INSTITUTE FOR CAPACITY DEVELOPMENT

L-1: The Real Business Cycle (RBC) Model

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Course on DSGE Models, Kigali

Learning Objectives

- **Understand the methodology** of building, calibrating, and solving a small DSGE model

- Interpret the impulse responses of the simulated RBC model and explain the transmission mechanism of a productivity shock

- Code, solve, and simulate the RBC model in Dynare (W-1)

- Discuss the historical role, building blocks, and limitations of the Real Business Cycle (RBC) model

  ⇒Not a good policy tool

Outline

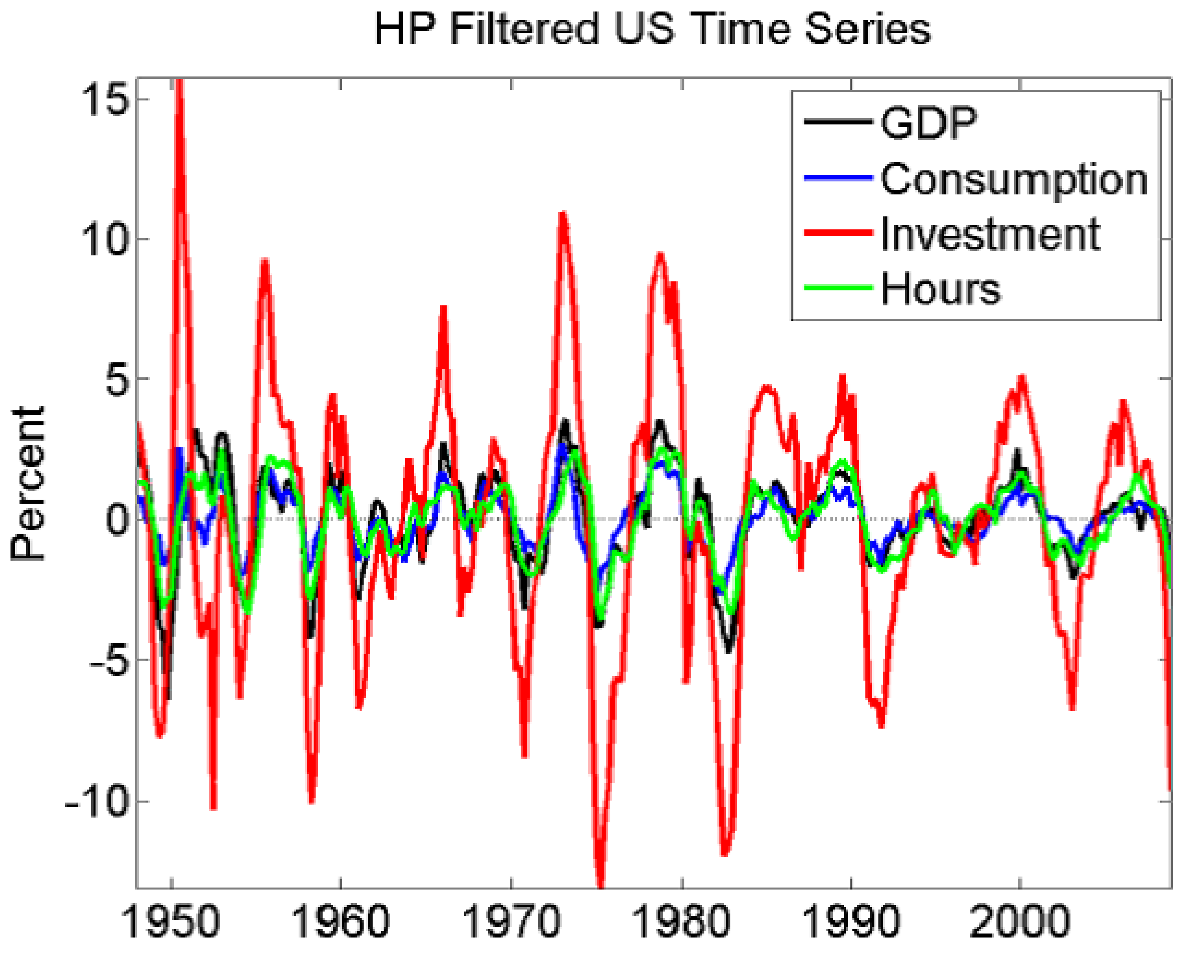
- The Real Business Cycle (RBC) Model and solution of forward-looking models

- The Classical Monetary (CM) model

$\Rightarrow$ Why the RBC model is not a good policy tool

- Workshop: How to code and simulate the RBC model in Dynare

Stylized Facts: US Quarterly Data (H-P Filtered), 1950:I – 2020:II



Stylized Facts: US Quarterly Data (H-P Filtered), 1950:I – 2020:II

		Correlation with GNP at t				
Variables	**Std**	t-2	t-1	t	t+1	t+2
GNP	1.72	0.63	0.85	1.00	0.85	0.63
Consumption	1.27	0.72	0.82	0.83	0.67	0.46
Investment	8.24	0.59	0.79	0.91	0.76	0.50
Hours	1.69	0.54	0.78	0.92	0.90	0.78
GNP/Hours	0.73	0.45	0.34	0.34	0.10	-0.09

The RBC Model and Neoclassical Growth models

- What drives business cycles? And key determinant of long-term growth?
- The RBC revolution–Kydland and Prescott (Econometrica, 1982) emphasized the role of technology shocks... The RBC revolution founded a framework for the future of macroeconomic models:
  - Established the use of DSGEs as central tools for macro analysis
  - Provided solid micro foundations
  - Used rational expectations
  - Stressed the quantitative aspects of macro modeling
    - Measured success as the model’s ability to replicate quantitatively some stylized “facts” of an economy
- The RBC is consistent with the neoclassical growth models:
  - Same framework is capable of analyzing cyclical shocks and trend productivity shocks.

Model: Household (HH) - Preferences

HH's problem

$$\underset{c_t, i_t, k_{t+1}, n_t}{Max} \quad \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \, U(c_t, 1 - n_t) \right\}$$

$s.t.$

$$c_t + i_t = w_t n_t + r_t k_t + \Omega_t$$

$$k_{t+1} = (1 - \delta) k_t + i_t$$

The Lagrangian Representation

Choose $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ to maximize the discounted Lagrangian

$$L_t = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{U(c_t, 1 - n_t) + \lambda_t [w_t n_t + (r_t^K + 1 - \delta)k_t + \Omega_t - c_t - k_{t+1}]\}$$

First-Order Conditions (FOCs)

$$\frac{\partial L_t}{\partial c_t} = 0 \Rightarrow \frac{\partial U_t}{\partial c_t} = \lambda_t \qquad (1)$$

$$\frac{\partial L_t}{\partial n_t} = 0 \Rightarrow \frac{\partial U_t}{\partial n_t} = \lambda_t w_t \qquad (2)$$

$$\frac{\partial L_t}{\partial k_{t+1}} = 0 \Rightarrow \lambda_t = \beta \mathbb{E}_t [\lambda_{t+1}(r_{t+1}^K + 1 - \delta)] \qquad (3)$$

The Intertemporal Condition or Euler Equation

- From equations (1) and (3)

$$\underbrace{\frac{\partial U_t}{\partial c_t}}_{\substack{\text{marginal cost of savings} \\ \text{at the present}}} = \underbrace{\beta \mathbb{E}_t \left[\frac{\partial U_{t+1}}{\partial c_{t+1}} (r_{t+1}^K + 1 - \delta) \right]}_{\substack{\text{marginal benefit of savings} \\ \text{in the next period}}}$$

- In the optimal case, the marginal cost of savings and benefit must be equal, otherwise the agents decide to reallocate the consumption-saving path. And it would not be optimal…

⇒Consumption smoothing: well-behaved preferences and the Euler-equation imply that in the rational equilibrium the consumption path is flat, and agents maintain stable consumption path. The shocks permanently shifts the consumption.

