2.

b) Write the sum of matrix product

$$-\frac{1}{2}\sum_{x,y}\phi_x \underline{M}_{xy}\phi_y + \sum_x J_x \phi_x$$

$$= -\frac{1}{2}\underline{\Phi}^T \underline{M}\,\underline{\Phi} + \underline{J}^T \underline{\Phi}$$

where $\underline{J}$, $\underline{\Phi}$ are vectors w/ components ϕ_x, J_x

NOTE:
I've absorbed factors of a^D as, $a^D M \to J$
$a^D J \to J$

M is invertible, so

$$\underline{\Phi} = y + M^{-1} J$$

$$M_{xy} = M_{yx} \quad M^T = M$$

$$= (M^T)^{-1} = M^{-1}$$

So,

$$-\frac{1}{2}\left[\underline{\Phi}^T M \underline{\Phi} + J^T \underline{\Phi}\right]$$

$$= -\frac{1}{2}\left(y^T + J^T [M^T]^{-1}\right) M \left(y + M^{-1} J\right)$$

$$+ J^T y + J^T M^{-1} y$$

$$= -\frac{1}{2}\Big[y^T M y + \cancel{y^T J + J^T y}$$

$$+ J^T M^{-1} J - 2 J^T y$$

$$- 2 J^T M^{-1} J$$

(as $J^T y$ is a scalar $J^T y = y^T J$)

$$= -\frac{1}{2}\left[y^T M y - J^T M^{-1} J \right]$$

So now the formula for integral can be applied,

$$Z = \int_{-\infty}^{\infty} \prod_i dy_i \exp\Big\{ -\frac{\Phi^D}{2}\big(y^T M y$$

$$- J^T M^{-1} J \big) \Big\}$$

$$= e^{\frac{1}{2} J^T M^{-1} J} \int_{-\infty}^{\infty} \prod_i dy_i \exp\left[-\frac{\Phi^D}{2} y^T M y \right]$$

$$= e^{\frac{1}{2} J^T M^{-1} J} \underbrace{\det\left(\frac{M}{2\pi}\right)^{-1/2}}_{K}$$

$$= \exp\left\{\frac{1}{2} J^T M^{-1} J\right\} K$$

For $Z(J=0)$,

$$Z(0) = K$$
$$= \det(M/2\pi)^{-1/2}$$

Now, $\partial_{J_x} \partial_{J_y} Z(\phi, J)\Big|_{J=0}$

$$= K \,\partial_{J_x} \partial_{J_y} \exp\left\{\frac{1}{2} J^T M^{-1} J\right\}$$

$$\partial_{J_y}\left(\exp\left\{\frac{1}{2} J^T M^{-1} J\right\}\right)$$

$$= \partial_{J_y}\left[\exp \frac{1}{2}\left\{J_{x'} M^{-1}_{x'y'} J_{y'}\right\}\right]$$

$$= \frac{1}{2}\sum_{x'y'}\Big[J_{x'} M^{-1}_{x'y'} \underbrace{\frac{\partial J_{y'}}{\partial J_y}}_{\delta_{yy'}}$$
$$+ \overbrace{\frac{\partial J_{x'}}{\partial J_y}}^{\delta_{x'y'}} M^{-1}_{x'y'} J_{y'}\Big]\Big)$$

$$= \Big(\sum_{x', y'} \big[J_{x'} M^{-1}_{x'y'} \delta_{yy'} + J_{y'} M^{-1}_{x'y'} \delta_{x'y} \big] \Big)$$

Now, next derivative,

$$\partial J_x \sum_{x'y'} \big[J_{x'} M^{-1}_{x'y'} \delta_{yy'} + J_{y'} M^{-1}_{x'y'} \delta_{x'y'} \big]$$

