0.30 labor hour is needed to attach the set of leges for either model after all parts are available. Estimated profit is 30 for each model A sold and 45 for each model B sold. Plenty of top material is on hand but the company wants to decide how to use the available 5000 feet of leg stock and 800 labor hours to maximize profit assuming that everything made can be sold.

(a) Formulate a LP to choose the optimal plan. Assume that the number of tables and legs manufactured can take fractional values.

(b) Solve the linear program using Gurobi.

(c) Can you justify the assumption that the variables can take fractional values?

6. (Extra Credit: 10 points) Any convex 3-dimensional object (i.e., a body such that the line segment between any two points in it falls entirely within the body) with flat sides can be described as the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying a series of linear constraints. For example, a 3-by 5-by 9-meter box with one corner at the origin can be modeled as

$$\{(x, y, z) : 0 \leq x \leq 3, 0 \leq y \leq 5, 0 \leq z \leq 9\}.$$

Suppose that a stationary object is described in this way by constraints

$$a_i x + b_i y + c_i z \leq d_i \ \ \forall 1 \leq i \leq 20$$

where $a_1, \ldots, a_{20}, b_1, \ldots, b_{20}, c_1, \ldots, c_{20}, d_1, \ldots, d_{20} \in \mathbb{R}$. We are also given that a link of a robot arm is described at its initial position by the constraints

$$p_j x + q_j y + r_j z \leq s_j \ \ \forall 1 \leq j \leq 12$$

in a similar manner.

The object and the link do not intersect at that initial position but the link is in motion. Its location is being translated from the initial location by growing a step $\alpha > 0$ in direction $(\delta x, \delta y, \delta z)$. Formulate an LP in terms of decision variables x, y, z and α to find the smallest step (if any) that will produce a collision between the object and the link, and indicate how the LP would detect the case when no collision will occur.