The Intratemporal Condition

- From equations (1) and (2)

$$\frac{\partial U_t/\partial n_t}{\partial U_t/\partial c_t} = MRS_t = w_t$$

- Intuition

$$\underbrace{\frac{\partial U_t}{\partial n_t}}_{\text{marginal disutility of working time}} = \underbrace{w_t \frac{\partial U_t}{\partial c_t}}_{\text{marginal benefit of working time}}$$

⇒ Implicit form of the labor supply: $n_t = n(\underset{+}{w_t}, \underset{-}{c_t})$

Model: Firms - Technology

- Technology: Aggregate production function

$$y_t = e^{z_t} F(K_t, N_t)$$

- Technology level z_t follows the law of motion

$$z_{t+1} = \rho z_t + \varepsilon_{t+1},$$

$$\rho \in (0,1) \ \text{ and } \ \varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

Model: Firms - Technology

- Firm's profit-maximization problem

$$\underset{K_t,N_t}{Max}\ \Omega_t = \underset{K_t,N_t}{Max}\ \underbrace{e^{z_t}F(K_t,N_t)}_{y_t} - r_t^K K_t - w_t N_t, \quad \forall t$$

- FOCs

$$r_t^K = \frac{\partial y_t}{\partial K_t} = e^{z_t}F_K(K_t,N_t)$$

$$w_t = \frac{\partial y_t}{\partial N_t} = e^{z_t}F_N(K_t,N_t)$$

⇒ Implicit form of the labor demand: $N_t = N(\underset{-}{w_t}, \underset{+}{K_t}, \underset{+}{z_t},)$

The Competitive Equilibrium (CE)

A CE is an allocation $\{c_t, i_t, k_t, n_t\}$ for the HH, an allocation $\{N_t, K_t, y_t\}$ for the firm, and the factor prices $\{w_t, r_t\}$ such that:

- Given the factor prices, the allocation $\{c_t, i_t, k_t, n_t\}$ solves the HH's problem
- Given the factor prices, the allocation $\{N_t, K_t, y_t\}$ solves the firm's problem
- Prices $\{w_t, r_t^K\}$ satisfy the firm's FOCs
- Markets clear

$$\begin{aligned} n_t &= N_t \\ k_t &= K_t \\ y_t &= c_t + i_t \end{aligned}$$

The Problem of the Social Planner (SP)

- The social planner's problem

$$\underset{c_t,\, k_{t+1},\, n_t}{Max} \;\; \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t) \right\}$$

s.t.

$$c_t + i_t \leq y_t$$

$$y_t = e^{z_t} F(k_t, n_t)$$

$$k_{t+1} = (1 - \delta)\; k_t + i_t$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}$$

The SP FOCs

Choose $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ to maximize the discounted Lagrangian

$$\mathfrak{L}_t = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{U(c_t, 1 - n_t) + \lambda_t [e^{z_t} F(k_t, n_t) - c_t - k_{t+1} + (1 - \delta)\, k_t]\}$$

FOCs

$$\frac{\partial U_t / \partial n_t}{\partial U_t / \partial c_t} = \frac{\partial y_t}{\partial n_t}$$

$$\frac{\partial U_t}{\partial c_t} = \beta \mathbb{E}_t \left[\frac{\partial U_t}{\partial c_{t+1}} \left(\frac{\partial y_{t+1}}{\partial k_{t+1}} + 1 - \delta \right) \right]$$

The CE Conditions

The first and second welfare theorems hold ⇒ CE = SP, since there are no distortions or externalities

CE Equilibrium Conditions

$$\frac{\partial U_t/\partial n_t}{\partial U_t/\partial c_t} = w_t$$

$$\frac{\partial U_t}{\partial c_t} = \beta \mathbb{E}_t\left[\frac{\partial U_{t+1}}{\partial c_{t+1}}(r_{t+1}^K + 1 - \delta)\right]$$

$$r_t^K = \frac{\partial y_t}{\partial k_t}$$

$$w_t = \frac{\partial y_t}{\partial n_t}$$

$$k_{t+1} = i_t + (1-\delta)k_t$$

$$y_t = e^{z_t}F(k_t, n_t) = c_t + i_t$$

(Given z_t and $n_t = N_t$ and $k_t = K_t$)

SP Equilibrium Conditions

$$\frac{\partial U_t/\partial n_t}{\partial U_t/\partial c_t} = \frac{\partial y_t}{\partial n_t}$$

$$\frac{\partial U_t}{\partial c_t} = \beta \mathbb{E}_t\left[\frac{\partial U_t}{\partial c_{t+1}}\left(\frac{\partial y_{t+1}}{\partial k_{t+1}} + 1 - \delta\right)\right]$$

$$k_{t+1} = i_t + (1-\delta)k_t$$

$$y_t = e^{z_t}F(k_t, n_t) = c_t + i_t$$

(Given z_t)

Parametrization

- Utility function: Constant Relative Risk Aversion (CRRA)

$$U(c_t, 1-n_t) = \frac{[c_t^{1-\alpha}(1-n_t)^{\alpha}]^{1-\sigma} - 1}{1-\sigma} \qquad if \;\; \sigma \neq 1$$

$$U(c_t, 1-n_t) = (1-\alpha)log(c_t) + \alpha log(1-n_t) \qquad if \;\; \sigma = 1$$

- Production function: Cobb-Douglas

$$y_t = e^{z_t}F(k_t, n_t) = e^{z_t}k_t^{\theta}n_t^{1-\theta}$$

CE Conditions and Parametrization

General CE Conditions

$$\frac{\partial U_t/\partial n_t}{\partial U_t/\partial c_t} = w_t$$

$$\frac{\partial U_t}{\partial c_t} = \beta \mathbb{E}_t\left[\frac{\partial U_{t+1}}{\partial c_{t+1}}(r_{t+1}^K + 1 - \delta)\right]$$

$$k_{t+1} = i_t + (1-\delta)k_t$$

$$y_t = e^{z_t}F(k_t, n_t)$$

$$r_t^K = \frac{\partial y_t}{\partial k_t}$$

$$w_t = \frac{\partial y_t}{\partial n_t}$$

$$y_t = c_t + i_t$$

With Parametrization

$$\left(\frac{\alpha}{1-\alpha}\right)\frac{c_t}{1-n_t} = w_t$$

$$\frac{1}{c_t} = \beta \mathbb{E}_t\left[\frac{1}{c_{t+1}}(r_{t+1}^K + 1 - \delta)\right]$$

$$k_{t+1} = i_t + (1-\delta)k_t$$

$$y_t = e^{z_t}(k_t)^{\theta}(n_t)^{1-\theta}$$

$$r_t^K = \theta\frac{y_t}{k_t}$$

$$w_t = (1-\theta)\frac{y_t}{n_t}$$

$$y_t = c_t + i_t$$

RBC and balanced growth path

Add long-term growth

- Labor-augmenting technology A_t grows at rate γ

$$y_t = e^{z_t} F(k_t, A_t n_t)$$

$$A_t = (1+\gamma)A_{t-1} = (1+\gamma)^t A_0 = (1+\gamma)^t$$

$$y_t = e^{z_t} F(k_t, A_t n_t) = e^{z_t} k_t^{\theta} [(1+\gamma)^t n_t]^{1-\theta}$$

- Population H_t grows at rate η

$$\beta^t H_t U(c_t, 1-n_t)$$

$$H_t = (1+\eta)H_{t-1} = (1+\eta)^t H_0 = (1+\eta)^t$$

$$\beta^t H_t U(c_t, 1-n_t) = \beta^t (1+\eta)^t U(c_t, 1-n_t)$$

Detrending

- The economy grows at a constant rate γ, called balanced growth path (BGP)

- To solve for the stationary CE, detrend the per-capita variables and real wage by the growth rate factor

$$\tilde{c}_t = \frac{c_t}{(1+\gamma)^t}, \qquad \tilde{\imath}_t = \frac{i_t}{(1+\gamma)^t}, \qquad \tilde{k}_t = \frac{k_t}{(1+\gamma)^t}, \qquad \tilde{y}_t = \frac{y_t}{(1+\gamma)^t}$$

$$\widetilde{w}_t = \frac{w_t}{(1+\gamma)^t}$$

Detrending RBC model

Simplification: $\sigma = 1$, the number of households (population) increase by $1 + \eta$

$$\underset{\tilde{c}_t,\tilde{k}_{t+1},n_t}{Max} \ \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} [\beta(1+\eta)]^t \left[(1-\alpha)log(\tilde{c}_t) + \alpha log(1-n_t)\right] \right\}$$

