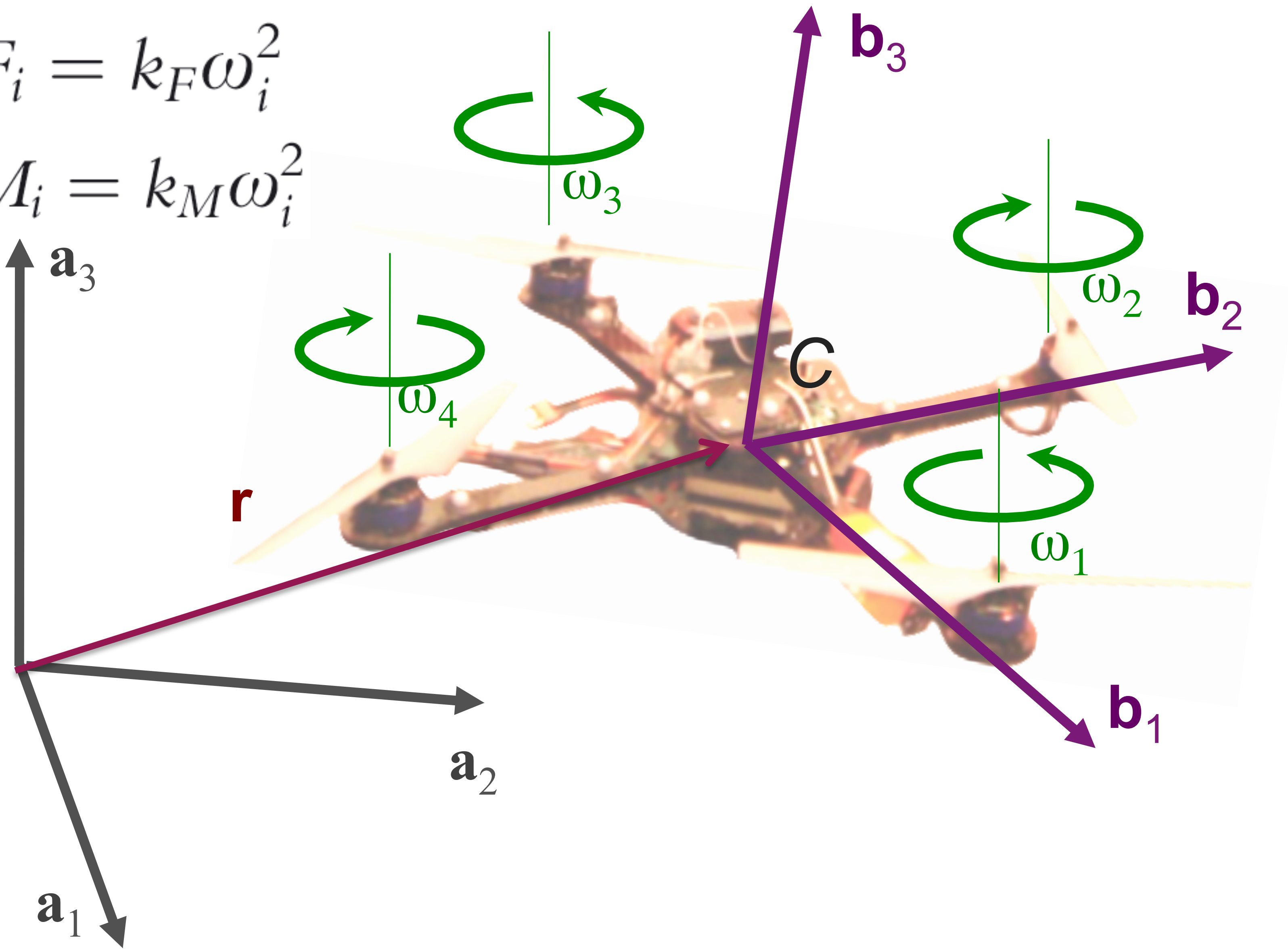
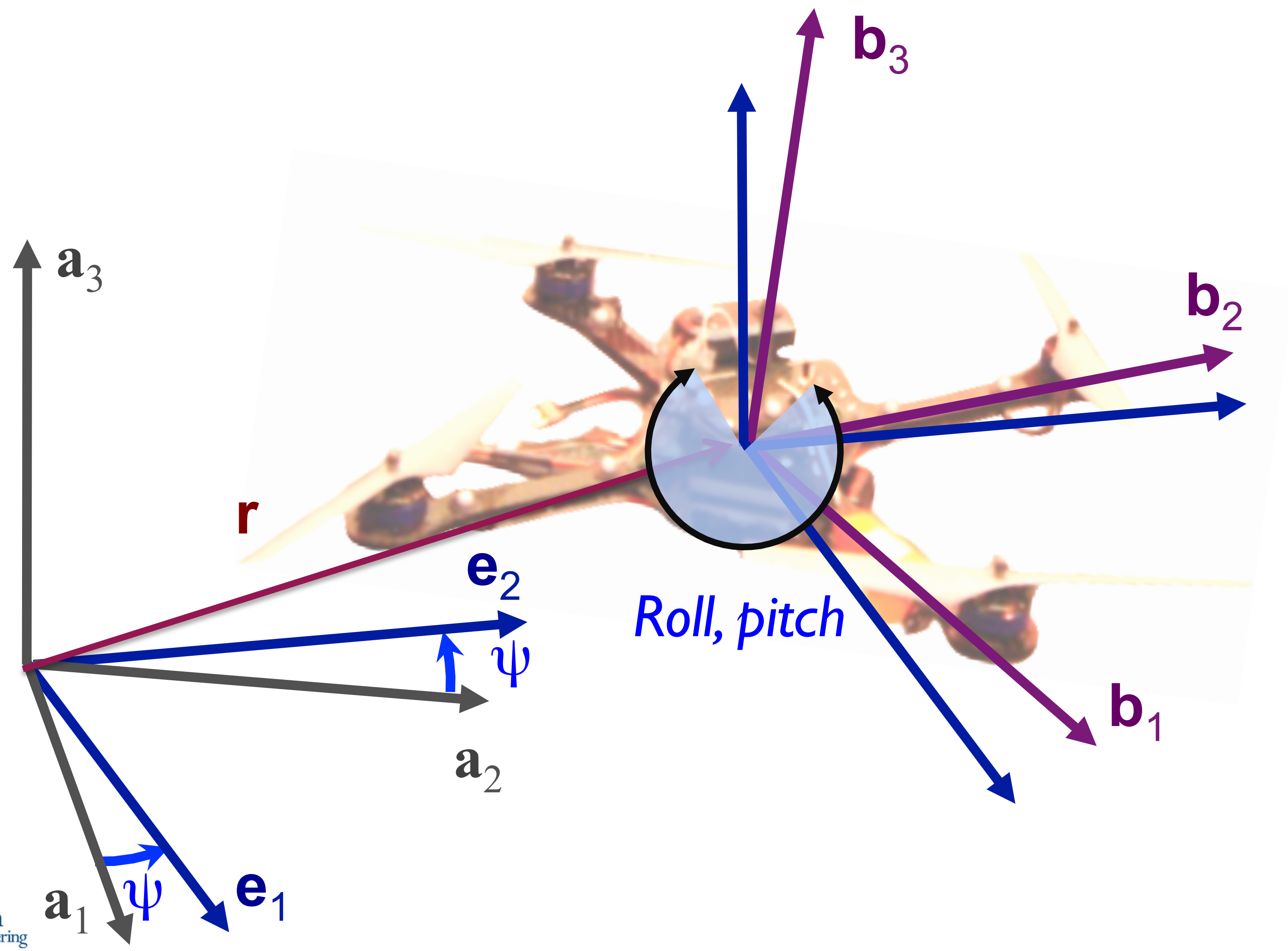
Dynamics of a Quadrotor

