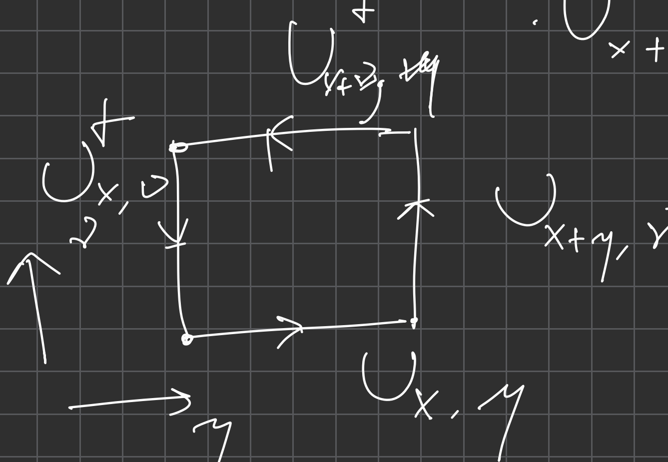
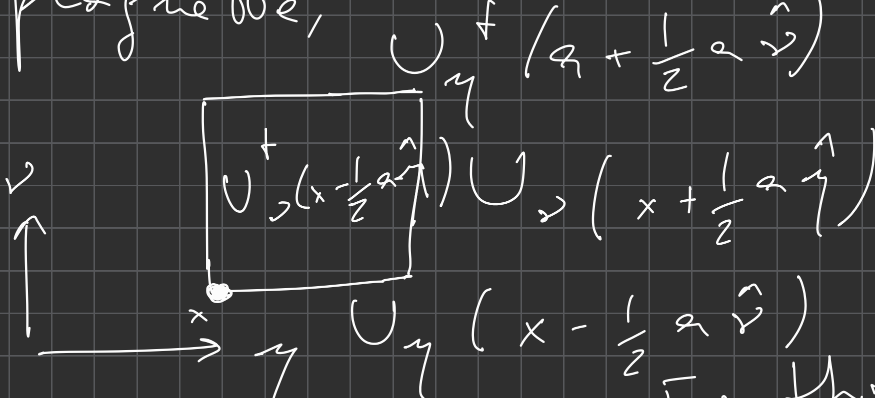
(1.) Lattice Action SU(N)

$$S = -\beta \sum_x \sum_{\mu>\nu} 1 - \frac{1}{N}\,\mathrm{Re\,Tr}\, U_{x,\mu\nu}$$

$$\cdot\, U_{x+\mu,\nu}\, U^\dagger_{x+\nu,\mu}\, U^\dagger_{x,\nu}$$



or, if we move to center of plaquette,



In this case, it places x at center of plaq.

$$U_\mu\left(x - \frac{1}{2}a\hat{\nu}\right) U_\nu\left(x + \frac{1}{2}a\hat{\mu}\right)$$

$$= e^{iga\left(A_\mu(x) - \frac{1}{2}a\partial_\nu A_\mu(x) + \mathcal{O}(a^2)\right)} \cdot e^{iga\left(A_\nu(x) + \frac{1}{2}a\partial_\mu A_\nu(x) + \mathcal{O}(a^2)\right)} \quad *$$

$$* = e^{iga\left(A_\mu + A_\nu + \frac{1}{2}a(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2}iga[A_\mu, A_\nu] + \mathcal{O}(a^3)\right)}$$

Gauge Field strength

$$= e^{iga\left(A_\mu + A_\nu + \frac{1}{2}aF_{\mu\nu} + \mathcal{O}(a^3)\right)}$$

Where,

$$e^{aA}e^{aB} = e^{a(A+B) + \frac{a^2}{2}[A,B] + \mathcal{O}(a^3)}$$

* higher orders are commutators of commutators

$$gA_\mu \longrightarrow A_\mu$$

Likewise,

$$U^\dagger_\mu\left(x+\frac{1}{2}a\hat{\nu}\right)U^\dagger_\nu\left(x-\frac{1}{2}a\hat{\mu}\right)$$

$$= e^{-iga\left(\underline{A_\mu}+\underline{A_\nu}+\frac{1}{2}a\overline{(\partial_\nu A_\mu-\partial_\mu A_\nu)}\right.}$$