$s.t.$

$$\tilde{c}_t + \tilde{\imath}_t = \widetilde{w}_t n_t + r_t^K \tilde{k}_t + \widetilde{\Omega}_t$$

$$(1+\gamma)(1+\eta)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{\imath}_t$$

Detrending RBC model

For firms (and abusing notation for labor and capital)

$$\underset{\tilde{k}_t, n_t}{Max} \;\; \tilde{y}_t - r_t^K \tilde{k}_t - \tilde{w}_t n_t, \quad \forall t$$

$s.t.$

$$\tilde{y}_t = e^{z_t} \tilde{k}_t^{\theta} n_t^{1-\theta}$$

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}, \;\; \varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

CE Conditions Under Parametrization and Rescaling

Given $z_{t+1} = \rho z_t + \varepsilon_{t+1}, \ \varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$

Endogenous variables: $\{\tilde{c}_t, \tilde{\imath}_t, \tilde{k}_t, \tilde{y}_t, n_t, \widetilde{w}_t, r_t, z_t\}$

With Parametrization	And Rescaling
$\left(\frac{\alpha}{1-\alpha}\right)\frac{c_t}{1-n_t} = w_t$	$\left(\frac{\alpha}{1-\alpha}\right)\frac{\tilde{c}_t}{1-n_t} = \widetilde{w}_t$
$\frac{1}{c_t} = \beta \mathbb{E}_t\left[\frac{1}{c_{t+1}}(r_{t+1}^K + 1 - \delta)\right]$	$\frac{1+\gamma}{\tilde{c}_t} = \beta \mathbb{E}_t\left[\frac{1}{\tilde{c}_{t+1}}(r_{t+1}^K + 1 - \delta)\right]$
$k_{t+1} = i_t + (1-\delta)k_t$	$(1+\gamma)(1+\eta)\tilde{k}_{t+1} = \tilde{\imath}_t + (1-\delta)\tilde{k}_t$
$y_t = e^{z_t}(k_t)^\theta (n_t)^{1-\theta}$	$\tilde{y}_t = e^{z_t}(\tilde{k}_t)^\theta (n_t)^{1-\theta}$
$r_t^K = \theta \frac{y_t}{k_t}$	$r_t^K = \theta \frac{\tilde{y}_t}{\tilde{k}_t}$
$w_t = (1-\theta)\frac{y_t}{n_t}$	$\widetilde{w}_t = (1-\theta)\frac{\tilde{y}_t}{n_t}$
$y_t = c_t + i_t$	$\tilde{y}_t = \tilde{c}_t + \tilde{\imath}_t$

The Steady State (SS)

Assuming no shock ε and $\tilde{x}_t = \tilde{x}_{t-1} = x$ at SS, solve for $\{c, i, k, y, n\}$:

$$\left(\frac{\alpha}{1-\alpha}\right)\frac{c}{1-n} = \underbrace{(1-\theta)\frac{y}{n}}_{w}$$

$$1+\gamma = \beta(\underbrace{\theta\frac{y}{k}}_{r^K} + 1 - \delta)$$

$$[(1+\gamma)(1+\eta) - 1 + \delta]k = i$$

$$c + i = y$$

$$y = k^{\theta} n^{1-\theta}$$

Parameters

Need to find values for parameters in

- Preferences

$$\alpha, \beta, \eta$$

- Technology

$$\theta, \delta, \gamma$$

- Shocks

$$\rho, \sigma_{\varepsilon}$$

Calibration

Information, policy experiments $\xrightarrow[\text{Calibration}]{}$ *Parameters values*

- It involves applying as much empirical information as possible to choose the parameters values
- It is based on information from national income accounts, input-output matrices, structural macro-econometric estimates and microeconomic evidence, among others
- Choose some parameters to match “great” macro ratios or moments
- It also may reflect policy experiments

Calibration

- Remember we chose CRRA coefficient $\sigma = 1 \Rightarrow$ log utility
- Some parameter values can be estimated from data, based on Cooley and Prescott (1995)

$$\rho = 0.95, \qquad \sigma_\varepsilon = 0.007, \qquad \theta = 0.4, \qquad \gamma = 1.56\% \text{ (annual)}, \qquad \eta = 1.2\% \text{ (annual)}$$

- Remaining three parameters can be calibrated
- Given the chosen parameter values, the deterministic BGP of the model economy should match certain long-run features of the US data
- Three parameters correspond to three “great” macro ratios

Calibration

- At the SS, the law of motion of k

$$(1+\gamma)(1+\eta)\frac{k}{y} = (1-\delta)\frac{k}{y} + \frac{i}{y}$$

In US data, $k/y = 3.32$, $i/y = 25.2\%$ (annual), and given values of γ and η, we determine $\delta = 0.048$ (or 0.012, quarterly)

- At the SS, the Euler equation is

$$\frac{1+\gamma}{\beta} + \delta - 1 = \theta\frac{y}{k}$$

Given θ, γ, k/y,and δ we have $\beta = 0.947$ (annual), or 0.987 (quarterly)

- At the SS, the intra-temporal condition

$$(1-\theta)\frac{y}{c} = \left(\frac{\alpha}{1-\alpha}\right)\frac{n}{1-n}$$

From micro data $n = 0.31$ and $c/y = 1 - i/y = 75\%$, then $\alpha = 0.64$

Calibration

Technology					Preferences			
θ	δ	ρ	σ_ε	γ	β	σ	α	η
0.40	0.012	0.95	0.007	0.004	0.987	1	0.64	0.003

Solving the RBC Model

- The model (equilibrium conditions) in its implicit non-linear form is

$$\mathbb{E}_t F(q_t, \varepsilon_t, \Theta) = 0$$

where x represents the endogenous variables, ε_t is an exogenous shock, and Θ represents the parameters

- A solution $x_t = h(\varepsilon_t, \Theta)$ is such that $\mathbb{E}_t F(h(\varepsilon_t, \Theta), \varepsilon_t, \Theta) = 0$
- How to find the solution?
  - No closed-form solution (except very few cases, e.g., $\sigma = \delta = 1$)
  - Use numerical methods
    - Heer, B. and A. Maussner (2009). *Dynamic General Equilibrium Modeling: Computational Methods and Applications*. Springer
    - Novales, A., E. Fernandez and J. Ruiz (2009). *Economic Growth: Theory and Numerical Solution Methods*. Springer

Solving the RBC Model

Perturbation methods are based on an approximation (1st, 2nd, 3rd,...) of $\mathbb{E}_t F(x_t, \varepsilon_t, \Theta) = 0$ around a steady state

- Solve for the steady state (SS)
- Solve for the dynamics caused by shocks around SS
  - Linearize or log-linearize the non-linear equilibrium conditions, e.g., the EE with $\delta = 1$

$$\frac{1+\gamma}{\tilde{c}_t} = \beta \mathbb{E}_t \left[\frac{r^K_{t+1}}{\tilde{c}_{t+1}} \right] \Rightarrow \frac{\tilde{c}_t - c}{c} = \mathbb{E}_t \frac{\tilde{c}_{t+1} - c}{c} - \mathbb{E}_t \frac{r^K_{t+1} - r^K}{r^K}$$

See the next slides

  - Solve the system of linear equations using methods to solve linear rational expectations models such as *Uhlig's Undetermined Coefficients Method* and *Sims' Eigenvalue-Eigenvector Decomposition Method. Blanchard-Kahn's can be used to check the uniqueness of solution*

How to Linearize?

