Ejemplo. Determinar w_0, w_1, x_0 y x_1 de tal forma que la fórmula:

$$\int_{-1}^{1} f(x)\,dx \simeq w_0 f(x_0) + w_1 f(x_1) = \sum_{i=0}^{1} w_i\, f(x_i).$$

$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{m} w_i\, f(x_i)$$

sea exacta para un polinomio del mayor grado posible.

Solución. Para este caso $m=1$ $\Rightarrow$ grado de exactitud: $2m+1 = 3$ o menor.

i.e. La fórmula será exacta si:

$$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \quad , \quad a_i = \text{cte.} \quad i = 0,1,2,3.$$

$$\int_{-1}^{1} f(x)\,dx = a_3 \int_{-1}^{1} x^3\,dx + a_2 \int_{-1}^{1} x^2\,dx + a_1 \int_{-1}^{1} x\,dx + a_0 \int_{-1}^{1} 1\,dx.$$

es equivalente a mostrar que la fórmula da resultados exactos para:

$$f(x) = 1,\ x,\ x^2 \text{ y } x^3.$$

Se requiere conocer w_0, w_1, x_0 y x_1.

$$w_0 \cdot 1 + w_1 \cdot 1 = \int_{-1}^{1} 1\,dx = 2$$

$$w_0 x_0 + w_1 x_1 = \int_{-1}^{1} x\,dx = 0$$

$$w_0 x_0^2 + w_1 x_1^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$w_0 x_0^3 + w_1 x_1^3 = \int_{-1}^{1} x^3\,dx = 0$$

$\Longrightarrow$ Sistema de ecuaciones

$$\begin{cases} w_0 + w_1 = 2 \\ w_0 x_0 + w_1 x_1 = 0 \\ w_0 x_0^2 + w_1 x_1^2 = \dfrac{2}{3} \\ w_0 x_0^3 + w_1 x_1^3 = 0. \end{cases}$$

$\downarrow$

resolver el sistema.

Solución: $w_0 = 1$, $w_1 = 1$, $x_0 = -\dfrac{\sqrt{3}}{3}$, $x_1 = \dfrac{\sqrt{3}}{3} \approx 0.5773502692$.

Por lo cual, la fórmula de aproximación:

