<u>Final Project:</u>
MCG5109

Karim Saadé – 8074914

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I. Introduction

In order to solve any engineering problem and to design and analyze engineering concepts, the first step towards the solution is the physical reality. By understanding the fundamental of physics, chemistry, and mathematics, one can build mathematical models to describe the situation and the problem ahead. One of these methods is called the Finite Element method. This method encompasses momentum equation, kinematics and the balance and imbalance laws [1] [2].

Finite Element method is a procedure in which the continuum behavior represented at an infinite number of points is estimated with a finite number of points. These points are called nodes and they are located at specific points in the domain. These nodes are connected in order to define regions, called finite elements. The elements enclose and are defined over the geometry where governing equations are approximated [3].

This report will firstly introduce and provide some general background information about stress and strain applied to linear elasticity and some terminology. Then, by explaining and laying down the steps and key features of finite element analysis, the 3D problem for linear elastostatic problem will be solved and programmed using Python. Finally, three test cases will put in practice the theory explained by showing the displacement, stress, and forces at each node. These results will be visualized using Paraview.

II. Preliminary Information

Firstly, some information must be discussed in order to have a general background about stress and strain. This theory includes “Stress and Strain Tensors”, “Generalized Hooke’s Law”, “Isotropic Linear Elasticity” and “Linear Elastostatics”.

a. Stress and Strain Tensors

- Displacement vector:

$$u_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{1.1}$$

- Stress tensor:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{1.2}$$

- Strain tensor:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{1.3}$$

b. Generalized Hooke’s Law

The matrix form of the generalized Hooke’s Law is derived from the tensor form using Voigt notation. Voigt notation maps a pair of indices ***(i, j)***, where *i, j* = 1, 2, 3 to a single index ***k***, where *k* = 1, …, 6.

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}D_{11}&D_{12}&D_{13}&D_{14}&D_{15}&D_{16}\\D_{21}&D_{22}&D_{23}&D_{24}&D_{25}&D_{26}\\D_{31}&D_{32}&D_{33}&D_{34}&D_{35}&D_{36}\\D_{41}&D_{42}&D_{43}&D_{44}&D_{45}&D_{46}\\D_{51}&D_{52}&D_{53}&D_{54}&D_{55}&D_{56}\\D_{61}&D_{62}&D_{63}&D_{64}&D_{65}&D_{66}\end{bmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\2\epsilon_4\\2\epsilon_5\\2\epsilon_6\end{bmatrix} \tag{1.4}$$

c. Isotropic Linear Elasticity

The isotropic linear elasticity is derived from the generalized Hooke's law where it encompasses material parameters using Lamé parameters, stress and strain shown below.

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}\lambda+2\mu&\lambda&\lambda&0&0&0\\\lambda&\lambda+2\mu&\lambda&0&0&0\\\lambda&\lambda&\lambda+2\mu&0&0&0\\0&0&0&G&0&0\\0&0&0&0&G&0\\0&0&0&0&0&G\end{bmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\2\epsilon_4\\2\epsilon_5\\2\epsilon_6\end{bmatrix} \tag{1.5}$$

Where,

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)} \tag{1.6}$$

And,

$$\mu=G=\frac{E}{2(1+\nu)} \tag{1.7}$$

Where,
ν, poisson ratio
E, Modulus of elasticity

d. Linear Elastostatics

The linear elastostatics equations are obtained from the balance of linear momentum in the domain, Ω, and on the edge of the domain, Γ, where its union is the domain. The set of equations to solve any elastostatic problem are shown below.

$$\begin{cases}\sigma_{ij,j}+l_i=0\ in\ \Omega\\ u_i=g_i\ on\ \Gamma_{u_i}\\ \sigma_{ij}n_j=h_i\ on\ \Gamma_{h_i}\end{cases} \tag{1.8}$$

This is called the strong form of the balance of linear momentum.

e. Parent Domain

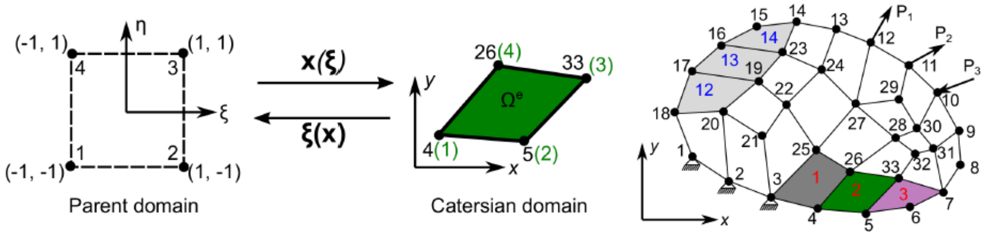

Figure 1. Parent and cartesian domain 2D [4]

In order to solve finite element problems for random element shape, there is a need to map all the quad elements with various shapes from the global domain into the same parent domain have coordinates of [-1, 1] x [-1, 1]. The (x, y) are called the Cartesian coordinates and (ζ, η) are called the natural coordinates. The key is that different quad elements have different coordinates and shapes.

Therefore, utilizing shape functions is key. These shapes functions are associated with each node, and they are as followed for 3D problems,

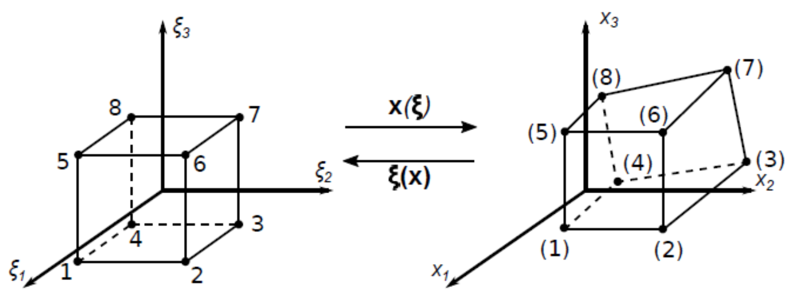

Figure 2. Parent and cartesian domain 3D [4]

$$N_1(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1+\xi_1)(1-\xi_2)(1-\xi_3) \tag{1.1}$$

$$N_2(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1+\xi_1)(1+\xi_2)(1-\xi_3) \tag{1.2}$$

$$N_3(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1-\xi_1)(1+\xi_2)(1-\xi_3) \tag{1.3}$$

$$N_4(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1-\xi_1)(1-\xi_2)(1-\xi_3) \tag{1.4}$$

$$N_5(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1+\xi_1)(1-\xi_2)(1+\xi_3) \tag{1.5}$$

$$N_6(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1+\xi_1)(1+\xi_2)(1+\xi_3) \tag{1.6}$$

$$N_7(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1-\xi_1)(1+\xi_2)(1+\xi_3) \tag{1.7}$$

$$N_8(\xi_1,\xi_2,\xi_3) = \frac{1}{8}(1-\xi_1)(1-\xi_2)(1+\xi_3) \tag{1.8}$$

There is a need to go back and forth from the parent domain to the cartesian domain and vice versa. This will allow to solve the problem in the parent domain and then map this answer into the cartesian domain.

Therefore, let $\boldsymbol{x}$ maps the parent domain to Ω^e according to the following form:

$$x(\xi) = \sum_{i=1}^{n_{en}} N_i(\xi)\boldsymbol{x}_i^e \tag{1.9}$$

Where $\boldsymbol{x}_i^e$ represents the Cartesian coordinate of the i-th node, and n_{en} represents the number of nodes in the element. Then the finite element solution inside Ω^e can be mapped as:

$$u^h(\xi) = \sum_{i=1}^{n_{en}} N_i(\xi)d_i^e \tag{1.10}$$

Where, d_i^e represents the solution at the i-th node.

III. Key Finite Element Features

In order to solve the 3D linear elastostatic problem, some formulations and key steps must be explained. The initial step towards the final solution starts with the weak form of the balance of linear momentum on the domain Ω and is shown in Equation (2.1).

$$\int_\Omega w_i(\sigma_{ij,j} + l_i)d\Omega = 0 \tag{2.1}$$

Where, w_i is the weighting function.

Equation (2.1) can be separated into volume and surface integral, integral over the domain Ω and over the surface Γ respectively; represented in equation (2.2).

$$-\int_\Omega w_{i,j}\sigma_{ij}d\Omega + \int_{\partial\Omega} w_i\sigma_{ij}n_j d\Gamma + \int_\Omega w_i l_i d\Omega = 0 \tag{2.2}$$

Since the union of the two edge domains, Γ, represent the surface over the domain, ∂Ω, the surface integral can be divided into the sum of the integral over the edge of the domain seen in Equation (2.3).

$$-\int_\Omega w_{i,j}\sigma_{ij}d\Omega + \sum_{i=1}^{n_{sd}}(\int_{\Gamma_{u_i}} w_i\sigma_{ij}n_j d\Gamma + \int_{\Gamma_{h_i}} w_i\sigma_{ij}n_j d\Gamma) + \int_\Omega w_i l_i d\Omega = 0 \tag{2.3}$$

Given that $w_i = 0$ on Γ_{ui} and that $\sigma_{ij}n_j = h_i$ on Γ_{hi}, Equation (2.3) can be simplified and rewritten as shown in Equation (2.4).

$$-\int_\Omega w_{i,j}\sigma_{ij}d\Omega + \sum_{i=1}^{n_{sd}}(\int_{\Gamma_{h_i}} w_i h_i d\Gamma) + \int_\Omega w_i l_i d\Omega = 0 \tag{2.4}$$

The weak form can be derived by finding solutions $u_i \in \boldsymbol{S}_i$ (trial solution space) such that for all $w_i \in \mathcal{V}_i$ (weighting function space).

Given the constitutive law of σ_{ij}, the weak form can be expressed and simplified.

$$a(\boldsymbol{w},\boldsymbol{u}) = (\boldsymbol{w},\boldsymbol{l}) + (\boldsymbol{w},\boldsymbol{h})_\Gamma \tag{2.5}$$

a. Galerkin Formulation

Prior to discretizing to find the solution of the problem, the Galerkin formulation needs to come in play in order to obtain an equation which can be discretized.

Define $\boldsymbol{S}^h$ and $\mathcal{V}^h$ be finite approximation to $\boldsymbol{S}$ and $\mathcal{V}$. Each member u^h of $\boldsymbol{S}^h$ can be written in the following equation.

$$\boldsymbol{u}^h = \boldsymbol{v}^h + \boldsymbol{g}^h \tag{2.6}$$

Where, $v^h \in \mathcal{V}^h$. Utilizing the simplified weak form obtained in Equation (2.5) and given **l**, **g** and **h**, the following equation can be obtained.

$$a\left(\boldsymbol{w}^h, \boldsymbol{v}^h\right) = \left(\boldsymbol{w}^h, \boldsymbol{l}\right) + \left(\boldsymbol{w}^h, \boldsymbol{h}\right)_\Gamma - a\left(\boldsymbol{w}^h, \boldsymbol{g}^h\right) \tag{2.7}$$

b. Discretization

Given the Galerkin formulation, the variables of Equation (2.7) can be discretized. Let η be the set of global node numbers and g_i-nodes be the nodes where $u_i^h = g_i$.

The following can be written:

$$w_i^h(\boldsymbol{x}) = \sum\nolimits_{A\in\eta-\eta_{g_i}} N_A(\boldsymbol{x}) c_{iA} \tag{2.8}$$

Where c_{iA} is an arbitrary constant.

$$v_i^h(\boldsymbol{x}) = \sum\nolimits_{A\in\eta-\eta_{g_i}} N_A(\boldsymbol{x}) d_{iA} \tag{2.9}$$

Where d_{iA} is the known at node A.

$$g_i^h(\boldsymbol{x}) = \sum\nolimits_{A\in\eta_{g_i}} N_A(\boldsymbol{x}) g_{iA} \tag{2.10}$$

Where $g_{iA} = g_i(\boldsymbol{x}_A)$ is known through the essential boundary condition at node A.

c. Linear System

Given the 3D case, let:

$$\boldsymbol{e_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \boldsymbol{e_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \boldsymbol{e_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Utilizing the discretized function, the weak form can be written as:

$$\begin{aligned}\sum_{j=1}^{n_{dof}}\left(\sum_{B\in\eta-\eta_{g_j}} a\left(N_A\boldsymbol{e_i}, N_B\boldsymbol{e_j}\right)d_{jB}\right) &= (N_A\boldsymbol{e_i}, l) + (N_A\boldsymbol{e_i}, \boldsymbol{h})_\Gamma \\ &\quad - \sum_{j=1}^{n_{dof}}\left(\sum_{B\in\eta_{g_j}} a\left(N_A\boldsymbol{e_i}, N_B\boldsymbol{e_j}\right)g_{jB}\right) \qquad \forall A\in\eta-\eta_{g_i}\end{aligned} \tag{2.11}$$

Where n_{dof} is equal to 3 for 3D problems.

A matrix form can be written for the linear system represented in the following equation.

$$\boldsymbol{Kd} = \boldsymbol{F} \text{ or } K_{PQ} d_Q = F_p \tag{2.12}$$

Where substituting Garelkin formulation and discretization gives the following,

$$K_{PQ} = a(N_A \boldsymbol{e_i}, N_B \boldsymbol{e_j}) \tag{2.13}$$

And

$$F_P = (N_A \boldsymbol{e_i}, \boldsymbol{l}) + (N_A \boldsymbol{e_i}, \boldsymbol{h})_\Gamma - \sum_{j=1}^{n_{dof}} \left(\sum_{B \in \eta_{g_j}} a\big(N_A \boldsymbol{e_i}, N_B \boldsymbol{e_j}\big) g_{jB} \right) \tag{2.14}$$

d. The B-Matrix

The B-Matrix is a matrix which is another step towards finding the finite element solution of the problem. The initial step towards finding the B-Matrix starts with the discretized gradient operator shown below,

$$\boldsymbol{\epsilon}(N_A \boldsymbol{e_1}) = \boldsymbol{\epsilon}\left(\begin{bmatrix} N_A \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} N_{A,1} \\ 0 \\ 0 \\ 0 \\ N_{A,3} \\ N_{A,2} \end{bmatrix}, \boldsymbol{\epsilon}(N_A \boldsymbol{e_2}) = \boldsymbol{\epsilon}\left(\begin{bmatrix} 0 \\ N_A \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ N_{A,2} \\ 0 \\ N_{A,3} \\ 0 \\ N_{A,1} \end{bmatrix}, \boldsymbol{\epsilon}(N_A \boldsymbol{e_3}) = \boldsymbol{\epsilon}\left(\begin{bmatrix} 0 \\ 0 \\ N_A \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ N_{A,3} \\ N_{A,2} \\ N_{A,1} \\ 0 \end{bmatrix}$$

Combining the above three gradient operator gives the following,

$$\boldsymbol{\epsilon}(N_A \boldsymbol{e_i}) = \boldsymbol{B_A} \boldsymbol{e_i} \tag{2.15}$$

Where,

$$\boldsymbol{B_A} = \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix} \tag{2.15}$$

IV. Application of the theory

Recall that solving the linear system Kd = F will provide the displacement solution. In order to obtain said linear system, the stiffness matrix, K, and the right-hand side vector or the force vector, F, must be calculated. These called the global stiffness matrix and force vector are obtained from the element stiffness matrix and force vector.

$$\boldsymbol{K} = \sum_{e=1}^{n_{el}} \boldsymbol{K}^e \tag{3.1}$$

Where, $\boldsymbol{K}^{(e)}$ is the element stiffness matrix.

