Examen : Eduardo Alberto Domínguez Fontes

1) La intersección de la esfera $x^2+y^2+z^2=1$ y el plano $x+y+z=1/2$ genera una curva con una forma cuadratica su proy. al plano xy. Hallar los ejes $x'y'$ en los cuales la forma cuadratica resulta de una matriz diagonal. Escribir la curva de dichos ejes.

$$x^2+y^2+z^2=1 \qquad x+y+z=\frac{1}{2}$$

$$z=\frac{1}{2}-x-y$$

Sust.

$$\Rightarrow x^2+y^2+\left(\left(\frac{1}{2}-x\right)-y\right)^2=1$$

$$\Rightarrow x^2+y^2+\left(\frac{1}{2}-x\right)^2-2\left(\frac{1}{2}-x\right)y+y^2=1$$

$$\Rightarrow \cancel{x^2}+\cancel{y^2}+\frac{1}{4}-x+\cancel{x^2}-y+2xy+\cancel{y^2}=1$$

$$\Rightarrow 2x^2+2y^2+2xy-y-x+\frac{1}{4}=1$$

$$A=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P_A(\lambda)=\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}=(2-\lambda)^2-1$$

$$\Rightarrow 4-4\lambda+\lambda^2-1=0$$

$$\Rightarrow \lambda^2-4\lambda+3=0$$

$$(\lambda-3)(\lambda-1)=0 \quad \Rightarrow \lambda_1=3\ ,\ \lambda_2=1$$

Para $\lambda_1=3$ encontramos V_1.

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x+y=0 \qquad x=y \qquad \Rightarrow V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x-y=0$$

Para $\lambda_2 = 1$ encontramos V_2

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + y = 0$$
$$x = -y \quad \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Entonces la matriz $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Ahora con $D = P^{-1} A P$

$$P^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow D = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -6 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

La forma cuadrática es de la forma:

$$3(x')^2 + (y')^2 + (-1 \ \ -1)\, P \begin{pmatrix} x' \\ y' \end{pmatrix} + \frac{1}{4} = 1$$

$$\Rightarrow 3(x')^2 + (y')^2 + (-1 \ \ -1)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + \frac{1}{4} = 1$$

$$\Rightarrow 3(x')^2 + (y')^2 + (-1 \ \ -1)\begin{pmatrix} x' + y' \\ x' - y' \end{pmatrix} + \frac{1}{4} = 1$$

$$\Rightarrow 3(x')^2 + (y')^2 - x' - y' - x' + y' + \frac{1}{4} = 1$$

$$\Rightarrow 3(x')^2 + (y')^2 - 2x' + \frac{1}{4} = 1 \quad //$$

Ahora graficamos

$$3(x')^2 + (y')^2 - 2x' + \frac{1}{4} = 1$$

$$\Rightarrow (x')^2 + \frac{(y')^2}{3} - \frac{2x'}{3} + \frac{1}{12} = \frac{1}{3}$$

$$\Rightarrow (x')^2 - \frac{2x'}{3} + \frac{1}{9} - \frac{1}{9} + \frac{(y')^2}{3} + \frac{1}{12} = \frac{1}{3}$$

$$\Rightarrow \left(x' - \frac{1}{3}\right)^2 + \frac{(y')^2}{3} = \frac{1}{3} - \frac{1}{12} + \frac{1}{9}$$

