FYS4411 - Project 1

Scrap I/O and friends

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Abstract

This is where we put the abstract. It will be so good, like, the best. The greatest, probably.

1 Introduction

Lorem ipsum dolor amet

2 Theory

Sample text without lorem ipsum (or is it?)

3 Method

Keeping the stuff here for examples. Might even be relevant.

3.1 Randomization

Randomizing the transaction factor ϵ, and the picking of financial agents, `agent_one` and `agent_two`, was done by initializing the following random number generators (RNGs):

```
std::random_device rd;
std::mt19937_64 gen(rd());
std::uniform_int_distribution<int> AgentPicker(0,NAgents-1);
std::uniform_real_distribution<double> TransactionFactorGenerator
    (0.0,1.0);
// Calling RNGs to initialize agents and transaction factor:
agent_one = AgentPicker(gen);
agent_two = AgentPicker(gen);
TransactionFactor = TransactionFactorGenerator(gen);
```

3.2 Conservation of money

A potential source of money "leaks" in the simulations is if `agent_one` = `agent_two`. In this case the system would "leak" an amount of money equal to $\epsilon(m_1 + m_2)$, propagating for each transaction where that agent is involved, and for each subsequent instance of the error. This was handled by a simple test

```
if (agent_one == agent_two){
    continue;
}
```

which throws away the transactions where this would happen.

4 Results

Keeping one figure as example

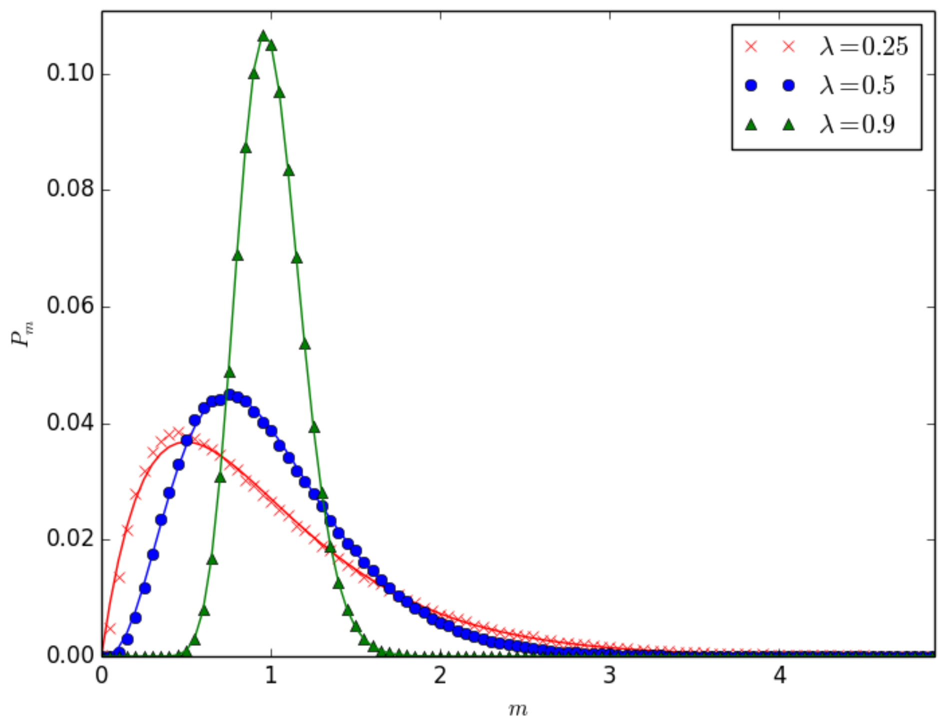

Figure 1: The wealth probability distribution of agents in the basic model where $\lambda = 0$ and $\eta = 0$

The data is normalized by dividing by the number of Monte Carlo cycles and the number of agents.

5 Discussion

Tekst

6 Conclusion

Herein lies the conclusions of yonder project, verily I say!