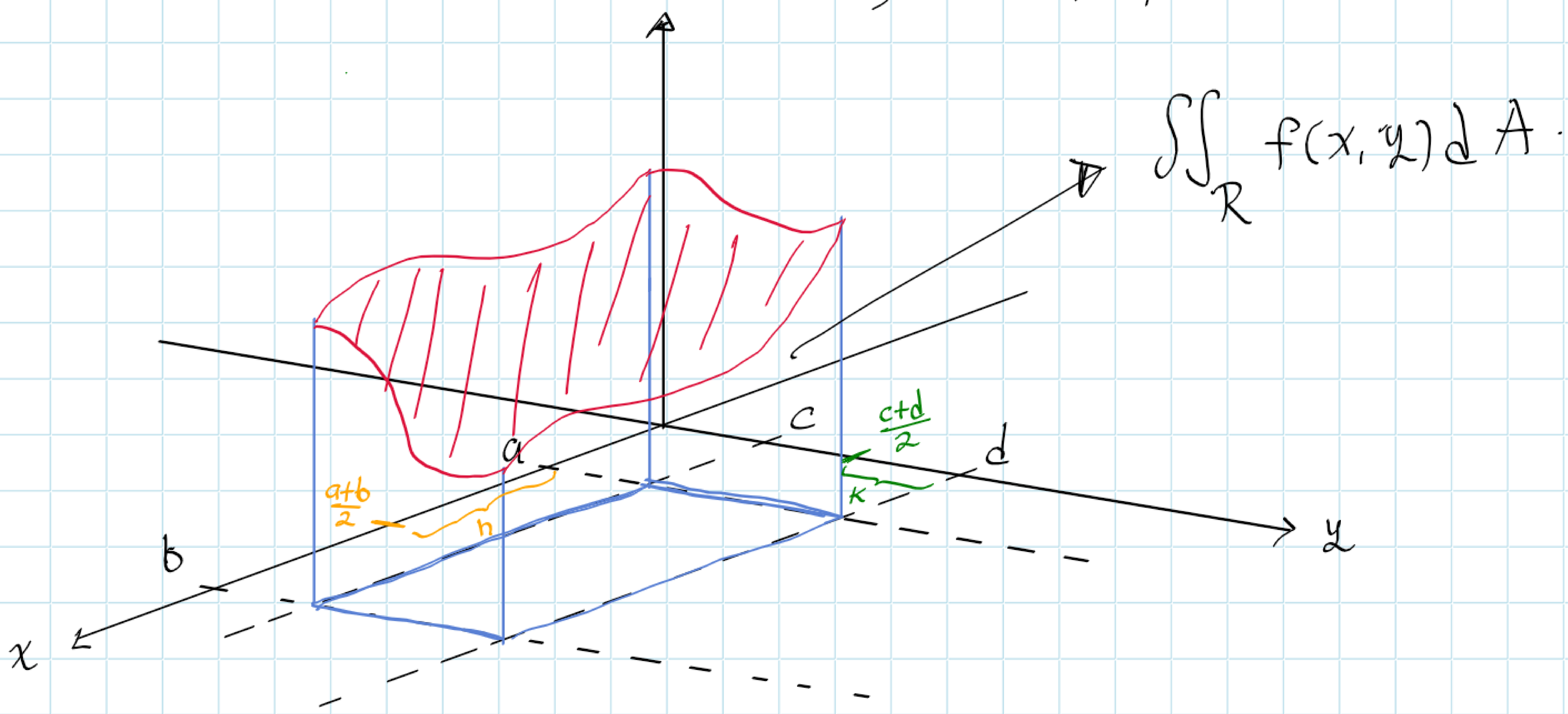
$$\int_{-1}^{1} f(x)\,dx \simeq f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

$\rightarrow$ exacta para polinomios de grado 3 o menor.

Integrales Multiples.

Consideremos la sig. integral $\iint_R f(x,y)\,dA$.

donde: $R = \{(x,y)\in \mathbb{R}^2 \mid x\in[a,b] \text{ y } y\in[c,d]\}$, a,b,c y d son constantes.



$$\iint_R f(x,y)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

Aplicaremos la regla compuesta del trapecio a la integral interior.

consideremos: $K = \frac{d-c}{2}$

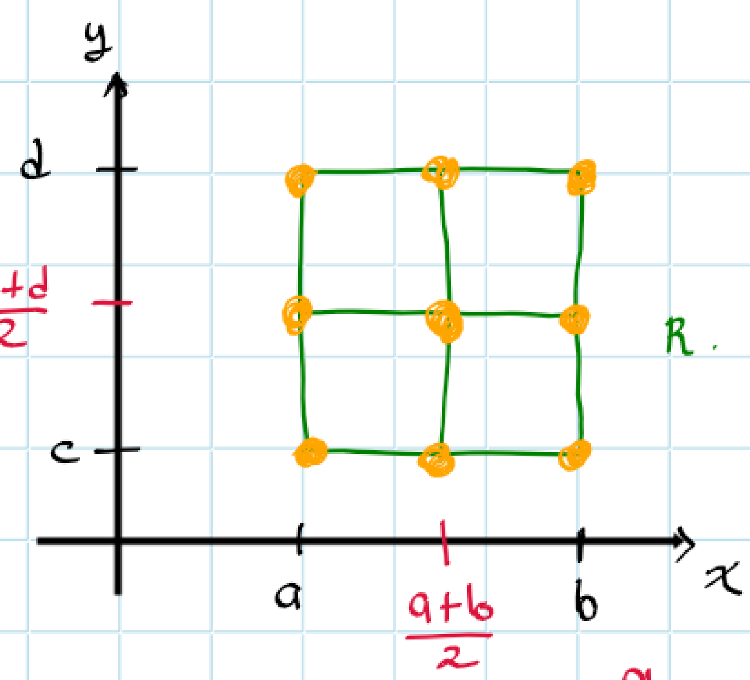
$$\int_c^d f(x,y)\,dy \simeq \frac{K}{2}\left(f(x,c) + f(x,d) + 2f\left(x, \frac{c+d}{2}\right) \right).$$

