Complex Exponentials

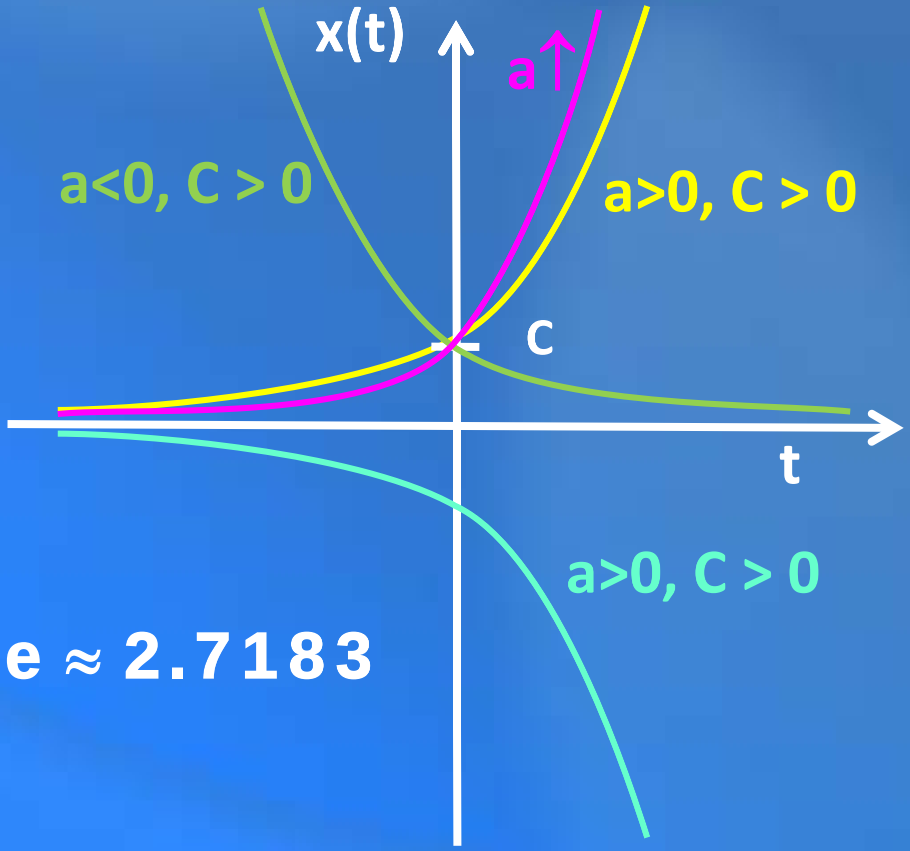
Exponential function

The exponential function is defined by the following power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Note that x can be real or complex.

Properties



$e \approx 2.7183$

(plots assume a is real valued)

$$x(t) = C \cdot e^{at}$$
$$= C \cdot \exp(at)$$

Properties:

$$e^{at} \cdot e^{bt} = e^{at+bt}$$
$$= e^{(a+b)t}$$

$$(e^{a \cdot t})^n = e^{n \cdot a \cdot t}$$

$$\frac{e^{a \cdot t}}{e^{b \cdot t}} = e^{a \cdot t - b \cdot t}$$
$$= e^{(a-b) \cdot t}$$