Appendix A - Problem 1

The local energy is given by

$$E_L(r) = \frac{1}{\Psi_T(r)}\hat{H}\Psi_T(r)$$

with trial wave equation

$$\Psi_T(r) = \prod_i g(\alpha,\beta,r_i)\prod_{i<j} f(a,|r_i - r_j|)$$

The Hamiltonian becomes

$$\hat{H} = \sum_i^N \left(\frac{-\hbar^2}{2m}\nabla_i^2 + V_{ext}(r_i)\right) + \sum_{i>j}^N V_{int}(r_i, r_j)$$

where

$$V_{ext}(r_i) = \begin{cases} \frac{1}{2}m\omega_{ho}^2 r^2 & (S) \\ \frac{1}{2}m[\omega_{ho}^2(x^2+y^2) + \omega_z^2 z^2] & (E) \end{cases}$$

and

$$v_{int}(|r_i - r_j|) = \begin{cases} \infty & |r_i - r_j| \leq a \\ 0 & |r_i - r_j| > a \end{cases}$$

First only harmonic oscillator (a=0) and we use $\beta = 1$ for one particle in 1D. for one particle the trial wave equation and Hamiltonian becomes

$$\Psi_T(r) = g(\alpha, x) = e^{-\alpha x_i^2},$$

$$\hat{H} = \left(\frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_{ho}^2 x^2\right)$$

$$E_L = \frac{1}{e^{-\alpha x^2}}\left(\frac{-\hbar^2}{2m}\nabla^2(e^{-\alpha x^2}) + \frac{1}{2}m\omega_{ho}^2 x^2(e^{-\alpha x^2})\right)$$

$$\begin{aligned}\nabla^2(e^{-\alpha x^2}) &= \frac{d^2 e^{-\alpha x^2}}{dx^2} \\ &= \frac{d}{dx}\left(\frac{de^{-\alpha x^2}}{dx}\right) \\ &= \frac{d}{dx}(-2\alpha x \cdot e^{-\alpha x^2}) \\ &= (4\alpha^2 x^2 - 2\alpha)e^{-\alpha x^2}\end{aligned}$$

$$E_L = \frac{e^{-\alpha x^2}}{e^{-\alpha x^2}}\left(\frac{-\hbar^2}{2m}(4\alpha^2 x^2 - 2\alpha) + \frac{1}{2}m\omega_{ho}^2 x^2\right)$$

$$E_L = \frac{-\hbar^2}{2m}(4\alpha^2 x^2 - 2\alpha) + \frac{1}{2}m\omega_{ho}^2 x^2$$

Solving the same problem for one particle in 2D and 3D we get something similar,

$$E_L = (\frac{-\hbar^2}{2m})4\alpha^2(x^2+y^2) - 4\alpha + \frac{1}{2}m\omega_{ho}^2(x^2+y^2),$$

$$E_L = (\frac{-\hbar^2}{2m})4\alpha^2(x^2+y^2+z^2) - 6\alpha + \frac{1}{2}m\omega_{ho}^2(x^2+y^2+z^2).$$

ADD LOCAL ENERGY FOR N PARTICLES IN 3D!!!

Now we can try to solve the complete problem. The trial wave function is now

$$\Psi_T(r) = \prod_i g(\alpha,\beta,r_i)\prod_{i<j} f(a,|r_i - r_j|)$$

and first we rewrite it using

$$g(\alpha,\beta,r_i) = \exp -\alpha(x_i^2 + y_i^2 + \beta z_i^2) = \phi(r_i)$$

and

$$f(r_{ij})\exp\left(\sum_{i<j} u(r_{ij})\right)$$

getting

$$\Psi_T(r) = \prod_i \phi(r_i)\exp\left(\sum_{i<j} u(r_{ij})\right).$$

The local energy for this problem becomes:

$$E_L = \frac{1}{\Psi_T(\boldsymbol{r})}\left(\sum_i^N\left(\frac{\hbar^2}{2m}\nabla_i^2\Psi_T(r) + \frac{1}{2}m\omega_{ho}^2 r^2\Psi_T(r)\right) + \sum_{i<j}^N V_{int}(r_i,r_j)\Psi_T(r)\right). \quad (1)$$

The difficulty in (1) is solving the derivatives of the wave equation given the complexity of the exponential. We begin with the first derivative.

$$\nabla_i^2 \prod_i \phi(r_i)\exp\left(\sum_{i<j} u(r_{ij})\right) = \nabla_i \cdot \nabla_i \prod_i \phi(r_i)\exp\left(\sum_{i<j} u(r_{ij})\right)$$