A 1st-order Taylor approximation around (x):

$$f(x_t) \cong f(x) + \left.\frac{\partial f}{\partial x}\right|_{(x)} (x_t - x)$$

around (x, y):

$$f(x_t, y_t) \cong f(x, y) + \left.\frac{\partial f}{\partial x}\right|_{(x,y)} (x_t - x) + \left.\frac{\partial f}{\partial y}\right|_{(x,y)} (y_t - y)$$

and so on

How to Linearize?

Consider as an example the Euler equation

$$(1+\gamma)c_t^{-1} = \beta \mathbb{E}_t\{c_{t+1}^{-1} r_{t+1}^K\}$$

A 1st-order approximation around (c, c, R, π) gives

$$(1+\gamma)c^{-1} - 1(1+\gamma)c^{-1-1}(c_t - c) \quad = \quad \beta c^{-1} r^K - \beta \mathbb{E}_t\{c^{-1-1} r^K (c_{t+1} - c)\} + \beta \mathbb{E}_t\{c^{-1}\,(r_{t+1}^K - r^K)\}$$

$$-(1+\gamma)c^{-1}\frac{c_t - c}{c} \quad = \quad -\beta \mathbb{E}_t\left\{c^{-1} r^K \frac{c_{t+1} - c}{c}\right\} + \beta \mathbb{E}_t\left\{c^{-1} r^K \frac{r_{t+1}^K - r^K}{r^K}\right\}$$

$$\frac{c_t - c}{c} = \mathbb{E}_t\left\{\frac{c_{t+1} - c}{c}\right\} - \mathbb{E}_t \frac{r_{t+1}^K - r^K}{r^K}$$

How to Log-Linearize?

Note that

$$x_t = xe^{\hat{x}_t} \quad \text{where} \quad \hat{x}_t = log\left(\frac{x_t}{x}\right)$$

Consider again

$$(1+\gamma)c_t^{-1} = \beta\mathbb{E}_t\{c_{t+1}^{-1}r_{t+1}^K\} \quad \text{written as} \quad (1+\gamma)c^{-1}e^{-\hat{c}_t} = \beta\mathbb{E}_t\left\{c^{-1}e^{-\hat{c}_{t+1}}r^K e^{\hat{r}_{t+1}^K}\right\}$$

A 1st-order approximation of $(\hat{c}_t, \hat{c}_{t+1}, \hat{r}_{t+1}^K)$ around $(0,0,0)$ gives

$$(1+\gamma)c^{-1} - (1+\gamma)c^{-1}\hat{c}_t \quad = \quad \beta c^{-1}r^K - \beta\mathbb{E}_t\{c^{-1}r^K\hat{c}_{t+1}\} + \beta\mathbb{E}_t\{c^{-1}r^K\hat{r}_{t+1}^K\}$$

$$\hat{c}_t = \mathbb{E}_t\{\hat{c}_{t+1}\} - \mathbb{E}_t\{\hat{r}_{t+1}^K\}$$

Linearizing versus Log-Linearizing

Comparison between linearizing and log-linearizing

$$\frac{c_t - c}{c} = \mathbb{E}_t \left\{ \frac{c_{t+1} - c}{c} \right\} - \mathbb{E}_t \frac{r_{t+1}^K - r^K}{r^K}$$

$$\hat{c}_t = \mathbb{E}_t \{ \hat{c}_{t+1} \} - \mathbb{E}_t \{ \hat{r}_{t+1}^K \}$$

where

$$\hat{x}_t = log \left(\frac{x_t}{x} \right)$$

Note that for small $\frac{x_t - x}{x}$

$$\hat{x}_t = log \left(\frac{x_t}{x} \right) = log \left(\frac{x_t}{x} - 1 + 1 \right) = log \left(\frac{x_t - x}{x} + 1 \right) \simeq \frac{x_t - x}{x}$$

Should be always do this???

No there are typical functional form, when we can easily (log-)linarize the equations:

1. Multiplicative terms: Cobb-Douglas function or Euler-equation

$$\tilde{y}_t = e^{z_t} \tilde{k}_t^{\theta} n_t^{1-\theta} \qquad\qquad (1+\gamma) c_t^{-1} = \beta \mathbb{E}_t \{ c_{t+1}^{-1} r_{t+1}^K \}$$

2. Additive terms: GDP identity

$$y_t = c_t + i_t$$

(Log)-linearizing equations with multiplicative terms

Steps:

1. Take the natural log of the equation
2. Take the steady-state version of the same equation
3. Subtract the steady-state version from 1.
4. Assume that $\hat{x}_t = \ln x_t - \ln x$ and simplify the equation

Cobb-Douglas function:

$$\tilde{y}_t = e^{z_t}\tilde{k}_t^{\theta} n_t^{1-\theta}$$

1. $$\ln \tilde{y}_t = z_t + \theta \ln \tilde{k}_t + (1-\theta) \ln n_t$$

2. $$\ln \tilde{y} = z + \theta \ln \tilde{k} + (1-\theta) \ln n$$

3. $$\ln \tilde{y}_t - \ln \tilde{y} = z_t - z + \theta(\ln \tilde{k}_t - \ln \tilde{k}) + (1-\theta)(\ln n_t - \ln n)$$

4. $$\hat{\tilde{y}} = \hat{z_t} + \theta \widehat{\tilde{k}_t} + (1-\theta)\widehat{n_t}$$

Euler-equation:

$$(1+\gamma)c_t^{-1} = \beta \mathbb{E}_t\{c_{t+1}^{-1} r_{t+1}^K\}$$

$$\ln(1+\gamma) - \ln c_t = \ln\beta - \mathbb{E}_t \ln c_{t+1} + \mathbb{E}_t \ln r_{t+1}^K$$

$$\ln(1+\gamma) - \ln c = \ln\beta - \ln c + \ln r^K$$

$$-(\ln c_t - \ln c) = -\mathbb{E}_t(\ln c_{t+1} - \ln c) + \mathbb{E}_t(\ln r_{t+1}^K - \ln r^K)$$

$$-\widehat{c_t} = -\mathbb{E}_t \widehat{c_{t+1}} + \mathbb{E}_t \widehat{r_{t+1}^K}$$

$$\widehat{c_t} = \mathbb{E}_t \widehat{c_{t+1}} - \mathbb{E}_t \widehat{r_{t+1}^K}$$

(Log)-linearizing equations with additive terms

Steps:

1. Take the steady-state version of the given equation
2. Subtract the steady-state version from 1.
3. Assume that $\hat{x}_t = (x_t - x)/x$ and try to simplify the equation

GDP identity:

$$y_t = c_t + i_t$$

1. $$y = c + i$$

2. $$y_t - \mathrm{y} = c_t - c + i_t - i$$

3. $$\frac{y}{y}(y_t - \mathrm{y}) = \frac{c}{c}(c_t - c) + \frac{i}{i}(i_t - i)$$

4. $$y\widehat{y_t} = c\widehat{c_t} + i\widehat{i_t}$$

$$\widehat{y_t} = \frac{c}{y}\widehat{c_t} + \frac{i}{y}\widehat{i_t}$$

Solving Forward-Looking Models

Solving the RBC Model

- With (log) linearization, re-express the model as

$$Ax_t = B\mathbb{E}_t x_{t+1} + Dx_{t-1} + N\varepsilon_t$$

where x_t is a vector of endogenous variables expressed as deviations form SS

- The solution has a VAR representation

$$\Gamma_0 x_t = \Gamma_1 x_{t-1} + \Psi\varepsilon_t$$

- The solution expressed as a minimal state variable representation

$$x_t^c = \Omega x_{t-1}^s + \mathrm{P}\varepsilon_t \qquad and \qquad x_t^s = \Upsilon x_{t-1}^s + \Phi\varepsilon_t$$

  - x_t^c is a vector of control (jump or non-predetermined) variables
  - x_t^s is a vector of state (predetermined) variables

- B,D,N,Ψ,Σ,Ω, P,Υ, and Φ are matrices

Solving the RBC Model

- Luckily, we do not have to do many of these steps
- We use Matlab and Dynare *(http://www.dynare.org/)* $\Rightarrow$ See W-1
- Input the non-linear model and Dynare (log)linearizes and writes it

$$\mathrm{A}x_t = B\mathbb{E}_t x_{t+1} + Dx_{t-1} + N\varepsilon_t$$

- Dynare finds the solution

$$x_t^c = \Omega x_{t-1}^s + \mathrm{P}\varepsilon_t \qquad and \qquad x_t^s = \Upsilon x_{t-1}^s + \Phi\varepsilon_t$$

  - x_t^c is a vector of control (jump or non-predetermined) variables
  - x_t^s is a vector of state (predetermined) variables
- B,D,N, Ω, P, Υ, and Φ are matrices

Solving the RBC Model

- For the RBC model, the model can be reduced further and written as a 3 by 3 system in $x_t = [\hat{c}_t, \hat{k}_t, z_t]'$

$$\mathbb{E}_t x_{t+1} = \Lambda x_t + \Xi \varepsilon_t$$

  where Λ and Ξ are matrices, $\hat{c}_t$ is a jump (non-predetermined) variable and $\hat{k}_t$ and z_t are state (predetermined) variables

- The stability of the system: What is the importance of the eigenvalues of Λ for the equilibrium determinacy?

