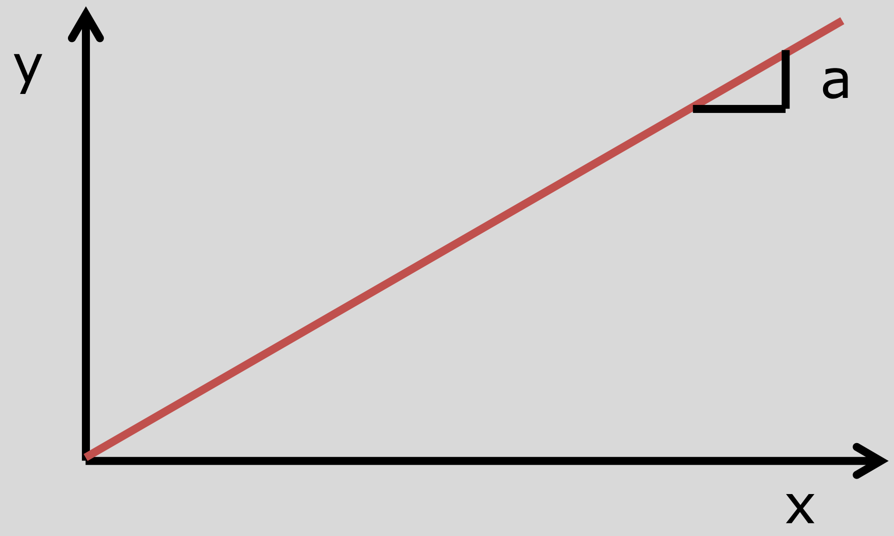
Linear Time Invariant Systems

Linear Functions

- **Function**: something that takes in an input number x and produces an output number y

$$x \longrightarrow \boxed{f(x)} \longrightarrow y$$

- A **linear function** has the form y = ax



Note: y = ax+b is not linear (unless b=0).

Properties of Linear Functions

- **Homogeneity:**

$$x \longrightarrow \boxed{ax} \longrightarrow y \quad \Rightarrow \quad cx \longrightarrow \boxed{ax} \longrightarrow cy$$

- **Additivity:**

$$\begin{array}{l} x_1 \longrightarrow \boxed{ax} \longrightarrow y_1 \\ x_2 \longrightarrow \boxed{ax} \longrightarrow y_2 \end{array} \quad \Rightarrow \quad x_1+x_2 \longrightarrow \boxed{ax} \longrightarrow y_1+y_2$$

Output of Linear Functions

- If you know
  - a function is linear, and
  - the output for <u>any</u> nonzero input

then, you can compute the output for any other input using **homogeneity** and **additivity**

Example: Suppose a linear function has output $y = 4$ for input $x = 2$.

