Model with Financial Frictions

1 Physical Capital Accumulation and Firms

1.1 Physical capital producers

Physical capital producers produce private capital used in the economy. They sell the capital produced to entrepreneurs at a nominal price equal to $Q_t = P_t^D q_t$.

The time t profit of physical capital producers are:

$$Profit_t^{Inv} = \left\{ Q_t K_t - (1-\delta) Q_t K_{t-1} - P_t Inv_t - P_t Inv_t S\left(\frac{Inv_t}{Inv_{t-1}}\right) \right\} \tag{1}$$

Subject to a capital accumulation process:

$$K_t = (1-\delta) K_{t-1} + Inv_t \tag{2}$$

where $S()$ the investment adjustment cost follows:

$$S\left(\frac{Inv_t}{Inv_{t-1}}\right) = \frac{\phi_{Inv}}{2}\left(\frac{Inv_t}{Inv_{t-1}(1+g_t)} - 1\right)^2$$

Each capital producer solves the following maximisation problem:

$$\mathcal{L} = E_t \sum_{n=0}^{\infty} \frac{1}{\prod_{k=1}^{n}(1+i_{t+k-1})} \left\{ Profit_{t+n}^{Inv} + Q_{t+n}\left[Inv_{t+n} + (1-\delta)K_{t+n-1} - K_{t+n}\right] \right\} \tag{3}$$

FOC only $[Inv_t]$:

$$\left\{ -P_t - P_t S\left(\frac{Inv_t}{Inv_{t-1}}\right) - P_t S^1\left(\frac{Inv_t}{Inv_{t-1}}\right) \right\} + Q_t + \frac{1}{1+i_t}\left\{ -P_{t+1}^D S^2\left(\frac{Inv_{t+1}}{Inv_t}\right) \right\} = 0 \tag{4}$$

where $S^1\left(\frac{Inv_t}{Inv_{t-1}}\right) \equiv \frac{\partial S\left(\frac{Inv_t}{Inv_{t-1}}\right)}{\partial Inv_t}$ is the partial derivative of $S\left(\frac{Inv_t}{Inv_{t-1}}\right)$ with respect to Inv_t and it is:

$$S^1\left(\frac{Inv_t}{Inv_{t-1}}\right) = \phi_{Inv}\left(\frac{Inv_t}{Inv_{t-1}(1+g_t)} - 1\right)\frac{Inv_t}{Inv_{t-1}(1+g_t)} \tag{5}$$

and $S^2\left(\frac{Inv_{t+1}}{Inv_t}\right) \equiv \frac{\partial S\left(\frac{Inv_{t+1}}{Inv_t}\right)}{\partial Inv_t}$ is the partial derivative of $S(\frac{Inv_{t+1}}{Inv_t})$ with respect to Inv_t:

$$S^2\left(\frac{Inv_{t+1}}{Inv_t}\right) = \phi_{Inv}\left(\frac{Inv_{t+1}}{Inv_t(1+g_{t+1})} - 1\right)\left(-\frac{Inv_{t+1}^2}{Inv_t^2(1+g_{t+1})}\right) \tag{6}$$

The FOC becomes after some manipulation:

$$1 + S\left(\frac{Inv_t}{Inv_{t-1}}\right) + S^1\left(\frac{Inv_t}{Inv_{t-1}}\right) + \frac{1}{1+i_t}\frac{P_{t+1}}{P_t}S^2\left(\frac{Inv_{t+1}}{Inv_t}\right) = \frac{Q_t}{P_t} \tag{7}$$

$$1+S\left(\frac{Inv_t}{Inv_{t-1}}\right)+S'\left(\frac{Inv_t}{Inv_{t-1}}\right)\frac{Inv_t}{Inv_{t-1}(1+g_t)} - \frac{1}{1+i_t}\frac{P_{t+1}}{P_t}S'\left(\frac{Inv_{t+1}}{Inv_t}\right)\left(\frac{Inv_{t+1}^2}{Inv_t^2(1+g_{t+1})}\right) = \frac{Q_t}{P_t} \tag{8}$$

where the $S'(\cdot)$:

$$S'\left(\frac{Inv_t}{Inv_{t-1}}\right) = \phi_{Inv}\left(\frac{Inv_t}{Inv_{t-1}(1+g_t)} - 1\right) \tag{9}$$