$$\left.\underline{-\frac{1}{2}iga[A_\mu, A_\nu]}\right)+\mathcal{O}(a)^3$$

$$= e^{iga\left(-A_\mu-A_\nu+\frac{1}{2}aF_{\mu\nu}\right)+\mathcal{O}(a^3)}$$

The full product,

$$e^{iga\left(\underline{A_\mu+A_\nu}+\underline{\frac{1}{2}aF_{\mu\nu}}\right)+\text{Chaff}}.$$

$$\cdot\, e^{iga\left(\underline{-A_\mu-A_\nu}+\underline{\frac{1}{2}aF_{\mu\nu}}\right)+\text{Chaff}}$$

$$= e^{iga^2\underline{F_{\mu\nu}}+\underbrace{iga^3[A_\mu+A_\nu, F_{\mu\nu}]}}$$

$$+\,\underline{\mathcal{O}_c(a^3)}$$

↳ However those are also all commutators!

$\mathrm{Tr}[A,B] = 0$

Expand,

$$= \mathbb{1}+iga^2F_{\mu\nu}+_c\mathcal{O}(a^3)$$

$$-\frac{1}{2}g^2a^4\,\mathrm{Tr}\,F^2_{\mu\nu}+\mathcal{O}(a^5)$$

$$+\ldots$$

$\mathrm{Tr}[A,[B,C]]=0$

So, now having the $\boxed{\mathrm{I}}$ term computed, we need to show,

Cont$_{\text{inuum}}$ = $1 - \frac{1}{N} \mathrm{Tr}\,\mathrm{Re}\, \mathrm{I}$

$F^{a}_{\eta\nu} F^{\eta\nu a}$

$$\underbrace{\mathbb{1}}_{\text{Real}} + \underbrace{i g a^2 F_{\eta\nu}}_{\text{Imaginary}} - \underbrace{\frac{1}{2} g^2 a^4 F_{\eta\nu} F^{\eta\nu}}_{\text{Real}}$$

$$\mathrm{Tr}\, \mathbb{1} = N$$

$$N + \mathrm{Tr}\left[-\frac{1}{2} g^2 a^4 F_{\eta\nu}^2\right]$$

Also, $\mathrm{Tr}(F_{\eta\nu}) = 0$

$$\mathrm{Tr}\,[A, B] = 0$$

$$= \mathrm{Tr}\, AB + \mathrm{Tr}\, BA$$

$$= 0 \Rightarrow \mathrm{Tr}\,(O_c(a^3)) = 0$$

$$\mathrm{Re}\,\mathrm{Tr}\ \square = N - \frac{1}{2} g^2 a^4\,\mathrm{Tr}(F_{\eta\gamma})^2$$

$$\to \sum_x \sum_{\eta<\nu} \left(1 - \frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\ \square\right)$$

$$\cancel{1} - \frac{1}{\cancel{N}}\left(\cancel{N} - \frac{1}{2} g^2 a^4\,\mathrm{Tr}(F_{\eta\gamma})^2\right)$$

$$\Rightarrow + \frac{1}{2N} g^2 a^4\,\mathrm{Tr}(F_{\eta\gamma})^2$$

$$= \beta \frac{1}{2} \underbrace{\sum_{\eta \neq \gamma}}_{\sum_{\eta<\nu}} \sum_x \left(\frac{a^4}{2N} g^2\,\mathrm{Tr}\,F_{\eta\gamma}^2\right)$$

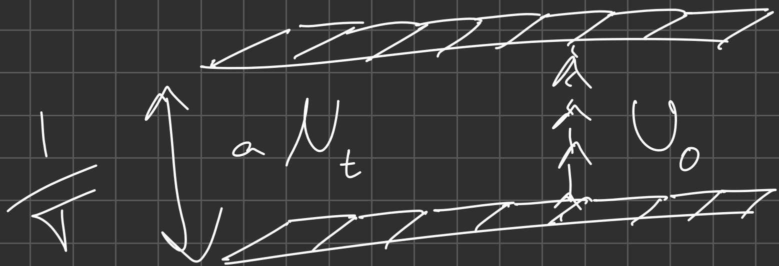
$$\sum_x a^4 \to \int d^4x$$

$$= \int d^4x\ \frac{1}{2}\mathrm{Tr}(F_{\eta\gamma}F^{\eta\gamma})$$

$\beta = \frac{2N}{g^2}$ is set to recover continuum.