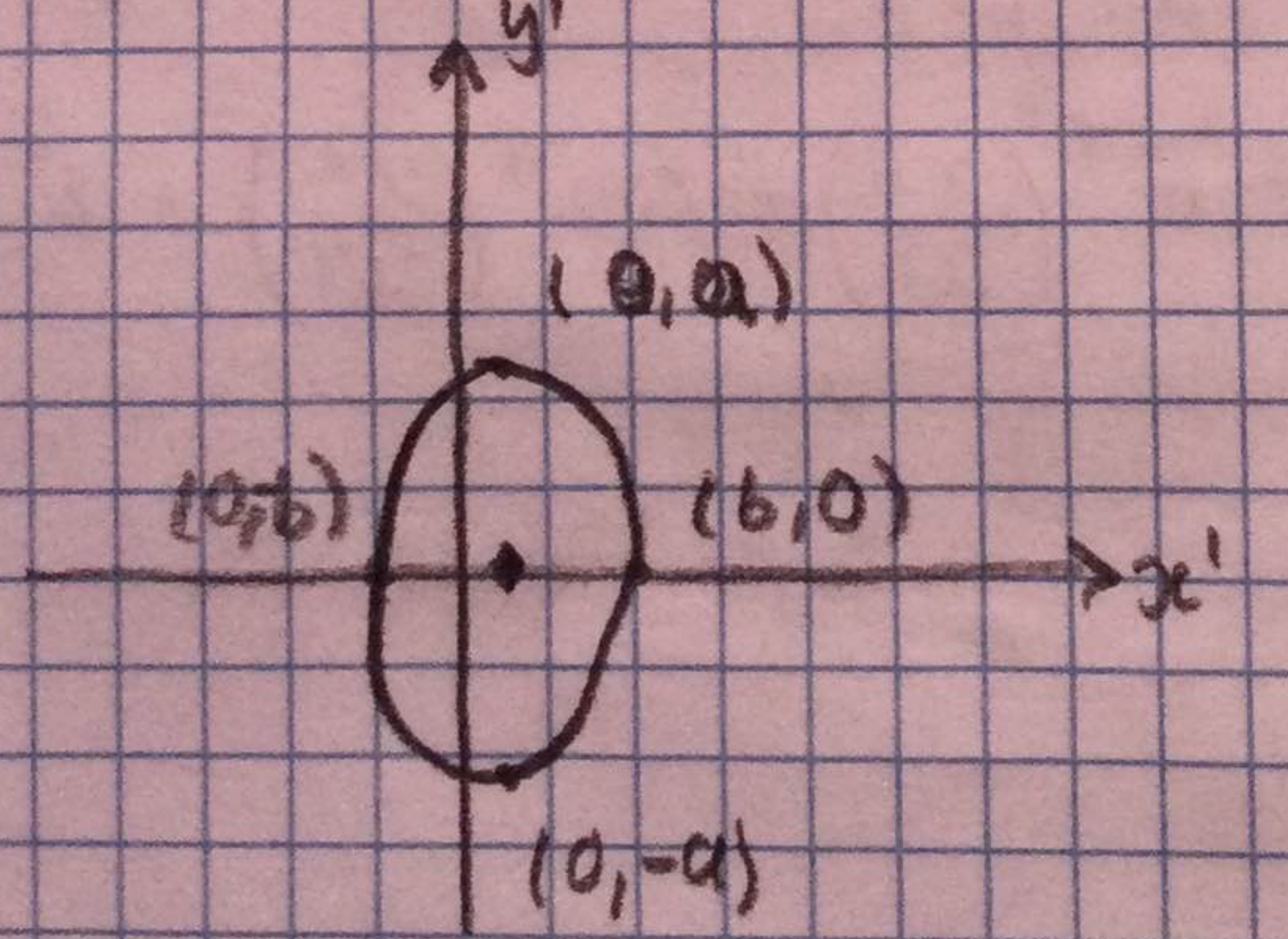
$$\Rightarrow \left(x' - \frac{1}{3}\right)^2 + \frac{(y')^2}{3} = \frac{13}{36}$$

$$\Rightarrow \frac{(x' - 1/3)^2}{\frac{13}{36}} + \frac{(y')^2}{\frac{39}{36}} = 1$$