$$F_i = k_F \omega_i^2$$

$$M_i = k_M \omega_i^2$$



Euler Angles



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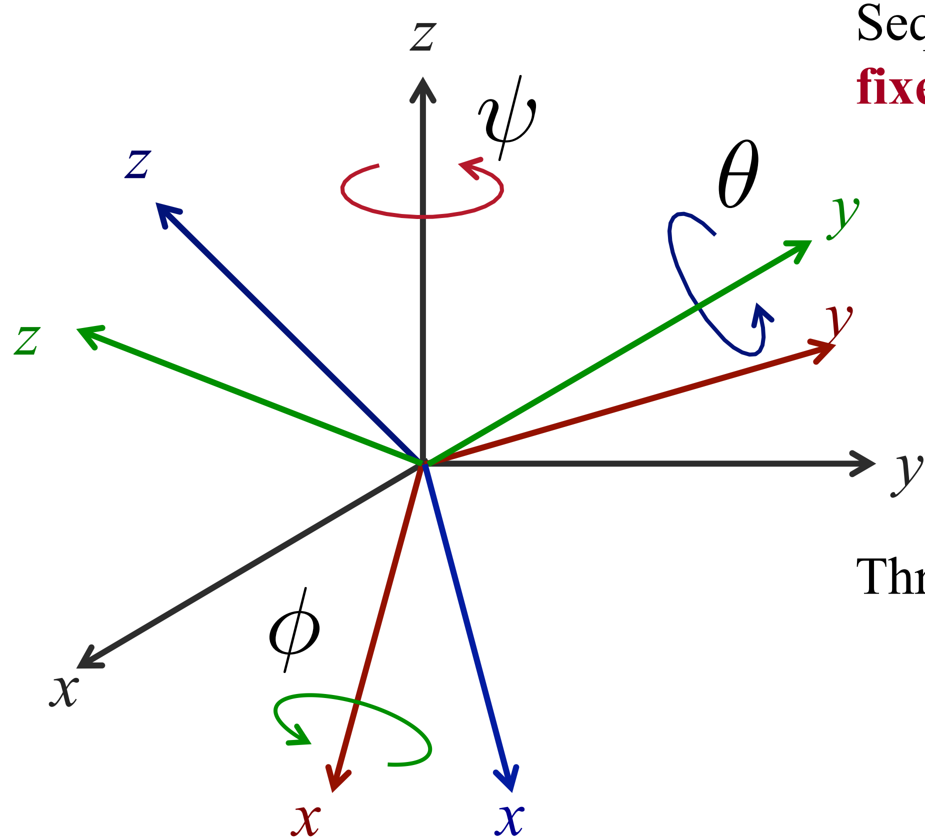
Z-X-Y Euler Angles