And,

$$\boldsymbol{F} = \sum_{e=1}^{n_{el}} \boldsymbol{F}^e \tag{3.2}$$

Where, $\boldsymbol{F}^{(e)}$ is the element force vector.

a. Element Stiffness Matrix

Recall the definition of the stiffness matrix and utilizing the B-Matrix, the following element stiffness matrix can be written

$$\begin{aligned} K_{PQ}^e &= a\left(N_A\boldsymbol{e_i}, N_B\boldsymbol{e_j}\right)^e = \int_\Omega \boldsymbol{\epsilon}(N_A\boldsymbol{e_i})^T\boldsymbol{D}\boldsymbol{\epsilon}\left(N_B\boldsymbol{e_j}\right)d\Omega \\ &= \int_\Omega (\boldsymbol{B}_A\boldsymbol{e}_i)^T\boldsymbol{D}\left(\boldsymbol{B}_B\boldsymbol{e}_j\right)d\Omega = \boldsymbol{e}_i^T(\int_\Omega \boldsymbol{B}_A^T\boldsymbol{D}\boldsymbol{B}_B d\Omega)\boldsymbol{e}_j \end{aligned} \tag{3.3}$$

Utilizing the local node number using *a* and *b*, the stiffness matrix inside an element is the following,

$$k_{pq}^e = a\left(N_a\boldsymbol{e_i}, N_b\boldsymbol{e_j}\right)^e = \boldsymbol{e}_i^T(\int_\Omega \boldsymbol{B}_A^T\boldsymbol{D}\boldsymbol{B}_B d\Omega)\boldsymbol{e}_j \tag{3.4}$$

Where,

$$p = n_{dof}a + i, \qquad q = n_{dof}b + j$$

b. Element Force Vector

Similarly, one can write the element force vector,

$$F_P^e = (N_A\boldsymbol{e_i}, \boldsymbol{l})^e + (N_A\boldsymbol{e}_i, \boldsymbol{h})_{\Gamma^e} - \sum_{j=1}^{n_{dof}}\left(\sum_{B\in\eta_{g_j}^e} a\left(N_A\boldsymbol{e_i}, N_B\boldsymbol{e_j}\right)^e g_{jB}^e\right) \tag{3.5}$$

And inside an element, the following can be utilized,

$$F_P^e = (N_A\boldsymbol{e_i}, \boldsymbol{l})^e + (N_A\boldsymbol{e}_i, \boldsymbol{h})_{\Gamma^e} - k_{pq}g_q^e = \int_{\Omega^e} N_a l_i d\Omega + \int_{\Gamma_{h_i}^e} N_a h_i d\Gamma - k_{pq}g_q^e \tag{3.6}$$

c. Quad Element

Let,

$$k_{ab}^e = \int_{\Omega^e} \boldsymbol{B}_a^T\boldsymbol{D}\boldsymbol{B}_b d\Omega \tag{3.7}$$

Where,

$$a = 1, \dots, 8; b = 1, \dots, 8$$

$\mathbf{B}^T_a$ is 6 x 3, **D** is 6 x 6, and $\mathbf{B}_b$ is 3 x 2, therefore k^e_{ab} is 24 x 24

The material matrix:

$$\boldsymbol{D} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}$$

Where,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{3.8}$$

And,

$$\mu = G = \frac{E}{2(1+\nu)} \tag{3.9}$$

The right-hand side force vector, $\boldsymbol{f}^e$ is a 24 x 1.

d. Assemble Global Quantities

By defining a data array to map local node numbers to global node numbers, the global quantities ***K*** and ***F*** can be assembled from the element quantities $\boldsymbol{k}^e$ and $\boldsymbol{f}^e$. Utilizing DOF number to map local DOF to global DOF, this is made possible by the following,

$$ID(e,a,i) = LM[e, a\times 2 + i] = \begin{cases} IEN(e,a)\times 2 & if\ i=0 \\ IEN(e,a)\times 2+1 & if\ i=1 \\ IEN(e,a)\times 2+2 & if\ i=2 \end{cases}$$

e. Dirichlet Boundary Conditions

Given the 3D problem, the unknown fields are $u_1(x, y, z)$, $u_2(x, y, z)$ and $u_3(x, y, z)$, there will be three sets of nodes assigned boundary conditions.

Utilizing the homogeneous Dirichlet boundary conditions method, the global K and F will be modified to incorporate these boundary conditions as followed,

$$\textstyle\sum_{B\in\eta} a(N_A, N_B) d_B = (N_A, l) + (N_A, h)_\Gamma \quad \forall A \in \eta \tag{3.10}$$

After assembling global ***K*** and ***F***, the rows and columns for Dirichlet boundary nodes will be zeroed out as well as zero rows.

The final system will be a **(n_{dof} x number of nodes, n_{dof} x number of nodes).**

f. Stress at Nodes

The next step of the problem is obtaining the stress at each node. In order to do so, the stress at element Gauss points must first be calculated.

Recall the following equation,

$$\sigma = \boldsymbol{D\epsilon}$$

This equation must be applied to the I-th Gauss point as follows,

$$\sigma(\zeta^l) = \boldsymbol{D}\boldsymbol{\epsilon}(\zeta^l) \tag{3.11}$$

Where,

$$\boldsymbol{\epsilon}(\zeta^l) = \sum_{a=1}^{n_{en}} \boldsymbol{B_a}(\zeta^l)\boldsymbol{u_a} \tag{3.12}$$

The nodal stress can be found using the following equation,

$$\boldsymbol{\sigma_B} = \frac{1}{N}\sum_{e=1}^{N} \boldsymbol{\sigma_b^e} \tag{3.13}$$

Where, N is the number of elements sharing the node $\boldsymbol{B}$ and $\boldsymbol{\sigma}^{e}{}_{b}$ is the stress at each element nodal points calculated using the following linear system,

$$\boldsymbol{M_{ab}^e}\boldsymbol{\sigma_b^e} = \boldsymbol{f_a^e} \tag{3.14}$$

Using the quadrature rule, $\boldsymbol{M_{ab}}^{e}$ can be found,

$$\boldsymbol{M_{ab}^e} = \int_{\Omega^e} N_a N_b d\Omega = \sum_{I=1}^{n_{int}} N_a(\zeta^l)N_b(\zeta^l)jac(\zeta^l)w(\zeta^l) \tag{3.15}$$

And $\boldsymbol{f_a}^{e}$,

$$\boldsymbol{f_a^e} = \int_{\Omega^e} N_a \sigma d\Omega = \sum_{I=1}^{n_{int}} N_a(\zeta^l)\sigma(\zeta^l)jac(\zeta^l)w(\zeta^l) \tag{3.16}$$

g. Force at Nodes

The sum of contributions from all elements containing a specific node is the total force at that node found using the following equation,

$$\boldsymbol{F_A} = \sum_{e=1}^{N} \boldsymbol{f_a^e} \tag{3.17}$$

Where N is the number of elements sharing the node A.

The force vector at an element node is obtained with the following equation,

$$\begin{aligned}\boldsymbol{f_a^e} &= \int_{\Omega^e} \boldsymbol{B_a^T}\boldsymbol{\sigma} d\Omega - \int_{\Omega^e} N_a \boldsymbol{l} d\Omega - \int_{\Gamma_h^e} N_a \boldsymbol{h} d\Gamma \\ &= \sum_{I=1}^{n_{int}} \left(\boldsymbol{B_a^T}\sigma(\zeta^l)jac(\zeta^l)w(\zeta^l) - N_a(\zeta^l)\boldsymbol{l}jac(\zeta^l)w(\zeta^l)\right)\end{aligned} \tag{3.18}$$