- Use additivity to determine the output if $x = 4$

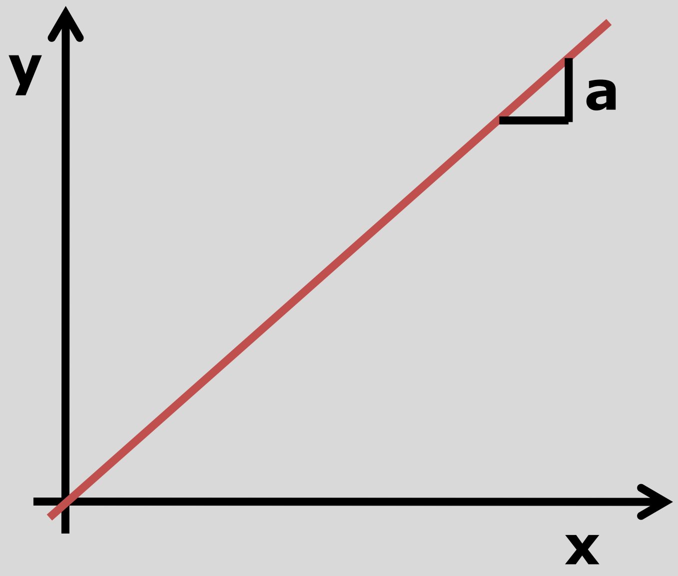
  Since $x = 2 + 2$,

  $y = 4 + 4 = 8$

- Use homogeneity to determine the output if $x = 6$

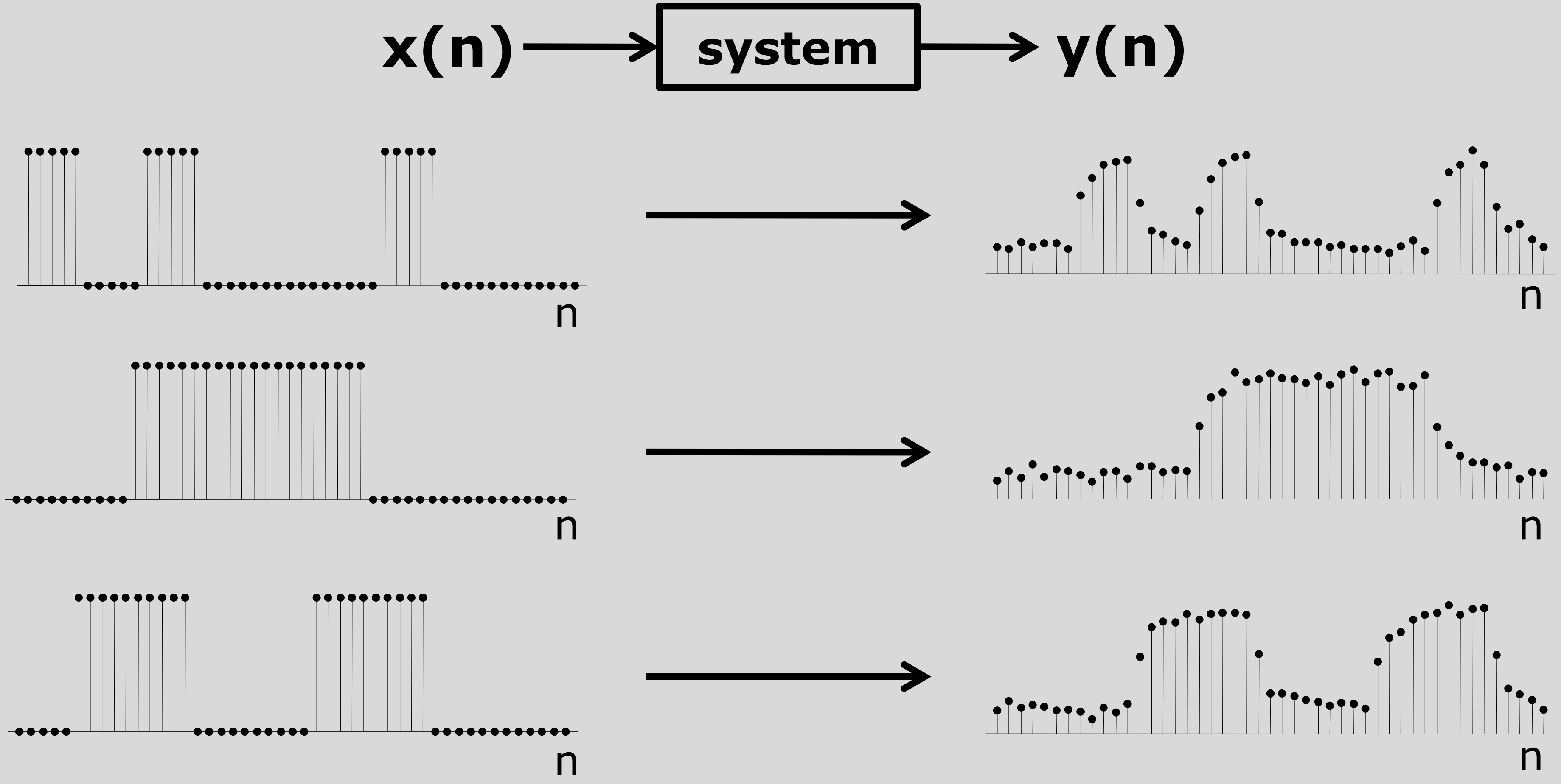
  Since $x = 3 \cdot 2$,

  $y = 3 \cdot 4 = 12$



Systems

System: something that takes in input waveform x(n) and produces an output waveform y(n).