- The Blanchard-Kahn Condition - The Rank Condition
  - # of explosive eigenvalues = # of jump variables ⇒Unique equilibrium
  - # of explosive eigenvalues < # of jump variables ⇒ Multiple equilibria
  - # of explosive eigenvalues > # of jump variables ⇒ No equilibrium

Don’t be scared, the Dynare solves the model automatically! ☺

Impulse Responses

Analyze the impulse responses using the equilibrium conditions. For simplicity ignore $\mathbb{E}_t(.)$

$$\frac{\tilde{c}_{t+1}}{\tilde{c}_t} = \frac{\beta}{1+\gamma}(MPK_{t+1} + 1 - \delta) \quad with \quad MPK_{t+1} = \theta\frac{\tilde{y}_{t+1}}{\tilde{k}_{t+1}} = r^K_{t+1}$$

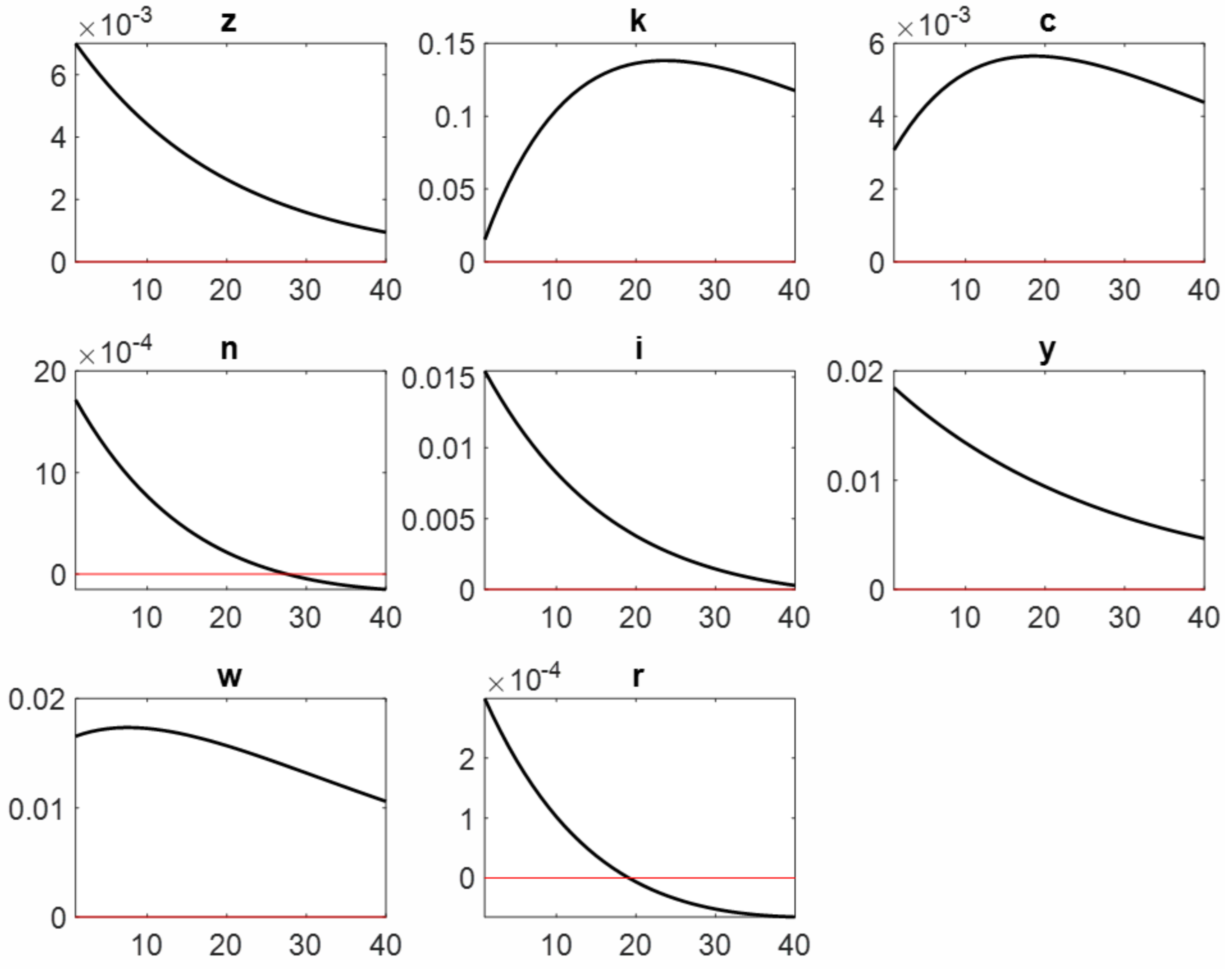
$$\tilde{w}_t = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\tilde{c}_t}{1-n_t}\right) \quad Labor \quad Supply$$

$$\tilde{w}_t = (1-\theta)e^{z_t}\left(\frac{\tilde{k}_t}{n_t}\right)^{\theta} \quad LaborDemand$$

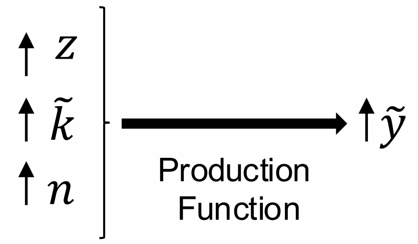
$$\tilde{y}_t = e^{z_t}\tilde{k}_t^{\theta}n_t^{1-\theta} \qquad and \qquad \tilde{c} + \tilde{\imath}_t = \tilde{y}_t$$