Con centro en $(1/3, 0)$

$$b^2 = \frac{13}{36} \quad , \quad a^2 = \frac{39}{36}$$

$$b = \pm\sqrt{\frac{13}{36}} \quad , \quad a = \pm\sqrt{\frac{39}{36}}$$



2) Dadas las coordenadas elípticas bidimensionales uv $(a>0)$

$$x^1 = a\sinh(u)\cos(v)$$
$$x^2 = a\cosh(u)\sin(v)$$

a) $$J = \begin{pmatrix} \frac{\partial x^1}{\partial u} & \frac{\partial x^1}{\partial v} \\ \frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \end{pmatrix} = \begin{pmatrix} a\cosh(u)\cos(v) & -a\sinh(u)\sin(v) \\ a\sinh(u)\sin(v) & a\cosh(u)\cos(v) \end{pmatrix}$$

$$\det J = a^2\cosh^2(u)\cos^2(v) + a^2\sinh^2(u)\sin^2(v)$$
$$= a^2\cosh^2(u)\cos^2(v) + a^2\sinh^2(u)(1-\cos^2(v))$$
$$= a^2\cosh^2(u)\cos^2(v) + a^2\sinh^2(u) - a^2\sinh^2(u)\cos^2(v)$$
$$= a^2\cos^2(v)[\cosh^2(u) - \sinh^2(u)] + a^2\sinh^2(u)$$
$$= a^2\cos^2(v) + a^2\sinh^2(u)$$
$$\det J = a^2(\sinh^2(u) + \cos^2(v))$$

los puntos donde la transf. no es válida se dan cuando

$$\det J = 0 = a^2(\sinh^2(u) + \cos^2(v))$$
$$\Rightarrow \sinh^2(u) = -\cos^2(v)$$

$$\begin{cases} \sinh^2(u) = 0 \text{ cuando } u = 0 \\ \cos^2(v) = 0 \text{ cuando } v = \frac{n\pi}{2} \text{ ; con } n \text{ impar} \end{cases}$$

no es válida en $\left(0, \frac{n\pi}{2}\right)$ //

$U=0$, $U=U_0$, $V=0$, $V=V_0$

Para $U=0$

$$\left.\begin{aligned} x^1 &= 0 \\ x^2 &= a \sin V \end{aligned}\right\} \text{ recta del eje } x^2$$

Para $U=U_0$

$x^1 = a \sinh(U_0)\cos(V) \quad = a C_1 \cos(V)$
$x^2 = a \cosh(U_0)\sin(V) \quad = a C_2 \sin(V)$

$V=0 \rightarrow x^1 = aC_1$, $x^2 = 0$, $V=\pi \rightarrow x^1 = -aC_1$, $x^2 = 0$

$V=\frac{\pi}{2} \rightarrow x^1 = 0$, $x^2 = 1$, $V=\frac{3\pi}{2} \rightarrow x^1 = 0$, $x^2 = -aC_2$

Si $C_1 = C_2$ es un circulo
Si $C_1 \neq C_2$ es una elipse.