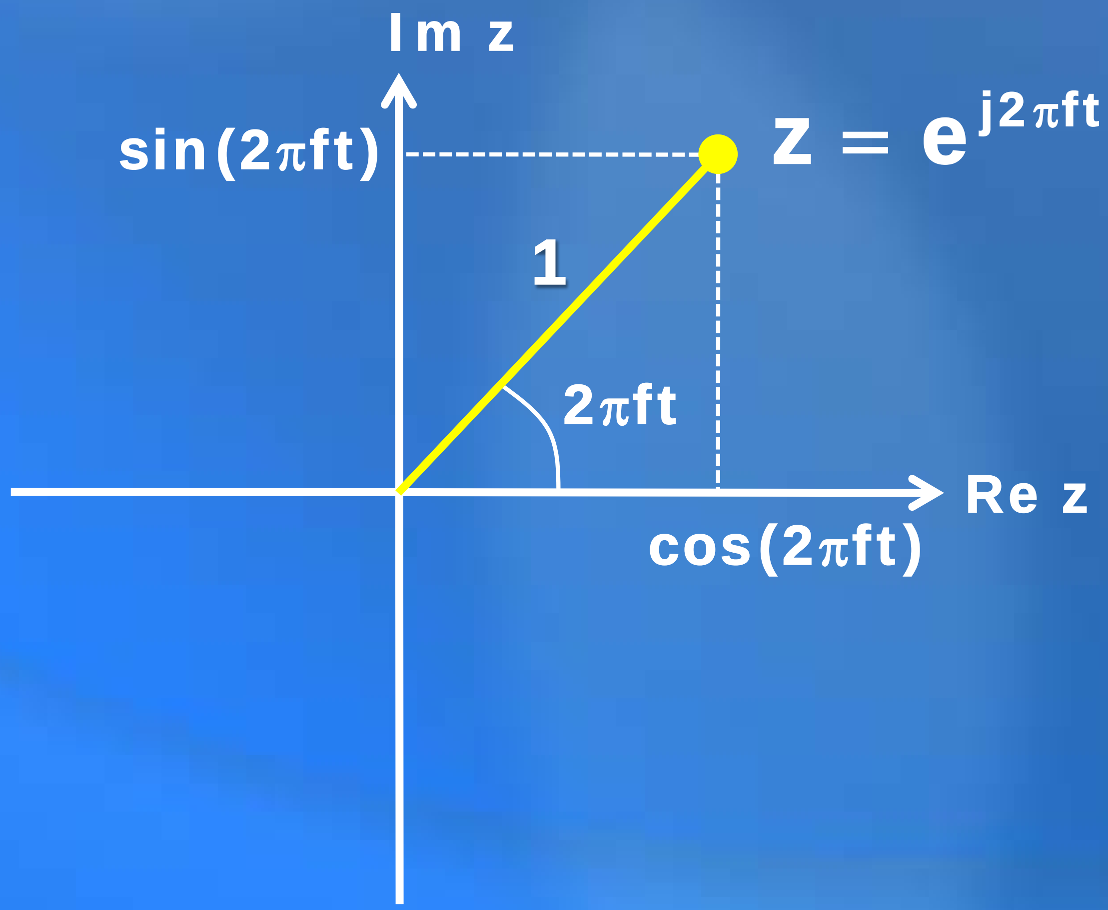
$$\frac{1}{e^{b \cdot t}} = e^{-b \cdot t}$$

Euler's identity

Euler's identity relates the exponential with imaginary input and trigonometric functions.

$$
\begin{aligned}
e^{j2\pi ft} &= 1 + (j2\pi ft) + \frac{(j2\pi ft)^2}{2!} + \frac{(j2\pi ft)^3}{3!} + \ldots \\
&= \left(1 - \frac{(2\pi ft)^2}{2!} + \ldots\right) + j\left(2\pi ft - \frac{(2\pi ft)^3}{3!} + \ldots\right) \\
&= \cos(2\pi ft) + j\sin(2\pi ft)
\end{aligned}
$$