V. Numerical Results

a. Test Problem 1

i. Displacement

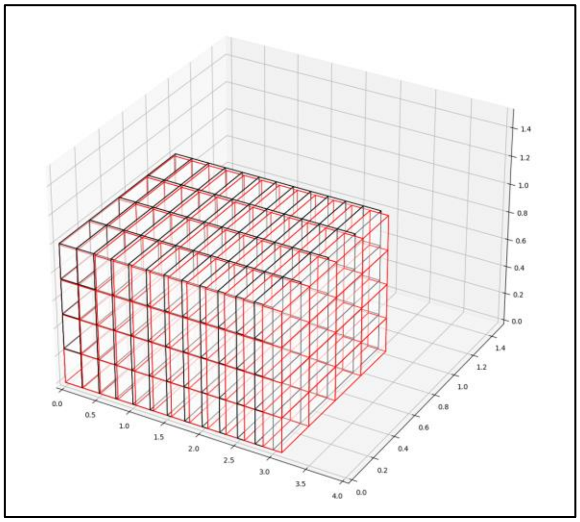

Figure 3. t = 0.04 s

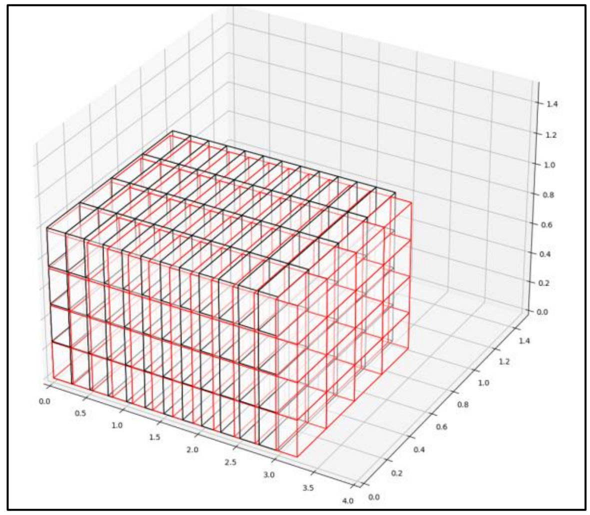

Figure 4. t = 0.08 s

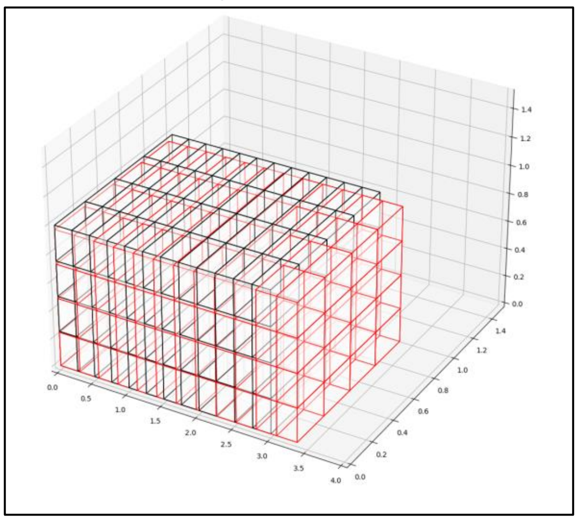

Figure 5. t = 0.12 s

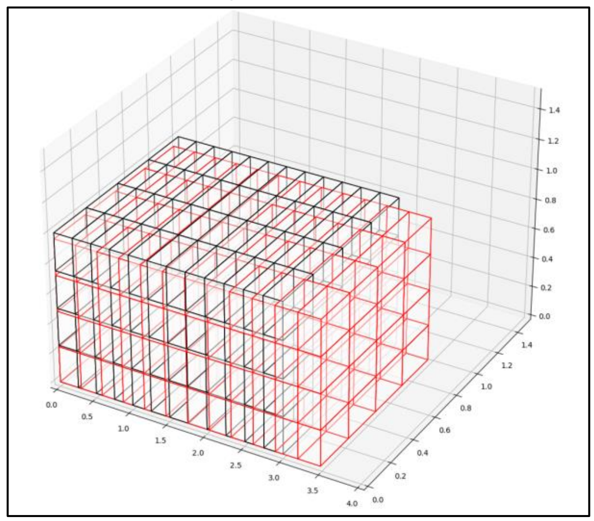

Figure 6. t = 0.16 s

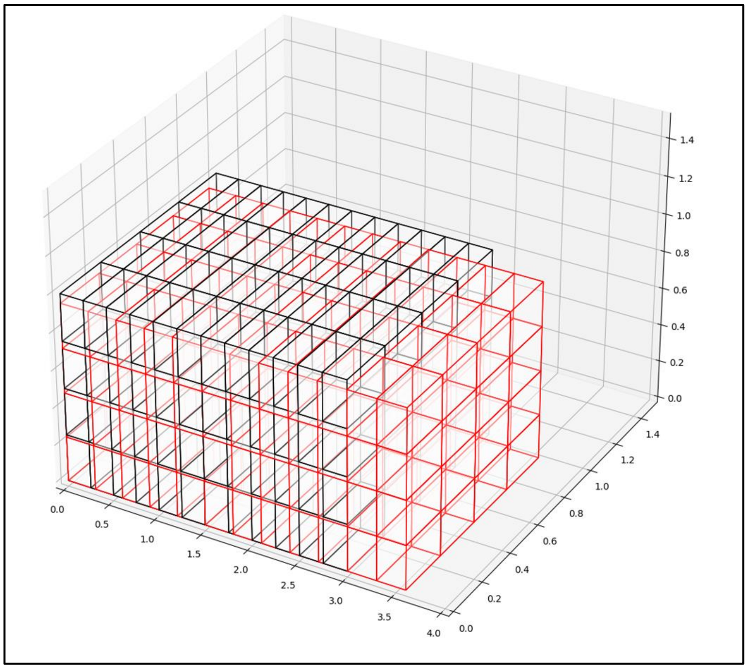

Figure 7. t = 0.2 s

ii. Stress

t = 0.04 s

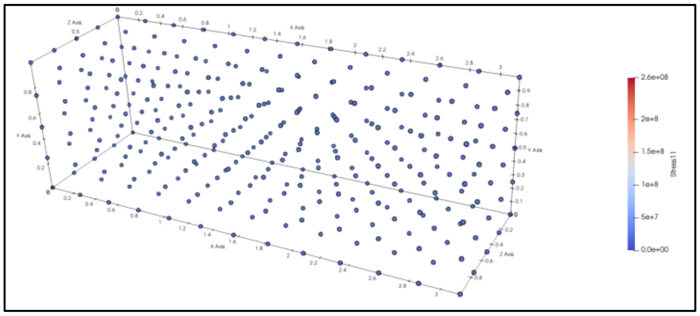

Figure 8. Stress11

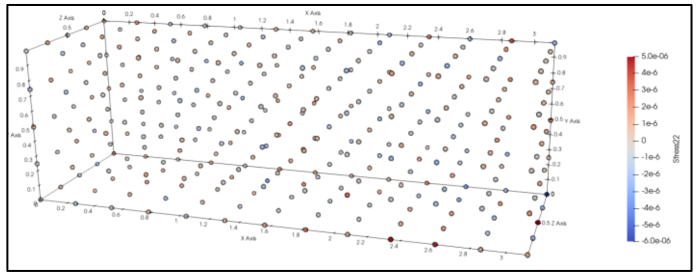

Figure 9. Stress22

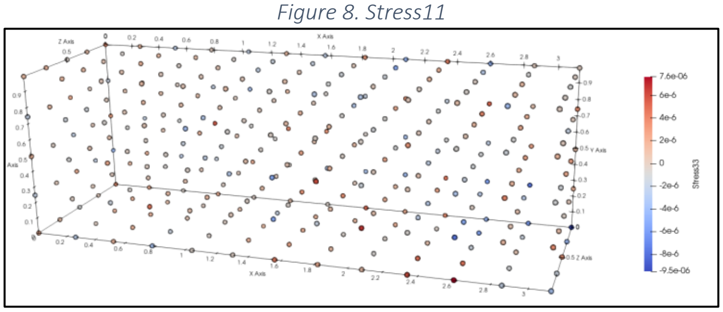

Figure 10. Stress33

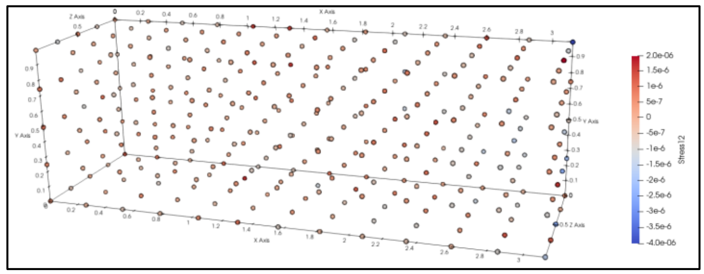

Figure 11. Stress12

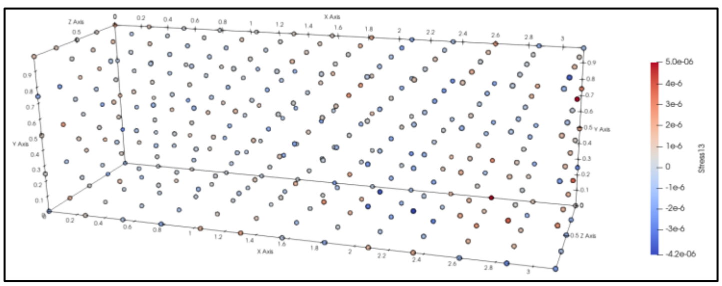

Figure 12. Stress13

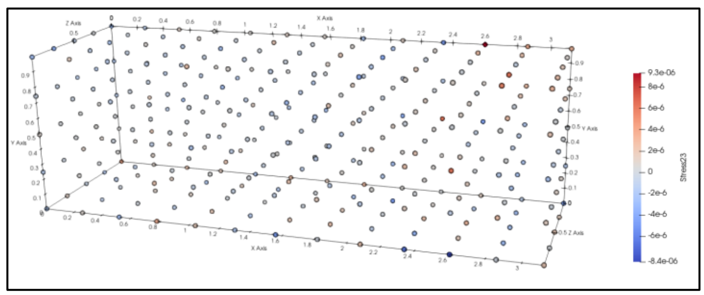

Figure 13. Stress23

t = 0.12 s

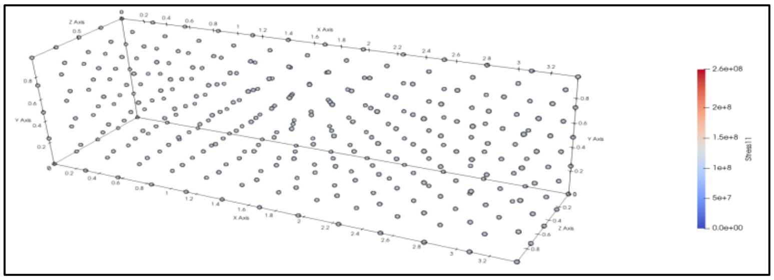

Figure 14. Stress11

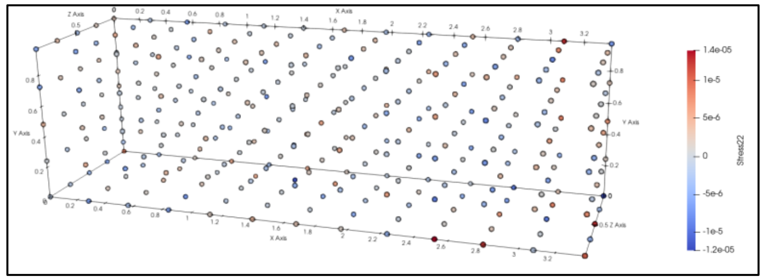

Figure 15. Stress22

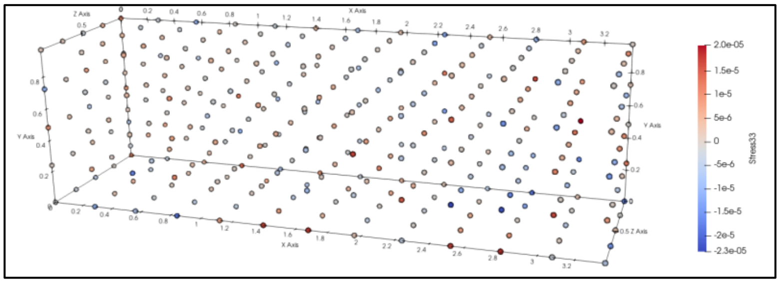

