HUDM 5123 - Linear Models and Experimental Design
Lab 04 - Inferences for Group Means

1 The Data

We will use the same data today as in Lab 03. A quick recap is that the data are related to an attempted replication of a study of positive psychology interventions as compared with a control; see the introduction to Lab 03 for more details and references. Once again we will use the AHI and CES-D total score differences (one month follow-up minus baseline) as the two primary outcomes.

1. SS; signature strengths
2. TGT; three good things
3. GV; gratitude visit
4. EM; early memories

Before running tests related to group means, let's get a table of necessary information, similar to the table that we created in the notes for the blood pressure example, which also had four group means. Group samples sizes can be summarized by making a table of the intervention factor variable:

```
table(rct_wide$int_fact)
```

```
 SS TGT  GV REM
 29  43  30  37
```

Table 1: Table of descriptive statistics related to group sample means for AHI gain scores (one month follow-up minus baseline AHI score)

Estimand	Estimator	Estimate	Est. Variance	Est. SD
μ_1	$\bar{Y}_{1\cdot}$	6.31	$\frac{116.79}{29} = 4.03$	$\sqrt{4.03} = 2.01$
μ_2	$\bar{Y}_{2\cdot}$	3.16	$\frac{116.79}{43} = 2.72$	$\sqrt{2.72} = 1.65$
μ_3	$\bar{Y}_{3\cdot}$	5.17	$\frac{116.79}{30} = 3.89$	$\sqrt{3.89} = 1.97$
μ_4	$\bar{Y}_{4\cdot}$	3.00	$\frac{116.79}{37} = 3.16$	$\sqrt{3.16} = 1.78$
σ^2	MSE	116.79		

The formula for the (1-α) confidence interval for the group i mean, μ_i, is

$$\bar{Y}_{i\cdot} \pm t_{n_T - r}^{1-\alpha/2} s_{\bar{Y}_{i\cdot}}$$

Here, $n_T = 139$ and $r = 4$ groups, so there are 135 df associated with MSE. For the (1 - .05) confidence interval, need to calculate the .975 quantile of the t distribution with 135 degrees of freedom. From R:

```
qt(p = .975, df = 135, lower.tail = TRUE)
[1] 1.977692
```

For group 1, the interval is given by $6.31 \pm 1.98(2.01) = 6.31 \pm 3.98 = (2.33, 10.29)$. We can verify these calculations by using package **emmeans** to display group means and descriptive statistics. In particular, output from running `emmeans()` provides the group sample means and the standard error (SE), which is $s_{\bar{Y}_{i\cdot}}$:

$$s_{\bar{Y}_{i\cdot}} = \sqrt{\frac{\text{MSE}}{n_i}}$$

along with 95% confidence interval bounds. For group 1, the confidence bounds are within .01 (i.e., rounding error) of the values we calculated 'by hand'.

```
library(emmeans)
emm1 <- emmeans(object = lm1,
                specs = ~ int_fact)
emm1

 int_fact emmean   SE  df lower.CL upper.CL
 SS         6.31 2.01 135   2.3416    10.28
 TGT        3.16 1.65 135  -0.0965     6.42
 GV         5.17 1.97 135   1.2646     9.07
 REM        3.00 1.78 135  -0.5136     6.51

Confidence level used: 0.95
```

Task 1 *Create a table like Table 1 above for the CES-D gain scores. Get MSE from the summary of a regression model. You may use* `emmeans()` *to get sample means. In addition, add the lower and upper bounds for 95% confidence intervals. Write out the confidence interval formula for group 1 and use it to verify the* `emmeans()` *output for* μ_1*. You only need to do this verification with the formula for* μ_1*, not for any of the other group means.*

2 Testing a Single Factor Level Mean

Suppose we wish to test that the average gain score (one month follow-up minus baseline) for the AHI outcome for the SS intervention is positive. That is,

$$\begin{aligned} H_0 :& \mu_1 \leq 0 \\ H_1 :& \mu_1 > 0 \end{aligned}$$

If we are thinking (and we should be), we should already know that the test will come out significant because the two-sided 95% confidence interval calculated above did not include zero. Nevertheless, we will carry out the calculations to get the p-value for the test. Recall that the test statistic $t*$ is given by

$$t^* = \frac{\bar{Y}_{i\cdot} - c}{s_{\bar{Y}_{i\cdot}}} = \frac{6.31 - 0}{2.01} = 3.14$$

The p-value is given by the area under the t distribution with 135 df to the right of 3.14. That is, $p = .001$.

```
pt(q = 3.14, df = 135, lower.tail = FALSE)
[1] 0.001037937
```

There is evidence that the mean change in happiness as measure by the AHI from baseline to one month post follow-up is positive ($t_{135} = 3.14; p = .001$). To extract this information from package **emmeans**, use the `text()` function. The argument `side` can be specified as "<", ">", or "!=" for left-tailed, right-tailed, and two-tailed, respectively. Here, our alternative is that $\mu_1 > 0$, so the test is right-tailed.

```
test(object = emm1, side = ">")
 int_fact emmean   SE  df t.ratio p.value
 SS         6.31 2.01 135 3.145   0.0010
 TGT        3.16 1.65 135 1.919   0.0285
 GV         5.17 1.97 135 2.619   0.0049
 REM        3.00 1.78 135 1.689   0.0468

P values are right-tailed
```