Complex Exponential



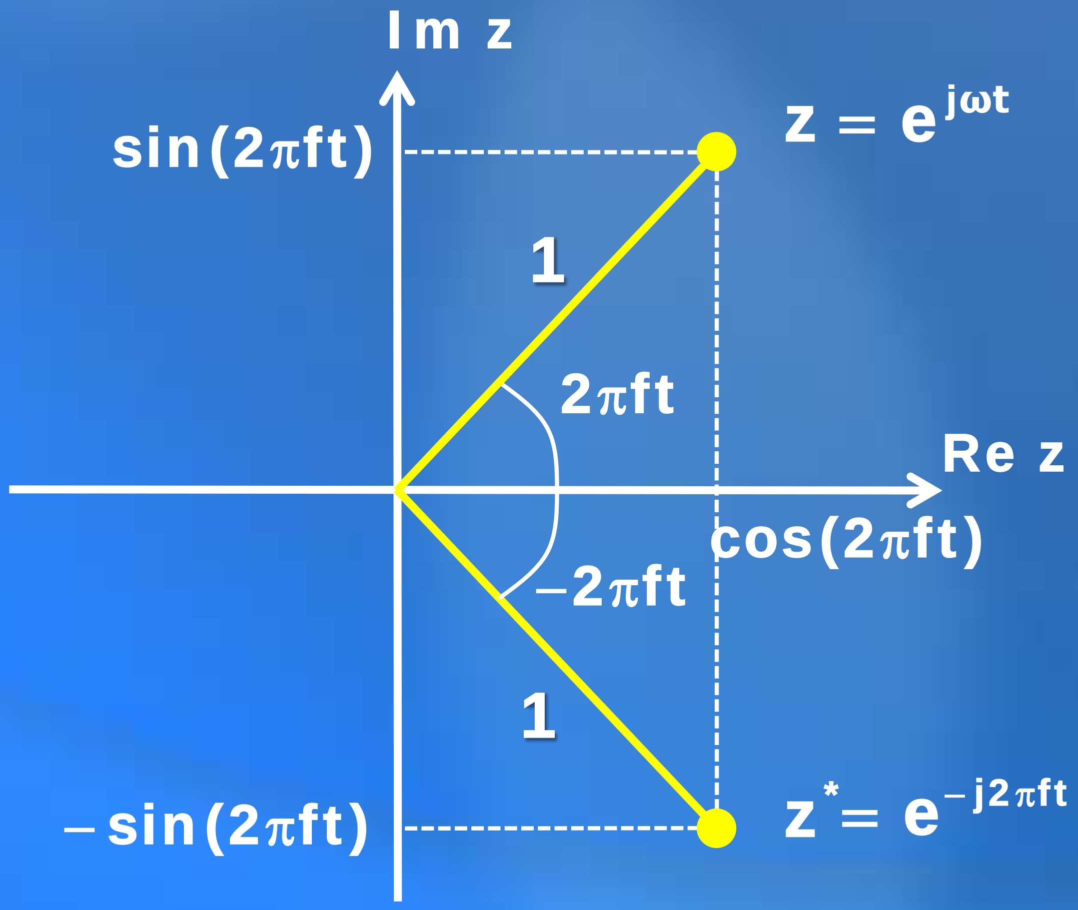
$$|z| = \sqrt{\cos^2(2\pi ft) + \sin^2(2\pi ft)} = 1$$