Figure 16. Stress33

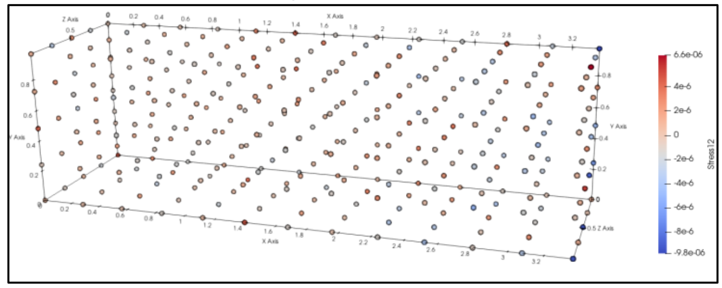

Figure 17. Stress12

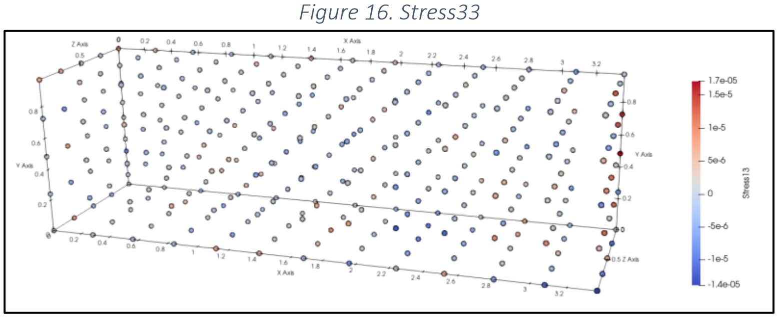

Figure 18. Stress13

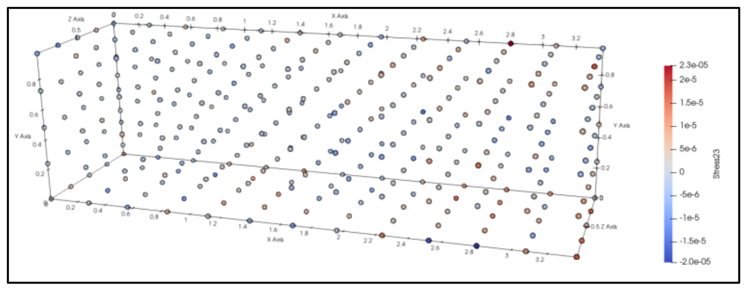

Figure 19. Stress23

t = 0.2 s

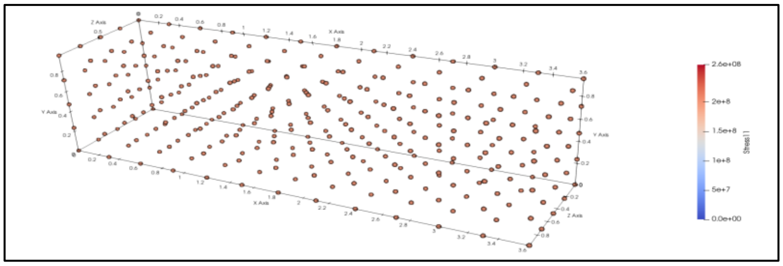

Figure 20. Stress11

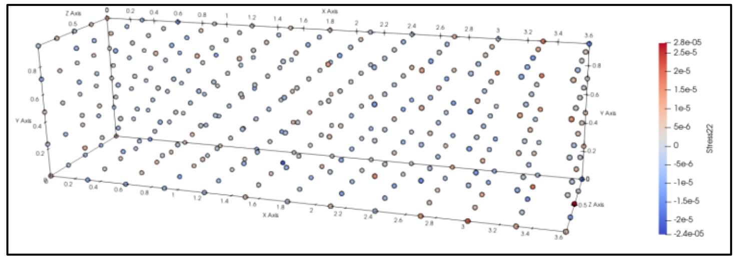

Figure 21. Stress22

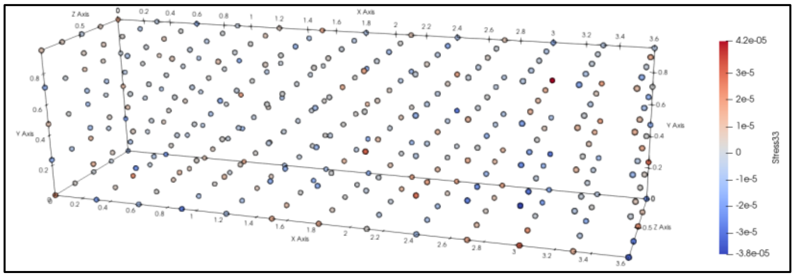

Figure 22. Stress33

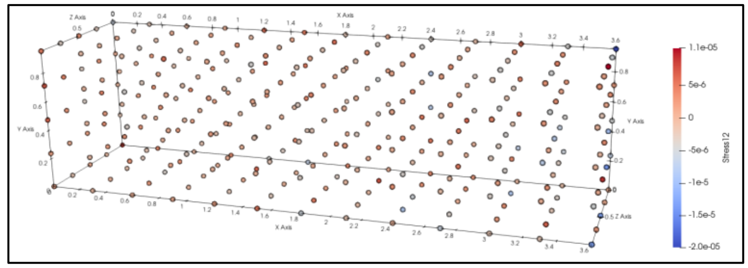

Figure 23. Stress12

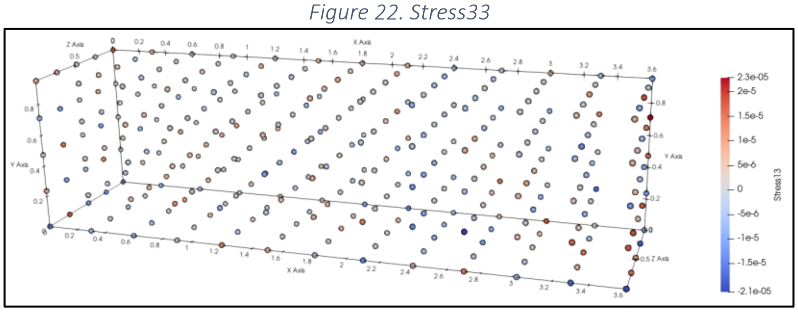

Figure 24. Stress13

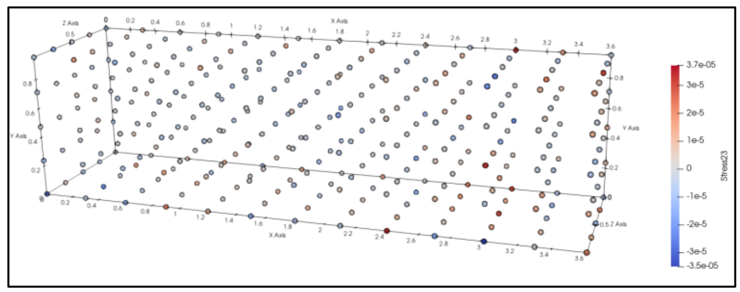

Figure 25. Stress23

iii. Forces at t = 0.2 s

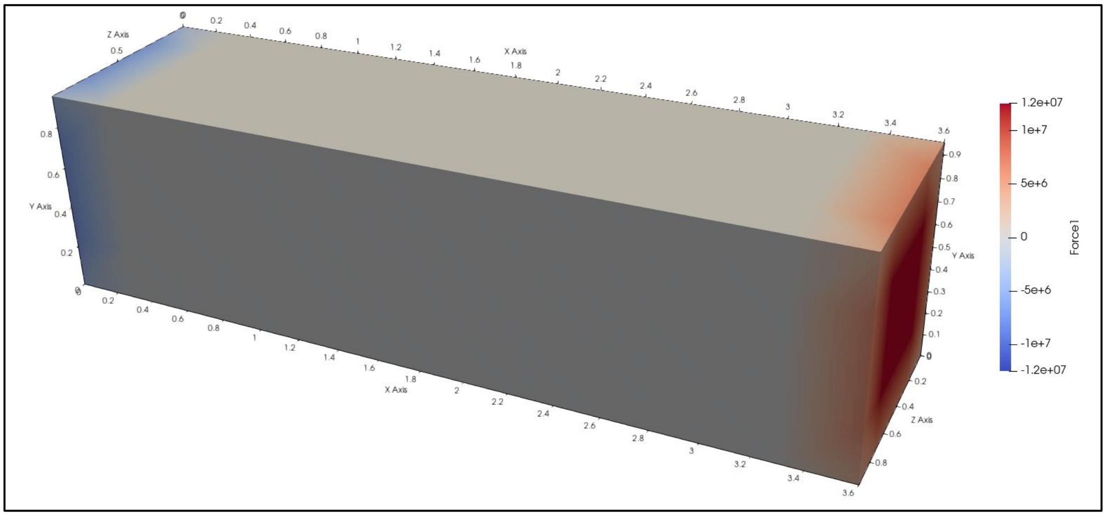

Figure 26. Force1

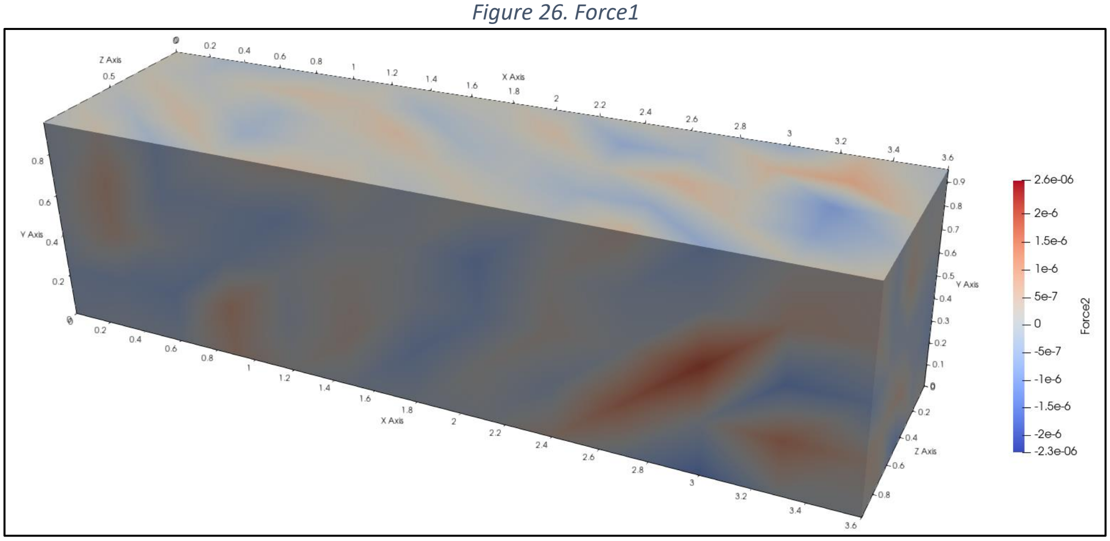

Figure 27. Force2

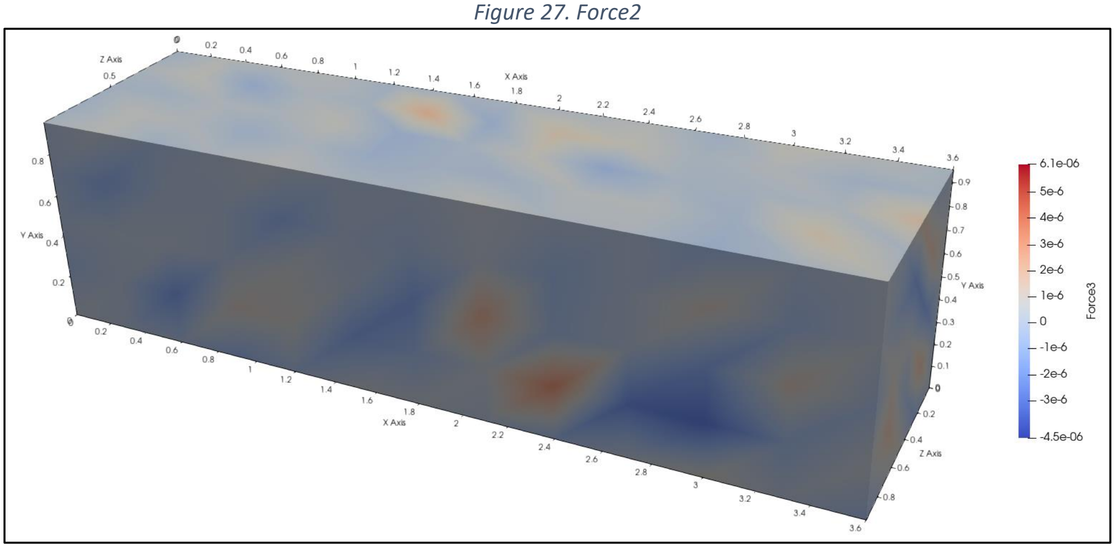

Figure 28. Force3

b. Test Problem 2

i. Displacement

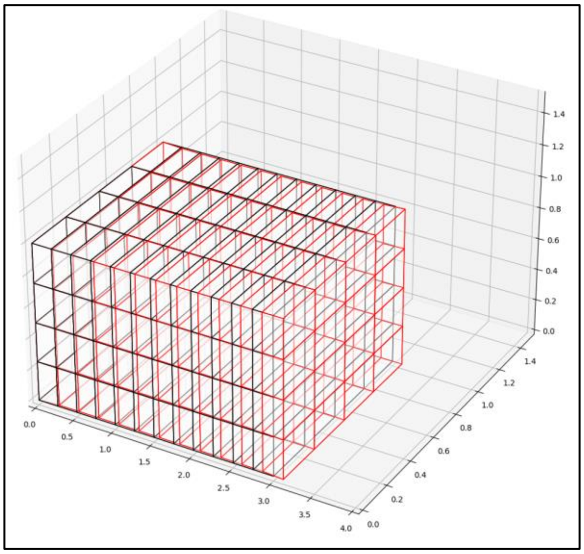

Figure 29. t = 0.04 s

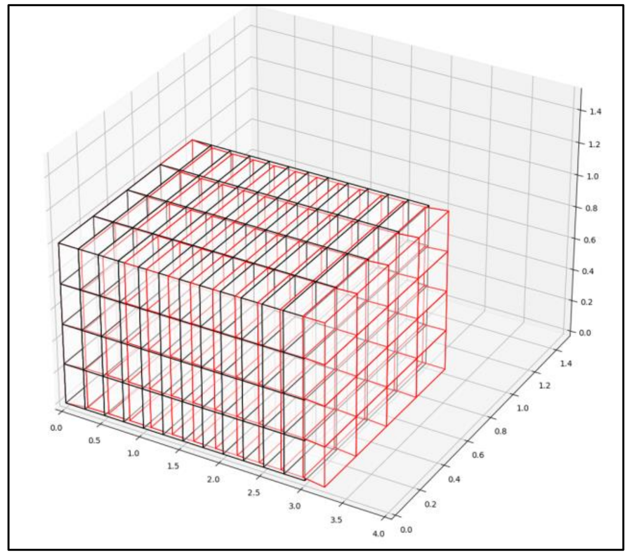