Sequence of three rotations about **body-fixed** axes

- Rot(z, ψ)
- Rot(x, ϕ)
- Rot(y, θ)

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

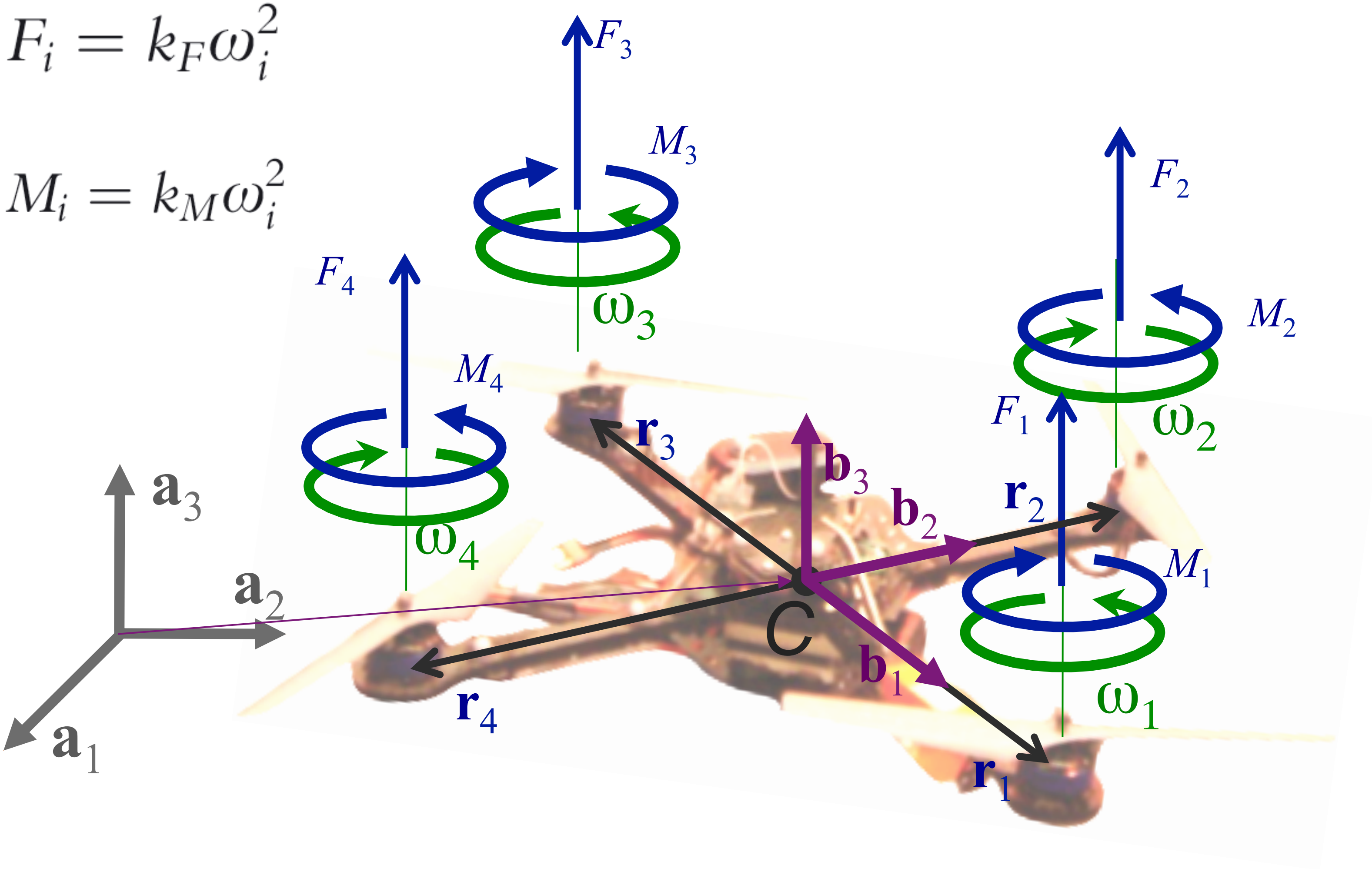
Note there are singularities!