$$\Rightarrow \iint_R f(x,y)\,dx \approx \int_a^b \left[\underbrace{\frac{K}{2}\left(f(x,c) + f(x,d) + 2f\left(x, \frac{c+d}{2}\right)\right)}_{F(x)} \right] dx.$$

Aplicamos la regla compuesta del trapecio a $\int_a^b \frac{K}{2} F(x)\,dx$, con $h = \frac{b-a}{2}$.

$$\frac{K}{2}\int_a^b F(x)\,dx \simeq \frac{K}{2}\frac{h}{2}\left[F(a) + F(b) + 2F\left(\frac{a+b}{2}\right)\right].$$

$$= \frac{Kh}{4}\left[f(a,c) + f(a,d) + 2f\left(a, \frac{c+d}{2}\right) + f(b,c) + f(b,d) + 2f\left(b, \frac{c+d}{2}\right) \right.$$
$$\left. + 2f\left(\frac{a+b}{2}, c\right) + 2f\left(\frac{a+b}{2}, d\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right].$$



se requieren 9 evaluaciones

$$\boxed{\begin{aligned}\iint_R f(x,y)\,dA \simeq \frac{(b-a)(d-c)}{16}\Big[& f(a,c) + f(a,d) + f(b,c) + f(b,d) \\ & + 2\left(f\left(a, \tfrac{c+d}{2}\right) + f\left(b, \tfrac{c+d}{2}\right) + f\left(\tfrac{a+b}{2}, c\right) + f\left(\tfrac{a+b}{2}, d\right)\right) \\ & + 4f\left(\tfrac{a+b}{2}, \tfrac{c+d}{2}\right)\Big]\end{aligned}}$$

Problema: Aplicar la regla compuesta de Simpson a la integral

$$\iint_R f(x,y)\,dA.$$

donde $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b \ \text{ y } \ c \le y \le d\}.$

Ejemplo: Usar la cuadratura de Gauss-Legendre con $\boxed{m=2}$ en ambas dimensiones para aproximar a la siguiente integral.

$$\int_{1.4}^{2}\int_{1}^{1.5} \ln(x+2y)\,dy\,dx.$$

Solución:

$$R = \{(x,y) \in \mathbb{R}^2 \mid 1.4 \le x \le 2, \quad 1 \le y \le 1.5\}.$$

lo primero que requiere hacer es transformar la región R. en $\overline{R}$

$$\overline{R} = \{(u,v) \in \mathbb{R}^2 \mid -1 \le u \le 1, \quad -1 \le v \le 1\}.$$

$$u = \frac{2x - 1.4 - 2}{2 - 1.4} \qquad\qquad v = \frac{2y - 1 - 1.5}{1.5 - 1}$$

$$\Rightarrow \quad x = \frac{2-1.4}{2}u + \frac{2+1.4}{2} \qquad\qquad y = \frac{1.5-1}{2}v + \frac{1.5+1}{2}$$

$$\Rightarrow \quad x = 0.3u + 1.7 \qquad\qquad y = 0.25v + 1.25$$

$$\Rightarrow \int_{1.4}^{2}\int_{1}^{1.5} \ln(x+2y)\,dy\,dx = (0.3)(0.25)\int_{-1}^{1}\int_{-1}^{1} \ln\big(0.3u + 1.7 + 2(0.25v + 1.25)\big)\,dv\,du.$$

$$= 0.075\int_{-1}^{1}\int_{-1}^{1} \ln(0.3u + 0.5v + 4.2)\,dv\,du.$$

Con $m=2$ se tiene la aprox. para la integral interior.