$$\angle z = \arctan\left(\frac{\sin(2\pi ft)}{\cos(2\pi ft)}\right) = 2\pi ft$$

Note that $e^{j2\pi ft}$ is periodic

with period $T = \dfrac{1}{f}$

Complex Conjugate

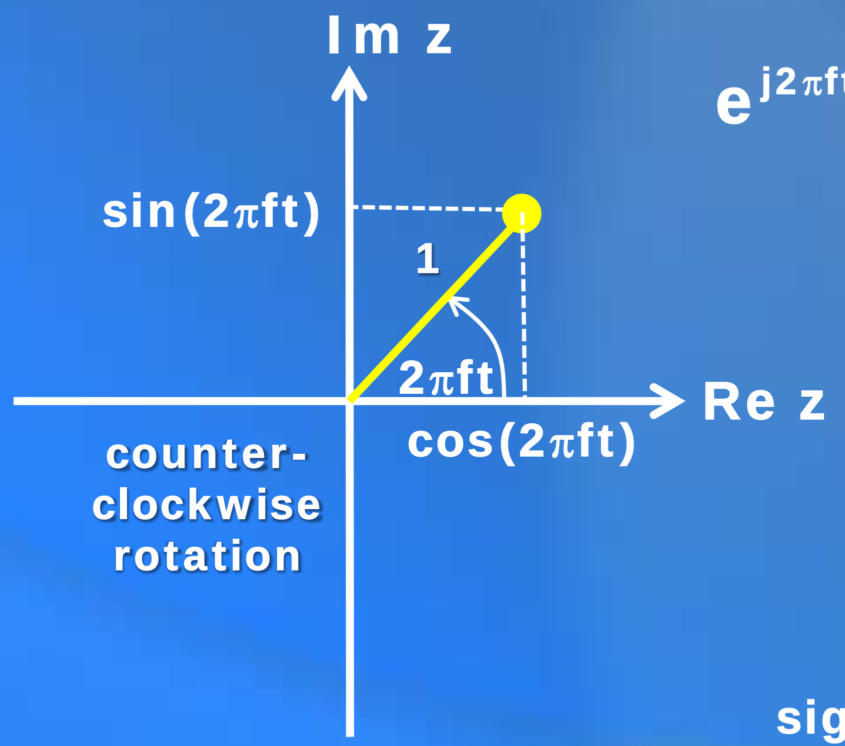


Thus,

$$\left(e^{j2\pi ft}\right)^* = e^{-j2\pi ft}$$

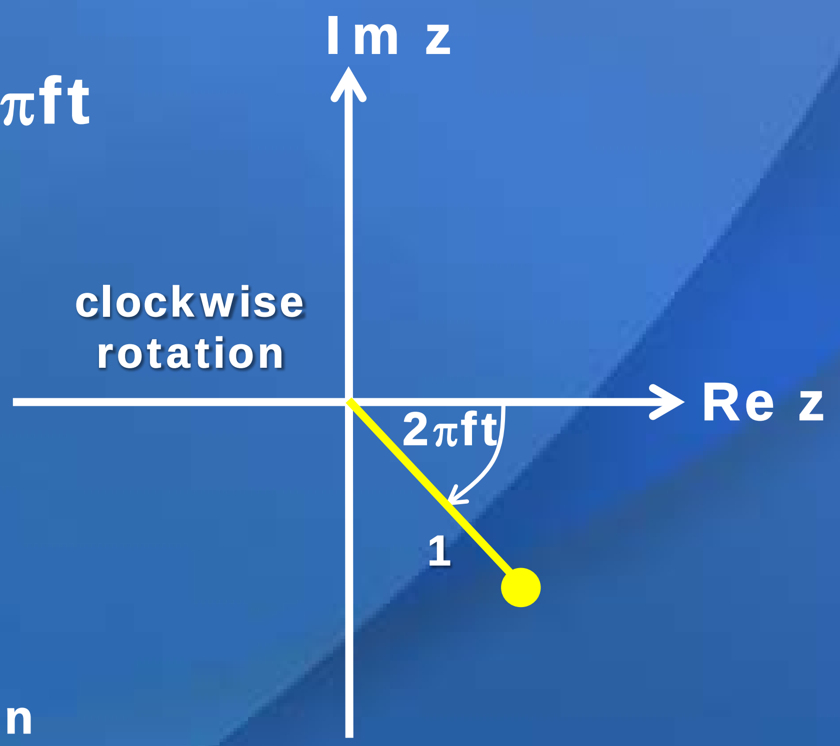
Positive / Negative Frequency

Positive (f > 0)