Figure 30. t = 0.08 s

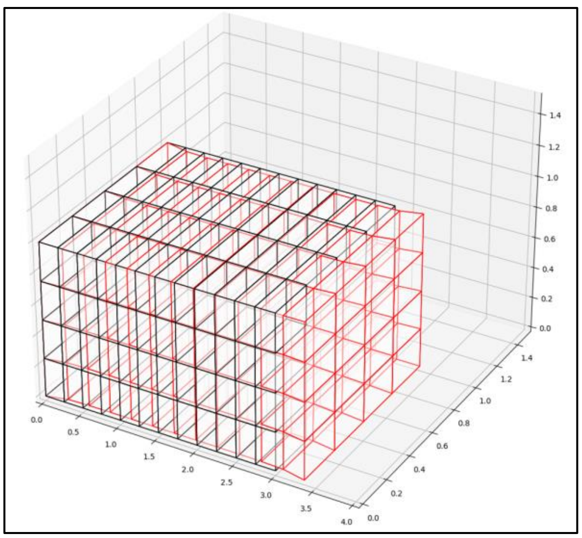

Figure 31. t = 0.12 s

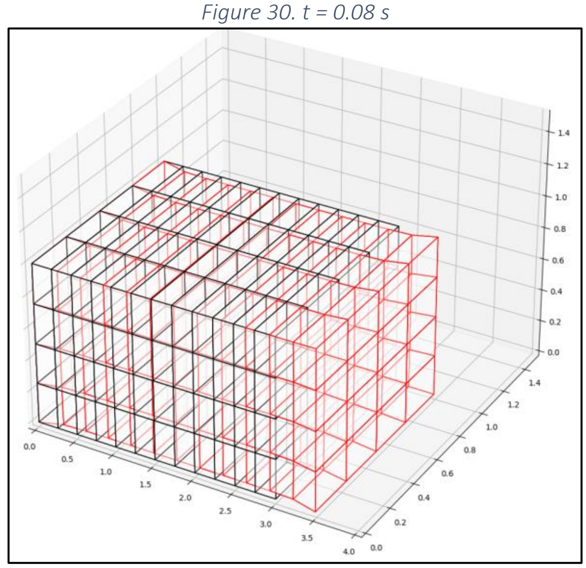

Figure 32. t = 0.16 s

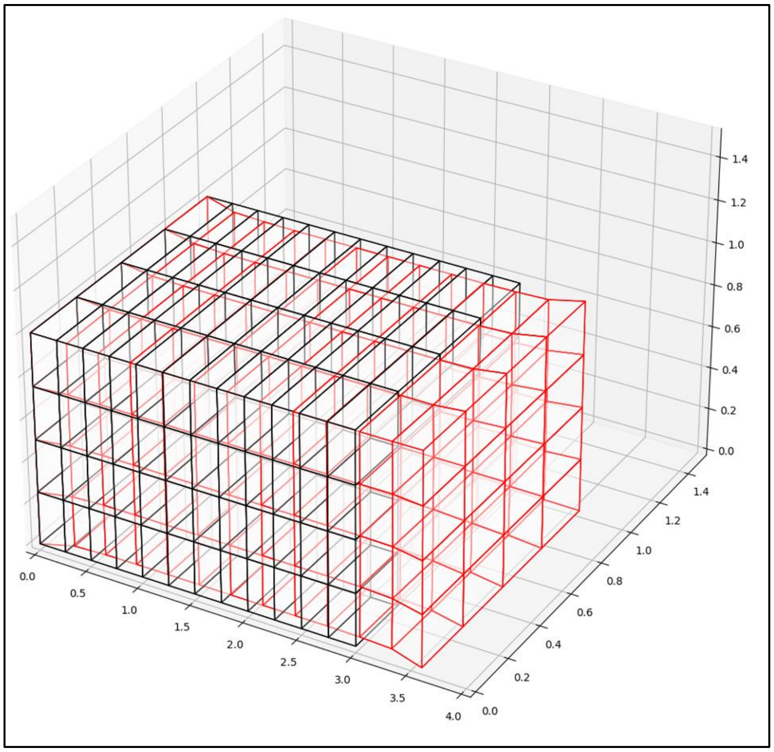

Figure 33. t = 0.2 s

ii. Stresses

t = 0.04 s

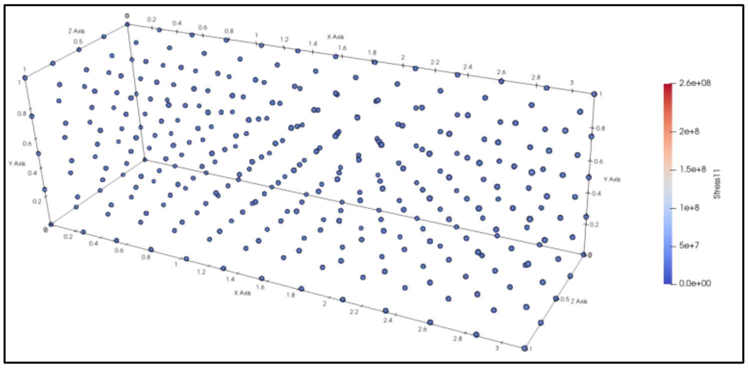

Figure 34. Stress11

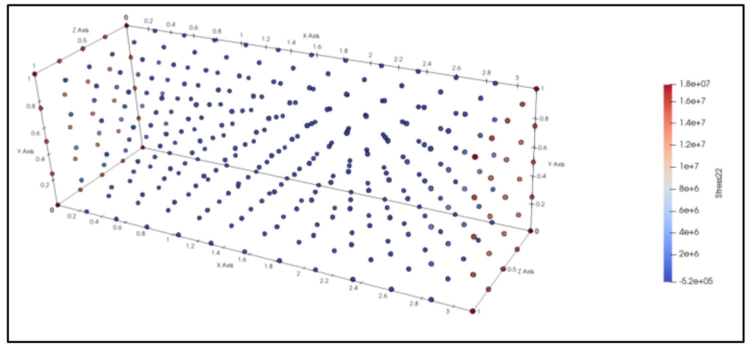

Figure 35. Stress22

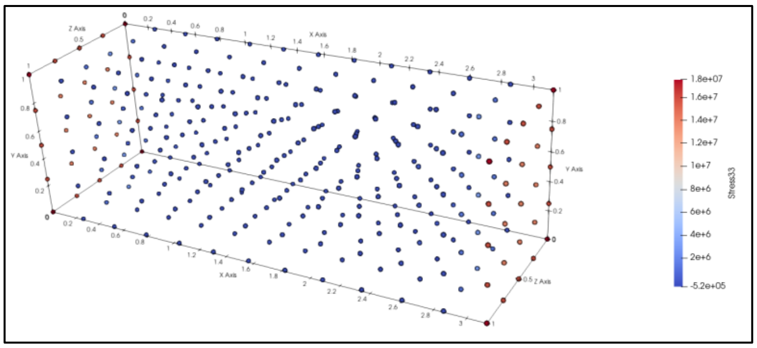

Figure 36. Stress33

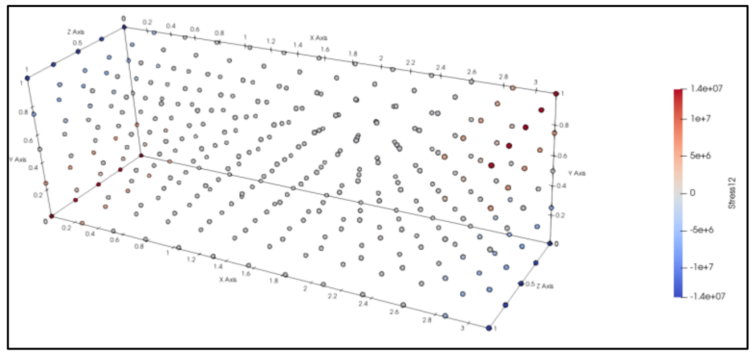

Figure 37. Stress12

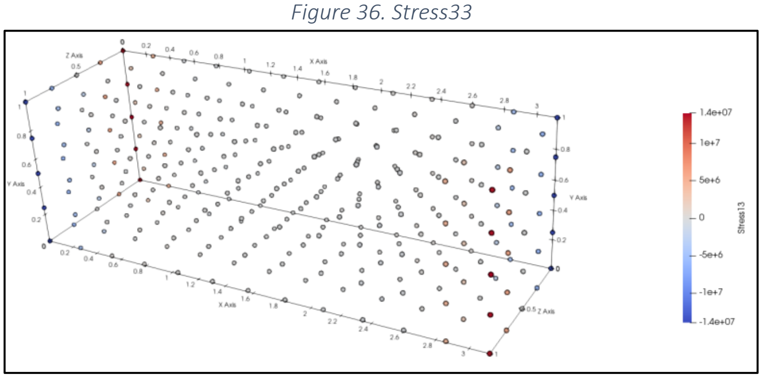

Figure 38. Stress13

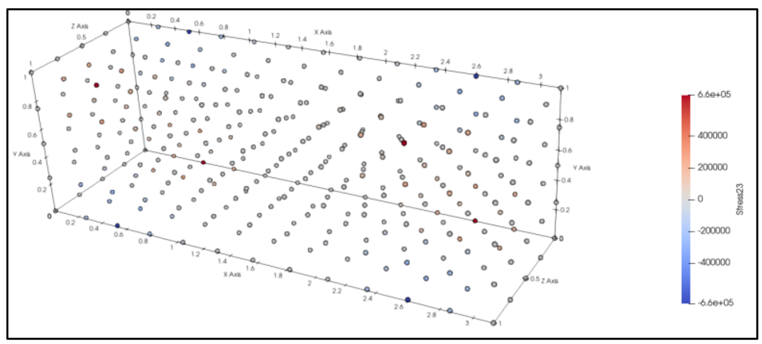

Figure 39. Stress23

t = 0.12 s

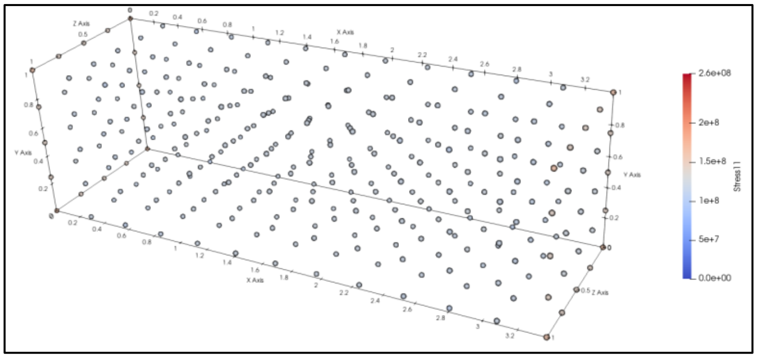

Figure 40. Stress11

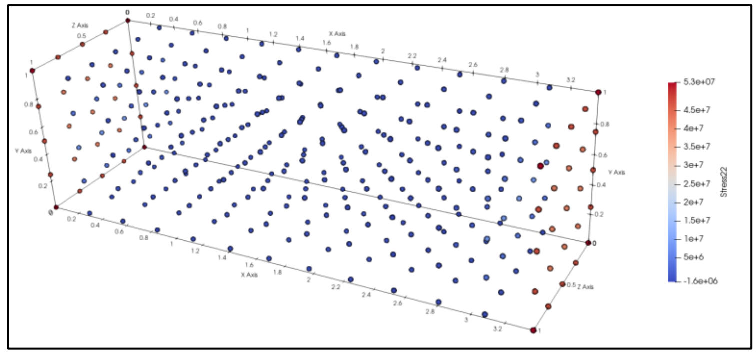

Figure 41. Stress22

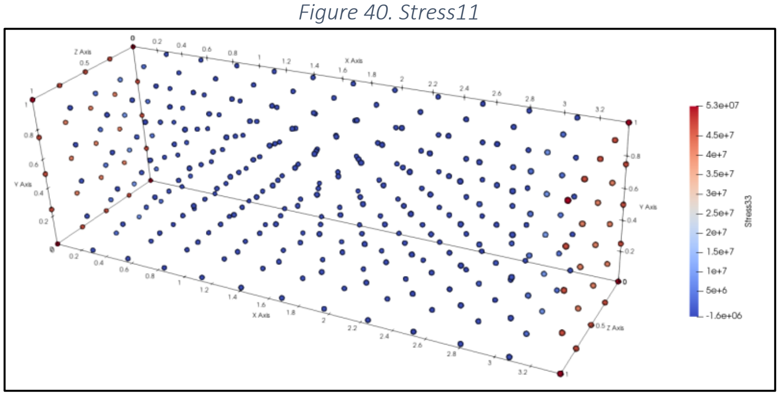

Figure 42. Stress33

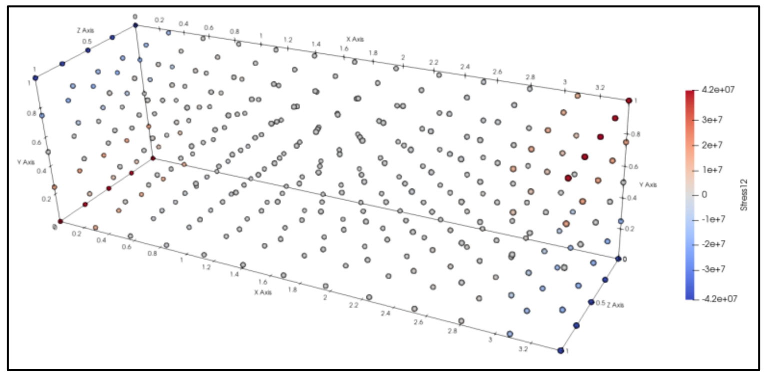

Figure 43. Stress12

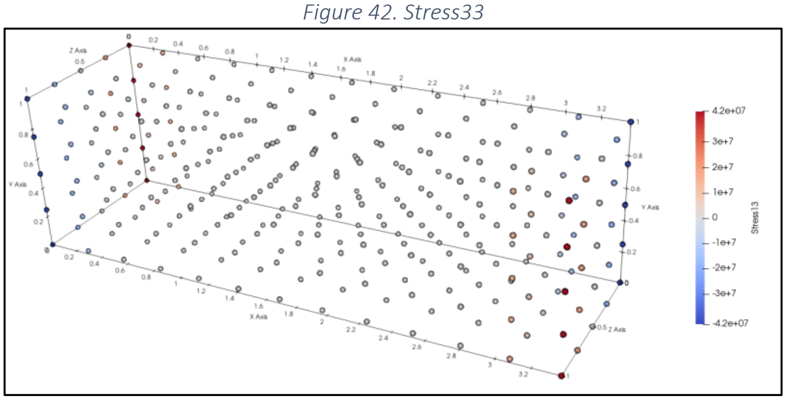

Figure 44. Stress13