The first derivative of particle k:

$$\nabla_k \prod_i \phi(r_i) \exp\left(\sum_{i<j} u(r_{ij})\right) = \nabla_k \left(\prod_i \phi(r_i)\right) \exp\left(\sum_{i<j} u(r_{ij})\right) + \nabla_k \left(\exp\left(\sum_{i<j} u(r_{ij})\right)\right) \prod_i \phi(r_i)$$

$$\begin{aligned}
\nabla_k \left(\prod_i \phi(r_i)\right) &= \nabla_k (\phi(r_1)(\phi(r_2)...(\phi(r_k)..(\phi(r_N)) \\
&= \nabla_k \left(e^{-\alpha(x_1^2+y_1^2+z_1^2)} e^{-\alpha(x_2^2+y_2^2+z_2^2)} ... e^{-\alpha(x_k^2+y_k^2+z_k^2)} ... e^{-\alpha(x_N^2+y_N^2+z_N^2)}\right) \\
&= \nabla_k \phi(r_k) \left[\prod_{i \neq k} \phi(r_i)\right] \\
\nabla_k \exp\left(\sum_{i<j} u(r_{ij})\right) &= \nabla_k \exp(u(r_{12}) + u(r_{13}) + ... + u(r_{23}) + ... + u(r_{kj}) + .. + u(r_{N-1,N})) \\
&= \exp\left(\sum_{i<j} u(r_{ij})\right) \sum_{i \neq k} \nabla_k u(r_{kj})
\end{aligned}$$

And the first derivative of the trial wave equation is

$$\nabla_k \Psi_T(r) = \nabla_k \phi(r_k) \left[\prod_{i \neq k} \phi(r_i)\right] \exp\left(\sum_{i<j} u(r_{ij})\right) + \prod_i \phi(r_i) \exp\left(\sum_{i<j} u(r_{ij})\right) \sum_{j \neq k} \nabla_k u(r_{kj}).$$

FINAL EXPRESSION FOR TRIAL WAVE FUNCTION?
Now we find the second derivative of the wave function.

$$\frac{1}{\Psi_T(r)}\nabla_k^2\Psi_T(r) = \frac{1}{\nabla_k \prod_i \phi(r_i)\exp\left(\sum_{i<j} u(r_{ij})\right)}\left(\nabla_k\left(\nabla_k\phi(r_k)\left[\prod_{i\neq k}\phi(r_i)\right]\right)\cdot\exp\left(\sum_{i<j}u(r_{ij})\right)\right.$$

$$+\nabla_k\phi(r_k)\left[\prod_{i\neq k}\phi(r_i)\right]\cdot\nabla_k\left(\exp\left(\sum_{i<j}u(r_{ij})\right)\right)$$

$$+\nabla_k\left(\prod_i\phi(r_i)\right)\cdot\exp\left(\sum_{i<j}u(r_{ij})\right)\sum_{j\neq k}\nabla_k u(r_{kj})$$

$$+\prod_i\phi(r_i)\cdot\nabla_k\left(\exp\left(\sum_{i<j}u(r_{ij})\right)\right)\sum_{j\neq k}\nabla_k u(r_{kj})$$

$$\left.+\prod_i\phi(r_i)\exp\left(\sum_{i<j}u(r_{ij})\right)\cdot\nabla_k\left(\sum_{j\neq k}\nabla_k u(r_{kj})\right)\right)$$

Solving these equations separately makes it easier.

$$\frac{\nabla_k^2\phi(r_k)\left[\prod_{i\neq k}\phi(r_i)\right]\cdot\exp\left(\sum_{i<j}u(r_{ij})\right)}{\nabla_k\prod_i\phi(r_i)\exp\left(\sum_{i<j}u(r_{ij})\right)} = \frac{\nabla_k^2\phi(r_k)}{\phi(r_k)}$$

$$\frac{\nabla_k\phi(r_k)\left[\prod_{i\neq k}\phi(r_i)\right]\cdot\nabla_k\left(\exp\left(\sum_{i<j}u(r_{ij})\right)\right)}{\nabla_k\prod_i\phi(r_i)\exp\left(\sum_{i<j}u(r_{ij})\right)} = \frac{\nabla_k\phi(r_k)}{\phi(r_k)}\sum_{j\neq k}\nabla_k u(r_{kj})$$