1.2 Entrepreneurs and Banks

1.2.1 Timings of Events

- At period t, entrepreneurs enter the period with net worth N_t (nominal).
- They make capital investment by purchasing physical capital at price Q_t (nominal) of amount K_t.
- They borrow $Loan_t$ at price Z_t.
- Hence, they combine their net worth with bank loans to make capital investment $Q_t K_t = N_t + Loan_t$.
- They are hit by a shock ω, which determines how productive the capital is. The productive capital can be used for renting out to intermediate producers is $\omega_{t+1} K_t$. The nominal rental rate is $P_{t+1} r^k_{t+1}$. $t+1$ for ω means that the shock is realised after capital is purchased, and it is affecting capital used at period $t+1$.
- ω also determines whether some entrepreneurs will default on their loans.
- They sell un-depreciated capital back to physical capital producer at price Q_{t+1}.
- The gross nominal return of holding a unit of capital from period t to $t+1$ is Ret^k_{t+1}, and it is defined as:

$$
\begin{aligned}
Ret^k_t Q_{t-1} &= R^k_t + (1-\delta) Q_t \\
Ret^k_t &= \frac{R^k_t + (1-\delta) Q_t}{Q_{t-1}}
\end{aligned}
$$

  where the R^k is the rental fee (marginal product) of capital. We can divide both side of the equation by the CPI:

$$
\begin{aligned}
\frac{Ret^k_t}{P_t} &= \frac{R^k_t/P_t + (1-\delta) Q_t/P_t}{Q_{t-1}} \frac{P_{t-1}}{P_{t-1}} \\
ret^k_t = Ret^k_t \frac{P_{t-1}}{P_t} &= \frac{r^k_t + (1-\delta) q_t}{q_{t-1}}
\end{aligned}
$$

  which says that holding one unit of capital leads to an earning of real rental rate r^k_{t+1} deductible by capital gain tax and a re-sell price of $(1-\delta) q_{t+1} P_{t+1}$ for the non-depreciated part of the capital, $q_t = Q_t/P_t$ is the relative price of purchasing a unit of capital to domestic final good.

1.2.2 Banks

The banks operate in a competitive market with free entry.

- Banks take deposit from households (which is D_t) and pay a nominal risk-free rate, $(1+i_t)$.
- Banks extend loans ($Loan_t$) to entrepreneurs at a price Z_t.
- Assume that $D_t = Loan_t$, i.e., the loan amount equals to the amount of deposit that the bank receives from the households.

For the amount of $Loan_t$, the banks receive:

- $(1 - F(\bar{\omega}))(1 + i^e_t) Loan_t$ from the non-bankrupted entrepreneurs;
- $(1-\mu) \int_0^{\bar{\omega}_{t+1}} \omega\, f(\omega)\, dw\, Ret^k_{t+1} Q_t K_t$ from bankrupted entrepreneurs subject to a monitoring cost measured by a parameter μ^e.

The profit of the bank is:

$$Profit_t^{bank} = [1 - F(\bar{\omega}_{t+1})]\,(1+i_t^e)Loan_t + (1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega\, f(\omega)\, dw\, Ret_{t+1}^k Q_t K_t - (1+i_t)D_t, \quad (10)$$

Zero profit condition implies

$$[1 - F(\bar{\omega}_{t+1})]\,(1+i_t^e)Loan_t + (1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega\, f(\omega)\, dw\, Ret_{t+1}^k Q_t K_t = (1+i_t)D_t, \quad (11)$$

where $1 - F(\bar{\omega})$ is the probability for the entrepreneurs above the threshold $(\bar{\omega})$. The banks try to charge the entrepreneurs as much as possible to maximize their profit, but the entrepreneurs in the optimum will report the lowest possible productivity level to save its profitability. Reporting the lowest possible profitablity means that in the optimum the firms are report their ω on the threshold, and the total revenue of the bank from that segment can be given as:

$$(1+i_t^e)Loan_t = \bar{\omega}_{t+1} Ret_{t+1}^k Q_t K_t \quad (12)$$

The total deposit of the bank is used for loans $D_t = Loan_t$, and the bank balance sheet can be given as

$$Loan_t = Q_t K_t - N_t \quad (13)$$

Combining these conditions and identities into the zero-profit conditions:

$$[1 - F(\bar{\omega}_{t+1})]\,\bar{\omega}_{t+1} Ret_{t+1}^k Q_t K_t + (1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega\, f(\omega)\, dw\, Ret_{t+1}^k Q_t K_t = (1+i_t)(Q_t K_t - N_t), \quad (14)$$