$$\int_{-1}^{1} \underbrace{\ln(0.3u + 0.5v + 4.2)}_{f(v)}\,dv \simeq \underbrace{w_0 \ln(0.3u + 0.5x_0 + 4.2) + w_1 \ln(0.3u + 0.5x_1 + 4.2) + w_2 \ln(0.3u + 0.5x_2 + 4.2)}_{G(u)}.$$

$$\int_{-1}^{1}\int_{-1}^{1} \ln(0.3u + 0.5v + 4.2)\,dv\,du \simeq \int_{-1}^{1} G(u)\,du \simeq w_0 G(x_0) + w_1 G(x_1) + w_2 G(x_2)$$

$$\int_{-1}^{1}\int_{-1}^{1} \ln(0.3u+0.5v+4.2)\,dv\,du \simeq \frac{8}{9}\left[\frac{8}{9}\ln(0.3(0)+0.5(0)+4.2) + \frac{5}{9}\ln\left(0.3(0)+0.5\left(\frac{3}{\sqrt{15}}\right)+4.2\right) + \frac{5}{9}\ln\left(0.3(0)+0.5\left(\frac{-3}{\sqrt{15}}\right)+4.2\right)\right]$$

$$+\frac{5}{9}\left[\frac{8}{9}\ln\left(0.3\left(\frac{3}{\sqrt{15}}\right)+0.5(0)+4.2\right) + \frac{5}{9}\ln\left(0.3\left(\frac{3}{\sqrt{15}}\right)+0.5\left(\frac{3}{\sqrt{15}}\right)+4.2\right) + \frac{5}{9}\ln\left(0.3\left(\frac{3}{\sqrt{15}}\right)+0.5\left(\frac{-3}{\sqrt{15}}\right)+4.2\right)\right]$$

$$+\frac{5}{9}\left[\frac{8}{9}\ln\left(0.3\left(\frac{-3}{\sqrt{15}}\right)+0.5(0)+4.2\right) + \frac{5}{9}\ln\left(0.3\left(\frac{-3}{\sqrt{15}}\right)+0.5\left(\frac{3}{\sqrt{15}}\right)+4.2\right) + \frac{5}{9}\ln\left(0.3\left(\frac{-3}{\sqrt{15}}\right)+0.5\left(\frac{-3}{\sqrt{15}}\right)+4.2\right)\right]$$

$$= 1.96032$$

$$= \frac{8}{9}(2.86542)$$

$$= 2.54704$$

$$= \boxed{5.727393749 \longrightarrow \text{Total}}$$

$$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} \ln(0.3u+0.5v+4.2)\,dv\,du = 5.727393749$$