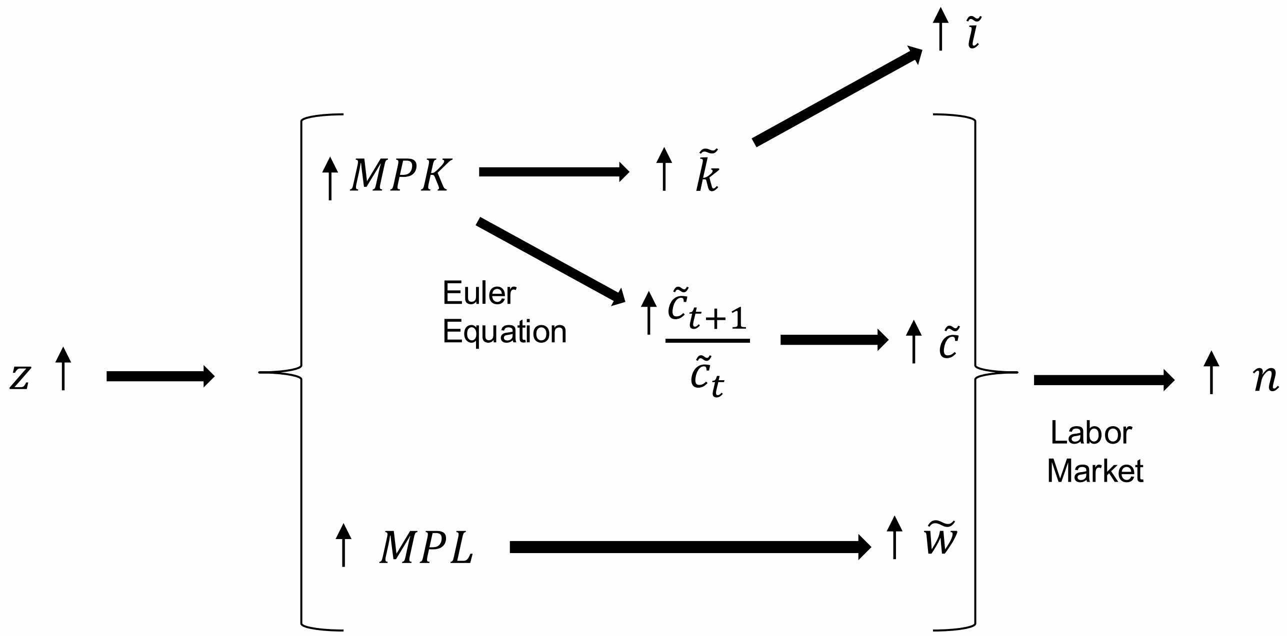
Impulse Responses

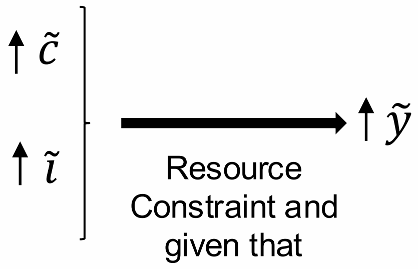
The impulse responses to one-time shock in z_t, i.e., $\varepsilon_t \uparrow$ by one std 0.007 at $t=0$ and then $\varepsilon_t = 0, \forall t \geq 1$. Variables expressed as deviations from SS levels



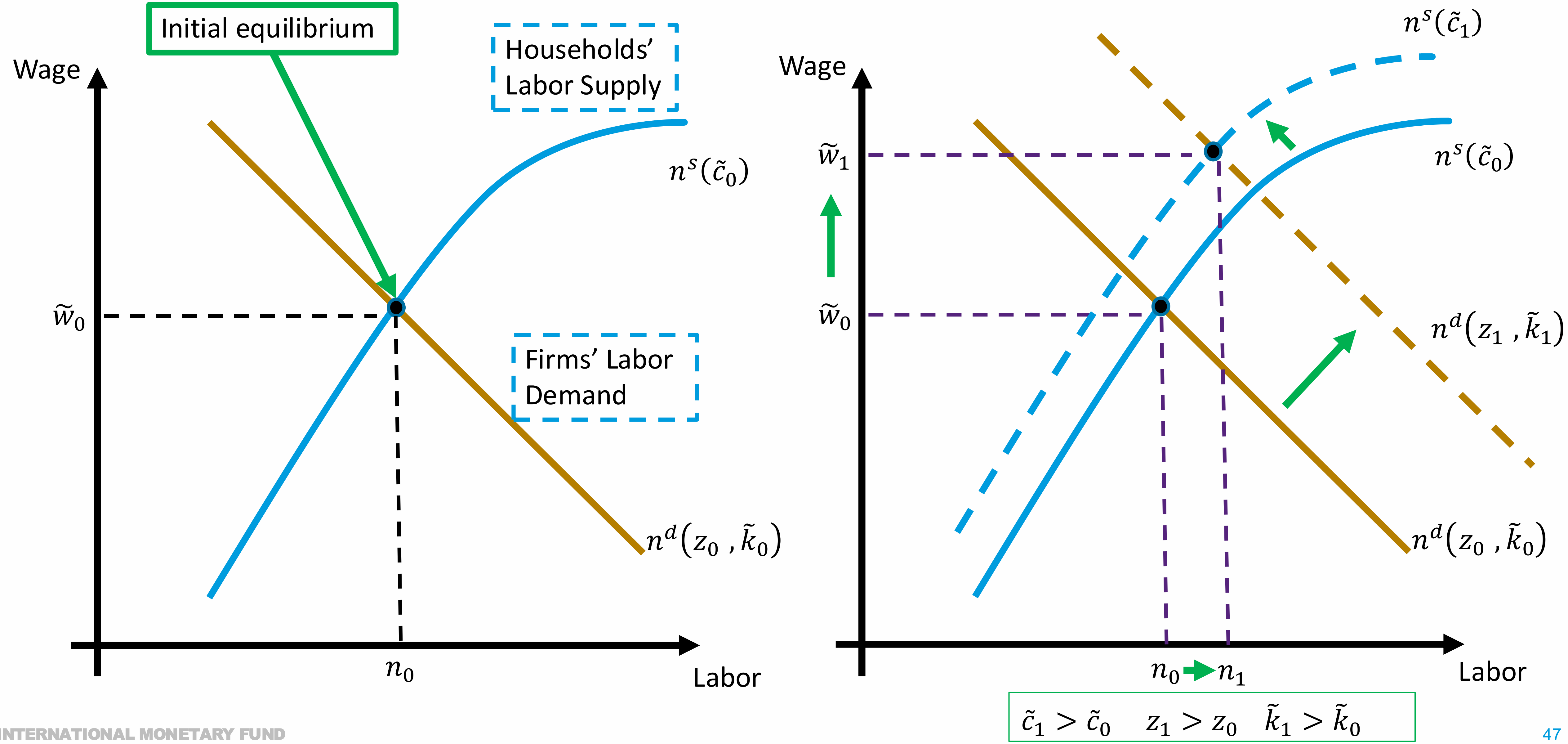
Transmission Mechanism







Labor Market Adjustment



Business Cycles: Model vs. Data

	Model		Data	
Variable	SD (%)	Corr(var, y)	SD (%)	Corr(var, y)
y	1.34	1.00	1.72	1.00
c	0.35	0.88	1.27	0.83
i	4.32	0.99	8.24	0.91
n	0.72	0.99	1.69	0.92
y/n	0.64	0.98	0.73	0.34

$$\frac{1.34}{1.72} = 0.78$$

Take-Aways

- Technology shocks are the main driving force behind business cycles!
- They account for about 78% of the volatility of output (Kydland and Prescott, 1982)
- Business cycles are an efficient equilibrium outcome resulting from the economy's response to exogenous variations in real forces
- No role of monetary factors! The RBC model is not relevant for monetary policy analysis

The Classical Monetary (CM) Model

- Add money into the RBC model

- A classical monetary model (based on Chapter 2 of Gali [2015])
  - Money only serves as a unit of account
  - Still features perfect competition and flexible prices

- Model delivers “Classical Dichotomy”: Money does not have real impact!

Log-linearized RBC model without Capital

Log-linear approximation ($x_t \equiv \log X_t$)

$$w_t - p_t = \sigma c_t + \varphi n_t \qquad (1)$$

$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho) \qquad (2)$$

$$w_t - p_t = a_t - \alpha n_t + \log(1-\alpha) \qquad (3)$$

$$y_t = c_t \qquad (4)$$

$$y_t = a_t + (1-\alpha) n_t \qquad (5)$$

with $i_t \equiv -\log Q_t \quad \pi_{t+1} \equiv p_{t+1} - p_t \quad \rho \equiv -\log \beta$

Append a demand for real money balances, **next lectures show example for this**:

$$m_t - p_t = c_t - \eta i_t \qquad (6)$$

We can substitute out real wage from eq1 and eq3, and using the good market equilibrium to substitute out consumption

We have 3 equations:

$$a_t - \alpha n_t + \log(1-\alpha) = \sigma y_t + \varphi n_t \qquad (1)$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \tfrac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho) \qquad (2)$$

$$y_t = a_t + (1-\alpha)n_t \qquad (5)$$

Now, we can substitute out the labor from eq1:

$$a_t + \log(1-\alpha) = \sigma y_t + \frac{(\alpha+\varphi)(y_t - a_t)}{1-\alpha}$$

$$a_t\left(1 + \frac{\alpha+\varphi}{1-\alpha}\right) + \log(1-\alpha) = y_t\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)$$

It means, that the technology shock determines the output, based on the output we can express the consumption, labor, real wage $(w-p)$. The equation 2 determines the real interest rate $(i-\pi)$.