$$\frac{\nabla_k\left(\prod_i\phi(r_i)\right)\cdot\exp\left(\sum_{i<j}u(r_{ij})\right)\sum_{j\neq k}\nabla_k u(r_{kj})}{\nabla_k\prod_i\phi(r_i)\exp\left(\sum_{i<j}u(r_{ij})\right)} = \frac{\nabla_k\phi(r_k)}{\phi(r_k)}\sum_{j\neq k}\nabla_k u(r_{kj})$$

$$\frac{\prod_i \phi(r_i)\cdot\nabla_k\left(\exp\left(\sum_{i<j}u(r_{ij})\right)\right)\sum_{j\neq k}\nabla_k u(r_{kj})}{\nabla_k\prod_i\phi(r_i)\exp\left(\sum_{i<j}u(r_{ij})\right)} = \sum_{i\neq k}\nabla_k u(r_{ki})\sum_{j\neq k}\nabla_k u(r_{kj})$$

$$\frac{\prod_i \phi(r_i)\exp\left(\sum_{i<j}u(r_{ij})\right)\cdot\nabla_k\left(\sum_{j\neq k}\nabla_k u(r_{kj})\right)}{\nabla_k\prod_i\phi(r_i)\exp\left(\sum_{i<j}u(r_{ij})\right)} = \sum_{j\neq k}\nabla_k^2 u(r_{kj})$$

Putting them together again we get the following

$$\frac{1}{\Psi_T(r)}\nabla_k^2\Psi_T(r) = \frac{\nabla_k^2\phi(r_k)}{\phi(r_k)}+2\frac{\nabla_k\phi(r_k)}{\phi(r_k)}\sum_{j\neq k}\nabla_k u(r_{kj})+\sum_{i\neq k}\nabla_k u(r_{ki})\sum_{j\neq k}\nabla_k u(r_{kj})+\sum_{j\neq k}\nabla_k^2 u(r_{kj}) \tag{2}$$

We solve the first and second derivatives of $u(r_{kj})$.

$$\nabla_k u(r_{kj}) = \left(\vec{i}\frac{\partial}{\partial x_k}+\vec{j}\frac{\partial}{\partial y_k}+\vec{k}\frac{\partial}{\partial z_k}\right)u(r_{kj}) \tag{3}$$

From Rottmann p. 128 we have that

$$\begin{aligned}
\frac{\partial u(r_{kj})}{\partial x_k}\,\vec{i} &= \frac{\partial u(r_{kj})}{\partial r_{kj}}\frac{\partial r_{kj}}{\partial x_k}\,\vec{i} \\
&= u'(r_{kj})\frac{\partial\sqrt{|x_k-x_j|^2+|y_k-y_j|^2+|z_k-z_j|^2}}{\partial x_k}\vec{i} \\
&= u'(r_{kj})\frac{1}{2}2|x_k-x_j|\frac{1}{r_{kj}}\vec{i} \\
&= \frac{u'(r_{kj})|x_k-x_j|}{r_{kj}}\vec{i} \\
\frac{\partial u(r_{kj})}{\partial y_k}\,\vec{j} &= \frac{u'(r_{kj})|y_k-y_j|}{r_{kj}}\vec{j} \\
\frac{\partial u(r_{kj})}{\partial z_k}\,\vec{k} &= \frac{u'(r_{kj})|z_k-z_j|}{r_{kj}}\vec{k}
\end{aligned}$$

where we have used the fact that $r_{kj} = \sqrt{|x_k-x_j|^2+|y_k-y_j|^2+|z_k-z_j|^2}$. Now equation (3) becomes

$$\begin{aligned}
\nabla_k u(r_{kj}) &= u(r_{kj})\frac{(|x_k-x_j|\vec{i}+|y_k-y_j|\vec{j}+|z_k-z_j|\vec{k})}{r_{kj}} \\
&= u(r_{kj})\frac{(\boldsymbol{r_k}-\boldsymbol{r_j})}{r_{kj}}.
\end{aligned}$$