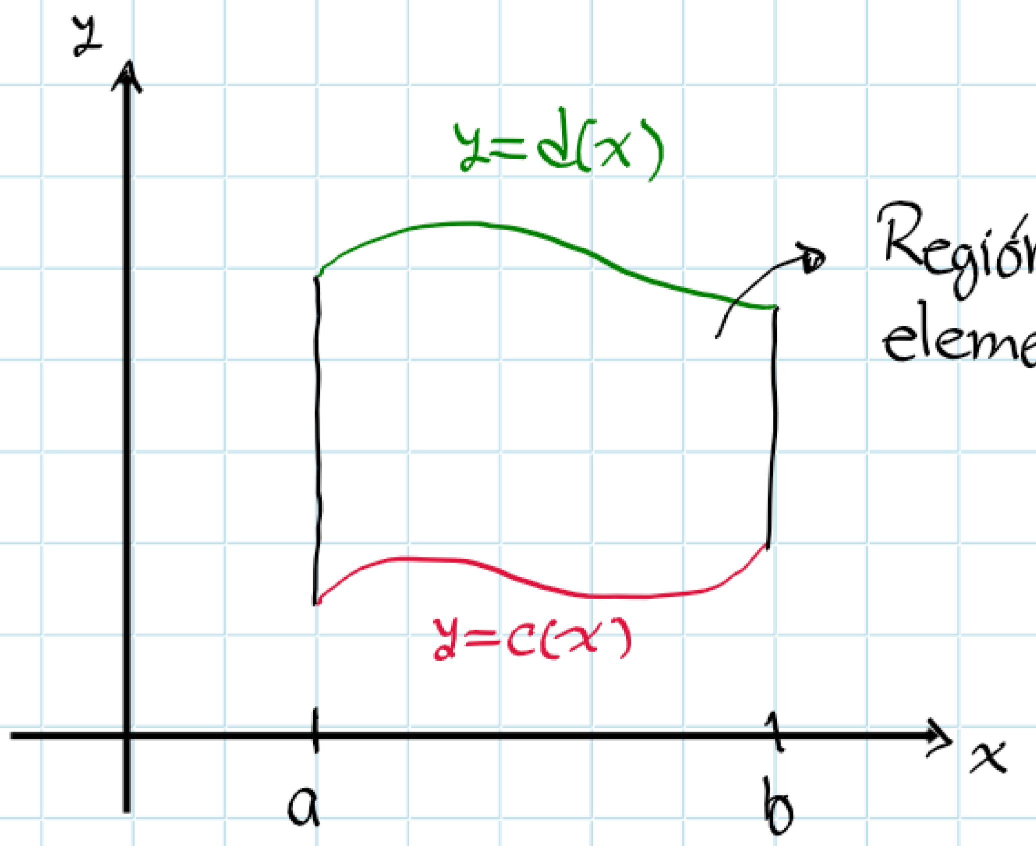
$$\Rightarrow \int_{1.4}^{2}\int_{1}^{1.5} \ln(x+2y)\,dy\,dx \simeq 0.075(5.727393749)$$