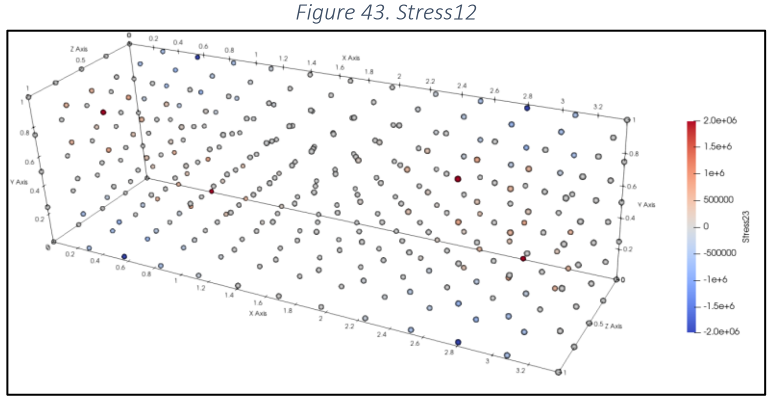

Figure 45. Stress23

t = 0.20 s

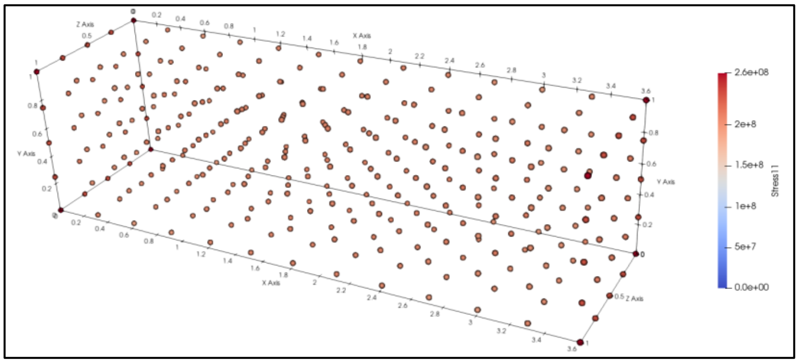

Figure 46. Stress11

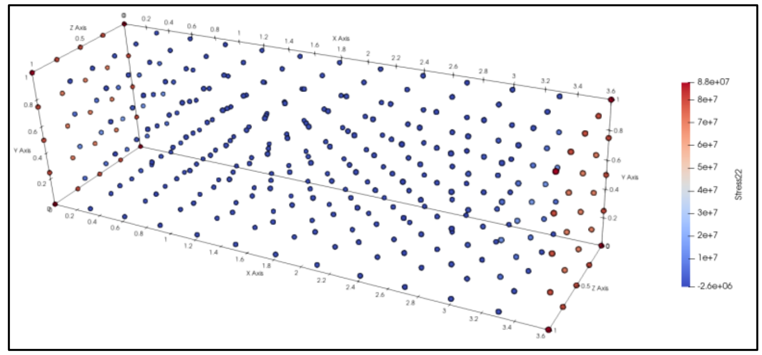

Figure 47. Stress22

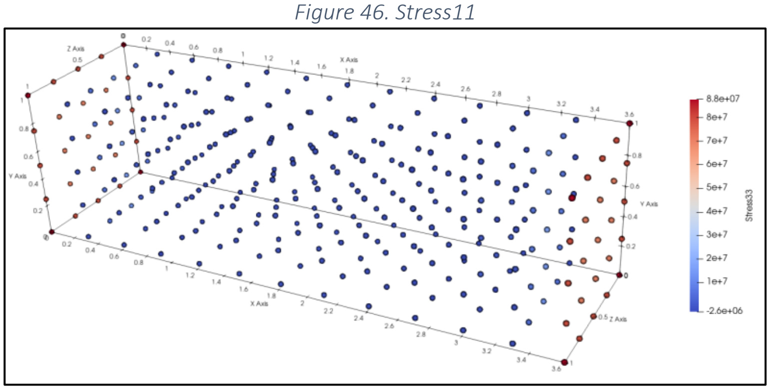

Figure 48. Stress33

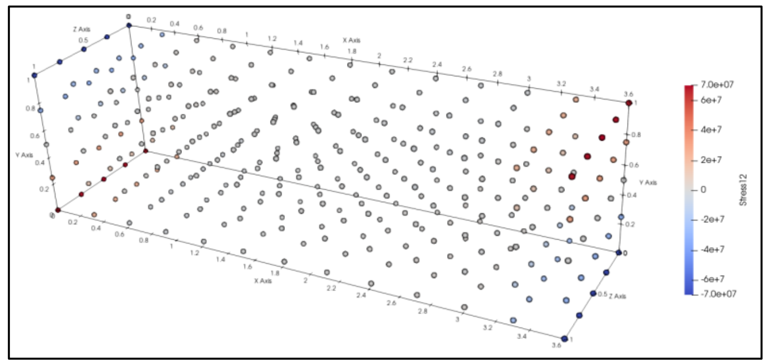

Figure 49. Stress12

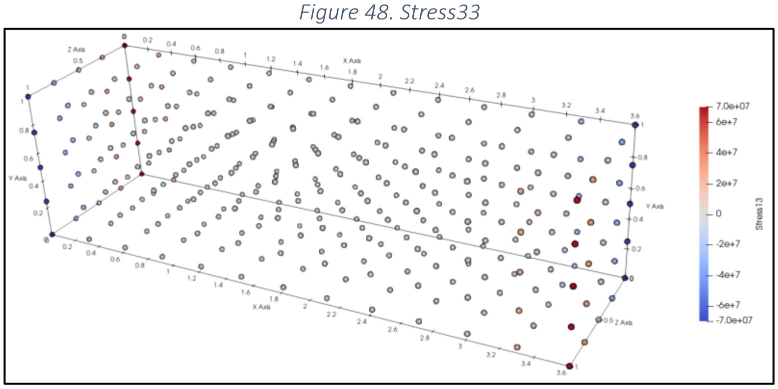

Figure 50. Stress13

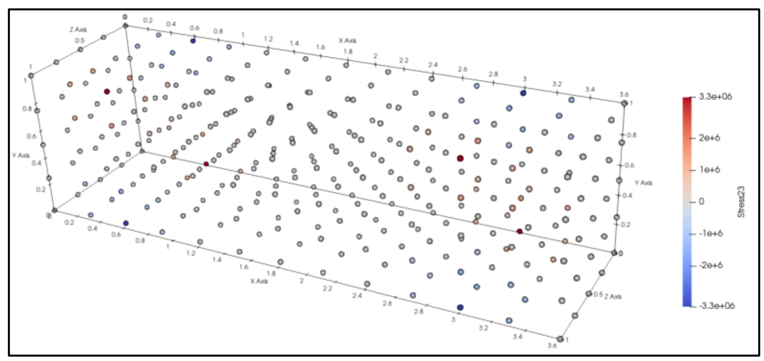

Figure 51. Stress23

iii. Forces

t = 0.08 s

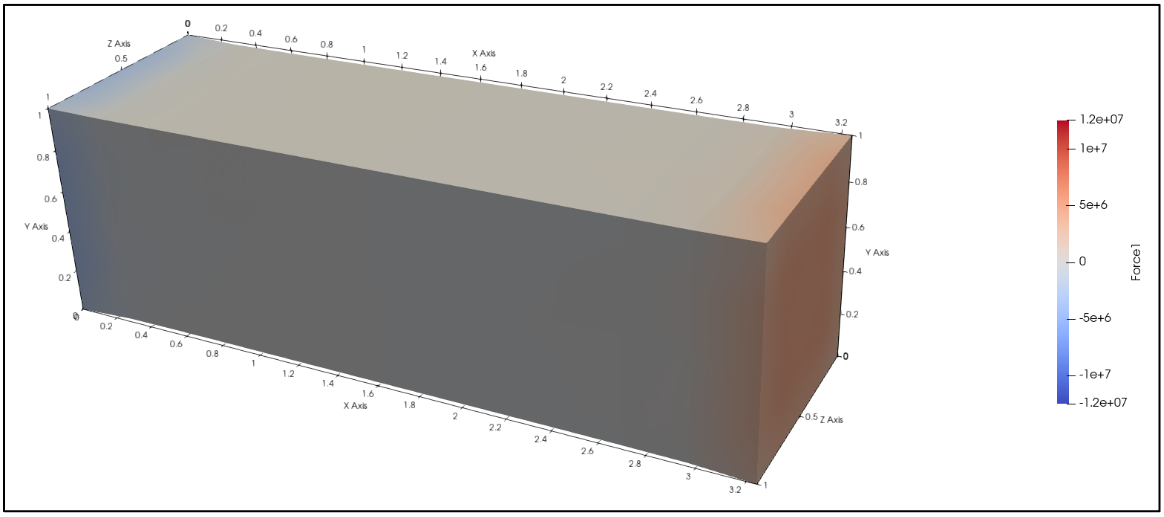

Figure 52. Force1

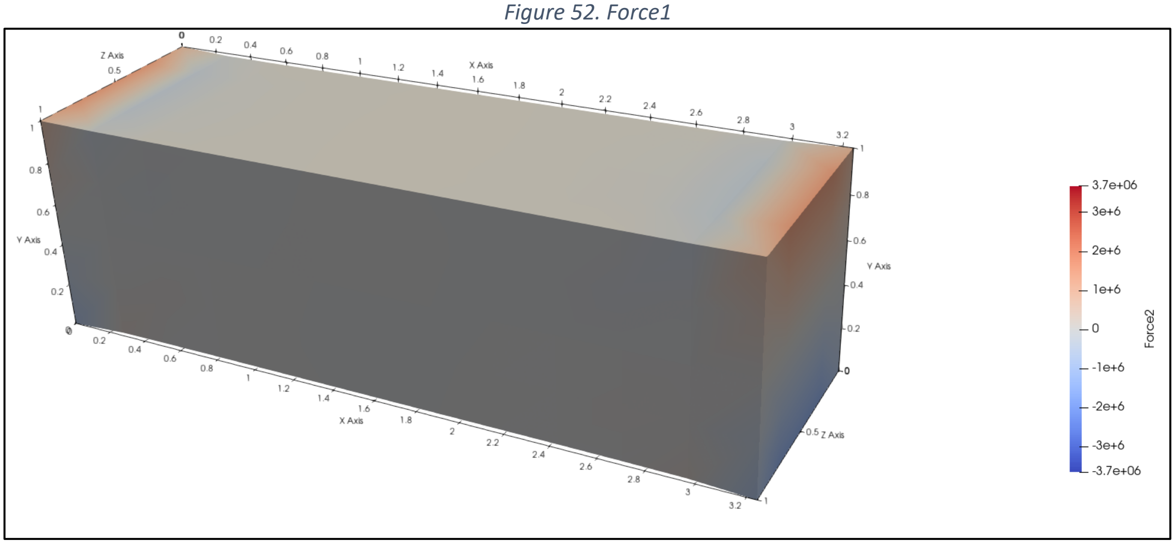

Figure 53. Force2

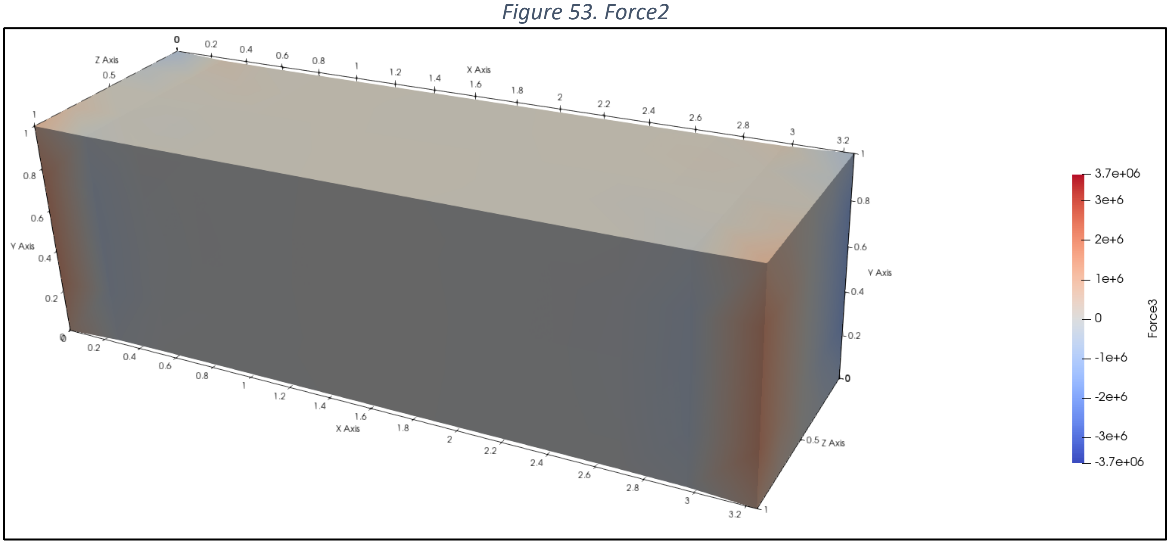

Figure 54. Force3

t = 0.20 s

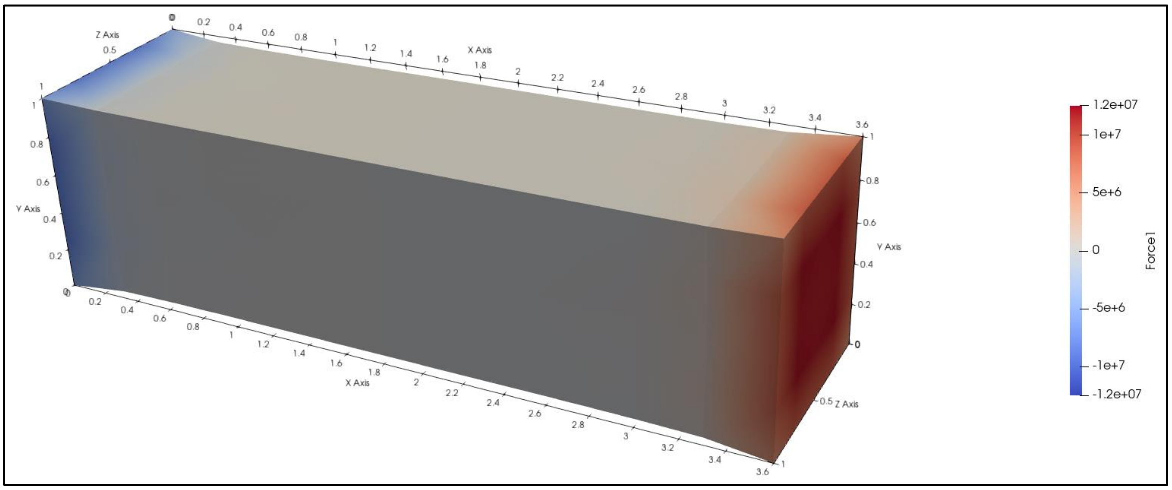

Figure 55. Force1

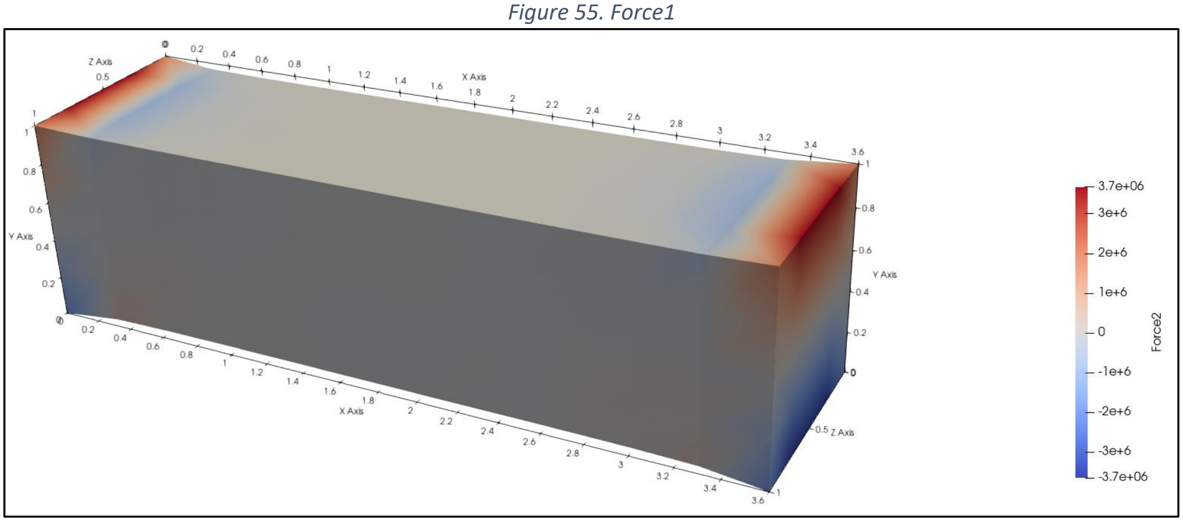

Figure 56. Force2

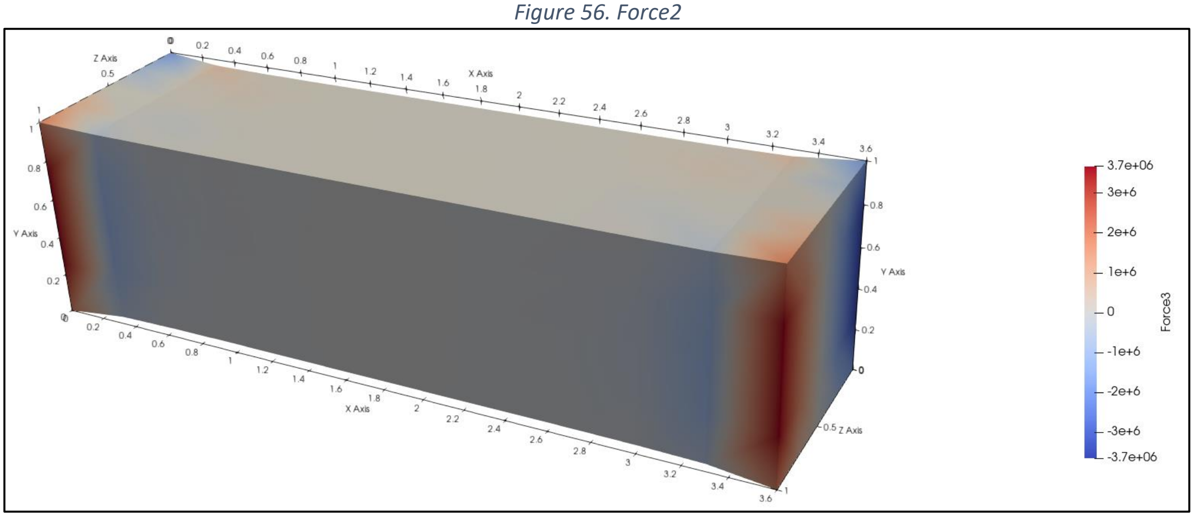

Figure 57. Force3

c. Test Problem 3

i. Displacement

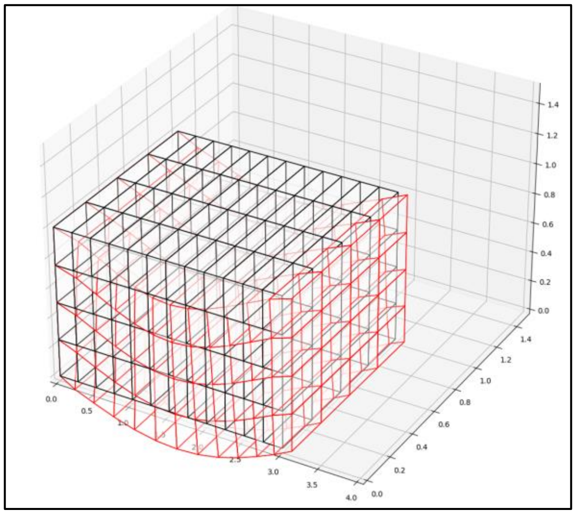

Figure 58. t = 0.04 s