Linear Systems

- A **linear system** is a system that satisfies the same two properties as a linear function.
- Homogeneity:

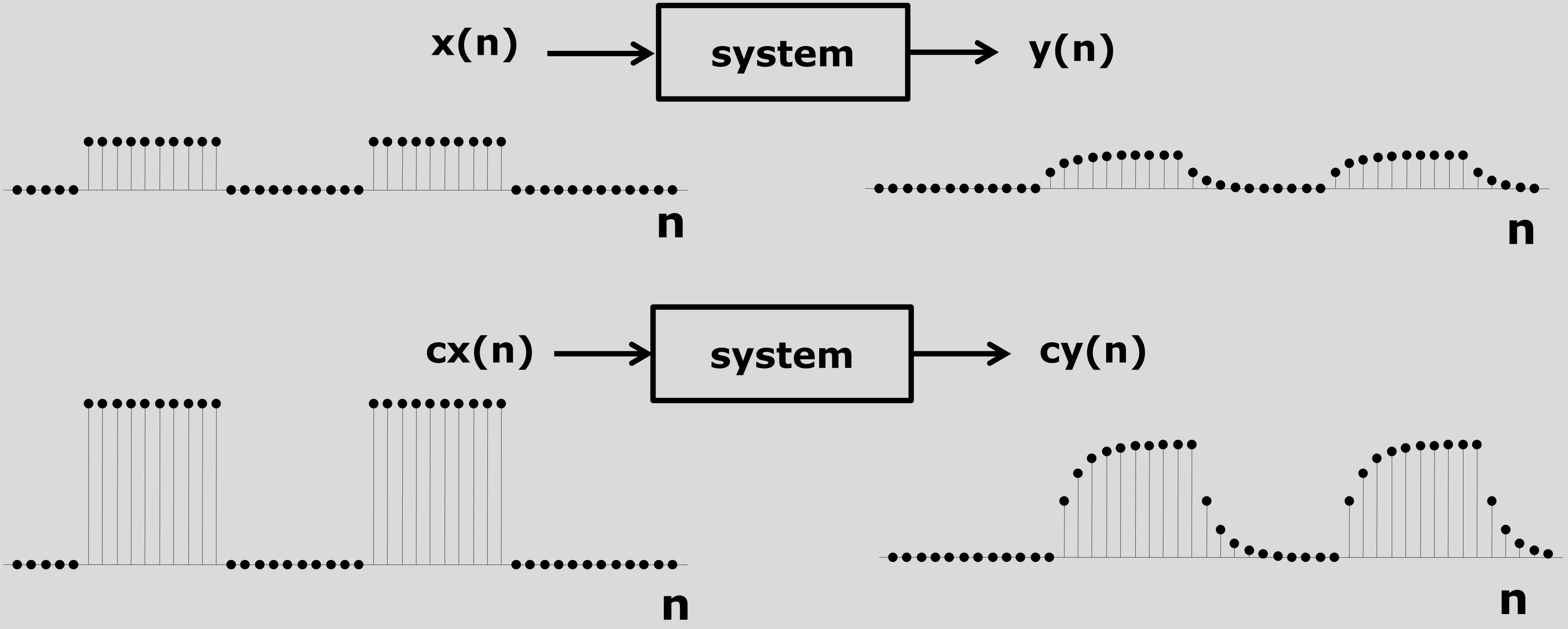
$$x(n) \longrightarrow \boxed{\text{system}} \longrightarrow y(n) \quad \Rightarrow \quad cx(n) \longrightarrow \boxed{\text{system}} \longrightarrow cy(n)$$

- Additivity

$$\begin{array}{l} x_1(n) \longrightarrow \boxed{\text{system}} \longrightarrow y_1(n) \\ x_2(n) \longrightarrow \boxed{\text{system}} \longrightarrow y_2(n) \end{array} \quad \Rightarrow \quad x_1(n)+x_2(n) \longrightarrow \boxed{\text{system}} \longrightarrow y_1(n)+y_2(n)$$