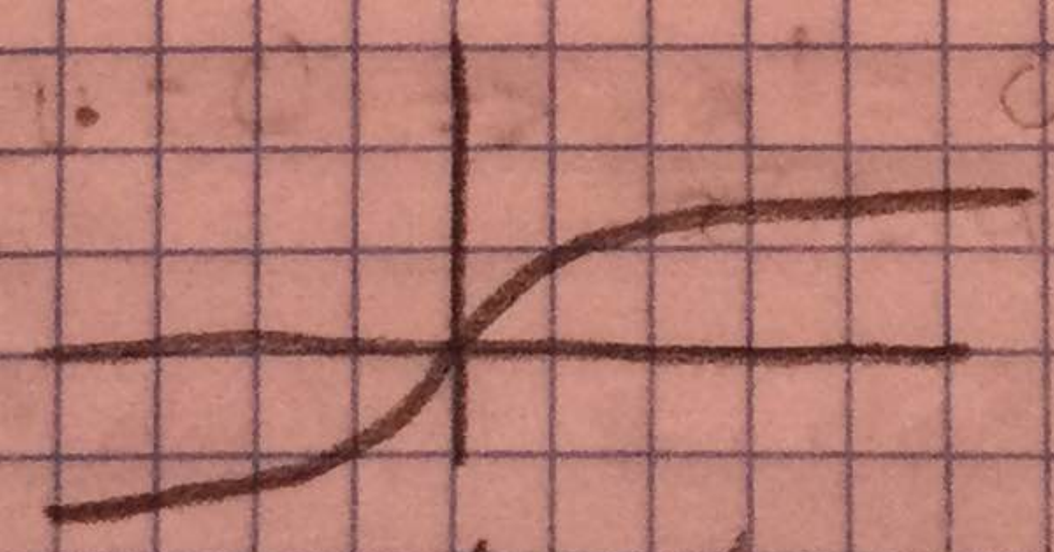
Para $V=0$

$$\left.\begin{aligned} x^1 &= a \sinh(U) \\ x^2 &= 0 \end{aligned}\right\} \text{ recta del eje } x^1$$

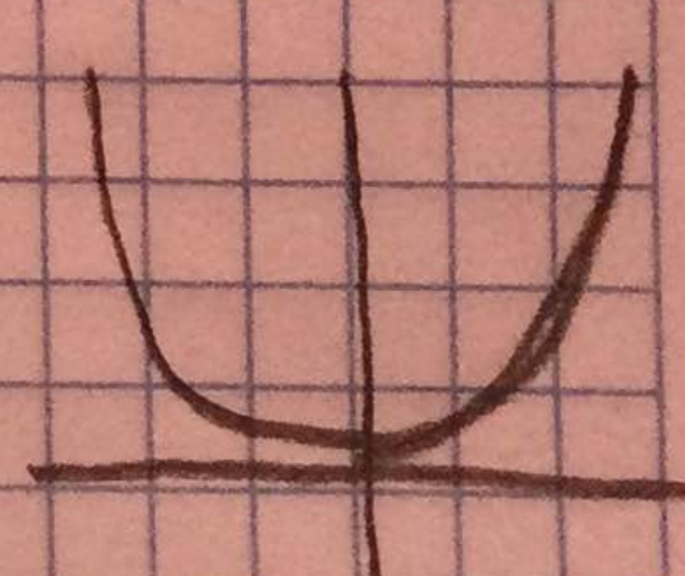
Para $V=V_0$

$x^1 = a \sinh(U)\cos(V_0) \quad = a D_1 \sinh(U)$
$x^2 = a \cosh(U)\sin(V_0) \quad = a D_2 \cosh(U)$