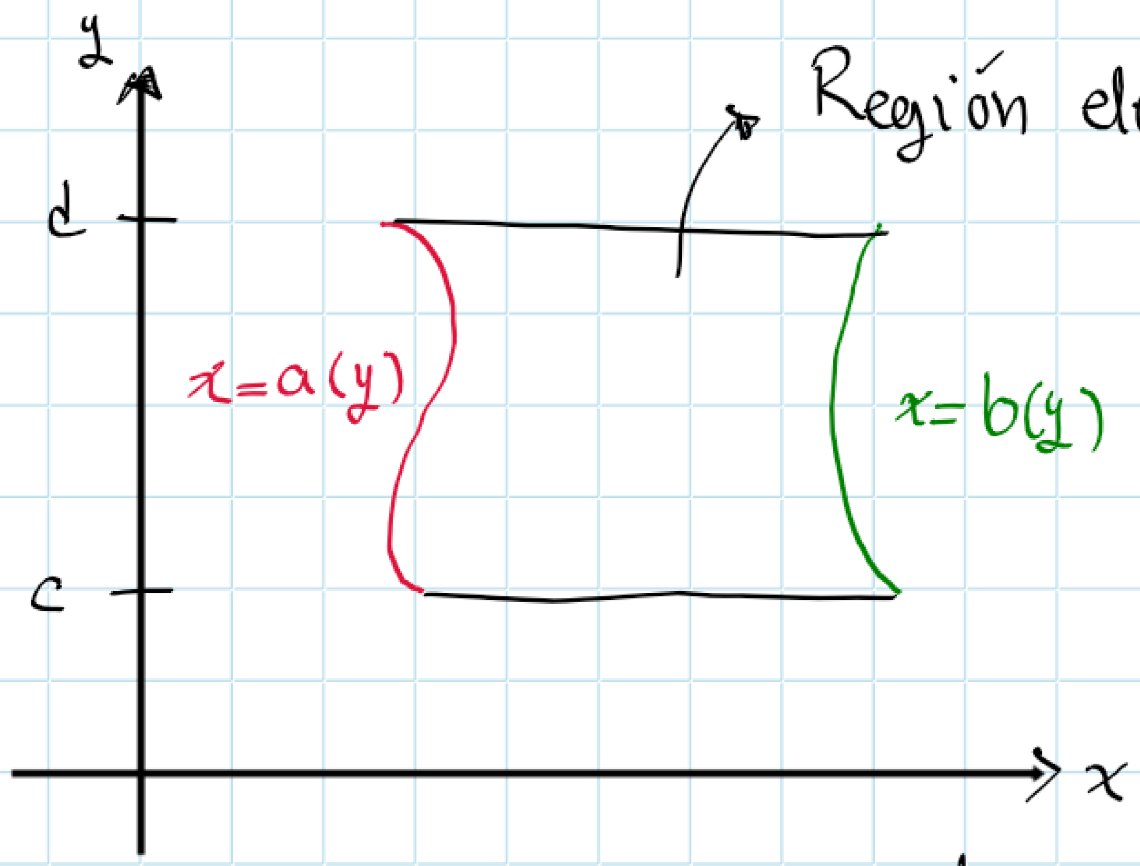
$$\simeq 0.4295545311 \checkmark$$

$$0.42955$$

* Regiones No rectangulares.

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \quad \text{o} \quad \int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy.$$