Price level and money neutrality

Money balance equation:

$$m_t - p_t = c_t - \eta i_t$$

where $r_t = i_t - \mathbb{E}_t(\pi_{t+1})$ has been already determined from the technology shock. And the consumption also driven by the technology shock only…

$$m_t - p_t = c_t - \eta(r_t + \mathbb{E}_t\,(\pi_{t+1}))$$

with $\pi_{t+1} \equiv p_{t+1} - p_t$. Green letters assign the already determined variables. The money balance equation tells us direct relationship between the money stock and price level. If we substitute out inflation, and rearrange the equation:

$$m_t - p_t = c_t - \eta r_t - \eta \mathbb{E}_t\, p_{t+1} + \eta p_t$$
$$p_t = 1/(1+\eta)m_t - 1/(1+\eta)(c_t - \eta r_t) + \eta/(1+\eta)\mathbb{E}_t\, p_{t+1}$$

The money supply affects the prices, but does it have a real effect?

NO: the real variables have been already determined independently from money supply!!

Non-neutrality of a Monetary Shock

- Does monetary policy play a role in data? What happens after a monetary policy shock?
- Christiano, Eichenbaum, and Evans (CEE, 1999, Handbook of Macro) provide answers to these questions
- In response to a contractionary monetary policy shock (residual from an estimated monetary policy rule followed by Fed)
  - Short-term interest rates (Fed Funds Rate) increase
  - Output and employment decline
  - Prices respond very slowly
  - Wages fall but little

CEE (1999)

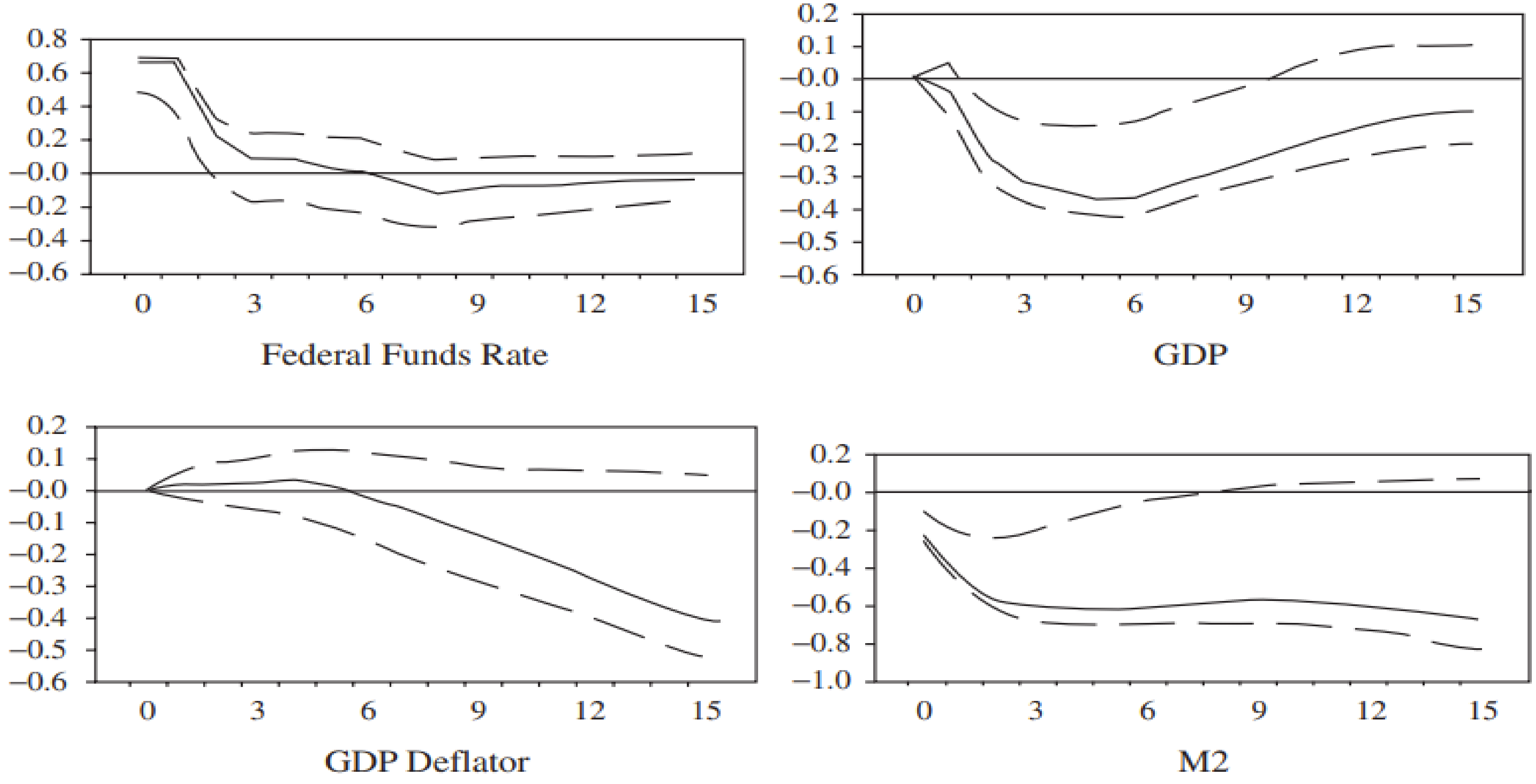


Figure 1.1 Estimated Dynamic Response to a Monetary Policy Shock

Source: Christiano, Eichenbaum, and Evans (1999).

Lessons from CEE (1999)

- Monetary policy has real impact
- The CM Model is at odds with empirical evidence
- How to fix it?
  - **Monetarist answer:** substitution between the real and monetary variable is limited: the households required to hold money for the purchase (Cash-in-Advance model) or due to the preferences Money in the Utility model with non-separable utility function for money.
  - **New-Keynesians:** Prices and wages are sticky

Conclusion

- The RBC Model builds the backbone of DSGE models, but monetary factors do not play a role
- The CM model introduces money as a unit of account into the RBC framework and finds that money is neutral
- Recent empirical evidence shows that monetary shocks are non-neutral
- A new model consistent with non-neutrality of money is needed to give a role to policy!

Thank you!

Annex: RBC model with Money!

The CM Model

HH's problem

$$\underset{C_t,\,B_t,\,N_t}{Max}\ \mathbb{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\left[\frac{C_t^{1-\sigma}}{1-\sigma}-\frac{N_t^{1+\varphi}}{1+\varphi}\right]\right\}$$

s.t.

$$P_tC_t + Q_tB_t \leq B_{t-1} + W_tN_t + D_t$$

and the non-Ponzi game constraint

$$\lim_{T\to\infty}\mathbb{E}_t\left\{\left[\beta^{T-t}\frac{\lambda_T}{\lambda_t}\right]B_T\right\} \geq 0$$

where B_t is one-period nominal riskfree bond. It pays one unit of money at maturity. Q_t is its price and satisfies $\frac{1}{Q_t} = e^{i_t}$ where i_t is the nominal interest rate

The CM Model

Firm's problem

$$\underset{N_t}{Max}\ P_t Y_t - W_t N_t$$

s.t.

$$Y_t = A_t N_t^{1-\alpha}$$

where

$$a_t = \log(A_t) = \rho_a a_{t-1} + \varepsilon_t^{\alpha}$$

Market clearing condition

$$Y_t = C_t$$

FOCs

From HH's problem

- The intratemporal condition

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi}$$

- The intertemporal (Euler) equation

$$Q_t = \beta \mathbb{E}_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}$$

