Estimation and Confidence Intervals with Python

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Contents

This chapter aims to show how to compute point estimates and confidence intervals for the mean, variance, and the difference if means. The second part of this chapter will show how to perform the most used and classical hypothesis testing.

We start by importing the data in the file `data1.csv`

[1]:
```
import pandas as pd
import numpy as np
df = pd.read_csv('data1.csv')
df.head(5)
```

[1]:
```
   Age Attrition     BusinessTravel  DailyRate              Department  \
0   41       Yes      Travel_Rarely       1102                   Sales
1   49        No  Travel_Frequently        279  Research & Development
2   37       Yes      Travel_Rarely       1373  Research & Development
3   33        No  Travel_Frequently       1392  Research & Development
4   27        No      Travel_Rarely        591  Research & Development

   DistanceFromHome  Education EducationField  EmployeeCount  EmployeeNumber  \
0                 1          2  Life Sciences              1               1
1                 8          1  Life Sciences              1               2
2                 2          2          Other              1               4
3                 3          4  Life Sciences              1               5
4                 2          1        Medical              1               7

   ...  RelationshipSatisfaction StandardHours  StockOptionLevel  \
0  ...                         1            80                 0
1  ...                         4            80                 1
```

```
2  ...                         2            80                  0
3  ...                         3            80                  0
4  ...                         4            80                  1

   TotalWorkingYears  TrainingTimesLastYear WorkLifeBalance  YearsAtCompany  \
0                  8                      0               1               6
1                 10                      3               3              10
2                  7                      3               3               0
3                  8                      3               3               8
4                  6                      3               3               2

  YearsInCurrentRole  YearsSinceLastPromotion  YearsWithCurrManager
0                  4                        0                     5
1                  7                        1                     7
2                  0                        0                     0
3                  7                        3                     0
4                  2                        2                     2

[5 rows x 35 columns]
```

1 The mean

1.1 The theory

Assume that we're interested in the variable age from the imported data, and we would like to know the following information:

- the average age of the employees in the survey?
- the probability that one given employee has an age higher than 50?
- the distribution of the employees' age between the different departments?

We will first assume that sequence of employees's age is a random sample. We will write as a sequence of random variables $X_1, \ldots, X_n$.

We assume also that $X_1, \ldots, X_n$ are generated from a Normal distribution with mean μ and with variance σ^2.

It's known that the $\overline{X}$ is an estimator of the mean μ. Since the sample mean is also a random variable with normal distribution with mean μ and variance $\frac{\sigma^2}{n}$, we can provide an interval that provides the error on the estimation. It's called the **Confidence Interval**.

We aim in chapter to show how to compute the confidence interval of the mean μ with level $(1-\alpha)$, $\alpha \in (0,1)$. We denoted by $\mathrm{CI}_{1-\alpha}(\mu)$

If σ^2 is **known**, $\mathrm{CI}_{1-\alpha}(\mu)$ is expressed as follows:

$$\left(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}},\ \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

where

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is the sample mean and $z_{1-\alpha/2}$ is the percentile associated to $(1-\alpha/2)$ from the standard normal distribution:

$$F_Z(z_{1-\alpha/2}) = 1-\alpha/2$$

where Z is a random variable with standard normal distribution.

If σ^2 is **unknown**, $\mathrm{CI}_{1-\alpha}(\mu)$ is expressed as follows:

$$\left(\overline{X} - t_{1-\alpha/2,n-1}\frac{S}{\sqrt{n-1}}, \ \overline{X} + t_{1-\alpha/2,n-1}\frac{S}{\sqrt{n-1}}\right)$$

where S^2 is the sample mean:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

and $t_{1-\alpha/2,n-1}$ is the percentile associated to $(1-\alpha/2)$ from the $t-$distribution with $n-1$ degrees of freedom:

$$F_{T_{n-1}}(t_{1-\alpha/2,n-1}) = 1-\alpha/2$$

where T_{n-1} is a random variable with $t-$distribution $n-1$ degrees of freedom.

