Proof of Equivalence

Proof of Equivalence

- The two channel models are equivalent
  - Model 1: The channel is an LTI system with step response
    $$s(n) = k(1-a^{n+1})u(n)$$
    where $0 < a < 1$
  - Model 2: The channel has input $x(n)$ and output $y(n)$ determined by
    $$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$
    where $0 < a < 1$
- We can establish equivalence by showing that
  - Model 2 is LTI (linear and time invariant)
  - Model 2 has a response to a unit step given by $s(n)$

Fact 1: Model 2 is LTI

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$

- **It is linear**
  - **Intuitively, the output is a linear function of the input and past outputs.**
  - **Formally, we will prove that it satisfies the two properties:**
    - **Homogeneity**
    - **Additivity**
- **It is time invariant**
  - **Intuitively, the parameters k and a do not change with *n*.**
  - **Formally, we prove that if we delay the input by d, the output is the same, just delayed by *d*.**

Proof of Homogeneity

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n) \Rightarrow cx(n) \rightarrow \boxed{\text{system}} \rightarrow cy(n)$$

1. **By the definition of model 2,**

   $$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$

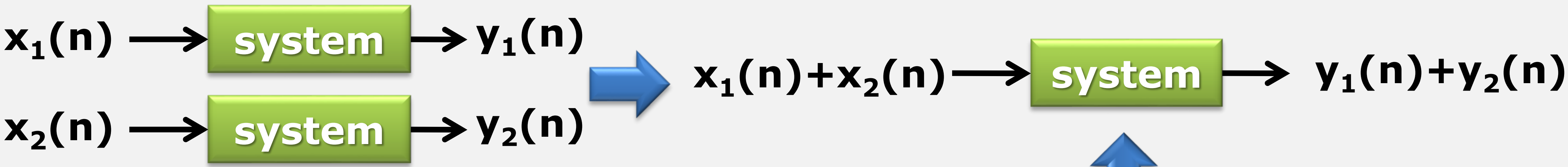
2. **Multiply left and right by c,**

   $$c \cdot y(n) = c \cdot [a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)]$$

3. **Distributive law of multiplication,**

   $$[c \cdot y(n)] = a \cdot [c \cdot y(n-1)] + (1-a) \cdot k \cdot [c \cdot x(n)]$$

Proof of Additivity



1. **By definition of model 2,**

$$y_1(n) = a \cdot y_1(n-1) + (1-a) \cdot k \cdot x_1(n)$$
$$y_2(n) = a \cdot y_2(n-1) + (1-a) \cdot k \cdot x_2(n)$$

2. **Adding the two equations above,**

$$[y_1(n) + y_2(n)] = a \cdot [y_1(n-1) + y_2(n-1)] + (1-a) \cdot k \cdot [x_1(n) + x_2(n)]$$

Fact 1: Model 2 is LTI

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$

- **It is linear**
  - **Intuitively, the output is a linear function of the input and past outputs.**
  - **Formally, we will prove that it satisfies the two properties:**
    - **Homogeneity**
    - **Additivity**
- **It is time invariant**
  - **Intuitively, the parameters k and a do not change with *n*.**
  - **Formally, we prove that if we delay the input by d, the output is the same, just delayed by *d*.**

Proof of Time Invariance

$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n) \Rightarrow x(n-d) \rightarrow \boxed{\text{system}} \rightarrow y(n-d)$

1. **By the definition of model 2,**

   $$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$

2. **Substitute n with n-d,**

   $$y(n-d) = a \cdot y(n-d-1) + (1-a) \cdot k \cdot x(n-d)$$

Fact 2: Same Step Response (example)

Assume: $y(-1) = 0,\ a = \frac{1}{2},\ k = 1$

n	Model 1 $s(n) = 1 - (\frac{1}{2})^{n+1}$	Model 2 $y(n) = \frac{1}{2} y(n-1) + \frac{1}{2}$
0	$s(0) = 1 - (\frac{1}{2})^1 = 1 - \frac{1}{2} = \frac{1}{2}$	$y(0) = \frac{1}{2} y(-1) + \frac{1}{2} = \frac{1}{2} \cdot 0 + \frac{1}{2} = \frac{1}{2}$
1	$s(1) = 1 - (\frac{1}{2})^2 = 1 - \frac{1}{4} = \frac{3}{4}$	$y(1) = \frac{1}{2} y(0) + \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{4} = \frac{3}{4}$
2	$s(2) = 1 - (\frac{1}{2})^3 = 1 - \frac{1}{8} = \frac{7}{8}$	$y(2) = \frac{1}{2} y(1) + \frac{1}{2} = \frac{1}{2} \cdot \frac{3}{4} + \frac{4}{8} = \frac{7}{8}$
3	$s(3) = 1 - (\frac{1}{2})^4 = 1 - \frac{1}{16} = \frac{15}{16}$	$y(3) = \frac{1}{2} y(2) + \frac{1}{2} = \frac{1}{2} \cdot \frac{7}{8} + \frac{8}{16} = \frac{15}{16}$
4	$s(4) = 1 - (\frac{1}{2})^5 = 1 - \frac{1}{32} = \frac{31}{32}$	$y(4) = \frac{1}{2} y(3) + \frac{1}{2} = \frac{1}{2} \cdot \frac{15}{16} + \frac{16}{32} = \frac{31}{32}$
5	$s(5) = 1 - (\frac{1}{2})^6 = 1 - \frac{1}{64} = \frac{63}{64}$	$y(5) = \frac{1}{2} y(4) + \frac{1}{2} = \frac{1}{2} \cdot \frac{31}{32} + \frac{32}{64} = \frac{63}{64}$

Summary

- The channel can be described (modeled) in two equivalent ways
  - Model 1:
    - Channel is linear and time invariant
    - Channel has step response

$$s(n) = k(1-a^{n+1})u(n)$$

  - Model 2:
    - If $x(n)$ is the channel input and $y(n)$ is the output

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$