We can rearrange it:

$$\left[[1 - F(\bar{\omega}_{t+1})]\,\bar{\omega}_{t+1} + (1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega\, f(\omega)\, dw\right] Ret_{t+1}^k Q_t K_t = (1+i_t)(Q_t K_t - N_t), \quad (15)$$

If we introduce the leverage ratio, in a compact version of the equation, we can show that the expected return on capital investment is the function of the benchmark rate and the spread. Leverage ratio is

$$Lev_t = \frac{Q_t K_t}{N_t}. \quad (16)$$

$$\left[[1 - F(\bar{\omega}_{t+1})]\,\bar{\omega}_{t+1} + (1-\mu^e)\int_0^{\bar{\omega}_{t+1}} \omega\, f(\omega)\, dw\right] Ret_{t+1}^k \frac{Q_t K_t}{N_t} = (1+i_t)\left(\frac{Q_t K_t}{N_t} - 1\right) \quad (17)$$

How can we treat the truncated joint distribution function in the squared bracket? The box below shows how to calculate the joint distribution function.

The ω follows log-normal distribution with the following properties:

$$\ln \omega \sim \mathcal{N}\left(-\frac{1}{2} \cdot \sigma^2, \sigma^2\right)$$

The corresponding probability density function is the following:

$$\begin{aligned} f(\omega) &= \frac{1}{\sqrt{2\pi}\omega\sigma} \exp\left\{-\frac{1}{2}\left(\frac{\ln \omega - E(\ln \omega)}{\sigma}\right)^2\right\} \\ f(\omega) &= \frac{1}{\sqrt{2\pi}\omega\sigma} \exp\left\{-\frac{1}{2}\left(\frac{\ln \omega + \frac{1}{2}\sigma^2}{\sigma}\right)^2\right\} \end{aligned}$$

The cumulative distribution function is denoted by $F(\omega)$. We want to express the cumulative distribution function with the normal distribution function. Integrating the density function give us the probability of the succes of the entrepreneurs:

$$\int_{\bar{\omega}}^{\infty} f(\omega) d\omega = \int_{\bar{\omega}}^{\infty} \frac{1}{\sqrt{2\pi}\omega\sigma} \exp\left\{-\frac{1}{2}\left(\frac{\ln \omega + \frac{1}{2}\sigma^2}{\sigma}\right)^2\right\} d\omega = \int_{\bar{\omega}}^{\infty} \frac{1}{\sqrt{2\pi}\omega\sigma} \exp\left\{-\frac{1}{2}y^2\right\} d\omega$$

Renaming power of the exponential function as

$$y = \frac{\ln \omega + \frac{1}{2}\sigma^2}{\sigma}$$

If the ω is at the cut-off, we recall y as $\bar{y}$:

$$\bar{y} = \frac{\ln \bar{\omega} + \frac{1}{2}\sigma^2}{\sigma}$$

We can rearrange it to express ω:

$$\omega = \exp\left\{y\sigma - \frac{1}{2}\sigma^2\right\}$$

The differential is

$$d\omega = \exp\left\{y\sigma - \frac{1}{2}\sigma^2\right\} \sigma dy$$

Plugging it back

$$\begin{aligned} \int_{\bar{\omega}}^{\infty} f(\omega) d\omega &= \int_{\bar{y}}^{\infty} \frac{1}{\sqrt{2\pi}\omega\sigma} \exp\left\{-\frac{1}{2}y^2\right\} \underbrace{\exp\left\{y\sigma - \frac{1}{2}\sigma^2\right\}}_{\omega} \sigma dy \\ &= \int_{\bar{y}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}y^2\right\} dy \end{aligned}$$