Firm's FOC

$$\frac{W_t}{P_t} = (1 - \alpha)\, A_t N_t^{-\alpha}$$

Log-linearization

Log-linear approximation ($x_t \equiv \log X_t$)

$$w_t - p_t = \sigma c_t + \varphi n_t \tag{1}$$

$$c_t = \mathbb{E}_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho) \tag{2}$$

$$w_t - p_t = a_t - \alpha n_t + \log(1-\alpha) \tag{3}$$

$$y_t = c_t \tag{4}$$

$$y_t = a_t + (1-\alpha)n_t \tag{5}$$

with $i_t \equiv -\log Q_t \quad \pi_{t+1} \equiv p_{t+1} - p_t \quad \rho \equiv -\log \beta$

Append a demand for real money balances, **next lectures show example for this**:

$$m_t - p_t = c_t - \eta i_t \tag{6}$$

We can substitute out real wage from eq1 and eq3, and using the good market equilibrium to substitute out consumption

We have 3 equations:

$$a_t - \alpha n_t + \log(1-\alpha) = \sigma y_t + \varphi n_t \qquad (1)$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - \rho) \qquad (2)$$

$$y_t = a_t + (1-\alpha)n_t \qquad (5)$$

Now, we can substitute out the labor from eq1:

$$a_t + \log(1-\alpha) = \sigma y_t + \frac{(\alpha+\varphi)(y_t - a_t)}{1-\alpha}$$

$$a_t\left(1 + \frac{\alpha+\varphi}{1-\alpha}\right) + \log(1-\alpha) = y_t\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right)$$

It means, that the technology shock determines the output, based on the output we can express the consumption, labor, real wage $(w-p)$. The equation 2 determines the real interest rate $(i-\pi)$.

Formally, all variables are function of technology shock

The variables can be given as

$$n_t = \psi_{na} a_t + \psi_n \quad y_t = \psi_{ya} a_t + \psi_y \quad \text{and} \quad w_t - p_t = \psi_{wa} a_t + \psi_w$$

where $\psi_{na} \equiv \frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha} \quad \psi_n \equiv \frac{\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha} \quad \psi_{ya} \equiv \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}$

$$\psi_y \equiv (1-\alpha)\psi_n \qquad \psi_{wa} \equiv \frac{\sigma+\varphi}{\sigma(1-\alpha)+\varphi+\alpha} \qquad \psi_w \equiv \frac{[\sigma(1-\alpha)+\varphi]\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha}$$

Using equation (2), solve for the real interest rate $r_t \equiv i_t - \mathbb{E}_t(\pi_{t+1})$

$$r_t = \rho + \sigma \mathbb{E}_t\{\Delta y_{t+1}\} = \rho + \sigma \psi_{ya} \mathbb{E}_t\{\Delta a_{t+1}\}$$

All variables and equations only depend on real technology, not monetary policy! What is the role of the money??? The money has no real effect in RBC/CM model.

Price level determination

Money balance equation:

$$m_t - p_t = c_t - \eta i_t$$

where $r_t = i_t - \mathbb{E}_t(\pi_{t+1})$ has been determined from the technology shock. The consumption also driven by the technology shock only…

$$m_t - p_t = c_t - \eta(r_t + \mathbb{E}_t\,(\pi_{t+1}))$$

with $\pi_{t+1} \equiv p_{t+1} - p_t$. Green letters assign the already determined variables. The money balance equation tells us direct relationship between the money stock and price level. If we substitute out inflation, and rearrange the equation:

$$m_t - p_t = c_t - \eta r_t - \eta \mathbb{E}_t\, p_{t+1} + \eta p_t$$

$$(1+\eta)p_t = m_t - (c_t - \eta r_t) + \eta \mathbb{E}_t\, p_{t+1}$$

$$p_t = 1/(1+\eta)m_t - 1/(1+\eta)(c_t - \eta r_t) + \eta/(1+\eta)\mathbb{E}_t\, p_{t+1}$$

The price level can be expressed from the forward-looking variables.

$$p_t = \sum\nolimits_{i=0}^{\infty} \left(\frac{\eta}{1+\eta}\right)^i \left[\frac{1}{1+\eta} m_{t+i} - \frac{1}{1+\eta}(c_{t+i} - \eta r_{t+i})\right]$$

If the monetary policy…

Assuming no technological shock, the real variables are fixed at their steady-state values. The monetary policy directly affect the nominal price level. Assuming the simple monetary policy rule:

$$m_t = m$$

The price level can be solved as an infinite geometric sequence:

$$p_t = m\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i\left[\frac{1}{1+\eta}\right] + u_t = m\frac{1}{1-\frac{\eta}{1+\eta}}\frac{1}{1+\eta} + u_t = m + u_t$$

If the monetary policy…

Assuming no technological shock, the real variables are fixed at their steady-state values. If money growth follows an AR(1) process and assuming, without loss of generality, that u=0:

$$\Delta m_t = \rho_m \Delta m_{t-1} + \varepsilon_t^m$$

Then the price level:

$$p_t = \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta}\right)^i \left[\frac{1}{1+\eta} m_{t+i}\right] = \frac{1}{1+\eta} m_t + \frac{\eta}{1+\eta}\frac{1}{1+\eta} m_{t+1} + \left(\frac{\eta}{1+\eta}\right)^2 \frac{1}{1+\eta} m_{t+2} + \cdots$$

$$= \frac{1}{1+\eta} m_t + \frac{\eta}{1+\eta}\frac{1}{1+\eta}(\Delta m_{t+1} + m_t) + \left(\frac{\eta}{1+\eta}\right)^2 \frac{1}{1+\eta}(\Delta m_{t+2} + \Delta m_{t+1} + m_t) + \cdots$$

$$= \frac{1}{1+\eta} m_t + \frac{\eta}{1+\eta}\frac{1}{1+\eta}(\rho_m \Delta m_t + m_t) + \left(\frac{\eta}{1+\eta}\right)^2 \frac{1}{1+\eta}(\rho_m \rho_m \Delta m_t + \rho_m \Delta m_t + m_t) + \cdots$$

$$= \frac{1}{1+\eta}\frac{1}{1-\frac{\eta}{1+\eta}} m_t + \frac{\eta}{1+\eta}\frac{1}{1-\frac{\eta}{1+\eta}}\frac{\rho_m}{1+\eta}\Delta m_t + \left(\frac{\eta}{1+\eta}\right)^2 \frac{1}{1-\frac{\eta}{1+\eta}}\frac{\rho_m \rho_m}{1+\eta}\Delta m_t + \cdots$$

$$= m_t + \frac{\eta}{1+\eta}\rho_m \Delta m_t + \left(\frac{\eta}{1+\eta}\right)^2 \rho_m \rho_m \Delta m_t + \cdots = m_t + \frac{\eta}{1+\eta}\rho_m \frac{1}{1-\frac{\eta}{1+\eta}\rho_m}\Delta m_t = m_t + \frac{\eta \rho_m}{1+\eta(1-\rho_m)}\Delta m_t$$

Nominal Variables

The price level p_t should respond more than one-for-one with the increase in money supply

$$p_t = m_t + \frac{\eta\rho_m}{1+\eta(1-\rho_m)}\Delta m_t \quad \Rightarrow \quad \frac{\partial p_t}{\partial \varepsilon_t^m} > 1$$

We can express the expected inflation:

$$\pi_{t+1} = p_{t+1} - p_t = m_{t+1} + \frac{\eta\rho_m}{1+\eta(1-\rho_m)}\Delta m_{t+1} - m_t - \frac{\eta\rho_m}{1+\eta(1-\rho_m)}\Delta m_t$$

$$= m_t + \rho_m \Delta m_t + \frac{\eta\rho_m}{1+\eta(1-\rho_m)}\rho_m \Delta m_t - m_t - \frac{\eta\rho_m}{1+\eta(1-\rho_m)}\Delta m_t = \frac{\rho_m}{1+\eta(1-\rho_m)}\Delta m_t$$

The nominal interest rate i_t also increases (no liquidity effect)

$$i_t = \frac{\rho_m}{1+\eta(1-\rho_m)}\Delta m_t \quad \Rightarrow \quad \frac{\partial i_t}{\partial \varepsilon_t^m} > 0$$