$$\mathbf{R} = \mathrm{Rot}(z, \psi) \times \mathrm{Rot}(x, \phi) \times \mathrm{Rot}(y, \theta)$$

External Forces and Moments

$$F_i = k_F \omega_i^2$$

$$M_i = k_M \omega_i^2$$



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 - mg\mathbf{a}_3$$

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4 + \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4$$

Newton-Euler Equations

System of Particles

Rigid Body

Newton-Euler Equations

System of Particles

Rigid Body

Newton-Euler Equations

Newton's Equations of Motion for a Single Particle of mass *m*

$$\mathbf{F} = m\mathbf{a}$$

Newton-Euler Equations

System of Particles

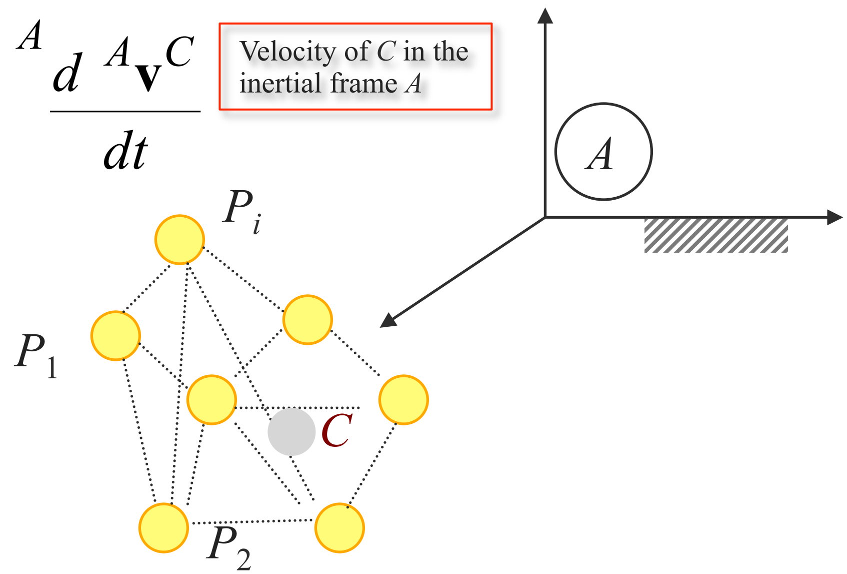
Rigid Body

Newton's Second Law for a System of Particles

The center of mass for a system of particles, *S*, accelerates in an inertial frame (*A*) as if it were a single particle with mass *m* (equal to the total mass of the system) acted upon by a force equal to the net external force.

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m \frac{{}^A d\, {}^A\mathbf{v}^C}{dt}$$

Velocity of C in the inertial frame A



Center of mass

$$\mathbf{r}_c = \frac{1}{m} \sum_{i=1,N} m_i \mathbf{p}_i$$

Rate of Change of Linear Momentum

Derivative in the inertial frame A

$$\mathbf{F} = \frac{{}^A d\, \mathbf{L}}{dt}$$

Linear momentum of the system of particles in the inertial frame A

Also true for a rigid body

Rotational equations of motion for a rigid body

The rate of change of angular momentum of the rigid body B relative to C in A is equal to the resultant moment of all external forces acting on the body relative to C