The last term is identical with the cumulative normal distribution function, with zero mean and one std deviation:

$$\int_{\bar{\omega}}^{\infty} f(\omega) d\omega = 1 - \Phi(\bar{y})$$

We can express the second term implicitly, first we derive the value of $\int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega$ and the rest of the interval between zero and $\bar{\omega}$ gives the probability of default. Let's introduce a new variable (later it helps us for simplifying the integration):

$$\tilde{y} = \frac{\ln \omega - \frac{1}{2}\sigma^2}{\sigma}$$

where the difference between y and $\tilde{y}$ is σ. Then the $y = \tilde{y} + \sigma$. We can rearrange it to express ω:

$$\omega = \exp\left\{\tilde{y}\sigma + \frac{1}{2}\sigma^2\right\}$$

The differential is

$$d\omega = \exp\left\{\tilde{y}\sigma + \frac{1}{2}\sigma^2\right\}\sigma d\tilde{y}$$

Turning to the integration

$$\begin{aligned}
\int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega &= \int_{\bar{\omega}}^{\infty} \frac{1}{\sqrt{2\pi}\omega\sigma}\omega \exp\left\{-\frac{1}{2}\left(\frac{\ln\omega + \frac{1}{2}\sigma^2}{\sigma}\right)^2\right\} d\omega \\
&= \int_{\bar{\omega}}^{\infty} \frac{1}{\sqrt{2\pi}\omega\sigma}\omega \exp\left\{-\frac{1}{2}y^2\right\} d\omega \\
&= \int_{\bar{\tilde{y}}}^{\infty} \frac{1}{\sqrt{2\pi}\omega\sigma}\omega \exp\left\{-\frac{1}{2}y^2\right\} \exp\left\{\tilde{y}\sigma + \frac{1}{2}\sigma^2\right\} \sigma d\tilde{y} \\
&= \int_{\bar{\tilde{y}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}y^2\right\} \exp\left\{\tilde{y}\sigma + \frac{1}{2}\sigma^2\right\} d\tilde{y} \\
&= \int_{\bar{\tilde{y}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\tilde{y}+\sigma)^2\right\} \exp\left\{\tilde{y}\sigma + \frac{1}{2}\sigma^2\right\} d\tilde{y} \\
&= \int_{\bar{\tilde{y}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\tilde{y}^2 + 2\tilde{y}\sigma + \sigma^2)\right\} \exp\left\{\tilde{y}\sigma + \frac{1}{2}\sigma^2\right\} d\tilde{y} \\
&= \int_{\bar{\tilde{y}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tilde{y}^2 - \tilde{y}\sigma - \frac{1}{2}\sigma^2 + \tilde{y}\sigma + \frac{1}{2}\sigma^2\right\} d\tilde{y} \\
&= \int_{\bar{\tilde{y}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tilde{y}^2\right\} d\tilde{y} = 1 - \Phi(\bar{\tilde{y}})
\end{aligned}$$

But as we know that $y = \tilde{y} + \sigma$, we can express the $\int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega$ as

$$\int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega \quad = \quad 1 - \Phi(\bar{y} - \sigma)$$

In the original problem the integration check the interval of 0 and $\bar{\omega}$ but based on the results above we can express it as

$$\int_{0}^{\bar{\omega}} \omega f(\omega) d\omega \quad = \quad \Phi(\bar{y} - \sigma)$$

Then the squared bracket can be given by the following way:

$$\begin{aligned}
\left[\left[1 - F(\bar{\omega}_{t+1})\right]\bar{\omega}_{t+1} + (1-\mu)\int_{0}^{\bar{\omega}_{t+1}} \omega\, f(\omega)\, dw\right] &= (1 - \Phi(\bar{y}_{t+1}))\bar{\omega}_{t+1} + (1-\mu)\Phi(\bar{y}_{t+1} - \sigma) \\
&= \underbrace{(1 - \Phi(\bar{y}_{t+1}))\bar{\omega}_{t+1} + \Phi(\bar{y}_{t+1} - \sigma)}_{\Gamma(\bar{\omega}_{t+1})} - \\
&- \mu \underbrace{\Phi(\bar{y}_{t+1} - \sigma)}_{G(\bar{\omega}_{t+1})} \\
&= \Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})
\end{aligned}$$