$$e^{j2\pi ft} = \cos 2\pi ft + j\sin 2\pi ft$$

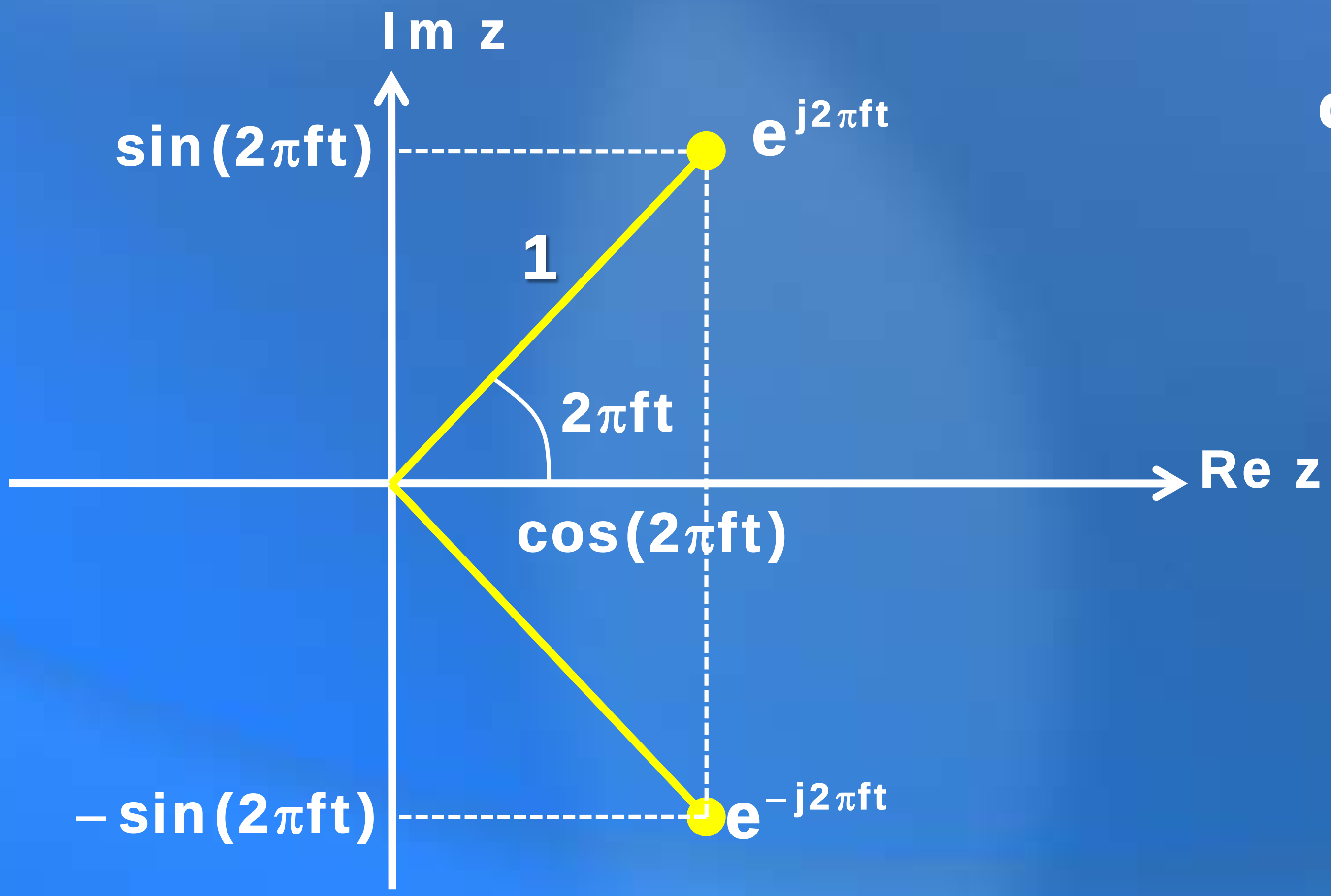
Negative (f < 0)