1.2 Practice with Python

[2]:
```
import numpy as np
from scipy.stats import norm,t
```

We will write two functions. A first one returns the $\mathrm{CI}_{1-\alpha}(\mu)$ when σ^2 is known and the second functions returns $\mathrm{CI}_{1-\alpha}(\mu)$ when σ^2 is unknown.

1st function

[3]:
```
def get_ci_known_variance(sigma, sample_mean, sample_size, alpha):
    margin_of_error = norm.ppf(1 - alpha/2)*sigma/np.sqrt(sample_size)
    return sample_mean - margin_of_error, sample_mean + margin_of_error
```

2nd function

[4]:
```
def get_ci_unknown_variance(sample_s, sample_mean, sample_size, alpha):
    margin_of_error = t.ppf(1 - alpha/2,sample_size-1)*sample_s/np.
 ↪sqrt(sample_size-1)
    return sample_mean - margin_of_error, sample_mean + margin_of_error
```

Practice

simulating data

[5]:
```
mu, sigma = 40, 2.5
random_sample = np.random.normal(mu, sigma, 100)
```

[6]:
```
random_sample[0:5]
```

[6]:
```
array([36.26312646, 40.93868526, 38.17728249, 47.47319864, 38.35240819])
```

Computing the sample mean

[7]:
```
sample_mean=np.average(random_sample)
```

[8]:
```
sample_mean
```

[8]:
```
39.64952413958712
```

[9]:
```
sample_size=len(random_sample)
sample_size
```

[9]:
```
100
```

$\mathrm{CI}_{1-\alpha}(\mu)$ with known variance and $\alpha = 0.05$

[10]:
```
get_ci_known_variance(sigma,sample_mean,sample_size,.05)
```

[10]:
```
(39.15953314345211, 40.139515135722135)
```

Computing the sample standard deviation

[11]:
```
sample_s=random_sample.std()
sample_s
```

[11]:
```
2.5822881766605192
```

$\mathrm{CI}_{1-\alpha}(\mu)$ with unknown variance and $\alpha = 0.05$

[12]:
```
get_ci_unknown_variance(sample_s,sample_mean,sample_size,.05)
```

[12]:
```
(39.13456085636115, 40.164487422813096)
```

We can also use a function already implemented in the library `scipy` to compute $\mathrm{CI}_{1-\alpha}(\mu)$ when the variance is unknown.

[13]:
```
confidence_level = 0.95
degrees_freedom = sample_size - 1
```

[14]:
```
import scipy
```

[15]:
```
sample_standard_error = scipy.stats.sem(random_sample)
```

[16]:
```
sample_standard_error
```

[16]:
```
0.2595297267440583
```

[17]:
```
sample_s/np.sqrt(sample_size-1)
```

[17]: 0.2595297267440583

[18]:
```
scipy.stats.t.interval(confidence_level, degrees_freedom, sample_mean,␣
 ↪sample_standard_error)
```

[18]: (39.13456085636115, 40.164487422813096)

We can also use `scipy.stats.norm.interval` to compute $\mathrm{CI}_{1-\alpha}(\mu)$ with known variance

[19]:
```
scipy.stats.norm.interval(.95,loc=sample_mean, scale=sigma/np.sqrt(sample_size))
```

[19]: (39.15953314345211, 40.139515135722135)

1.3 Practice with data

Application: comparing the average of the Daily rate between Men and Women.