② Polyakov Loop

$$P(x) = \mathrm{Tr}\left[U_0(x,1)\, U_0(x,2) \ldots U_0(x,N_t) \right]$$



a) $U_0(x,i) \longrightarrow g(x,i)\, U_0(x,i) \cdot g^\dagger(x, i+1)$

where $g \in SU(N)$

$$P(x) =$$

$$= \mathrm{Tr}\left[g(x,1)\, U_0(x,1)\, \cancel{g^\dagger(x,2)} \cdot \cancel{g(x,2)}\, U_0(x,2)\, g^\dagger(x,3) \cdot \ldots g(x,N_t)\, U_0(x,N_t)\, g^\dagger(x,1) \right]$$

$$= P(x)$$

b) <u>Center elements</u>

$\propto \mathbb{1}$, $c\,\mathbb{1}$, $c \in \mathbb{C}$

Must commute w/ any $U \in SU(N)$

$$(c\,\mathbb{1})^\dagger = (c\,\mathbb{1})^{-1}$$

$$\Rightarrow c^* = \frac{1}{c} \Rightarrow c^* c = 1$$

$$\Rightarrow |c| = 1$$

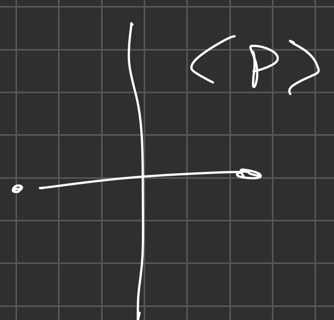
$$\det(c\,\mathbb{1}) = c^N = 1$$

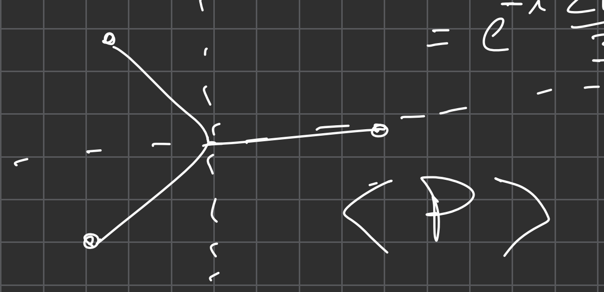
$$\Rightarrow c = e^{i 2\pi n / N}$$

$n = 0, 1, \ldots (N-1)$

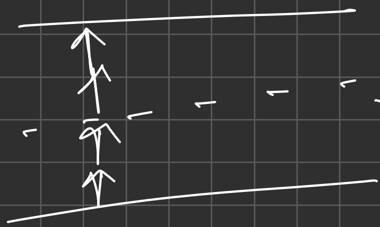
$\Rightarrow SU(2) = c\ \pm 1$

$$SU(3) = 1,\ e^{i 2\pi/3},\ e^{i \frac{4\pi}{3}} = e^{-i\frac{2\pi}{3}}$$





c)



$\Rightarrow U_0 = z U_0$

on this slice

Gauge action consists of (small) closed loops (traced)!
Now if a loop includes $U_0(x, t_{fix})$, it must also include $U^\dagger(x, t_{fix})$

$$\mathrm{Tr}\left[U_1 U_2 \dots U_0(x, t_{fix}) \dots U_0^\dagger(x, t_{fix}) \dots \right]$$

$$\to \mathrm{Tr}\left[U_1 U_2 \dots \underline{z} U_0(x, t_{fix}) \dots U_0^\dagger(x, t_{fix}) \underline{z}^\dagger \dots \right]$$

= exactly the same as before = Action is invariant

Now for Polyakov loop,

$$P(x) = \mathrm{Tr}\left[\ldots U_0(x, t_{fix}) \ldots\right]$$

$$\rightarrow \mathrm{Tr}\left[\ldots z\, U_0(x, t_{fix}) \ldots\right]$$

$$= c\, \mathrm{Tr}\left[\ldots U_0(x, t_{fix}) \ldots\right]$$

$$= c\, P(x)$$

where $c = z\,\mathbb{1}$. Thus $P(x)$ is rotated on complex plane.

If this center symmetry is a symmetry of the action, states with $\langle P \rangle$ and $z\langle P \rangle$ are equally likely.