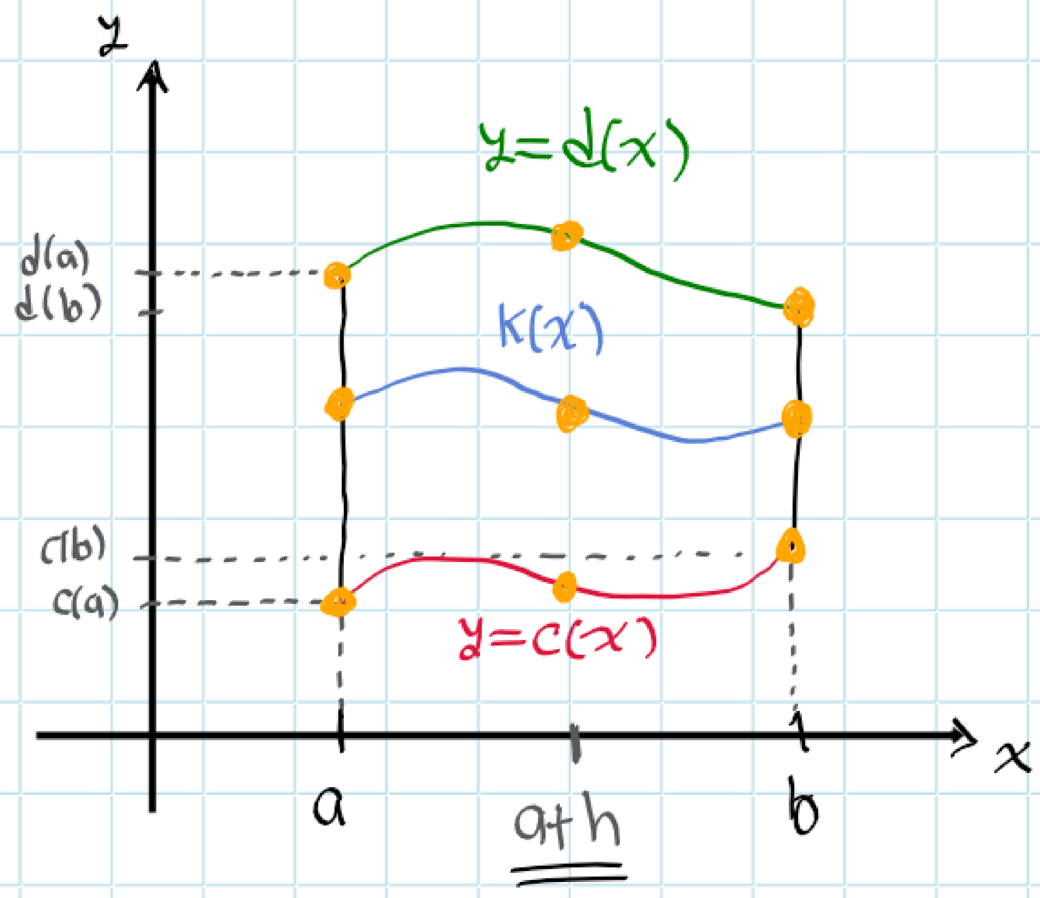




Obs los métodos que se conocen se pueden aplicar a regiones mas generales.

Usaremos la regla básica de Simpson para calcular:

$$\rightarrow \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx.$$



Consideremos: $h = \dfrac{b-a}{2}$, $k(x) = \dfrac{d(x)-c(x)}{2}$

$\dfrac{d(x)+c(x)}{2}$

Primero analizamos la integral interior:

$$\int_{c(x)}^{d(x)} f(x,y)\,dy \simeq \frac{k(x)}{3}\Big[\underbrace{f(x,c(x)) + 4f(x,c(x)+k(x)) + f(x,d(x))}_{G(x).}\Big]$$

$$\frac{d(x)+c(x)}{2} = c(x) + \frac{d(x)-c(x)}{2}$$

$$= c(x) + k(x)$$

$$\Rightarrow \int_a^b \frac{1}{3}k(x)\,G(x)\,dx \simeq \frac{h}{3}\Big[\frac{1}{3}k(a)\,G(a) + \frac{4}{3}k(a+h)\,G(a+h) + \frac{1}{3}k(b)\,G(b)\Big]$$

$$\Rightarrow \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \simeq \frac{h}{3}\Big\{ \frac{k(a)}{3}\Big[f(a,c(a)) + 4f(a,c(a)+k(a)) + f(a,d(a))\Big]$$

$$+ \frac{4}{3}k(a+h)\Big[f(a+h,c(a+h)) + 4f(a+h,c(a+h)+k(a+h)) + f(a+h,d(a+h))\Big]$$

$$+ \frac{k(b)}{3}\Big[f(b,c(b)) + 4f(b,c(b)+k(b)) + f(b,d(b))\Big]\Big\}.$$

Cuadratura Gaussiana. (Gauss-Legendre)

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx.$$

$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{m} w_i\, f(x_i).$$

Polinomios de Legendre.
$\hookrightarrow [-1,1]$.

$x_i :=$ ceros $L_{m+1}(x)$

$$w_i = \frac{2}{(1-x_i^2)(L'_{m+1}(x_i))^2} > 0$$

$$\int_a^b \phi_1(x)\cdot\phi_2(x)\,dx = 0$$

obs. $x \in [a,b]$, $y \in [c(x), d(x)]$.

hacemos el siguiente cambio de variable.

$$v = \frac{2y - c(x) - d(x)}{d(x) - c(x)}$$

o

$$y = \frac{d(x)-c(x)}{2}\,v + \frac{d(x)+c(x)}{2}$$

$$\Rightarrow \int_{c(x)}^{d(x)} f(x,y)\,dy = \int_{-1}^{1} f\left(x, \frac{d(x)-c(x)}{2}\,v + \frac{d(x)+c(x)}{2}\right)\left(\frac{d(x)-c(x)}{2}\right)dv.$$

$$= \frac{d(x)-c(x)}{2}\left[\sum_{i=0}^{m} w_i\, f\left(x, \frac{d(x)-c(x)}{2}\,x_i + \frac{d(x)+c(x)}{2}\right)\right]$$

$$= F(x).$$

$$\Rightarrow \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \simeq \int_a^b F(x)\,dx = \int_{-1}^{1} F\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}\,dt.$$

$$= \frac{b-a}{2}\int_{-1}^{1} F\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)dt.$$