And for the second derivative have that

$$\nabla_k^2 u(r_{kj}) = \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} + \frac{\partial^2}{\partial z_k^2} \right) u(r_{kj}) \tag{4}$$

and use Rottmann p. 128 again and see that

$$\begin{aligned}
\frac{\partial^2 u(r_{kj})}{\partial x_{kj}^2} &= \frac{\partial^2 u(r_{kj})}{\partial r_{kj}^2} \left(\frac{\partial r_{kj}}{\partial x_{kj}} \right)^2 + \frac{\partial u(r_{kj})}{\partial r_{kj}} \frac{\partial^2 r_{kj}}{\partial x_{kj}^2} \\
&= u''(r_{kj}) \frac{(x_k - x_j)^2}{r_{kj}^2} + u'(r_{kj}) \frac{\partial^2 r_{kj}}{\partial x_{kj}^2} \\
\frac{\partial^2 r_{kj}}{\partial x_{kj}^2} &= \frac{\partial^2 \sqrt{|x_k - x_j|^2 + |y_k - y_j|^2 + |z_k - z_j|^2}}{\partial x_{kj}^2} \\
&= \frac{\partial \frac{x_k - x_j}{\sqrt{|x_k - x_j|^2 + |y_k - y_j|^2 + |z_k - z_j|^2}}}{\partial x_{kj}} \\
&= \frac{r_{kj} - \frac{1}{2}(x_k - x_j) \cdot \frac{2(x_k - x_j)}{r_{kj}}}{r_{kj}^2} \\
&= \frac{1}{r_{kj}} - \frac{(x_k - x_j)^2}{r_{kj}^3}.
\end{aligned}$$

These equations put together and solving with respect to y and z we get

$$\begin{aligned}
\frac{\partial^2 u(r_{kj})}{\partial x_{kj}^2} &= u''(r_{kj}) \frac{(x_k - x_j)^2}{r_{kj}^2} + u'(r_{kj}) \left(\frac{1}{r_{kj}} - \frac{(x_k - x_j)^2}{r_{kj}^3} \right), \\
\frac{\partial^2 u(r_{kj})}{\partial y_{kj}^2} &= u''(r_{kj}) \frac{(y_k - y_j)^2}{r_{kj}^2} + u'(r_{kj}) \left(\frac{1}{r_{kj}} - \frac{(y_k - y_j)^2}{r_{kj}^3} \right), \\
\frac{\partial^2 u(r_{kj})}{\partial z_{kj}^2} &= u''(r_{kj}) \frac{(z_k - z_j)^2}{r_{kj}^2} + u'(r_{kj}) \left(\frac{1}{r_{kj}} - \frac{(z_k - z_j)^2}{r_{kj}^3} \right).
\end{aligned}$$

Now we can add them all together and equation (4) becomes

$$
\begin{aligned}
\nabla_k^2 u(r_{kj}) &= \frac{u''(r_{kj})}{r_{kj}^2}((x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2) \\
&+ u'(r_{kj}) \left(\frac{3}{r_{kj}} - \frac{(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2}{r_{kj}^3} \right) \\
&= \frac{u''(r_{kj}) r_{kj}^2}{r_{kj}^2} + u'(r_{kj}) \left(\frac{3}{r_{kj}} - \frac{r_{kj}^2}{r_{kj}^3} \right) \\
&= u''(r_{kj}) + u'(r_{kj}) \frac{2}{r_{kj}}
\end{aligned}
$$

Now we can write out the complete second derivative, equation (2)

$$
\begin{aligned}
\frac{1}{\Psi_T(r)} \nabla_k^2 \Psi_T(r) &= \frac{\nabla_k^2 \phi(r_k)}{\phi(r_k)} + 2 \frac{\nabla_k \phi(r_k)}{\phi(r_k)} \sum_{j \neq k} \frac{(\boldsymbol{r_k} - \boldsymbol{r_j})}{r_{kj}} u'(r_{kj}) \\
&+ \sum_{ij \neq k} \frac{(\boldsymbol{r_k} - \boldsymbol{r_i})}{r_{ki}} \frac{(\boldsymbol{r_k} - \boldsymbol{r_j})}{r_{kj}} u'(r_{ki}) u'(r_{kj}) + \sum_{j \neq k} \left(u''(r_{kj}) + \frac{2}{r_{kj}} u'(r_{kj}) \right).
\end{aligned}
$$

For the full analytical solution of the interacting problem we solve each term by it self.