[20]:
```
df['Gender'].value_counts()
```

[20]:
```
Male      882
Female    588
Name: Gender, dtype: int64
```

[21]:
```
df.groupby("Gender").agg({"DailyRate": [np.mean, np.std, np.size]})
```

[21]:

	DailyRate		
	mean	std	size
Gender			
Female	808.273810	408.241680	588
Male	798.626984	400.509021	882

[22]:
```
x=df.groupby("Gender").agg({"DailyRate": [np.mean, np.std, np.size]})
```

[23]:
```
x=np.array(x)
```

[24]:
```
x
```

[24]:
```
array([[808.27380952, 408.24167967, 588.        ],
       [798.62698413, 400.50902101, 882.        ]])
```

The female sample size

[25]:
```
x[0,2]
```

[25]: 588.0

The female sample mean

[26]:
```
x[0,0]
```

[26]: 808.2738095238095

The female standard error

[27]:
```
x[0,1]/np.sqrt(x[0,2]-1)
```

[27]: 16.84993739332325

The `DailyRate` Female CI(95%)

[28]:
```
CI_F=scipy.stats.t.interval(.95, x[0,2] -1, x[0,0], x[0,1]/np.sqrt(x[0,2]-1))
```

[29]:
```
CI_F
```

[29]: (775.1803043941346, 841.3673146534844)

We compute then the The DailyRate Female CI(95%)

We select now the `DailyRate`sample for Men and Women separatly

[30]:
```
CI_M=scipy.stats.t.interval(.95, x[1,2] -1, x[1,0], x[1,1]/np.sqrt(x[1,2]-1))
```

[31]:
```
CI_M
```

[31]: (772.1438431601089, 825.1101250938593)

[32]:
```
CI_M[0]
```

[32]: 772.1438431601089

We can then visualize these Confidence intervals together to see the difference between the `DailyRate` means.

[33]:
```
ci_dailrate = {}
ci_dailrate['Gender'] = ['Male','Female']
ci_dailrate['lb'] = [CI_M[0],CI_F[0]]
ci_dailrate['ub'] = [CI_M[1],CI_F[1]]
df_ci= pd.DataFrame(ci_dailrate)
```

[34]:
```
df_ci
```

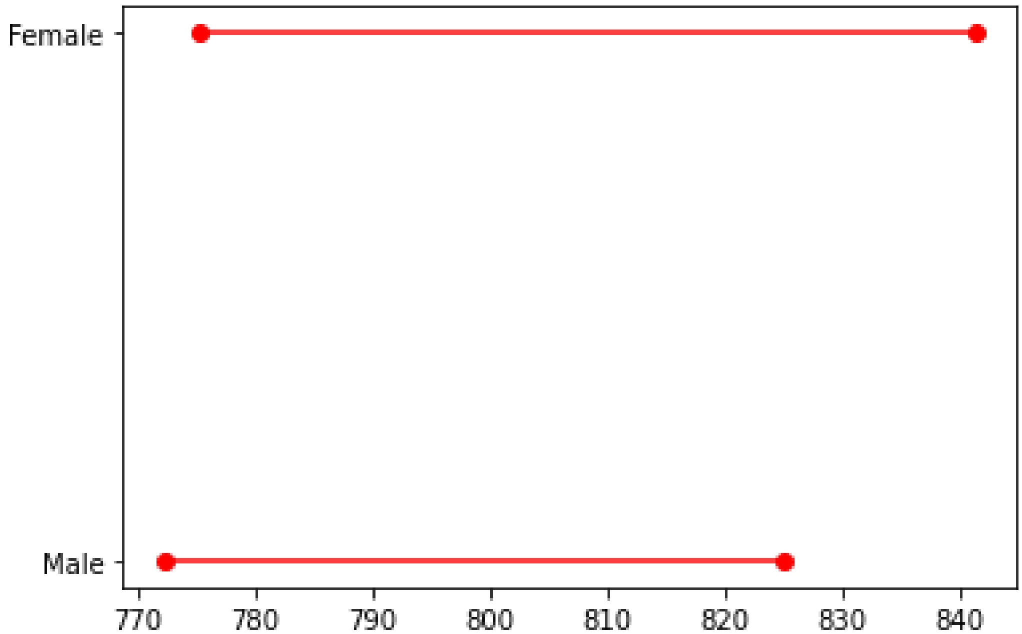
[34]:

	Gender	lb	ub
0	Male	772.143843	825.110125
1	Female	775.180304	841.367315

[35]:
```
import matplotlib.pyplot as plt
```

[36]:
```
for lb,ub,y in zip(df_ci['lb'],df_ci['ub'],range(len(df_ci))):
    plt.plot((lb,ub),(y,y),'ro-')
plt.yticks(range(len(df_ci)),list(df_ci['Gender']))
```

[36]:
```
([<matplotlib.axis.YTick at 0x14386816d30>,
  <matplotlib.axis.YTick at 0x143868165b0>],
 [Text(0, 0, 'Male'), Text(0, 1, 'Female')])
```

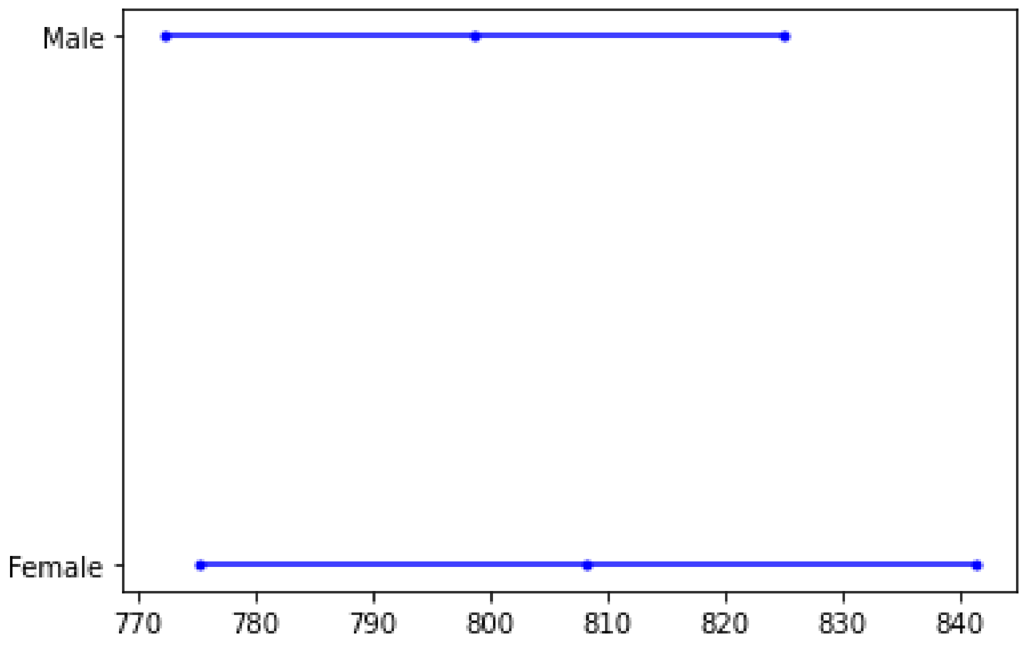


We will now write a Python function that can compare the Confidence Intervals of the means for a given continuous variable according to groups defined by a categorical variable.

[37]:
```
import pandas as pd
import numpy as np
import scipy.stats as st

def plot_diff_in_means(data: pd.DataFrame,alpha, col1: str, col2: str):
    """
    given data, plots difference in means with confidence intervals across groups
    col1: categorical data with groups
    col2: continuous data for the means
    alpha: is the level of significance, it's usualy equal to .95
    """
    n = data.groupby(col1)[col2].count()
    # n contains a pd.Series with sample size for each category

    cat = list(data.groupby(col1, as_index=False)[col2].count()[col1])
    # cat has names of the categories, like 'category 1', 'category 2'

    mean = data.groupby(col1)[col2].agg('mean')
    # the average value of col2 across the categories
```