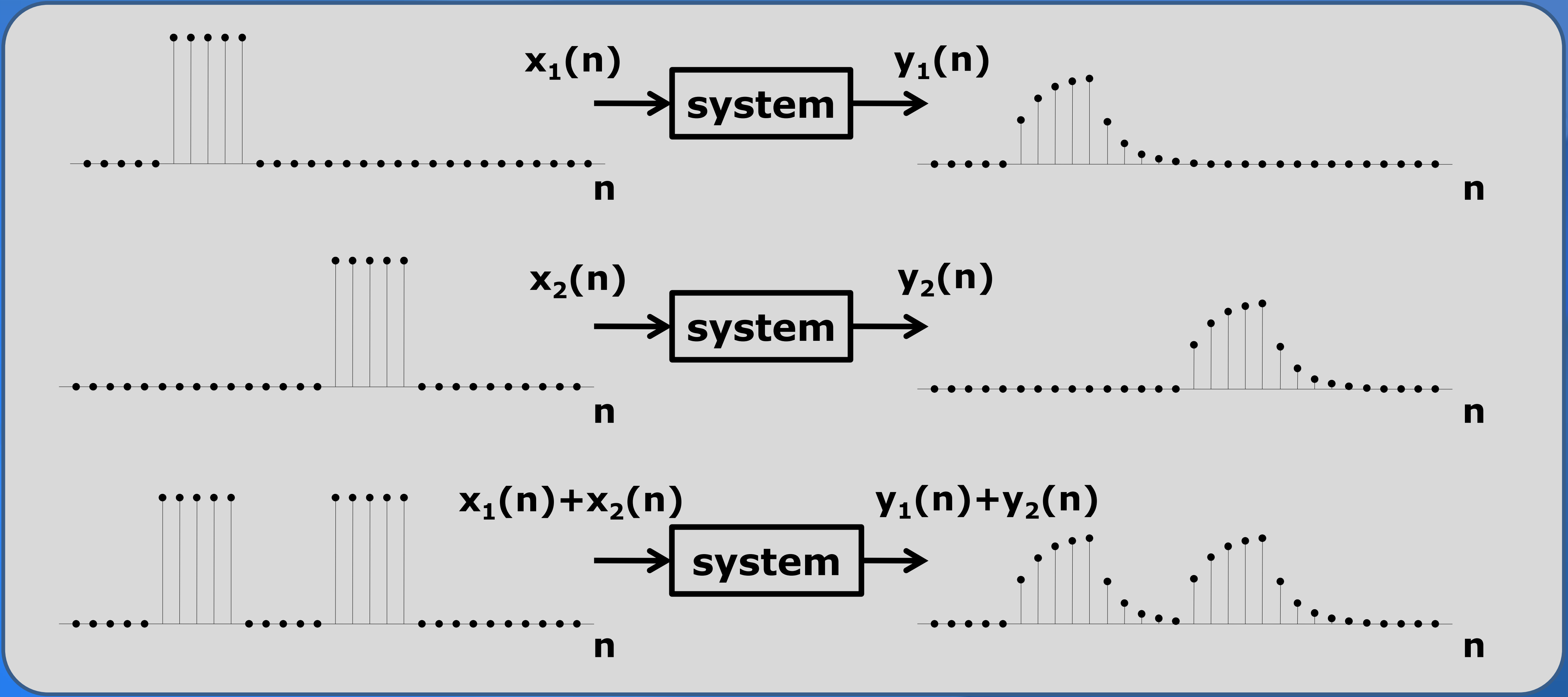
Homogeneity

- If you scale (multiply) the input by c, the output scales by c.



Additivity

- The output to the sum of two inputs is the sum of the outputs to each input applied individually.