Derivative in the inertial frame A

Angular momentum of the rigid body B with the origin C in the inertial frame A

$$\frac{{}^{A}d\,{}^{A}\mathbf{H}_C^S}{dt} = \mathbf{M}_C^S$$

Net moment from all external forces and torques about the reference C

Angular momentum

angular velocity of B in A

$$\boxed{{}^{A}\mathbf{H}_C^S = \mathbf{I}_C \cdot {}^{A}\omega^{B}}$$

inertia tensor with C as the origin

Principal Axes and Principal Moments

Principal axis of inertia

u is a unit vector along a principal axis if **I . u** is parallel to **u**

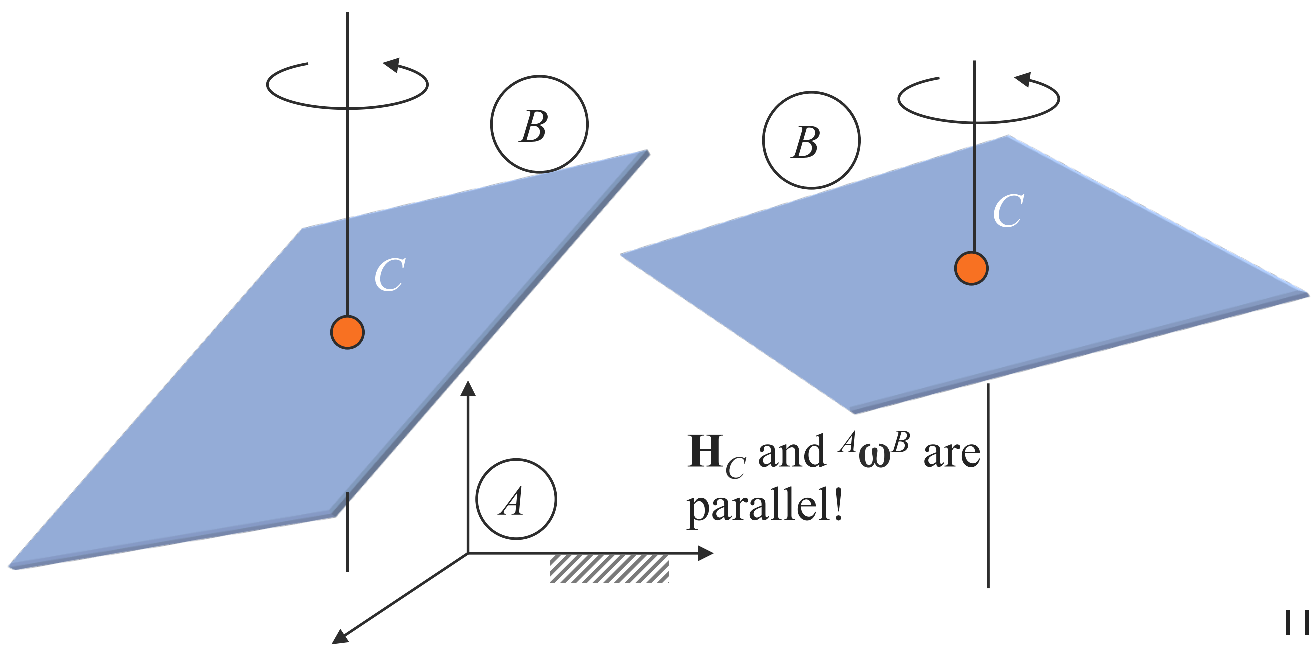
There are 3 independent principal axes!

Principal moment of inertia

The moment of inertia with respect to a principal axis, **u . I . u**, is called a principal moment of inertia.

Physical interpretation

$\mathbf{H}_C$ and ${}^A\boldsymbol{\omega}^B$ are not parallel!



Euler’s Equations

1

$$\frac{{}^{A}d\mathbf{H}_C}{dt} = \mathbf{M}_C$$

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}\begin{matrix} p \\ q \\ r \end{matrix} = \begin{bmatrix} M_{C,1} \\ M_{C,2} \\ M_{C,3} \end{bmatrix}$$

2

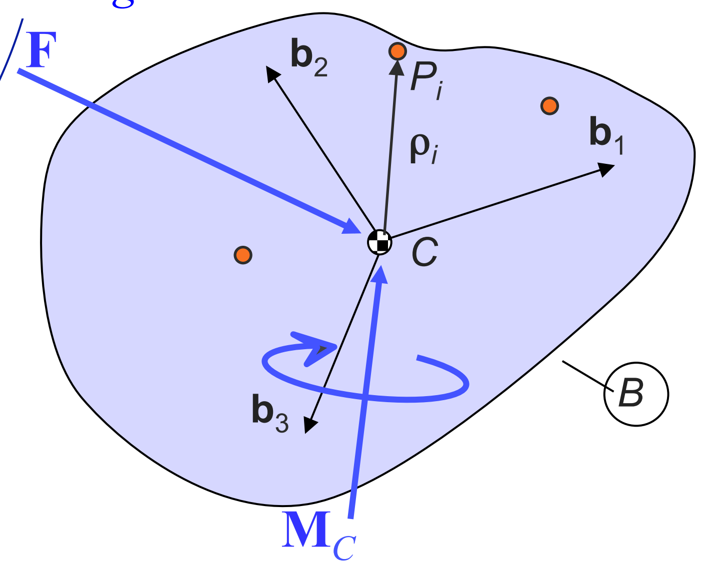
differentiating

Let $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, be along principal axes and

$${}^{A}\boldsymbol{\omega}^{B} = \omega_1\,\mathbf{b}_1 + \omega_2\,\mathbf{b}_2 + \omega_3\mathbf{b}_3$$

$$\frac{{}^{B}d\mathbf{H}_C}{dt} + {}^{A}\omega^{B}\times\mathbf{H}_C = \mathbf{M}_C$$

$$\frac{{}^{B}d\mathbf{H}_C}{dt} = I_{11}\dot{\omega}_1\mathbf{b}_1 + I_{22}\dot{\omega}_2\mathbf{b}_2 + I_{33}\dot{\omega}_3\mathbf{b}_3$$