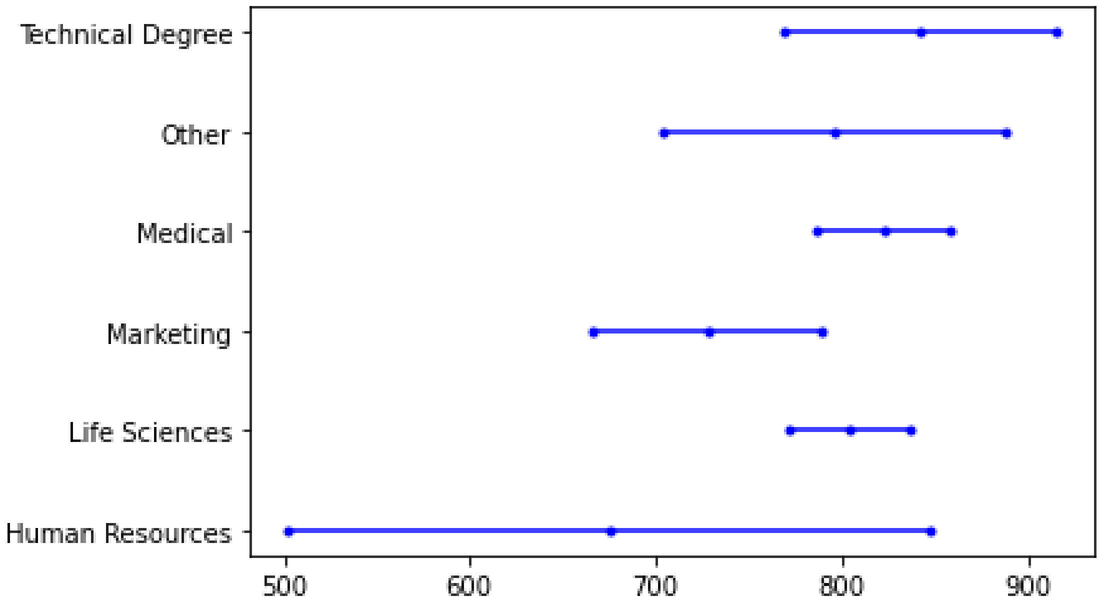
```
    std = data.groupby(col1)[col2].agg(np.std)
    se = std / np.sqrt(n)
    # standard deviation and standard error

    lower = st.t.interval(alpha = alpha, df=n-1, loc = mean, scale = se)[0]
    upper = st.t.interval(alpha = alpha, df =n-1, loc = mean, scale = se)[1]
    # calculates the upper and lower bounds using scipy

    for upper, mean, lower, y in zip(upper, mean, lower, cat):
        plt.plot((lower, mean, upper), (y, y, y), 'b.-')
        # for 'b.-': 'b' means 'blue', '.' means dot, '-' means solid line
    plt.yticks(
        range(len(n)),
        list(data.groupby(col1, as_index = False)[col2].count()[col1])
        )
```

[38]:
```
plot_diff_in_means(data = df,alpha=.95, col1 = 'Gender', col2 = 'DailyRate')
```



