Lattice Field Theory - Exercise Session 15.2-22.2

February 14, 2024

This exercise session will be centered around gauge fields on a lattice. This one is purely analytical and will focus on gauge invariance on the lattice and on reproduction of continuum limits.

For an $SU(N)$ non-Abelian symmetry, the lattice action reads,

$$S = -\beta \sum_x \sum_{\mu > \nu} 1 - \frac{1}{N} ReTrU_{x,\mu} U_{x+\mu,\nu} U^*_{x+\nu,\mu} U^*_{x,nu} \tag{1}$$

Again, the gauge matrix is a special unitary matrix, and due to the non-abelian nature of the symmetry, $[A_{x,\mu}, A_{y,\nu}] \neq 0$.

1. Show that this abovementioned plaquette action, generates the correct continuum limit,

$$S = \int dx Tr F_{x,\mu\nu} F^{x,\mu\nu} \tag{2}$$

   Tip: Remember to take into account the Baker-Campbell-Hausdorff formula.

2. Let us now assume the lattice theory above "lives" on an Euclidean lattice at non-zero temperature, where the lattice extent in imaginary time direction is $1/T = aN_t$, and the boundaries are all periodic. The Polyakov loop is defined as the trace of the product of link matrices along a closed path in the t-direction:

$$P(x) = Tr[U_t(x,1)U_t(x,2)...U_t(x,N_t)] \tag{3}$$

   (a) Show me that this quantity is gauge invariant.

   (b) The center of a group is the set of group elements which commute with all elements of the group. What are the centers of $SU(2)$ and $SU(3)$?

   Tip: Center elements must be proportional to the unit matrix, and belong to the group.

(c) Let z be some member of the center of the gauge group, $z \in \mathbb{Z}_{2,3}$. Under the transformation,

$$U_0(x,t) \to zU_0(x,t) \tag{4}$$

for all x and some fixed t, argue that the action is invariant but that $P(x)$ is not.

Although we won't further pusue this, the breaking of these center symmetries at high temperatures, along with its restoration at low temperatures is reflected in the order parameter $\langle P \rangle$, which will either be nill (low temperature, confinement) or different from zero (high-temperature, deconfinement).