Lattice Field Theory - Exercise Session 01.2-08.2

February 6, 2024

This exercise is a bit more in the analytical vein than the previous ones, we will merely focus on scalar field discretization and on the correlators. This is in preparation for the next exercise sheet, which will indeed be more computational.

1. Let us begin by studying discretization erros. Consider the one-dimensional action S for a massless free scalar field theory ($\phi \in R$) and it's lattice counterpart S_L,

$$S = \int dx \frac{1}{2}(\partial\phi)^2 \to S_L = \sum_x a\frac{1}{2}\left(\frac{\phi(x+a)-\phi(x)}{a}\right)^2 \tag{1}$$

   (a) Show me the lattice action can be re-written in a more elegant way as,

$$S = -\sum_x a\frac{1}{2}\phi(x)\frac{\phi(x+a)-2\phi(x)+\phi(x-a)}{a^2} \tag{2}$$

   Tip: $\sum_x f(x+a) = \sum_x f(x)$.

   (b) Show that the discretization errors of the action are of order $\mathcal{O}(a^2)$. Tip: Taylor expand $\phi(x+a)$

2. Alright, let's move on to studying correlation function $\langle\phi_x\phi_y\rangle$. Generically in d-dimensions, and with a mass m, the lattice action will read,

$$S = \frac{a^d}{2}\sum_{x,y}\phi_x M_{x,y}\phi_y \tag{3}$$

   where $M_{x,y}$ is given by,

$$M_{x,y} = -\sum_\mu \frac{1}{a^2}(\delta_{x+\mu,y} - 2\delta_{x,y} + \delta_{x-\mu,y}) + m^2\delta_{x,y} \tag{4}$$

   where μ is a basis vector (length a), and x, y are vectors ie. $x_\mu = an_\mu$. We have encountered the 1D massless specialization in the previous exercise. The two point propagator can be written,

$$\langle \phi_x \phi_y \rangle = \frac{1}{Z(0)} \partial_{J_x} \partial_{J_y} Z(J) \Big|_{J=0} \tag{5}$$

(a) Find a source (and cite it) for an n-dimensional Gaussian integral.

(b) Assuming $M_{x,y}$ is invertible and diagonalizable, complete the square and find $\langle \phi_x \phi_y \rangle$.

(c) Find the momentum space propagator $G(k)$ using the Green's function method, ie. take the Fourier transform,

$$a^d \sum_y G_{xy} M_{yz} = \delta_{xz}. \tag{6}$$

The result should be,

$$G(k) = \frac{1}{\sum_\mu \left[\frac{4}{a^2} sin^2(k_\mu \frac{a}{2}) + m^2 \right]}. \tag{7}$$