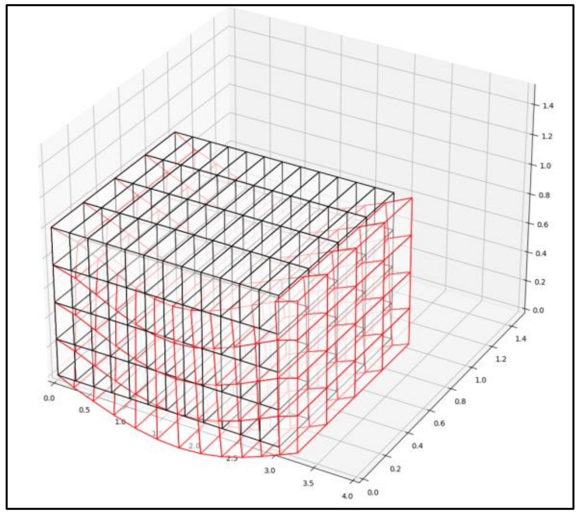

Figure 59. t = 0.08 s

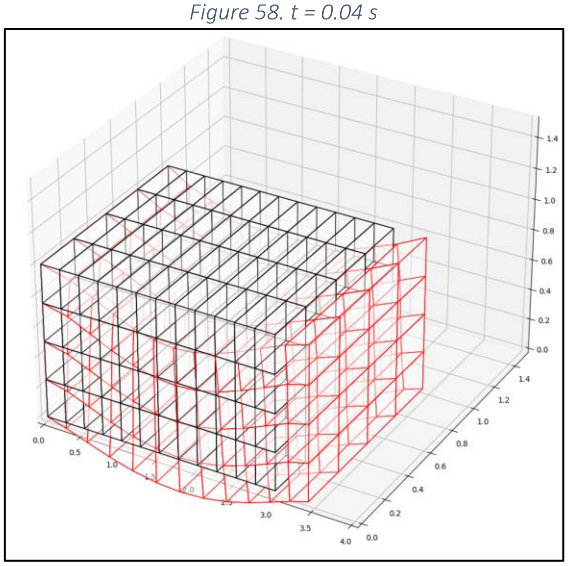

Figure 60. t = 0.12 s

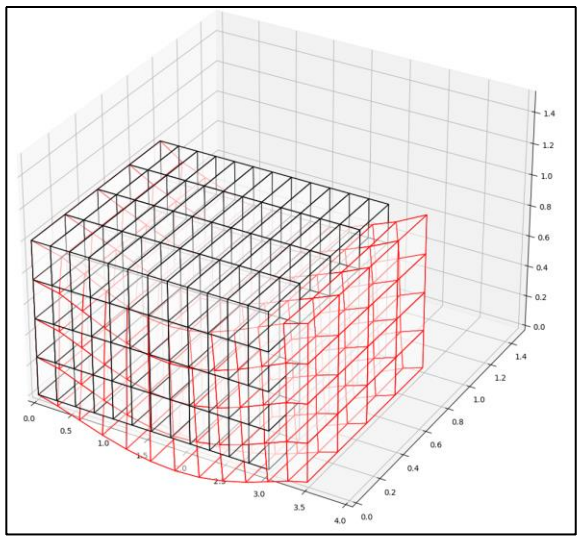

Figure 61. t = 0.16 s

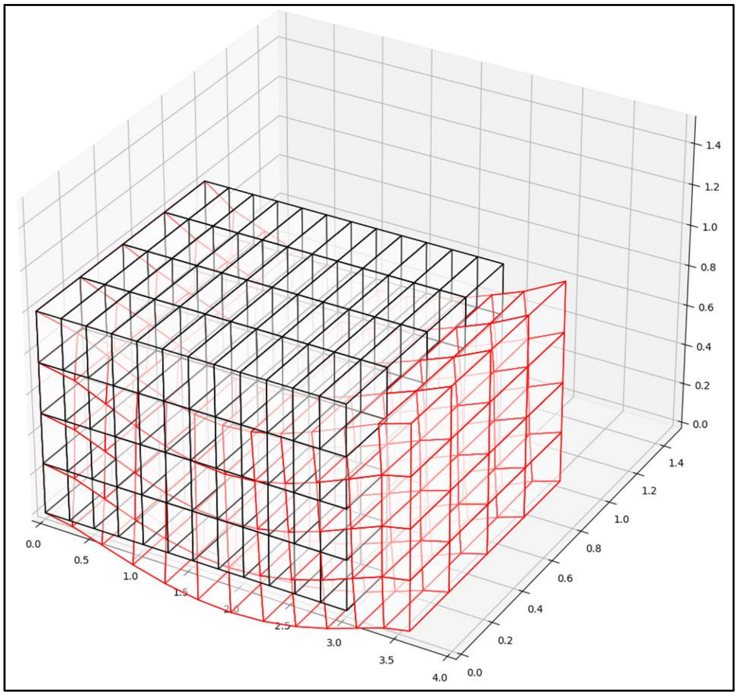

Figure 62. t = 0.20 s

ii. Stresses

t = 0.04 s

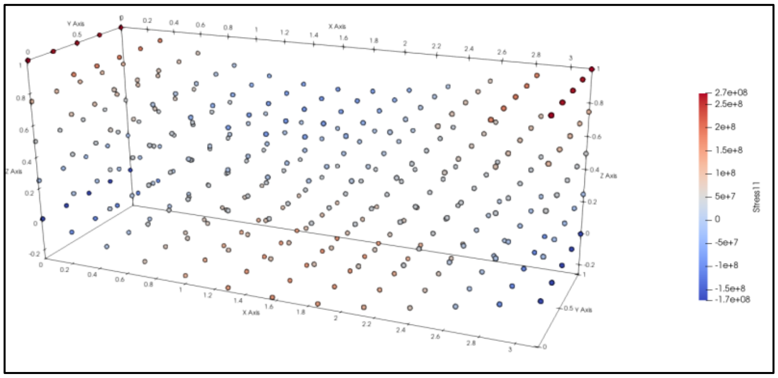

Figure 63. Stress11

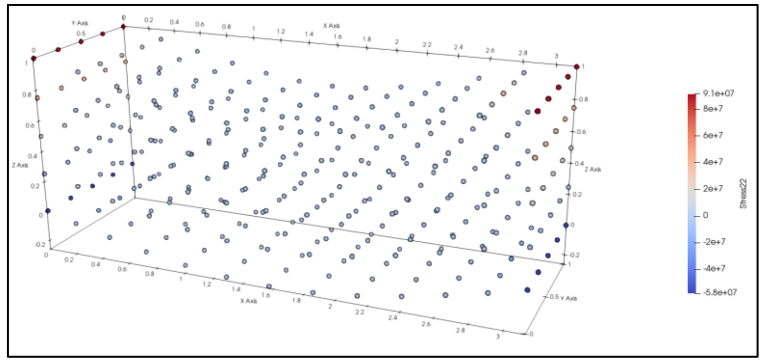

Figure 64. Stress22

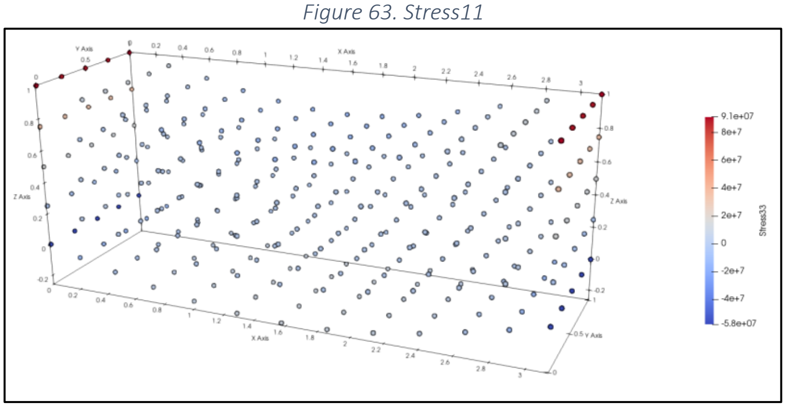

Figure 65. Stress33

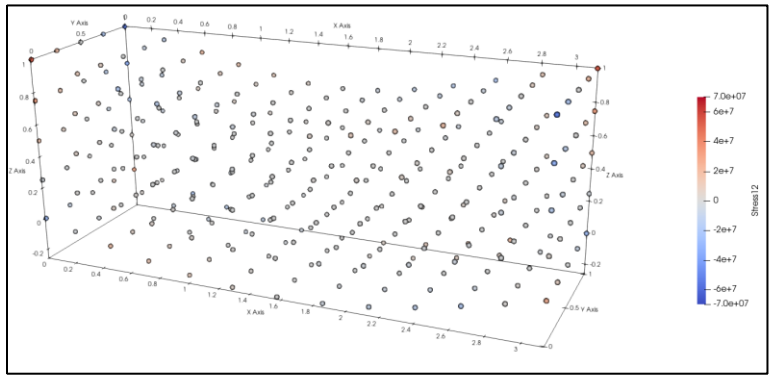

Figure 66. Stress12

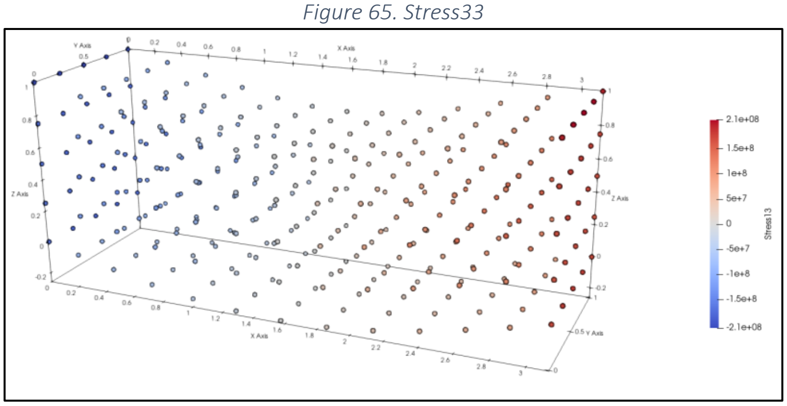

Figure 67. Stress13

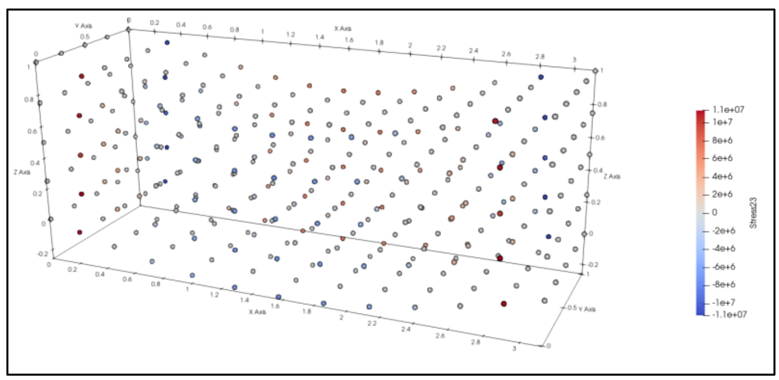

Figure 68. Stress23

t = 0.12 s

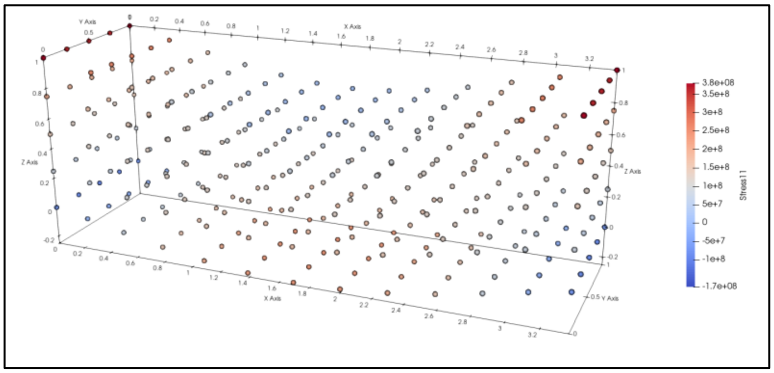

Figure 69. Stress11

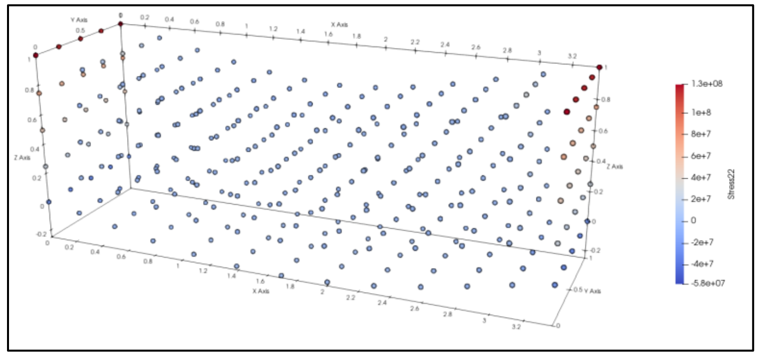

Figure 70. Stress22

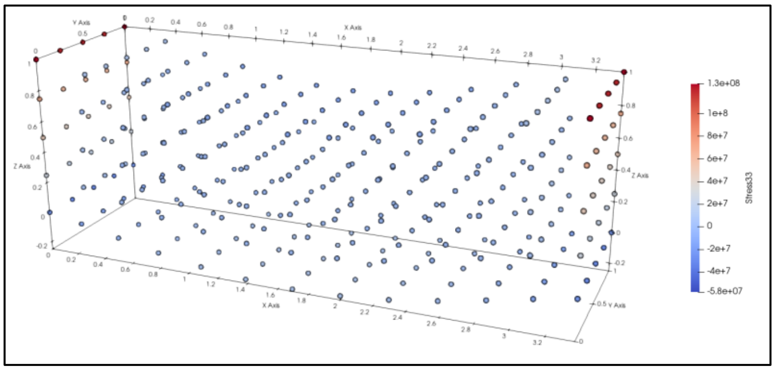

Figure 71. Stress33

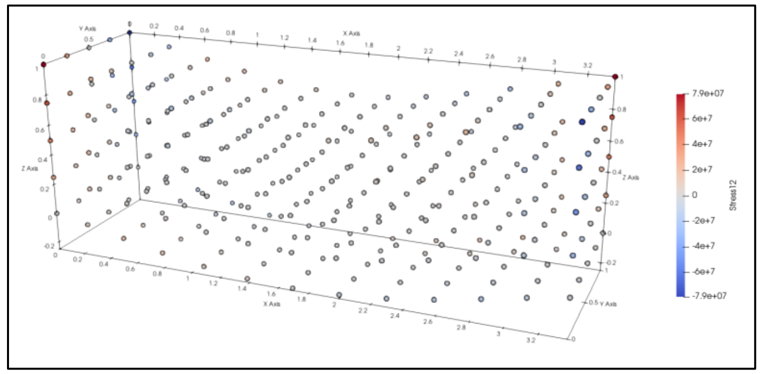

Figure 72. Stress12