Consideremos

$$\Delta_{z_j} = \frac{b-a}{2} z_j + \frac{b+a}{2}$$

$$\simeq \frac{b-a}{2}\left[\sum_{j=0}^{n} \phi_j\, F\left(\frac{b-a}{2} z_j + \frac{b+a}{2}\right)\right]$$

$$= \frac{b-a}{2} \sum_{j=0}^{n} \phi_j\, \frac{d\left(\frac{b-a}{2} z_j + \frac{b+a}{2}\right) - c\left(\frac{b-a}{2} z_j + \frac{b+a}{2}\right)}{2}$$

$$\left(\sum_{i=0}^{m} w_i\, f\left(\Delta_{z_j}, \frac{d(\Delta_{z_j}) - c(\Delta_{z_j})}{2}\,x_i + \frac{d(\Delta_{z_j}) + c(\Delta_{z_j})}{2}\right)\right)\Bigg\}$$

Obs w_i, ϕ_j son los pesos de los polinomios de Legendre.
z_j, x_i son los ceros de $L_{m+1}(x)$.

$$w_i = \int_{-1}^{1} \underbrace{\prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{x - x_j}{x_i - x_j}}\, dx.$$

$f = 1.$

$$\boxed{\int_{0.1}^{0.5} \int_{x^3}^{x^2} e^{y/x}\, dy\, dx.}$$

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx.$$

$$\int_{c(x)}^{d(x)} f(x,y)\,dy\,dx = \frac{d(x)-c(x)}{2}\int_{-1}^{1} f\left(x, \frac{d-c}{2}v + \frac{d+c}{2}\right) dv.$$

$$y = \boxed{\frac{d-c}{2}}\,v + \boxed{\frac{d+c}{2}} \qquad \simeq \frac{d(x)-c(x)}{2}\sum_{j=1}^{n} \underset{\substack{\uparrow \\ \text{pesos} \\ L_n(x)}}{\phi_j}\, f\left(x, \frac{d-c}{2}\underset{\substack{\uparrow \\ \text{ceros} \\ L_n(x)}}{z_j} + \frac{d+c}{2}\right) = F(x).$$

$$\int_a^b F(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} F\left(\frac{b-a}{2}u + \frac{b+a}{2}\right) du.$$

$$x = \boxed{\frac{b-a}{2}}\,u + \boxed{\frac{b+a}{2}}$$

$$\simeq \frac{b-a}{2}\sum_{i=1}^{m} \underset{\substack{\uparrow \\ \text{pesos} \\ L_m(x)}}{w_i}\, F\left(\frac{b-a}{2}\underset{\substack{\uparrow \\ \text{ceros} \\ L_m(x)}}{x_i} + \frac{b+a}{2}\right)$$

filas → i, columnas → 1

$m \to A = (i, 1) \longrightarrow$ pesos

$A = (i, 2) \longrightarrow$ ceros

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \qquad h_1 = \frac{b-a}{2}, \quad h_2 = \frac{b+a}{2} \qquad K_1 = \frac{d-c}{2}, \quad K_2 = \frac{d+c}{2}$$

$$\simeq \frac{b-a}{2}\sum_{i=1}^{m} w_i\left(\frac{d\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right) - c\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right)}{2}\sum_{j=1}^{n}\phi_j\, f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}, \frac{d-c}{2}z_j + \frac{d+c}{2}\right)\right)$$

$$= h_1\sum_{i=1}^{m} A(i,1)\left(\underbrace{\frac{d(h_1A(i,2)+h_2) - c(h_1A(i,2)+h_2)}{2}}_{K_1}\underbrace{\sum_{j=1}^{n} B(j,1)\, f\left(h_1A(i,2)+h_2, K_1B(j,2)+K_2\right)}_{JX}\right)$$

$i = 1$

$A = [\text{pesos} \quad \text{ceros}]$

Tarea: Aplicar la regla de Simpson compuesta

a una integral de la forma:

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx$$

y realizar el programa.