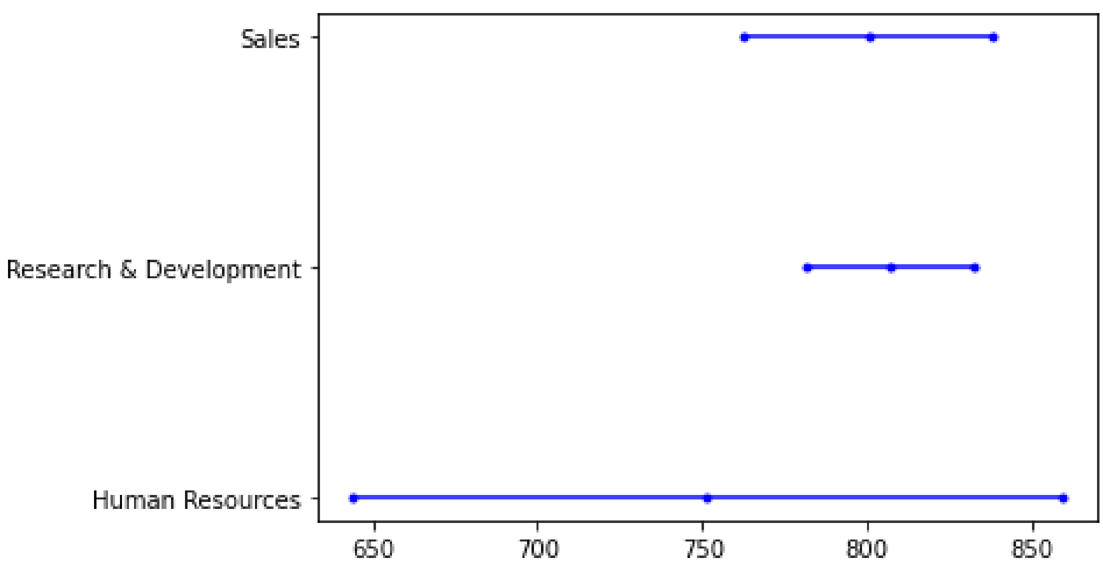
DailyRate and EducationField

[39]:
```
plot_diff_in_means(data = df,alpha=.95, col1 = 'EducationField', col2 =␣
 ↪'DailyRate')
```



DailyRate and Department

[40]:
```
plot_diff_in_means(data = df,alpha=.95, col1 = 'Department', col2 = 'DailyRate')
```



2 The proportion

Assume that we would like to estimate the probability p to win an election for candidate A. We randomly select n and consider X the number of people reported that will vote for A.

The probability or the proportion p is then estimated by

$$\widehat{p} = \frac{X}{n}$$

In most of the cases the number n, the sample size, is large and the probability distribution of $\widehat{p}$ is approximated with a Normal distribution with mean $\mu = p$ and variable $\sigma^2 = \dfrac{p(1-p)}{n}$. We can provide then, for a given $\alpha \in (0,1)$, a **Confidence interval** with level $1-\alpha$:

$$\mathrm{CI}_{1-\alpha}(p) = \left(\widehat{p} - z_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}},\ \widehat{p} + z_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}\right)$$

where $z_{1-\alpha/2}$ is the $z-$score associate to $1-\alpha/2$. It means, if Z is a standard normal distribution, then

$$F_Z(z_{1-\alpha/2}) = 1 - \alpha/2$$

Where F_Z is the CDF of Z.

Example: A Survey was conducted to estimate the probability p to vote a candidate A. Among the 1200 participated in the Survey, 560 reported that will vote for the candidate A. Find an estimation of p and its 95%-Confidence Interval.

[41]:
```
import statsmodels.api as sm
```

Importing the Function for computing proportion confidence intervals

[42]:
```
from statsmodels.stats.proportion import proportion_confint
```

[43]:
```
proportion_confint(count=560,    # Number of "successes"
                   nobs=1200,    # Number of trials
                   alpha=(1 - 0.95),
                   method='normal')  # when we use asymptotic normal␣
 ↪approximation
# Alpha, which is 1 minus the confidence level
```

[43]:
```
(0.4384399591955059, 0.49489337413782747)
```

There's also four oothers methods to compute the proportion confidence interval:

- agresti_coull : Agresti-Coull interval
- beta : Clopper-Pearson interval based on Beta distribution
- wilson : Wilson Score interval
- jeffreys : Jeffreys Bayesian Interval
- binom_test : experimental, inversion of binom_test

Example: We would like to compare the proportion of frequently traveling between the three departments. We start by computing first the contingency table between the variables BusinessTravel and Departments.