$\sinh(U)$

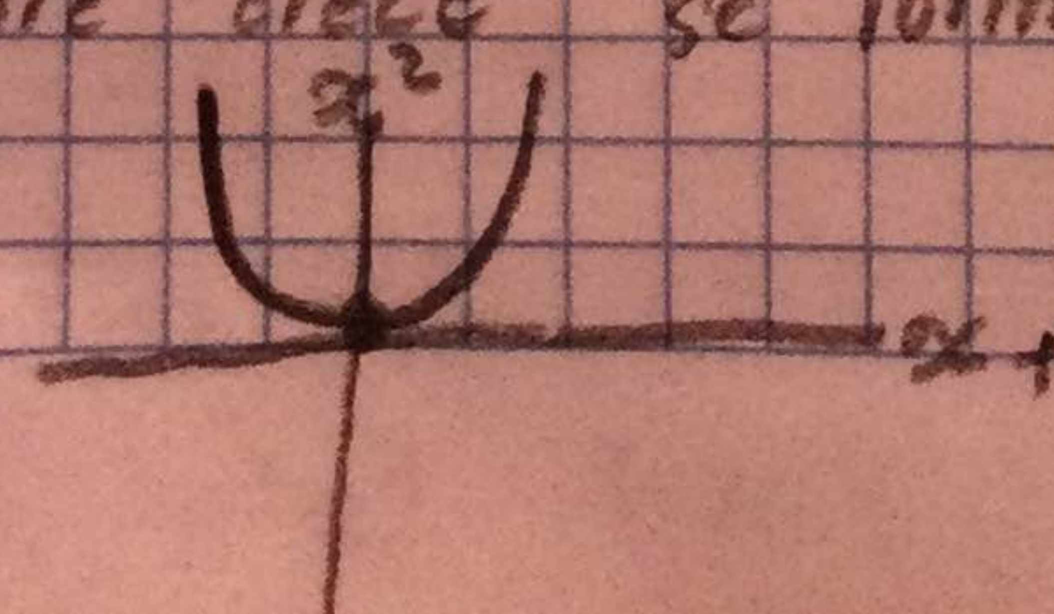


$\cosh(U)$



cuando x^2 toma val positivos igual que x^1 ambos crecen
cuando x^2 toma val positivos y x^1 negativos

Como x^2 siempre crece se forma una parábola que pasa por el origen



b) Hallar la métrica en las coordenadas UV

$$g_{ij}(y) = \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j}$$

$$\Rightarrow g_{uu} = \left(\frac{\partial x^1}{\partial u}\right)^2 + \left(\frac{\partial x^2}{\partial u}\right)^2 = a^2\cosh^2(u)\cos^2(v) + a^2\sinh^2(u)\sin^2(v)$$

$$g_{uv} = \frac{\partial x^1}{\partial u}\frac{\partial x^1}{\partial v} + \frac{\partial x^2}{\partial u}\frac{\partial x^2}{\partial v}$$

$$= a\cosh(u)\cos(v)[-a\sinh(u)\sin(v)] + a\sinh(u)\sin(v)[a\cosh(u)\cos(v)]$$

$$= 0$$

$$g_{vu} = g_{uv} = 0$$

$$g_{vv} = \left(\frac{\partial x^1}{\partial v}\right)^2 + \left(\frac{\partial x^2}{\partial v}\right)^2 = a^2\sinh^2(u)\sin^2(v) + a^2\cosh^2(u)\cos^2(v)$$

$$g_{ij} = \begin{pmatrix} a^2\cosh^2(u)\cos^2(v) + a^2\sinh^2(u)\sin^2(v) & 0 \\ 0 & a^2\sinh^2(u)\sin^2(v) + a^2\cosh^2(u)\cos^2(v) \end{pmatrix}$$

3) Demostrar

$$V_1(x^1, x^2, x^3) = -\frac{x^2}{(x^1)^2 + (x^2)^2}$$

$$V_2(x^1, x^2, x^3) = \frac{x^1}{(x^1)^2 + (x^2)^2} + x^3$$

$$V_3(\vec{x}) = x^2$$

Para dem que son comp. de un vector cov. devemos encontrar una función $\varnothing$ que al derivar parcialmente de V_1, V_2, V_3.

$$V_1 = \frac{\partial \varnothing}{\partial x^1} \Rightarrow -\int \frac{x^2}{(x^1)^2 + (x^2)^2} dx^1 = -\frac{x^2}{x^2} \tan^{-1}\left(\frac{x^1}{x^2}\right)$$

$$V_2 = \frac{\partial \varnothing}{\partial x^2} \Rightarrow \int \frac{x^1 \, dx^2}{(x^1)^2 + (x^2)^2} + \int x^3 dx^2 = x^2 x^3 + \frac{x^1}{x^1} \tan^{-1}\left(\frac{x^2}{x^1}\right)$$

$$V_3 = \frac{\partial \varnothing}{\partial x^3} \Rightarrow \int x^2 dx^3 = x^2 x^3$$

La función $\varnothing$ es la suma de los resultados, y los que se repiten se colocan una vez.

$$\varnothing(x^1, x^2, x^3) = -\tan^{-1}\left(\frac{x^1}{x^2}\right) + \tan^{-1}\left(\frac{x^2}{x^1}\right) + x^2 x^3$$