Task 2 *Test whether depression went down, on average, in the SS group as measured by change in the CES-D. Show work and verify your calculations with output from (emmeans).*

3 Testing Pairwise Comparisons

Let D be the difference between two factor level means, say μ_i and $\mu_{i'}$: $D = \mu_i - \mu_{i'}$. Then, $\hat{D} = \bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}$ is an estimator for D and the standard deviation of the estimator may be estimated by $s_{\hat{D}} = \sqrt{\text{MSE}\left(\frac{1}{n_i} + \frac{1}{n_{i'}}\right)}$. Begin by testing the first pairwise comparison. The test is two-tailed because there was no strong *a priori* hypothesis about which group should perform better.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Recall that

$$t* = \frac{\hat{D} - 0}{s_{\hat{D}}} = \frac{6.31 - 3.16}{\sqrt{116.79\left(\frac{1}{29} + \frac{1}{43}\right)}} = \frac{3.15}{2.60} = 1.21$$

The p-value is given by the two times the area under the t distribution with 135 df to the right of 1.21. That is, $p = .23$.

```
2*pt(q = 1.21, df = 135, lower.tail = FALSE)
[1] 0.2283933
```

The null hypothesis cannot be rejected; there is insufficient evidence to conclude the mean of group 1 differs from the mean of group 2 ($t_{135} = 1.21; p = .23$). To extract this information from package **emmeans**, use the `pairs()` function. So far as I can tell, the `pairs` function only produces two-tailed p-values, so you would have to convert them manually to one-sided if desired. Confirm that our calculations for the estimate, it's SE, the t value, and p value match with output from **emmeans**.

```
pairs(emm1, adjust = "none")
 contrast  estimate   SE  df t.ratio p.value
 SS - TGT     3.148 2.60 135  1.212  0.2276
 SS - GV      1.144 2.81 135  0.406  0.6851
 SS - REM     3.310 2.68 135  1.235  0.2189
 TGT - GV    -2.004 2.57 135 -0.779  0.4371
 TGT - REM    0.163 2.42 135  0.067  0.9465
 GV - REM     2.167 2.66 135  0.816  0.4159
```

Task 3 *Test for a pairwise difference between SS and TGT as above for the CES-D gain scores, showing the formula and substitutions. Interpret your conclusion in context. Use output from package* ***emmeans*** *to verify that your calculations are correct.*

4 Testing Linear Contrasts

Here we will test if the three positive psychology interventions led to more happiness, on average, than the control group. Let L be the following linear contrast:

$$L = \frac{1}{3}\mu_1 + \frac{1}{3}\mu_2 + \frac{1}{3}\mu_3 - \mu_4$$

The hypotheses:

$$H_0 : L \leq 0$$
$$H_1 : L > 0$$

Begin by calculating the value of the t statistic

$$t* = \frac{\hat{L} - 0}{s_{\hat{L}}}$$

where $s_{\hat{L}} = \sqrt{MSE \sum_{i=1}^{r} \left(\frac{c_i^2}{n_i}\right)}$

$$t* = \frac{\hat{L} - 0}{s_{\hat{L}}} = \frac{\frac{1}{3}6.31 + \frac{1}{3}3.16 + \frac{1}{3}5.17 - 3.00}{\sqrt{116.79\left(\frac{\left(\frac{1}{3}\right)^2}{29} + \frac{\left(\frac{1}{3}\right)^2}{43} + \frac{\left(\frac{1}{3}\right)^2}{30} + \frac{(-1)^2}{37}\right)}} = \frac{1.88}{2.08} = 0.90$$

The p-value is given by the area under the t distribution with 135 df to the right of 0.90. That is, $p = .18$.

```
pt(q = 0.9, df = 135, lower.tail = FALSE)
[1] 0.1848614
```

The null hypothesis cannot be rejected; there is insufficient evidence to conclude the three positive psychology interventions led to more happiness, on average, than the control group ($t_{135} = 0.90; p = .18$). To extract this information from package **emmeans**, use the `(contrast)` function. Pass the contrast coefficients to the argument `method` as a list. Confirm that our calculations for the estimate, it's SE, the t value, and p value match with output from **emmeans**.

```
contrast(emm1, method = list(c(1/3, 1/3, 1/3, -1)))
contrast                                     estimate   SE  df t.ratio p.value
c(0.333333333, 0.333333333, 0.333333333, -1     1.88 2.08 135 0.903   0.3684
```

Task 4 *Replicate the above contrast analysis for the CES-D change score with the exception that the hypothesis is that the positive psychology interventions decreased depression relative to the control. Show calculations and interpret results and verify your calculations by comparing to output from* ***emmeans****.*

Task 5 *Finally, test whether the average of TGT and GV (ignoring SS) decreased depression more than the control (REM). Interpret results.*