Solving partial differential equations with neural networks

FYS-STK4155: Project 3

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github.com/mortele/FYS-STK4155

November 21, 2018

Abstract

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I. INTRODUCTION

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II. THEORY

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A. The heat equation

The heat equation is a partial differential equation (in space x and time t) which describes the evolution of temperature differences in a region of space over a time interval. It is based on *Fourier's law*; the rate of heat flow through a surface is proportional to the temperature gradient across the surface, i.e.

$$\mathbf{q} = -k\nabla T = -k\frac{\partial T}{\partial x}, \tag{1}$$

where $\mathbf{q}$ denotes the heat flux density, k is the thermal conductivity of the surface material, and T represents the temperature. Changes in temperature are proportional to changes in internal energy, with the proportionality constant being the specific heat capacity c_p. With the arbitrary energy zero point placed at absolute zero, this can be written as

$$Q = c_p \rho T, \tag{2}$$

with Q being the internal energy and ρ denoting the mass density. This is essentially just a restatement of (a shifted) *first law of thermodynamics*, in the absence of applied work. The total heat energy contained in a region $[a, b]$ is given by the integral

$$\int_a^b \mathrm{d}x\, c_p \rho T(x,t). \tag{3}$$

Integrating over a small region of space and considering the change in internal energy over a short time interval (assuming c_p and ρ are both time-independent and spatially homogenous) gives

$$\begin{aligned}\Delta Q =& c_p\rho \int_x^{x+\Delta x} \mathrm{d}\chi \left[T(\chi, t+\Delta t) - T(\chi, t)\right] \\ =& c_p\rho \int_x^{x+\Delta x} \mathrm{d}\chi \int_t^{t+\Delta t} \mathrm{d}\tau \, \frac{\partial T}{\partial \tau}. \end{aligned} \tag{4}$$

Over a short time period Δt, the change in internal energy of a short segment of length Δx must be entirely due to the heat flux in/out of the boundaries,

$$\begin{aligned}\Delta Q &= k \int_t^{t+\Delta t} \mathrm{d}\tau \left[\frac{\partial T(x+\Delta x, \tau)}{\partial x} - \frac{\partial T(x,\tau)}{\partial x}\right] \\ &= k \int_t^{t+\Delta t} \mathrm{d}\tau \int_x^{x+\Delta x} \mathrm{d}\chi \, \frac{\partial^2 T}{\partial \chi^2}. \end{aligned} \tag{5}$$

By conservation of energy, the difference between Eq. (4) and Eq. (5) must obviously vanish. Since we are integrating over the same spatial and temporal regions in both equations, this means that the integrand must vanish identically:

$$\frac{k}{c_p\rho}\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}. \tag{6}$$

This is known as the *heat equation* and is a special case of the more general diffusion equation.

B. Closed form solution

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C. Solving differential equations with neural networks

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III. RESULTS AND DISCUSSION

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IV. CONCLUSION

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