where the Φ assigns the normal distribution cumulative function with 0 mean and 1 variance, and the additional variable can be given as the function of ω:

$$\bar{y}_t = \frac{\ln\bar{\omega}_t + \frac{1}{2}\sigma^2}{\sigma}$$

The banks' zero profit condition can be rewritten as in real terms:

$$[\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{ret^k_{t+1}}{1+r_t} \frac{q_t K_t}{n_t} = \frac{q_t K_t}{n_t} - 1 \tag{18}$$

$$Lev_t = \frac{q_t K_t}{n_t} \tag{19}$$

1.2.3 Entrepreneurs

At period t,

- Entrepreneurs have net worth N_t.
- They borrow $Loan_t$ at price Z_t.
- They purchase capital from physical capital producer at nominal price Q_t.
- They rent out capital to intermediate good producers at a real rental rate r^k_{t+1}.
- They sell non-depreciated capital back to physical capital producers at nominal price Q_{t+1}.
- They are hit by a shock $\omega_{t+1} \rightarrow$ total capital stock available for rent becomes $\omega_{t+1} Q_t K_t$.
- ω_{t+1} is from a distribution and its CDF is $F(\omega; \sigma_t)$
- There is a threshold $\bar{\omega}$ such that $\bar{\omega}_{t+1} Q_t Ret^k_{t+1} K_t = Z_t Loan_t$, i.e., just the right amount of shock for them to pay back the loan.

Entrepreneurs maximise their expected wealth at the end of the loan contract subject to the zero-profit condition from the bank, i.e, equation (15).

We express entrepreneurs expected wealth as a ratio to the amount an entrepreneur could receive by depositing its net worth in a bank, following Christiano et al 2011. It is thus:

$$E_t \int_{\bar{\omega}_{t+1}}^{\infty} (\omega_{t+1} - \bar{\omega}_{t+1}) f(\omega_{t+1}) \frac{Ret^K_{t+1}}{1+i_t} Q_t K_t \, d\omega_{t+1} \tag{20}$$

We can rearrange the integral:

$$E_t \left\{ \int_{\bar{\omega}_{t+1}}^{\infty} (\omega_{t+1} - \bar{\omega}_{t+1}) f(\omega_{t+1}) \, d\omega_{t+1} \right\} \frac{Ret^K_{t+1}}{1+i_t} Q_t K_t \tag{21}$$

$$E_t \left\{ \int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} f(\omega_{t+1}) \, d\omega_{t+1} - \bar{\omega}_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) \, d\omega_{t+1} \right\} \frac{Ret^K_{t+1}}{1+i_t} Q_t K_t \tag{22}$$

$$E_t \left\{ 1 - \Phi(\bar{y}_{t+1} - \sigma) - \bar{\omega}_{t+1}(1 - \Phi(\bar{y}_{t+1})) \right\} \frac{Ret^K_{t+1}}{1+i_t} Q_t K_t \tag{23}$$

$$E_t \left\{ 1 - \Gamma(\bar{\omega}_{t+1}) \right\} \frac{Ret^K_{t+1}}{1+i_t} Q_t K_t \tag{24}$$

The Lagrangian representation of entrepreneurs' problem is:

$$\mathcal{L} = \max_{K_t, \bar{\omega}_{t+1}} E_t \left\{ [1 - \Gamma(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} Q_t K_t + \lambda_t \left([\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} Q_t K_t - (Q_t K_t - N_t) \right) \right\}$$

The Lagrangian multiplier is defined for period $t+1$ state of nature. Assuming that $\lambda_{t+1} > 0$, we have the following first order conditions:

$$\frac{\partial \mathcal{L}}{\partial K_t} = E_t \left\{ [1 - \Gamma(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} Q_t + \lambda_t \left([\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} Q_t - Q_t \right) \right\} = 0 \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\omega}_{t+1}} = -\Gamma'(\bar{\omega}_{t+1}) \frac{Ret^k_{t+1}}{1+i_t} Q_t K_t + \lambda_t \left[\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1}) \right] \frac{Ret^k_{t+1}}{1+i_t} Q_t K_t = 0 \tag{26}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} Q_t K_t - (Q_t K_t - N_t) = 0 \tag{27}$$