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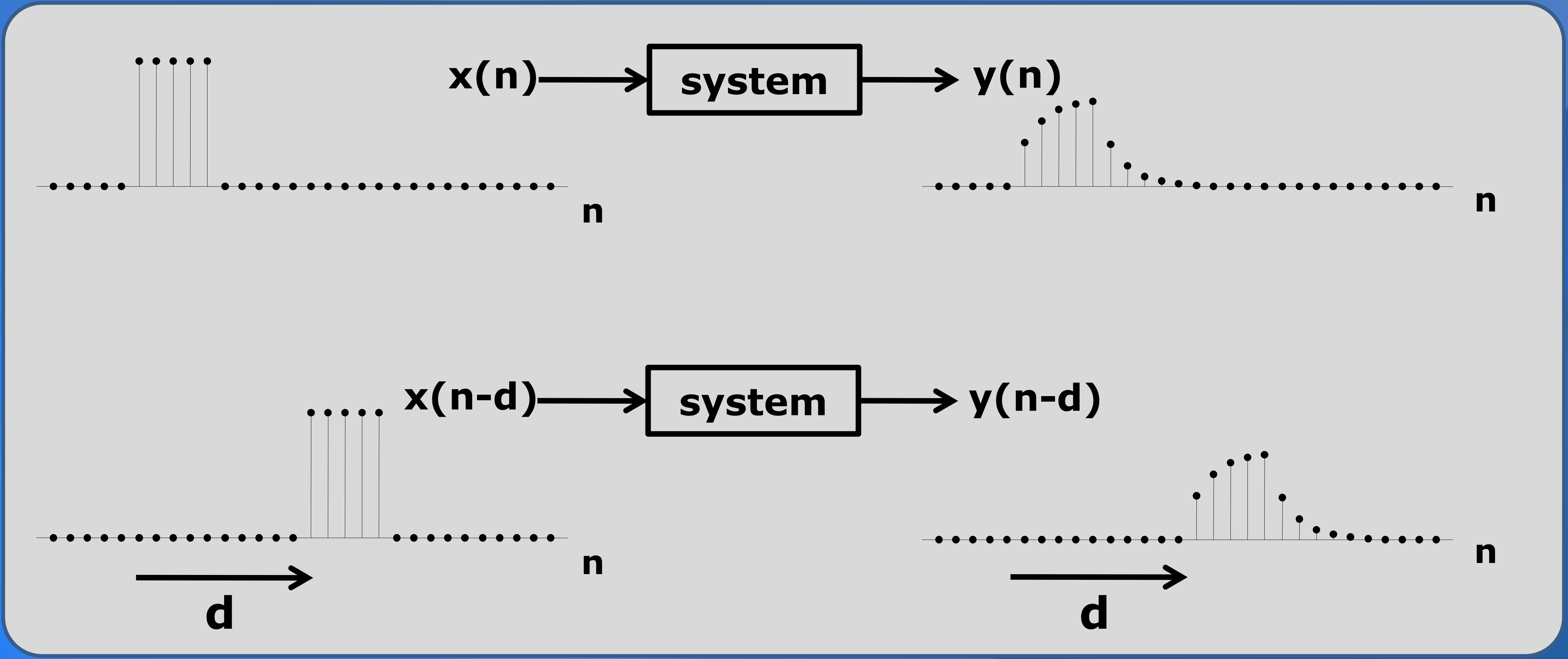
Time Invariance

- A system is **time invariant** if when you delay the input by *d*, the output is the same, just delayed by *d*.