(Remember that any terms left w/ J_x, J_y after derivative will go to zero : $J = 0$ at end)

$$= \sum_{x'y'} \big[M^{-1}_{x'y'} \underline{\delta_{xx'}} \, \underline{\delta_{yy'}} + M^{-1}_{x'y'} \delta_{x'y} \delta_{y'x} \big]$$

$$= 2 M^{-1}_{xy}$$ (symmetric and transpose)

$$\partial J_x \partial J_y Z(\phi, J) = M^{-1}_{xy}$$

$$\Rightarrow \langle \phi_x \phi_y \rangle = \frac{K}{Z(0)} \partial_{J_{x'}} \partial_{J_{y'}} \left[\cdot e^{\frac{1}{2} \sum_{x',y'} J_{x'} M^{-1}_{x'y'} J_{y'}} \right]$$

$$= \frac{K}{Z(0)} \partial_{J_{x'}} \partial_{J_{y'}} \left[e^{\frac{1}{2} \sum_{x'y'} J_{x'} M^{-1} J_{y'}} \right]$$

$$= \frac{\cancel{K} \cancel{Z}}{\cancel{K} \cancel{Z}} M^{-1}_{xy} \; e^{\underbrace{\frac{1}{2} J_{x'} M^{-1}_{x'y'} J_{y'}}} \Big|_{J=0} \; J=0$$

$$= M^{-1}_{xy} \Rightarrow D$$

Let's substitute back a^D,

$$\langle \phi_x \phi_y \rangle = M^{-1}_{xy} \, a^{-D}$$

c) $$G(K) = \int_{-\pi/a}^{\pi/a} \frac{d^n K}{(2\pi)^n} \langle \phi_x \phi_y \rangle \cdot e^{iK(x-y)}$$

Use Green's function method,

$$a^d \sum_y G_{xy} M_{yz} = \delta_{xz}$$

On Lattice,

$$\boxed{\tilde{f}(k) = \sum_x f(x) e^{ikx}}$$

recall, $G(x,y) = G(x-y)$

$G(x) e^{-ikx}$

RHS (easy bit),

$$a^d \sum_x \delta_{xz} e^{-ik(x-z)} = a^d$$

$\left(G(x-z) \, e^{-ik(x-z)} \right.$

Conversely m^2 term of M_{yz},

$$a^d \sum_y G_{xy} \, m^2 \sum_y \delta_{yz} \, e^{ik(x-y)}$$

$$= \underbrace{m^2 a^d \sum_y G(x-z) \delta_{xz} e^{ik(x-z)}}$$

$$= m^2 a^d \tilde{G}(k)$$

for terms in the edges of Laplacian stencil,

$$a^d \sum_y G_{xy} \sum_\eta \left(\delta_{y+\eta, z} + \delta_{y-\eta, z} \right)$$

$$\boxed{y = x \pm \eta} \quad G(x,y)\, e^{ik(x-y)}$$

$$= a^d \sum_{x \pm z} \sum_\eta \Big[G(x-\eta-z) \cdot e^{ik(x-\eta-z)}$$

$$+ \; G(x+\eta-z) \cdot e^{ik(x+\eta-z)} \quad (X - \eta - z) \quad (X + \eta - z)$$

$$G(x-z)$$

$$= a^d \sum_x \sum_\eta \Big[G(x) e^{ik(x-\eta)} + G(x) e^{ik(x+\eta)} \Big] \quad X \to (z, x, \eta)$$

$$= a^d \sum_x \sum_\eta G(x) e^{ikx} \cdot \left[e^{ik\eta} + e^{-ik\eta} \right]$$

$$= a^d \sum_x G(x) e^{ikx} .$$

$$\cdot \sum_\eta 2\cos(k\eta)$$

$$= a^d G(\tilde{k}) \sum_\eta 2\cos(k\eta)$$

Combined with the $2\delta_{x,y}$ terms in Lagragian

$$a^d G(\tilde{k}) \sum_\eta [2 - 2\cos(k\hat{\eta})]$$

$$\cong a^d G(\tilde{k})\, 2\sin^2\left(\frac{k\hat{\eta}}{2}\right)$$

$$k\hat{\eta} = k\eta a$$

$$\longrightarrow a^d \sum_\eta G(\tilde{k}) \left[2\sin^2\left(\frac{k\hat{\eta}}{2}\right) + m^2\right] = a^d \sum_\eta G(\tilde{k}) \cdot (\hat{k}^2 + m^2)$$

RHS + LHS combined,

$$\cancel{a^d} \sum_\eta (\hat{k} + m^2) = \cdot \frac{\cancel{a^d}}{G(\tilde{k})}$$

$$\Rightarrow G(\hat{k}) = \frac{1}{\hat{k} + m^2}$$