We can rearrange the equation 26 to express λ:

$$\begin{aligned}
\Gamma'(\bar{\omega}_{t+1}) \frac{Ret^k_{t+1}}{1+i_t} Q_t K_t &= \lambda_t [\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} Q_t K_t \\
\lambda_t &= \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})}
\end{aligned}$$

Combining the equations 25 and 26:

$$E_t \left\{ [1 - \Gamma(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} + \lambda_t \left([\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} - 1 \right) \right\} = 0$$

$$E_t \left\{ [1 - \Gamma(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} + \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})} \left([\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{Ret^k_{t+1}}{1+i_t} - 1 \right) \right\} = 0$$

$$E_t \left\{ [1 - \Gamma(\bar{\omega}_{t+1})] \frac{ret^k_{t+1}}{1+r_t} + \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})} \left([\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{ret^k_{t+1}}{1+r_t} - 1 \right) \right\} = 0$$

We can rearrange the equation above to show the difference between the expected return and nominal interest rate:

$$E_t \left\{ \left[1 - \Gamma(\bar{\omega}_{t+1}) + \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})} [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \right] \frac{ret^k_{t+1}}{1+r_t} - \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})} \right\} = 0$$

$$E_t \left[1 - \Gamma(\bar{\omega}_{t+1}) + \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})} [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \right] ret^k_{t+1} = (1+r_t) E_t \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})}$$

The spread between the expected return and real interest rate can be given as

$$\begin{aligned}
ret^k_{t+1} &= Spread_t (1+r_t) \\
Spread_t &= \left[1 - \Gamma(\bar{\omega}_{t+1}) + \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})} [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \right]^{-1} \frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G'(\bar{\omega}_{t+1})}
\end{aligned}$$

The derivatives of distribution function can be given as:

$$\Gamma'(\bar{\omega}_{t+1}) = \frac{\partial \Gamma(\bar{\omega}_{t+1})}{\partial \bar{\omega}_{t+1}} = \frac{\partial [\bar{\omega}_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) d\omega_{t+1} + \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1}]}{\partial \bar{\omega}_{t+1}} = \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) d\omega_{t+1} = 1 - \Phi(\bar{y}_{t+1})$$

$$G'(\bar{\omega}_{t+1}) = \frac{\partial G(\bar{\omega}_{t+1})}{\partial \bar{\omega}_{t+1}} = \frac{\partial \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1}}{\partial \bar{\omega}_{t+1}} = \bar{\omega}_{t+1} f(\bar{\omega}_{t+1})$$

for $t = 0, 1, 2, ...\infty$.

Net worth of entrepreneurs do not enter the first order conditions following the above equations, and thus Lev_t and $\bar{\omega}_{t+1}$ are the same for all entrepreneurs.[1]

1.2.4 Aggregation across all entrepreneurs

The total revenue of the firms:

$$V_t = (1 - \Gamma(\bar{\omega}_t)) Ret^k_t Q_{t-1} K_{t-1} = Ret^k_t Q_{t-1} K_{t-1} - \Gamma(\bar{\omega}_t) Ret^k_t Q_{t-1} K_{t-1}$$

The banks' zero profit conditions:

$$\begin{aligned}
[\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)] Ret^k_t Q_{t-1} K_{t-1} &= (1 + i_{t-1})(Q_{t-1} K_{t-1} - N_{t-1}) \\
\Gamma(\bar{\omega}_t) Ret^k_t Q_{t-1} K_{t-1} &= \mu G(\bar{\omega}_t) Ret^k_t Q_{t-1} K_{t-1} + (1 + i_{t-1})(Q_{t-1} K_{t-1} - N_{t-1})
\end{aligned}$$

[1] Suppose that entrepreneurs are subject to idiosyncratic shocks and have different histories. Solving for aggregate variables would require solving for a distribution of entrepreneurs according to their characteristics and solving for the law of motion for the distribution. Following assumptions made by Christiano et al (2011), results in same interest rate of bank loans for all entrepreneurs no matter their net worth, and additionally, loan amount is proportional to their net worth.