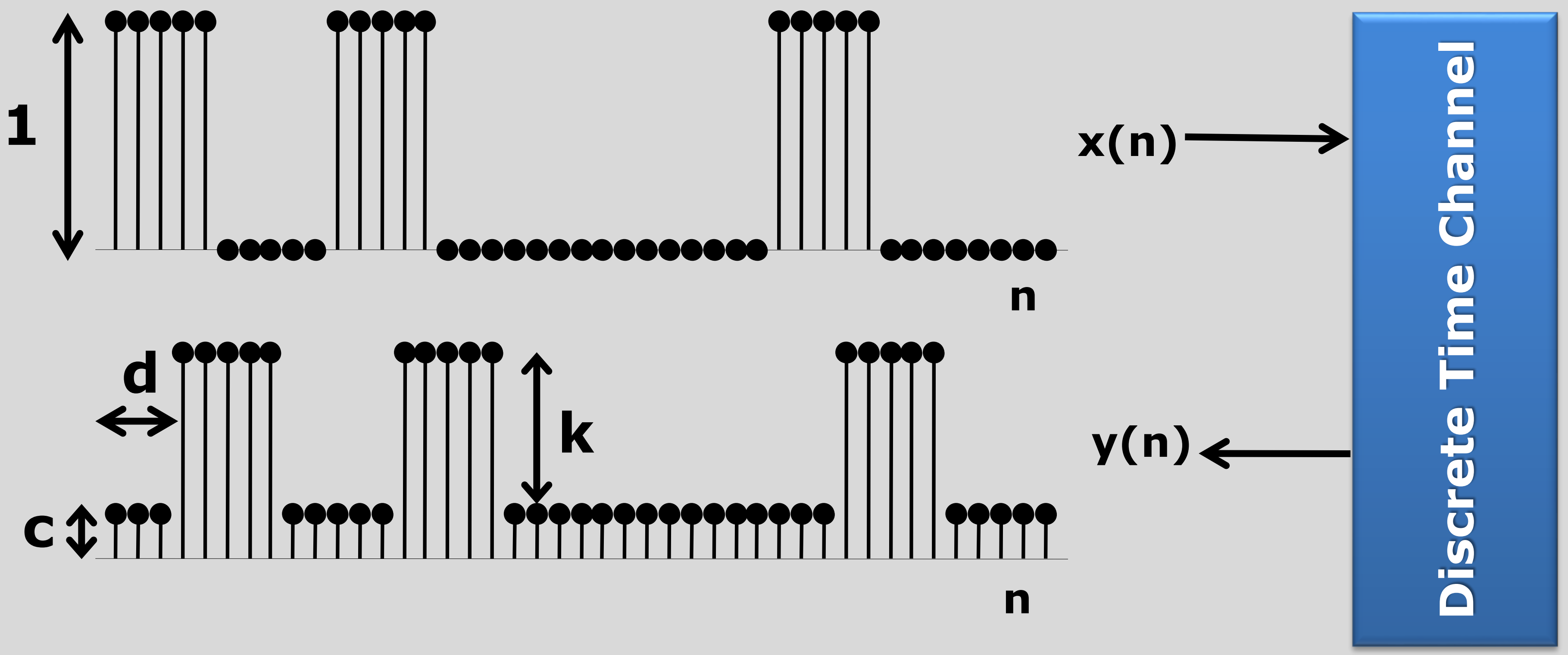


Linear Time Invariant (LTI) Systems

- **LTI system**: A system that is both linear and time invariant

Question: Is the channel shown below LTI?

$$y(n) = k \cdot x(n-d) + c$$



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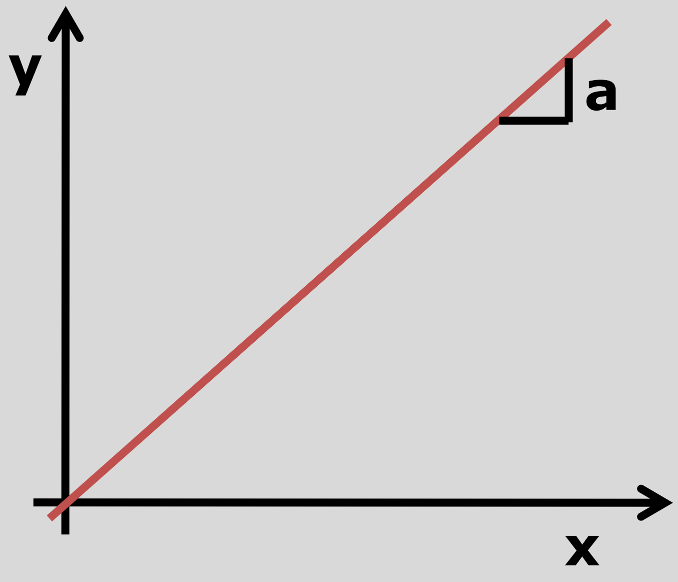
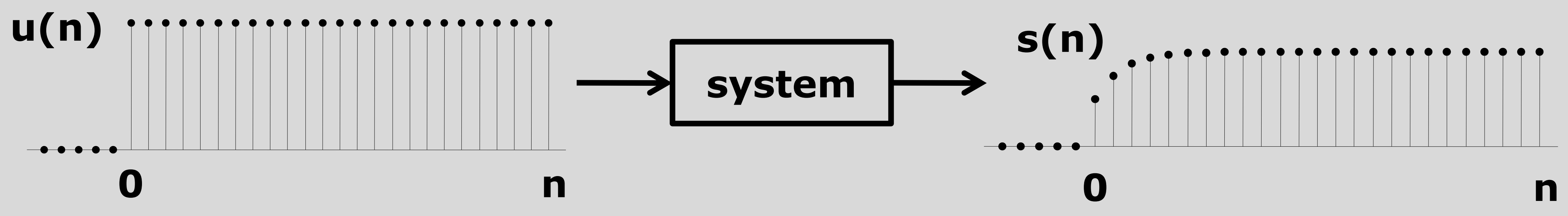
  Since x = 3 · 2,