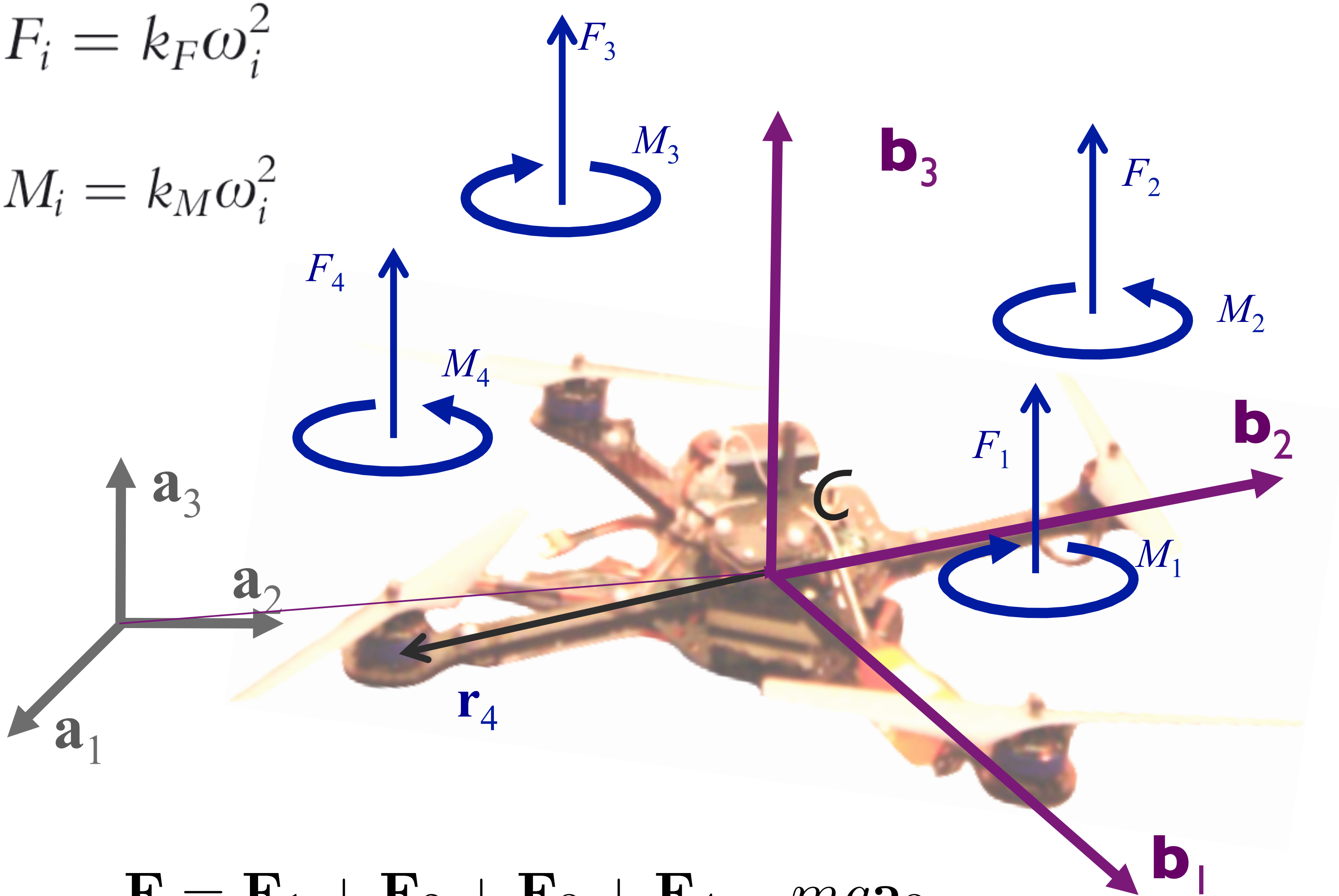


net external moment

Quadrotor Equations of Motion

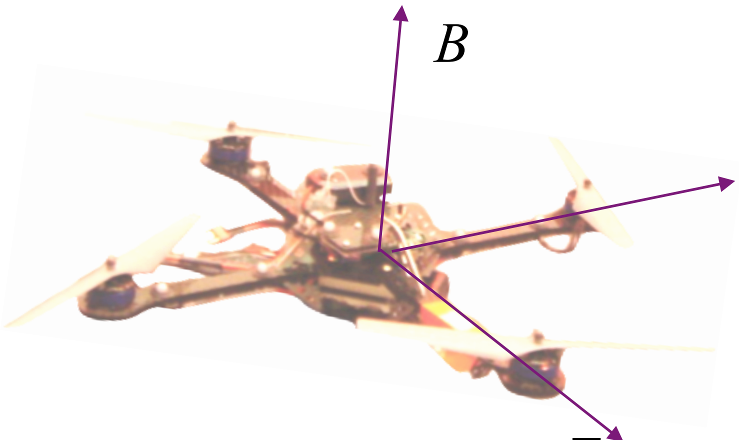
$$F_i = k_F \omega_i^2$$

$$M_i = k_M \omega_i^2$$



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 - mg\mathbf{a}_3$$

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4 + \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4$$