Combining the two conditions and substituting out the term with Γ:

$$\begin{aligned}
V_t &= Ret_t^k Q_{t-1}K_{t-1} - \mu G(\bar{\omega}_t) Ret_t^k Q_{t-1}K_{t-1} - (1+i_{t-1})(Q_{t-1}K_{t-1} - N_{t-1}) \\
&= (1-\mu G(\bar{\omega}_t)) Ret_t^k Q_{t-1}K_{t-1} - (1+i_{t-1})(Q_{t-1}K_{t-1} - N_{t-1}) \\
&= Ret_t^k Q_{t-1}K_{t-1} - \left[(1+i_{t-1}) + \frac{\mu^e G(\bar{\omega}_t) Ret_t^k Q_{t-1}K_{t-1}}{Q_{t-1}K_{t-1} - N_{t-1}^e}\right](Q_{t-1}K_{t-1} - N_{t-1}^e) \\
&= Ret_t^k Q_{t-1}K_{t-1} - \left[(1+i_{t-1}) + \frac{\mu G(\bar{\omega}_t) Ret_t^k Lev_{t-1}}{Lev_{t-1} - 1}\right](Q_{t-1}K_{t-1} - N_{t-1})
\end{aligned}$$

Each entrepreneur faces an identical and independent probability $(1-\gamma_t)$ to exit the economy.[2] With probability γ_t, the entrepreneur remains. The probability is random, and thus the net worth of entrepreneurs who survive is $\gamma_t \bar{V}_t$, where upper bar over a letter is the aggregate average value.

A fraction $1-\gamma_t$ of new entrepreneurs arrive in the economy. Together with the remaining entrepreneurs, they receive a transfer W_t^e to ensure that all entrepreneurs can obtain loans. Net worth across all entrepreneurs after the transfer, entry and exit is:

$$N_t = \gamma_t V_t + W_t^e \tag{28}$$

Substituting in V_t, we have:

$$N_t = \gamma_t Ret_t^k Q_{t-1}K_{t-1} - \gamma_t \left[(1+i_{t-1}) + \frac{\mu G(\bar{\omega}_t) Ret_t^k Lev_{t-1}}{Lev_{t-1} - 1}\right](Q_{t-1}K_{t-1} - N_{t-1}) + W_t^e \tag{29}$$

Expressing them in real-terms:

$$\begin{aligned}
N_t/P_t &= \gamma_t Ret_t^k Q_{t-1}K_{t-1}/P_t - \gamma_t \left[(1+i_{t-1}) + \frac{\mu G(\bar{\omega}_t) Ret_t^k Lev_{t-1}}{Lev_{t-1} - 1}\right](Q_{t-1}K_{t-1} - N_{t-1})/P_t + W_t^e/P_t \\
N_t/P_t &= \gamma_t Ret_t^k Q_{t-1}K_{t-1}/P_t \frac{P_{t-1}}{P_{t-1}} - \gamma_t \left[(1+i_{t-1}) + \frac{\mu G(\bar{\omega}_t) Ret_t^k Lev_{t-1}}{Lev_{t-1} - 1}\right](Q_{t-1}K_{t-1} - N_{t-1})/P_t \frac{P_{t-1}}{P_{t-1}} + W_t^e/
\end{aligned}$$

$$\begin{aligned}
N_t/P_t &= \gamma_t \left(Ret_t^k \frac{P_{t-1}}{P_t}\right)(Q_{t-1}/P_{t-1})K_{t-1} - \\
&- \gamma_t \left[(1+i_{t-1})\frac{P_{t-1}}{P_t} + \frac{\mu G(\bar{\omega}_t) Ret_t^k \frac{P_{t-1}}{P_t} Lev_{t-1}}{Lev_{t-1} - 1}\right]((Q_{t-1}/P_{t-1})K_{t-1} - (N_{t-1}/P_{t-1})) + W_t^e/P_t
\end{aligned}$$

$$n_t = \gamma_t ret_t^k q_{t-1}K_{t-1} - \gamma_t \left[(1+r_{t-1}) + \frac{\mu G(\bar{\omega}_t) ret_t^k Lev_{t-1}}{Lev_{t-1} - 1}\right](q_{t-1}K_{t-1} - n_{t-1}) + w_t^e$$

[2]This is to prevents entrepreneurs from accumulating net worth beyond a point at which they can self-finance the operation.