$$
\begin{aligned}
\frac{1}{\phi(r_k)} \nabla_k^2 \phi(r_k) &= \frac{\nabla_k^2 \ exp(-\alpha(x_k^2 + y_k^2 + \beta z_k^2))}{exp(-\alpha(x_k^2 + y_k^2 + \beta z_k^2))} \\
&= ((2\alpha(2\alpha x_k^2 - 1)) + (2\alpha(2\alpha y_k^2 - 1)) + (2\alpha\beta(2\alpha\beta z_k^2 - 1))) \cdot \frac{\phi(r_k)}{\phi(r_k)} \\
&= -4\alpha^2 - 2\alpha\beta + 4\alpha^2(x_k^2 + y_k^2 + \beta z_k^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\phi(r_k)} \nabla_k \phi(r_k) &= \frac{\nabla_k \ exp(-\alpha(x_k^2 + y_k^2 + \beta z_k^2))}{exp(-\alpha(x_k^2 + y_k^2 + \beta z_k^2))} \\
&= (2\alpha x_k \vec{i} + 2\alpha y_k \vec{j} + 2\alpha\beta z_k \vec{k}) \cdot \frac{\phi(r_k)}{\phi(r_k)} \\
&= (2\alpha x_k \vec{i} + 2\alpha y_k \vec{j} + 2\alpha\beta z_k \vec{k})
\end{aligned}
$$

Wolfram alpha gives to solution the first and second derivatives of function $u(r_{kj})$.

$$
\begin{aligned}
u'(r_{kj}) &= \frac{d(\ln f(r_{kj}))}{dr_{kj}} \\
&= \frac{d\ \ln\left(1 - \frac{a}{(r_k - r_j)}\right)}{dr_{kj}} \\
&= -\frac{a}{ar_{kj} - r_{kj}^2}
\end{aligned}
$$

$$
\begin{aligned}
u''(r_{kj}) &= \frac{d^2(\ln f(r_{kj}))}{dr_{kj}^2} \\
&= \frac{a(a-2r_{kj})}{r_{kj}^2(a-r_{kj})^2}
\end{aligned}
$$

Putting all of this together we end up with the following expression for the second derivative of the wave equation divided by it self:

$$
\begin{aligned}
\frac{1}{\Psi_T(r)}\nabla_k^2\Psi_T(r) = &-4\alpha^2 - 2\alpha\beta + 4\alpha^2(x_k^2 + y_k^2 + \beta z_k^2) \\
&+ 2((2\alpha x_k\vec{i} + 2\alpha y_k\vec{j} + 2\alpha\beta z_k\vec{k}))\sum_{j\neq k}\left(\frac{(x_k - x_j)\vec{i} + (y_k - y_j)\vec{j} + (z_k - z_j)\vec{k}}{r_{kj}}\left(\frac{-a}{ar_{kj} - r_{kj}^2}\right)\right) \\
&+ \sum_{j\neq k}\left(\frac{(x_k - x_i)\vec{i} + (y_k - y_i)\vec{j} + (z_k - z_i)\vec{k}}{r_{ki}}\right)\left(\frac{(x_k - x_j)\vec{i} + (y_k - y_j)\vec{j} + (z_k - z_j)\vec{k}}{r_{kj}}\right) \\
&* \left(\frac{-a}{ar_{ki} - r_{ki}^2}\right)\left(\frac{-a}{ar_{kj} - r_{kj}^2}\right) \\
&+ \sum_{j\neq k}\left(\frac{a(a-2r_{kj})}{r_{kj}^2(a-r_{kj})^2} + \frac{2}{r_{kj}} - \frac{a}{ar_{kj} - r_{kj}^2}\right)
\end{aligned}
$$

Drift force:
Add calculations later.... OBS OBS

$$
\nabla u(r_{kj}) = -\frac{a(\vec{r_k} - \vec{r_j})}{ar_{kj}^2 - r_{kj})^3}
$$

$$
F = (2\alpha x_k\vec{i} + 2\alpha y_k\vec{j} + 2\alpha\beta z_k\vec{k}) + \sum_{j\neq k}\left(-\frac{a(\vec{r_k} - \vec{r_j})}{ar_{kj}^2 - r_{kj})^3}\right)
$$