Newton-Euler Equations

$$^{A}\boldsymbol{\omega}^{B} = p\,\mathbf{b}_1 + q\,\mathbf{b}_2 + r\,\mathbf{b}_3$$

Rotation of thrust vector from B to A

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ \boxed{F_1 + F_2 + F_3 + F_4} \end{bmatrix}$$

Components in the inertial frame along $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$

u_1

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \boxed{\begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$\mathbf{u}_2$

Components in the body frame along $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, the principal axes

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How do we estimate all the parameters in this model?

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

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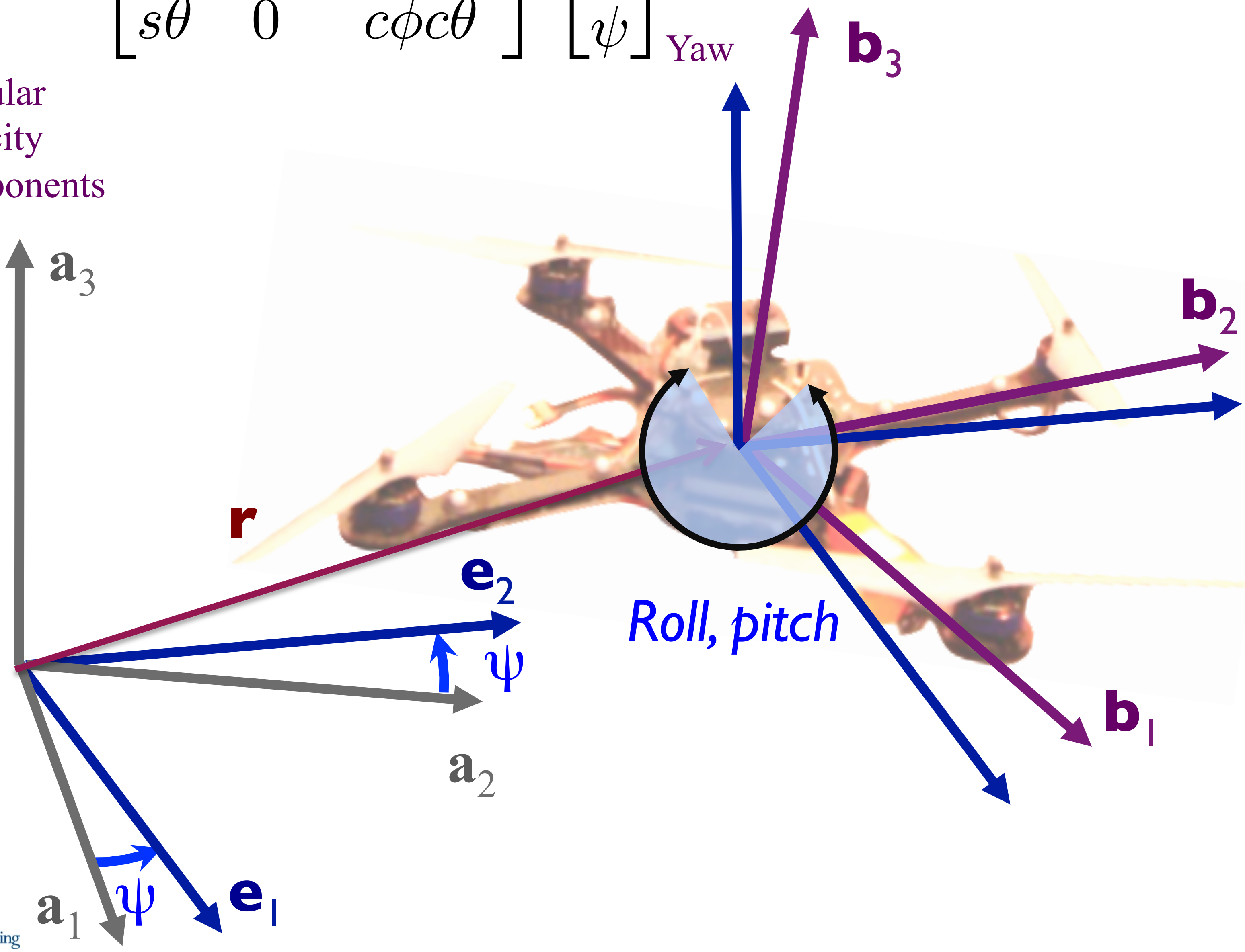
$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -c\phi s\theta \\ 0 & 1 & s\phi \\ s\theta & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \begin{array}{l} \text{Roll} \\ \text{Pitch} \\ \text{Yaw} \end{array}$$