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4) Demostrar que:

$$B_{ik}(x) = g_{i\beta}(x)\, A^{\beta}_{k}(x)$$

transforma como tensor $\binom{0}{2}$

$$g_{i\beta}(x) = \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial y^{\mu}}{\partial x^{\beta}} g_{\alpha\mu}(y)$$

$$A^{\beta}_{k}(x) = \frac{\partial y^{\nu}}{\partial x^{k}} \frac{\partial y^{\beta}}{\partial x^{\lambda}} C^{\lambda}_{\nu}(y)$$

$$\Rightarrow B_{ik}(x) = \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial y^{\mu}}{\partial x^{\beta}} g_{\alpha\mu}(y) \frac{\partial y^{\nu}}{\partial x^{k}} \frac{\partial y^{\beta}}{\partial x^{\lambda}} C^{\lambda}_{\nu}(y)$$

$$\Rightarrow B_{ik}(x) = \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial y^{\mu}}{\partial x^{\beta}} \frac{\partial y^{\nu}}{\partial x^{k}} \frac{\partial y^{\beta}}{\partial x^{\lambda}} g_{\alpha\mu}(y)\, C^{\lambda}_{\nu}(y)$$

$$\text{con} \quad \frac{\partial y^{\mu}}{\partial x^{\beta}} \frac{\partial y^{\beta}}{\partial x^{\lambda}} = \delta^{\mu}_{\lambda}$$

$$\Rightarrow B_{ik}(x) = \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial y^{\nu}}{\partial x^{k}} \delta^{\mu}_{\lambda} F^{\lambda}_{\nu\alpha\mu}(y)$$

$$B_{ik}(x) = \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial y^{\nu}}{\partial x^{k}} F^{\mu}_{\nu\alpha\mu}(y)$$

$$B_{ik}(x) = \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial y^{\nu}}{\partial x^{k}} F_{\nu\alpha}(y)$$

//

5)

a) $x^3 = \left[(x^1)^2 + (x^2)^2 \right]^{3/2}$ cilindricas

$x^1 = r\cos\theta$
$x^2 = r\sin\theta$
$x^3 = z$

para $ds^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$

$z = r^3$

$dz = 3r^2 dr$

Entonces: $ds^2 = (dr)^2 + r^2(d\theta)^2 + 9r^4(dr)^2$

$ds^2 = (1+9r^4)(dr^2) + r^2(d\theta)^2$

y $g_{ij} = \begin{pmatrix} 1+9r^4 & 0 \\ 0 & r^2 \end{pmatrix}$ //

$g^{-1} = \frac{1}{r^2(1+9r^4)} \begin{pmatrix} r^2 & 0 \\ 0 & 1+9r^4 \end{pmatrix}$

$g^{ij} = \begin{pmatrix} \frac{1}{1+9r^4} & 0 \\ 0 & 1/r^2 \end{pmatrix}$ //

b) $x^2 = x^1 x^3$ cartesianas.

$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$dx^2 = \frac{\partial x^2}{\partial x^1} dx^1 + \frac{\partial x^2}{\partial x^3} dx^3$

$= x^3 dx^1 + x^1 dx^3$

$\Rightarrow ds^2 = (dx^1)^2 + [x^3 dx^1 + x^1 dx^3]^2 + (dx^3)^2$

$= (dx^1)^2 + (x^3)^2(dx^1)^2 + 2x^3x^1 dx^1 dx^3 + (x^1)^2(dx^3)^2 + (dx^3)^2$

$= (1+(x^3)^2)(dx^1)^2 + (1+(x^1)^2)(dx^3)^2 + 2x^1x^3 dx^1 dx^3$

$g_{ij} = \begin{pmatrix} 1+(x^3)^2 & x^3x^1 \\ x^3x^1 & 1+(x^1)^2 \end{pmatrix}$

$\det g_{ij} = [1+(x^1)^2][1+(x^3)^2] - (x^3)^2(x^1)^2$

$$g^{ij} = \frac{1}{(1+(x^1)^2)(1+(x^3)^2) - (x^3)^2(x^1)^2} \begin{pmatrix} 1+(x^1)^2 & -x^3x^1 \\ -x^3x^1 & 1+(x^3)^2 \end{pmatrix} //$$