[44]:
```
import pandas as pd
```

[45]:
```
tab = pd.crosstab(df['BusinessTravel'], df['Department'],margins=True)
tab
```

[45]:
```
Department         Human Resources  Research & Development  Sales   All
BusinessTravel
Non-Travel                       6                      97     47   150
Travel_Frequently               11                     182     84   277
Travel_Rarely                   46                     682    315  1043
All                             63                     961    446  1470
```

[46]:
```
table = sm.stats.Table(tab)
table.table
```

[46]:
```
array([[   6.,   97.,   47.,  150.],
       [  11.,  182.,   84.,  277.],
       [  46.,  682.,  315., 1043.],
       [  63.,  961.,  446., 1470.]])
```

[47]:
```
CI_HR=proportion_confint(count=table.table[1,0],nobs=table.table[3,0],alpha=(1-.
↪95))
CI_HR
```

[47]:
```
(0.08086094446182195, 0.2683454047445272)
```

[48]:
```
CI_RD=proportion_confint(count=table.table[1,1],nobs=table.table[3,1],alpha=(1-.
↪95))
CI_RD
```

[48]:
```
(0.16461369335918247, 0.214158419023752)
```

[49]:
```
CI_SL=proportion_confint(count=table.table[1,3],nobs=table.table[3,3],alpha=(1-.
↪95))
CI_SL
```

[49]:
```
(0.16844448112059657, 0.20842626717872315)
```

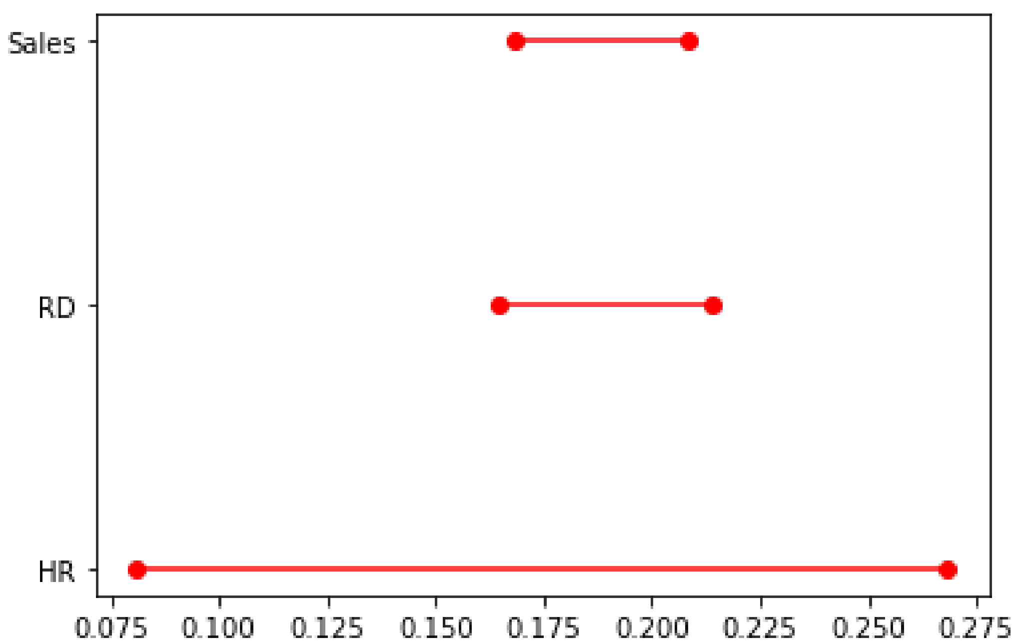
[50]:
```
ci_travel = {}
ci_travel['Department'] = ['HR','RD','Sales']
ci_travel['lb'] = [CI_HR[0],CI_RD[0],CI_SL[0]]
ci_travel['ub'] = [CI_HR[1],CI_RD[1],CI_SL[1]]
df_ci= pd.DataFrame(ci_travel)
df_ci
```