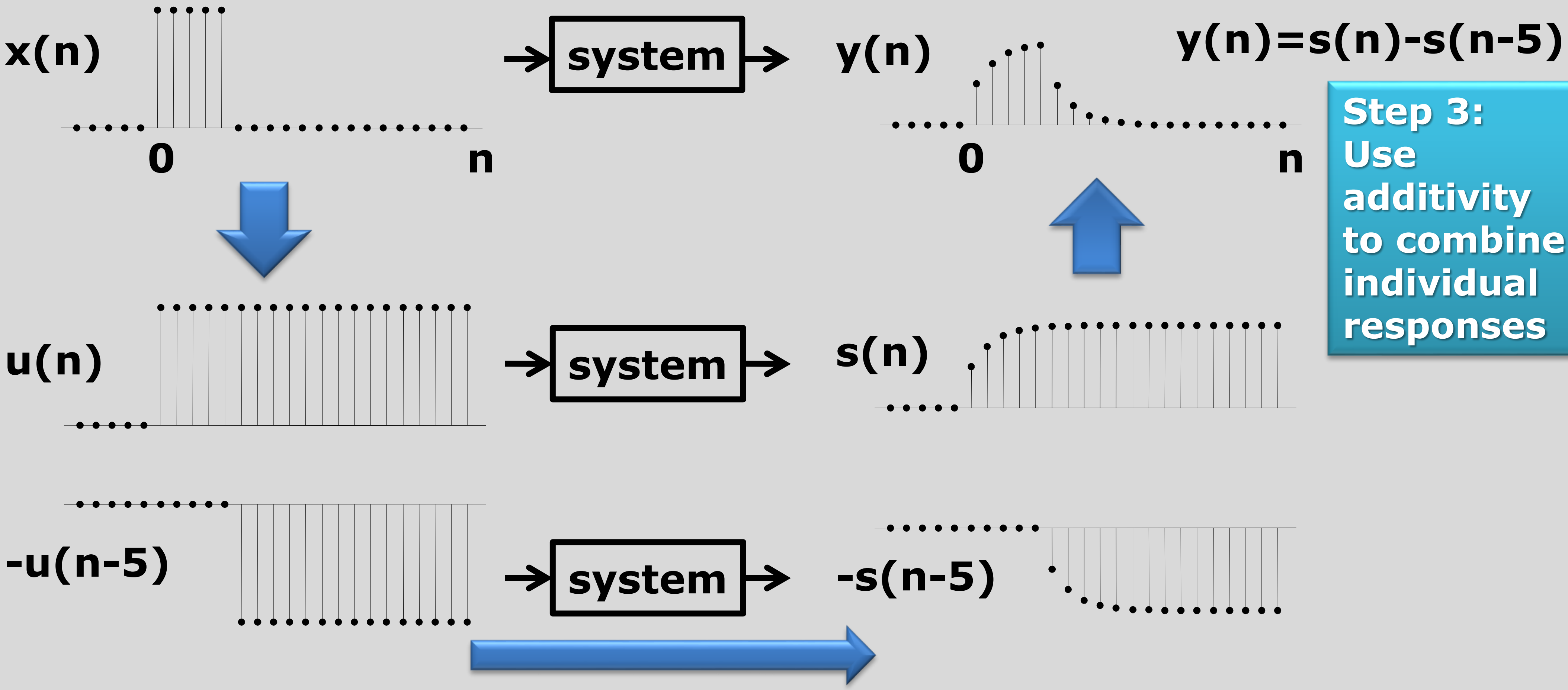
Step Response

- If a system is LTI, then you can find the output just by knowing the output to almost any non-zero input function.
- We choose the unit step function as the input.
- step response s(n): the output to the unit step input



Computing the Output of an LTI System

Step 1:
Write the input as the sum of scaled unit step functions



Step 3:
Use additivity to combine individual responses

$x(n)=u(n) - u(n-5)$

Step 2:
Use homogeneity and time invariance to compute responses to individual steps