We have considered two models of our communication channel

- Model 1: The channel is a linear time invariant (LTI) system with an exponential step response

$$s(n) = k(1-a^{n+1})u(n) \quad \text{where } 0 < a < 1$$

- Model 2: The channel has input x(n) and output y(n) determined by

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n) \quad \text{where } 0 < a < 1$$

In order to establish equivalence, it is sufficient to show that Model 2 is LTI and that Model 2 has an exponential step response. In the lecture video, we showed that Model 2 is LTI, but only gave an example showing that the response $y(n)$ of Model 2 when the input $x(n)$ is a unit step is the same as $s(n)$ for a particular choice of a and k and limited number of values of n,

In this document, we prove that the response $y(n)$ of Model 2 when $x(n) = u(n)$ is given by $s(n)$ for all values of $a \in (0, 1)$, $k \in (-\infty, \infty)$ and $n \in (-\infty, \infty)$.

<u>Proof:</u>

To start, we prove that $y(n) = 0$ for all $n < 0$. Since $u(n) = 0$ for all $n < 0$,

$$y(n) = a \cdot y(n-1) \quad \text{for all } n < 0$$

Since $a \neq 0$, this is satisfied if and only if $y(n) = 0$ for all $n < 0$. Since $s(n) = 0$ for all $n < 0$, we have that $y(n) = s(n)$ for all $n < 0$.

We then prove that $y(n) = s(n) = k(1-a^{n+1})$ for all $n \geq 0$ using mathematical induction.

First note that since $u(n) = 1$ for all $n \geq 0$, $y(n) = ay(n-1) + k(1-a)$ for all $n \geq 0$.

<u>Base step:</u> Since $y(-1) = 0$,

$$\begin{aligned} y(0) &= ay(-1) + k(1-a) \\ &= k(1-a) \\ &= k(1-a^{0+1}) \\ &= s(0) \end{aligned}$$

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Inductive step: Assume that for some $n \geq 0$, $y(n) = s(n)$, then

$$\begin{aligned} y(n+1) &= ay(n) + k(1-a) \\ &= ak(1-a^{n+1}) + k(1-a) \\ &= k(a - a^{n+2} + 1 - a) \\ &= k(1-a^{n+2}) \\ &= s(n+1) \end{aligned}$$

Thus, by mathematical induction, $y(n) = s(n)$ for all $n \geq 0$.