[50]:
```
  Department        lb        ub
0         HR  0.080861  0.268345
1         RD  0.164614  0.214158
2      Sales  0.168444  0.208426
```

[51]:
```python
import matplotlib.pyplot as plt
```

[52]:
```python
for lb,ub,y in zip(df_ci['lb'],df_ci['ub'],range(len(df_ci))):
    plt.plot((lb,ub),(y,y),'ro-')
plt.yticks(range(len(df_ci)),list(df_ci['Department']))
```

[52]:
```
([<matplotlib.axis.YTick at 0x14387d4d1f0>,
  <matplotlib.axis.YTick at 0x14387d47b50>,
  <matplotlib.axis.YTick at 0x14387d43a00>],
 [Text(0, 0, 'HR'), Text(0, 1, 'RD'), Text(0, 2, 'Sales')])
```



3 The difference of two proportions

We observe now two independents samples with different sizes n_1 and n_2. We estimate from each sample a proportion. We aim to provide a **confidence interval** of the difference between these proportions. It can be expressed as follows:

$$\mathrm{CI}_{1-\alpha}(p_1 - p_2) = \left(\widehat{p}_1 - \widehat{p}_2 - z_{1-\alpha/2}\sqrt{\frac{\widehat{p}_1(1-\widehat{p}_1)}{n_1} + \frac{\widehat{p}_2(1-\widehat{p}_2)}{n_2}},\ \widehat{p}_1 - \widehat{p}_2 + z_{1-\alpha/2}\sqrt{\frac{\widehat{p}_1(1-\widehat{p}_1)}{n_1} + \frac{\widehat{p}_2(1-\widehat{p}_2)}{n_2}} \right)$$

We will write the following Python function

[53]:
```python
import pandas as pd
import numpy as np
import scipy.stats as stats

def two_proprotions_confint(success_a, size_a, success_b, size_b, significance =␣
↪0.05):
    """
    A/B test for two proportions;
    given a success a trial size of group A and B compute
    its confidence interval;
    resulting confidence interval matches R's prop.test function

    Parameters
    ----------
    success_a, success_b : int
        Number of successes in each group

    size_a, size_b : int
        Size, or number of observations in each group

    significance : float, default 0.05
        Often denoted as alpha. Governs the chance of a false positive.
        A significance level of 0.05 means that there is a 5% chance of
        a false positive. In other words, our confidence level is
        1 - 0.05 = 0.95

    Returns
    -------
    prop_diff : float
        Difference between the two proportion

    confint : 1d ndarray
        Confidence interval of the two proportion test
    """
    prop_a = success_a / size_a
    prop_b = success_b / size_b
    var = prop_a * (1 - prop_a) / size_a + prop_b * (1 - prop_b) / size_b
    se = np.sqrt(var)

    # z critical value
    confidence = 1 - significance
    z = stats.norm(loc = 0, scale = 1).ppf(confidence + significance / 2)

    # standard formula for the confidence interval
    # point-estimtate +- z * standard-error
```

```
    prop_diff = prop_b - prop_a
    confint = prop_diff + np.array([-1, 1]) * z * se
    return prop_diff, confint
```

The Confidence interval of the difference between the proportions of frequently traveling between R&D and HR departments

[54]:
```
two_proprotions_confint(success_a=table.table[1,0],size_a=table.
 ↪table[3,0],success_b=table.table[1,1],size_b=table.table[3,1])
```

[54]:
```
(0.014782881588292635, array([-0.08217729,  0.11174306]))
```

The Confidence interval of the difference between the proportions of frequently traveling between R&D and Sales departments

[55]:
```
two_proprotions_confint(success_a=table.table[1,2],size_a=table.
 ↪table[3,2],success_b=table.table[1,1],size_b=table.table[3,1])
```

[55]:
```
(0.001045249016579347, array([-0.04289047,  0.04498097]))
```