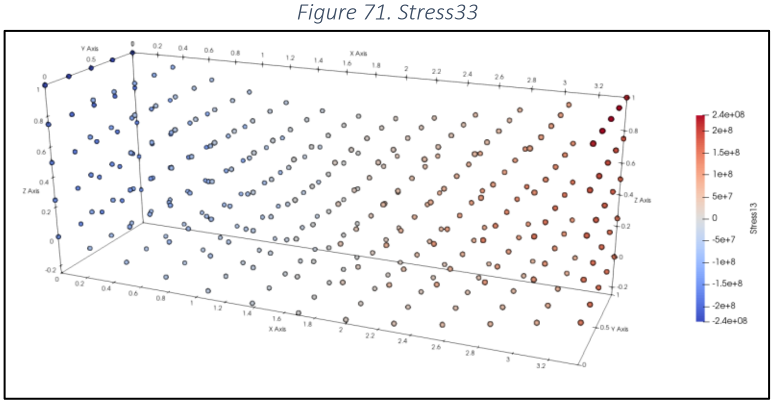

Figure 73. Stress13

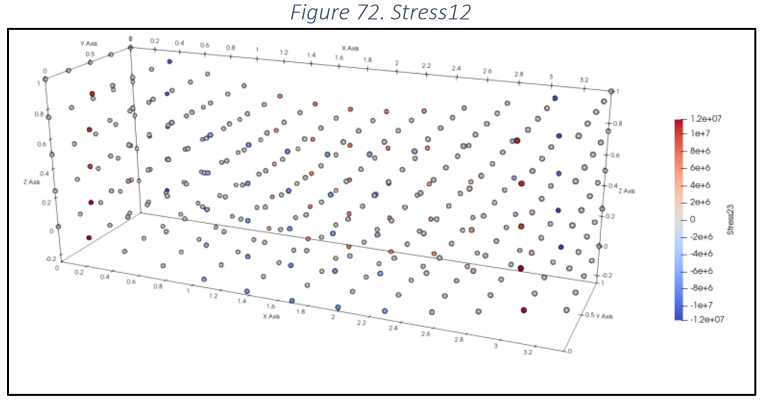

Figure 74. Stress23

t = 0.20 s

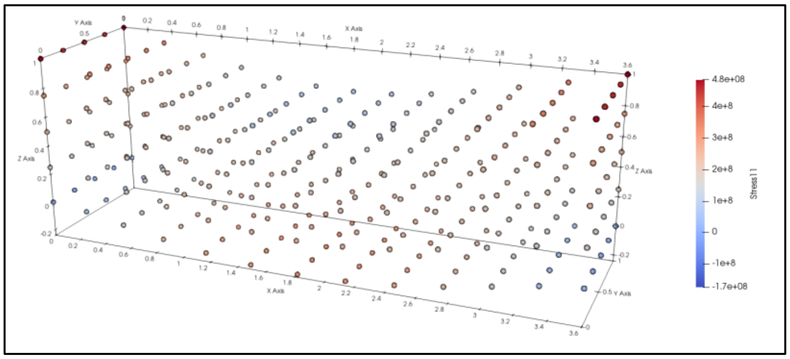

Figure 75. Stress11

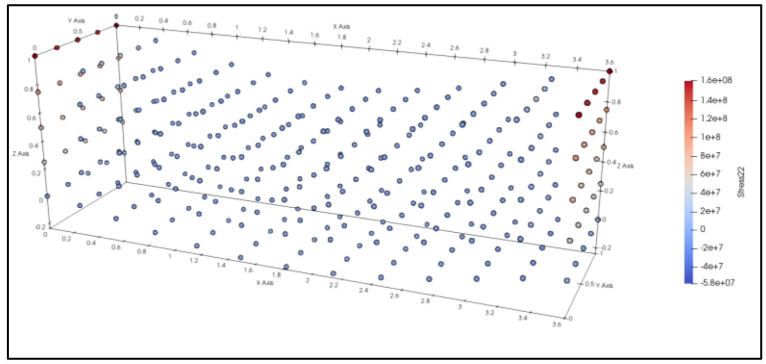

Figure 76. Stress22

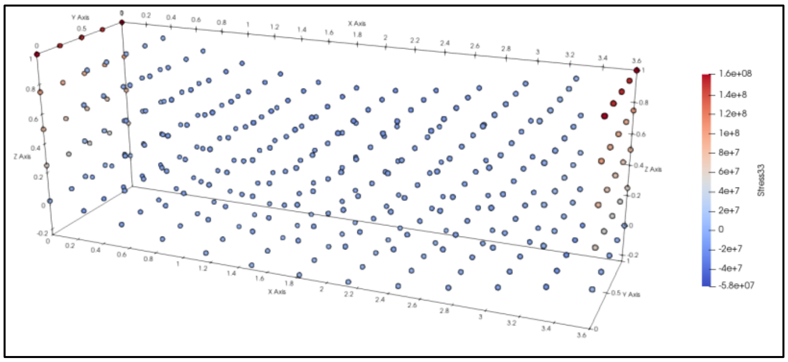

Figure 77. Stress33

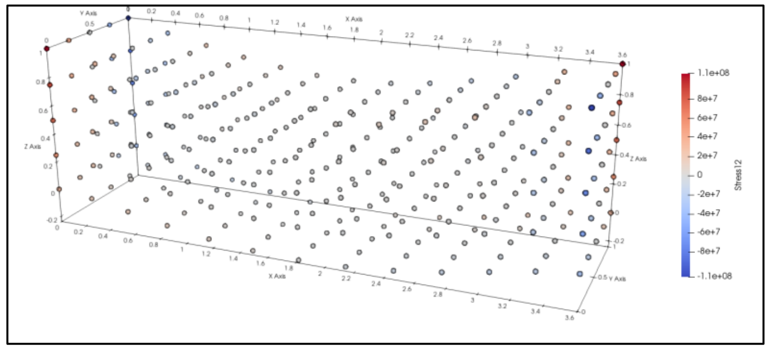

Figure 78. Stress12

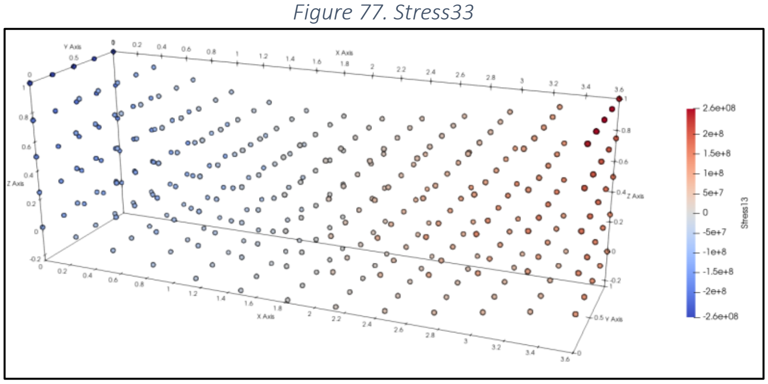

Figure 79. Stress13

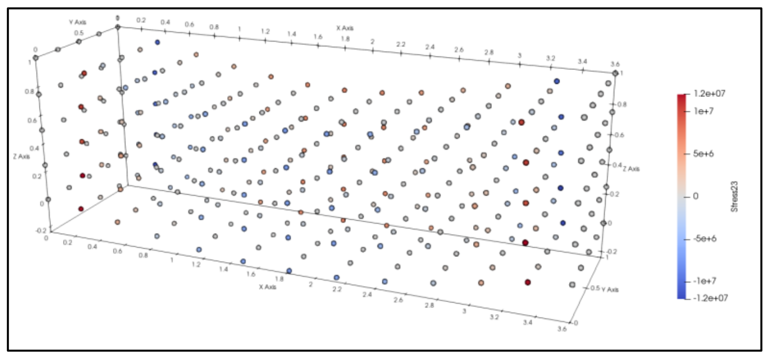

Figure 80. Stress23

iii. Forces

t = 0.04 s

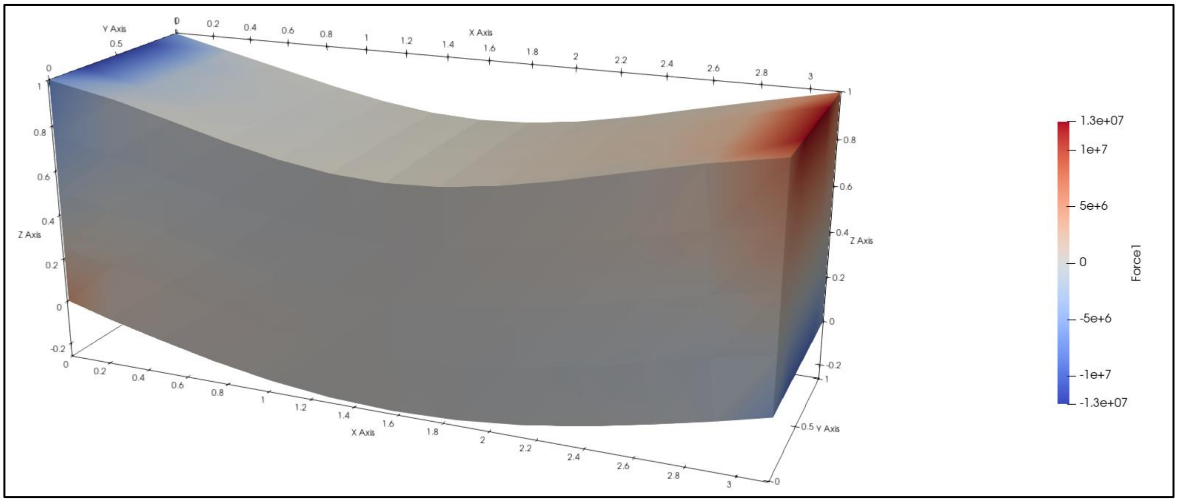

Figure 81. Force1

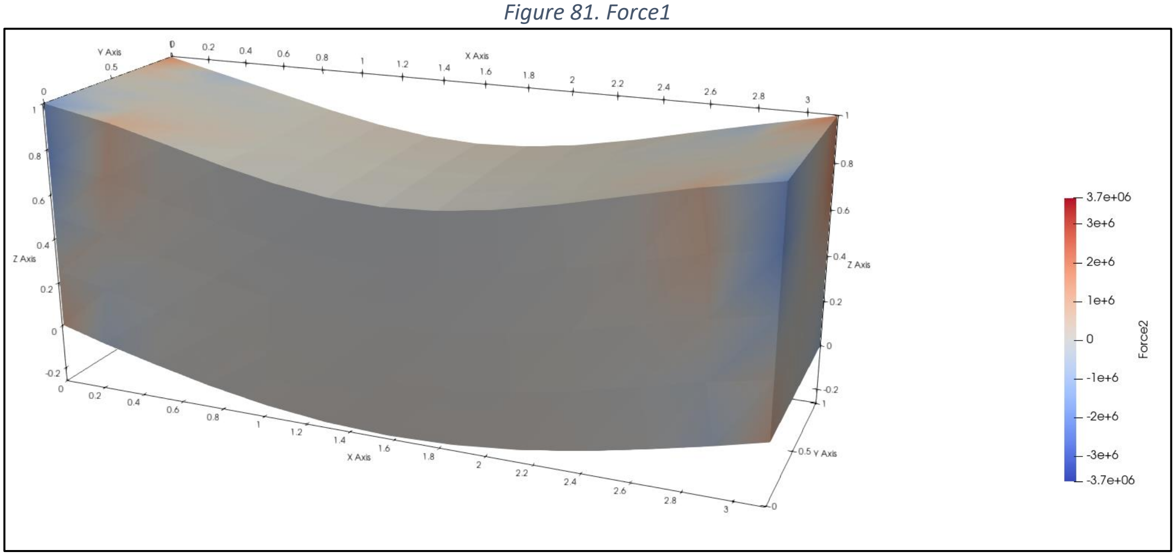

Figure 82. Force2

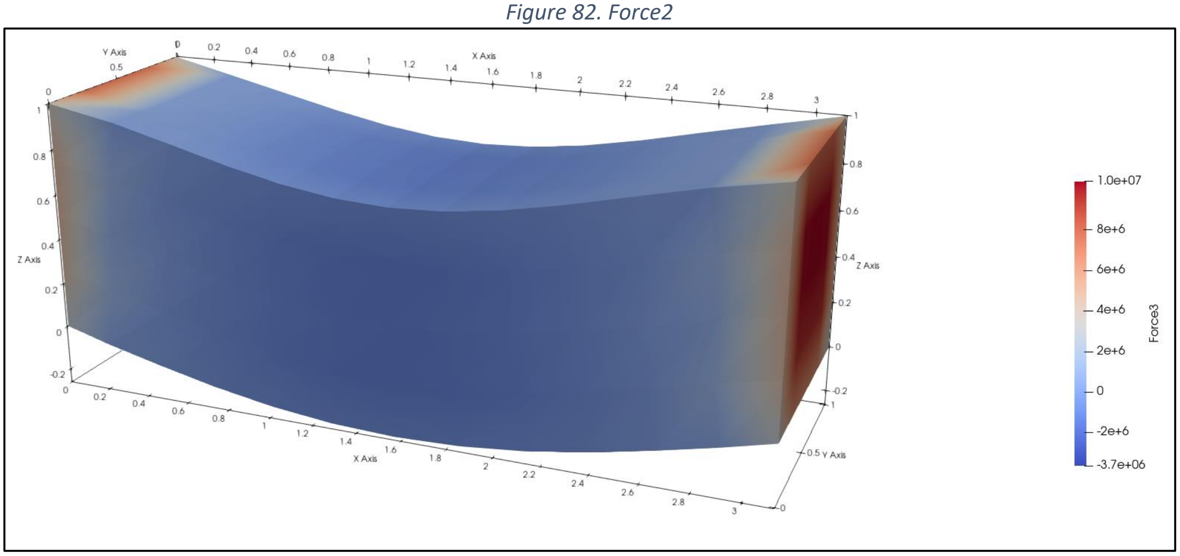

Figure 83. Force3

t = 0.20 s

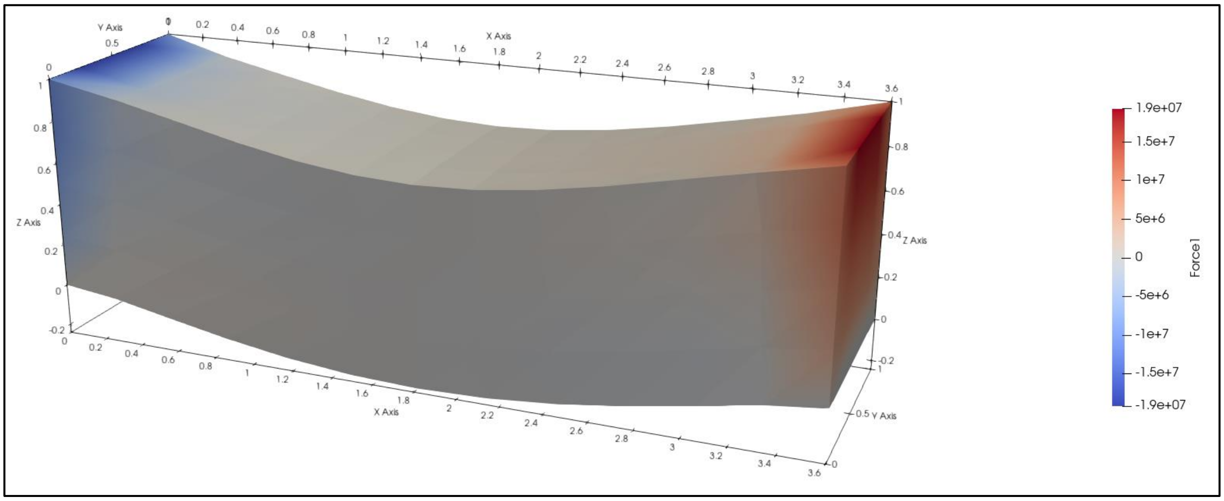

Figure 84. Force1

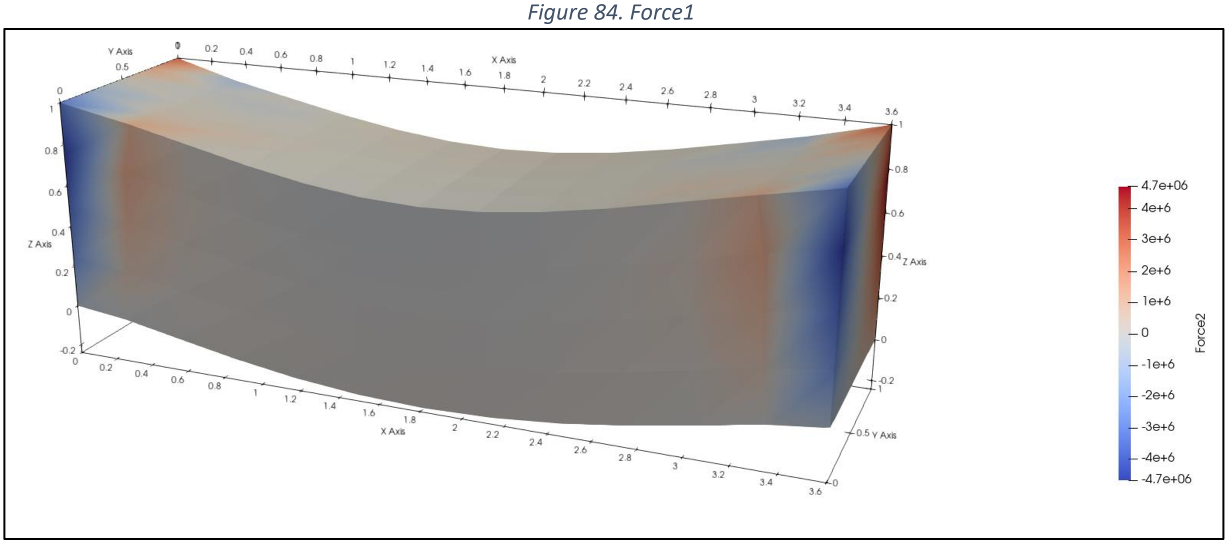

Figure 85. Force2

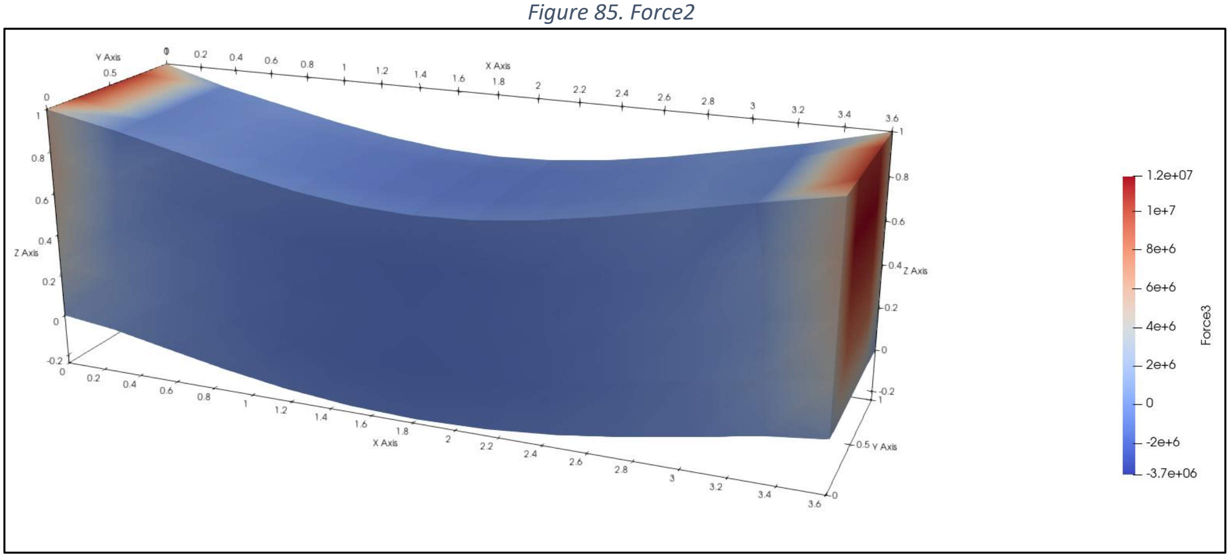

Figure 86. Force3

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