Angular velocity components in B



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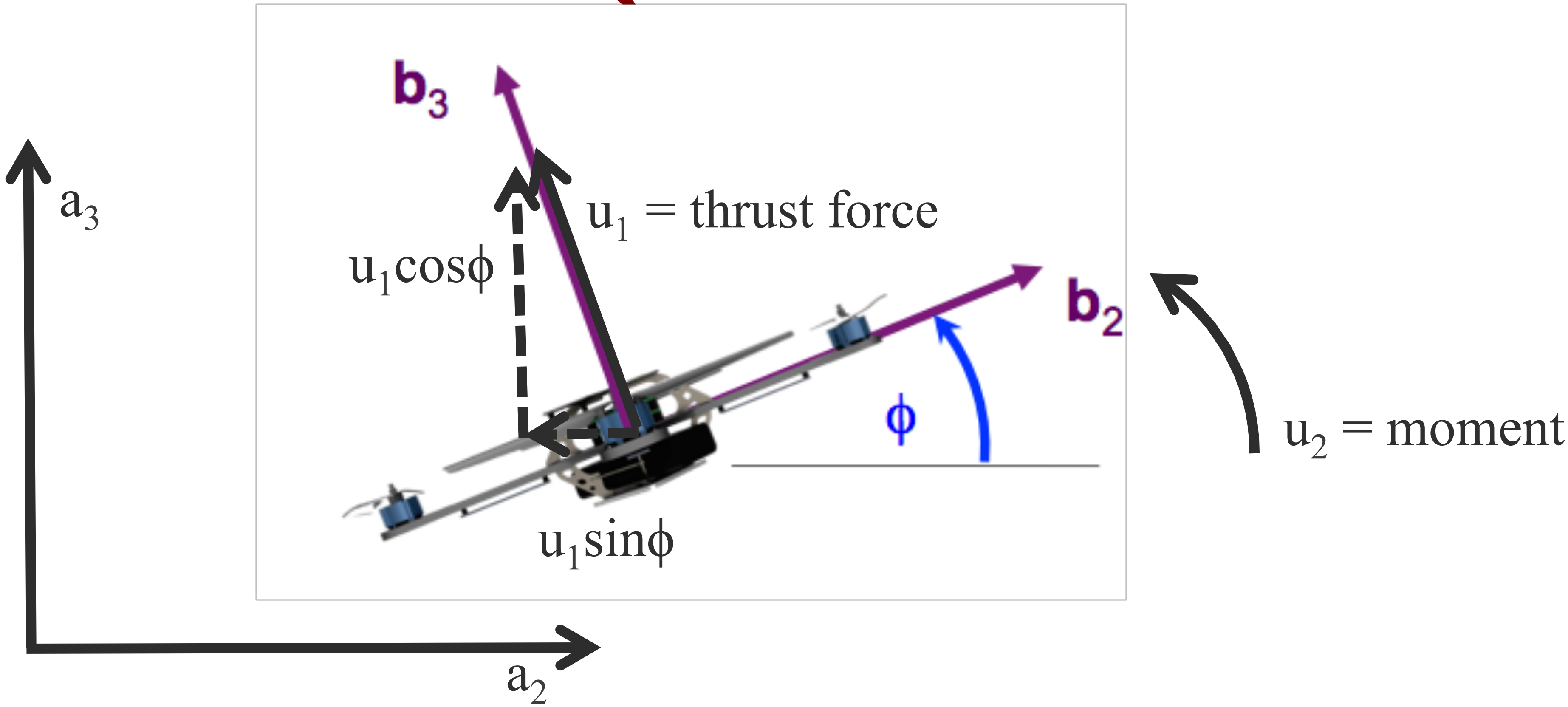
Planar Quadrotor Model



$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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State Space for Quadrotors

State Vector

- q describes the configuration (position) of the system
- x describes the state of the system

$$q = \begin{bmatrix} x \\ y \\ z \\ \varphi \\ \theta \\ \psi \end{bmatrix}, \ x = \begin{bmatrix} q \\ \text{--} \\ \dot{q} \end{bmatrix}$$

Planar Quadrotor

$$q = \begin{bmatrix} y \\ z \\ \varphi \end{bmatrix}, \ x = \begin{bmatrix} q \\ \text{--} \\ \dot{q} \end{bmatrix}$$

Equilibrium at Hover

- q_e describes the equilibrium configuration of the system
- x_e describes the equilibrium state of the system

$$q_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 0 \\ 0 \\ \psi_0 \end{bmatrix}, \ x_e = \begin{bmatrix} q_e \\ \text{-- --} \\ 0 \end{bmatrix}$$

$$q_e = \begin{bmatrix} y_0 \\ z_0 \\ 0 \end{bmatrix}, \ x_e = \begin{bmatrix} q_e \\ \text{-- --} \\ 0 \end{bmatrix}$$

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