$|f|$ determines speed

sign(f) determines direction

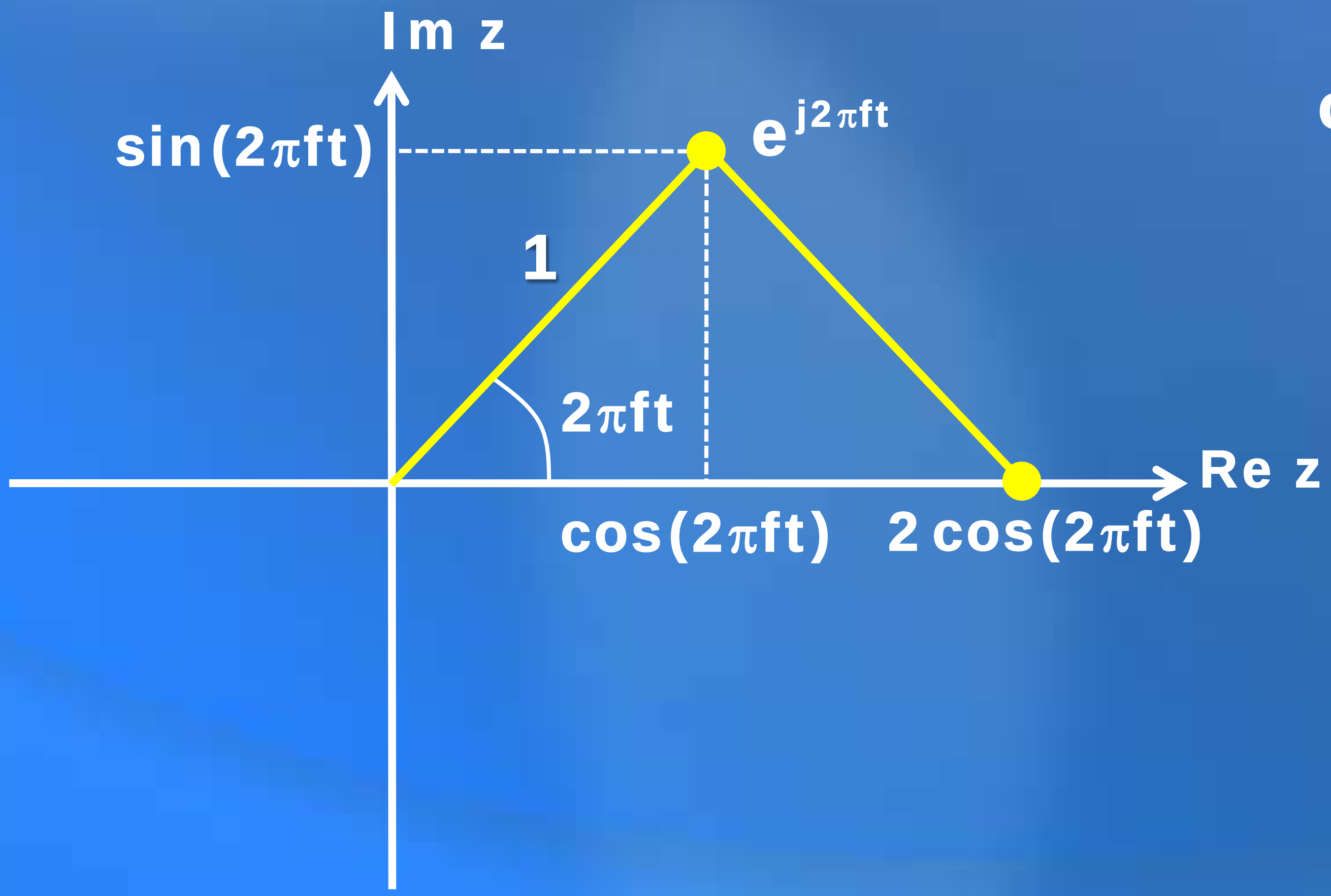
From $e^{j2\pi ft}$ to $\cos(2\pi ft)$



$$\cos(2\pi ft) = \mathrm{Re}\left\{e^{j2\pi ft}\right\}$$

$$e^{j2\pi ft} + e^{-j2\pi ft}$$

From $e^{j2\pi ft}$ to $\cos(2\pi ft)$



$$\cos(2\pi ft) = \mathrm{Re}\left\{e^{j2\pi ft}\right\}$$
$$= \frac{1}{2}\left(e^{j2\pi ft} + e^{-j2\pi ft}\right)$$

Other useful formulas

Sine: $\sin 2\pi ft = \mathrm{Im}\left\{e^{j2\pi ft}\right\}$

$$= \frac{1}{2j}\left(e^{j2\pi ft} - e^{-j2\pi ft}\right)$$

Phase shift: $\cos(2\pi ft + \theta) = \mathrm{Re}\left\{e^{j(2\pi ft+\theta)}\right\}$

$$= \mathrm{Re}\left\{e^{j\theta} \cdot e^{j2\pi ft}\right\}$$

$$= \frac{1}{2}\left(e^{j\theta} \cdot e^{j2\pi ft} + e^{-j\theta} \cdot e^{-j2\pi ft}\right)$$