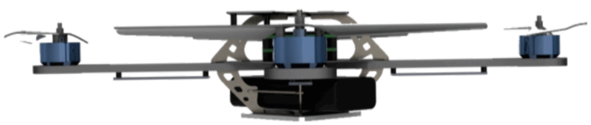
Rigid Body Transformations



Two distinct positions and orientations of the same rigid body

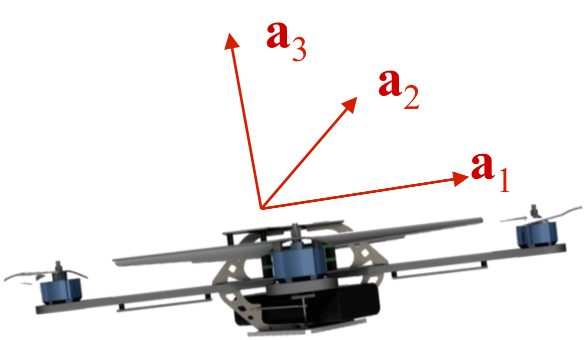
Reference Frames

We associate with any position and orientation a *reference frame*

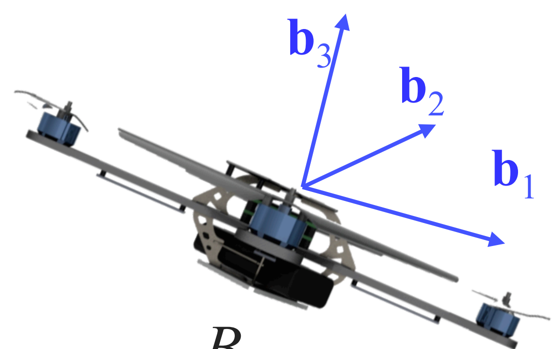
In reference frame $\{A\}$, we can find three linearly independent vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ that are basis vectors.

We can write any vector as a linear combination of the basis vectors in either frame.

$$\mathbf{v} = v_1\,\mathbf{a}_1 + v_2\,\mathbf{a}_2 + v_3\mathbf{a}_3$$



A



B

Notation

Vectors

- **x**, **y**, **a**, …
- $^{A}\mathbf{x}$
- u, v, p, q, …

Potential for Confusion!

Reference Frames

- A, B, C, …
- a, b, c, …

Matrices

- **A**, **B**, **C**, …

Transformations

- $^{A}\mathbf{A}_{B}$ $^{A}\mathbf{R}_{B}$ $^{A}\boldsymbol{\xi}_{B}$
- $\mathbf{A}_{ab}$ $\mathbf{R}_{ab}$
- g_{ab}, h_{ab}, …

Rigid Body Displacement

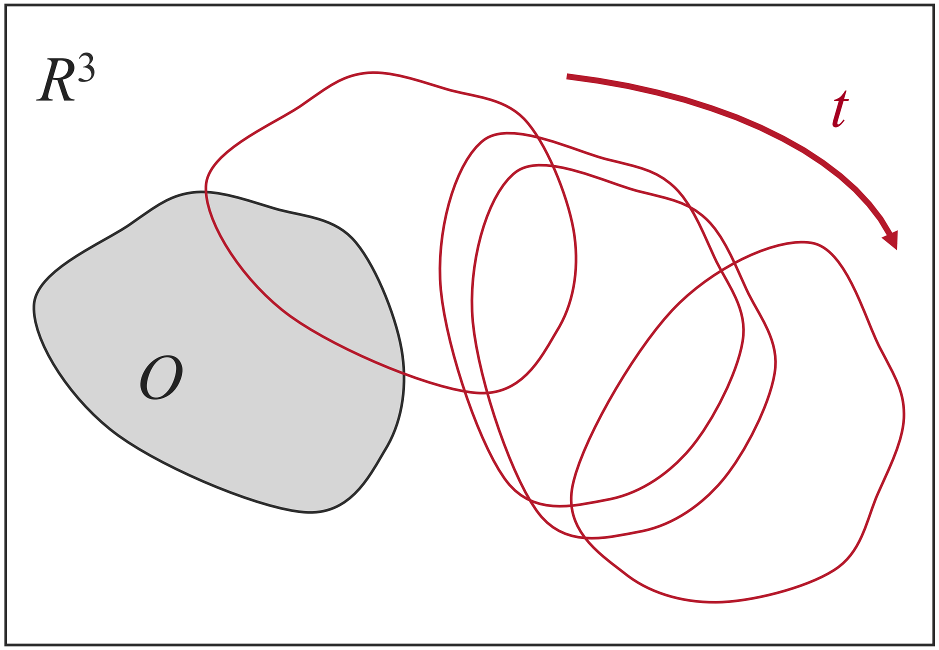
Object

$$O \subset R^3$$

Rigid Body Displacement

Map

$$g : O \to R^3$$

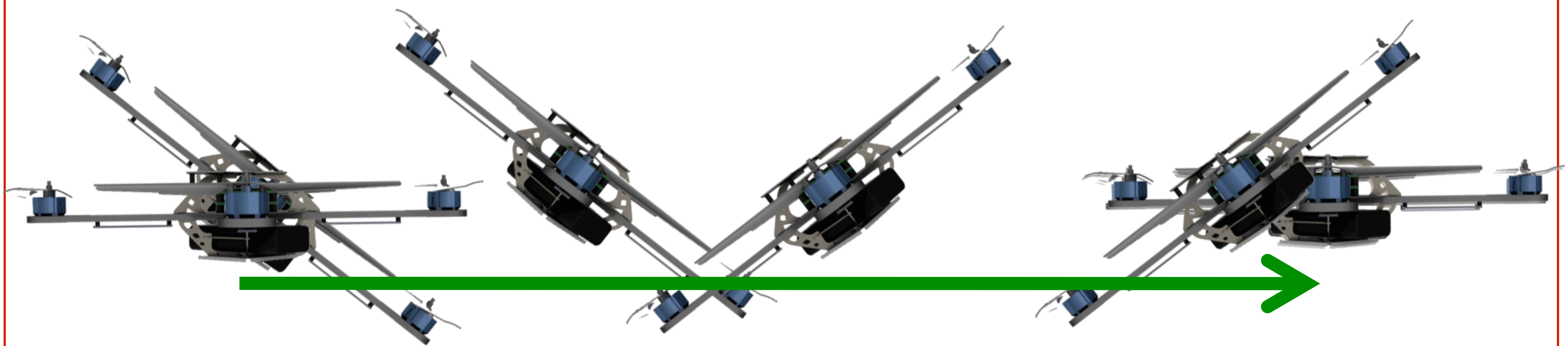


Rigid Body Motion

Continuous family of maps

$$g(t) : O \to R^3$$

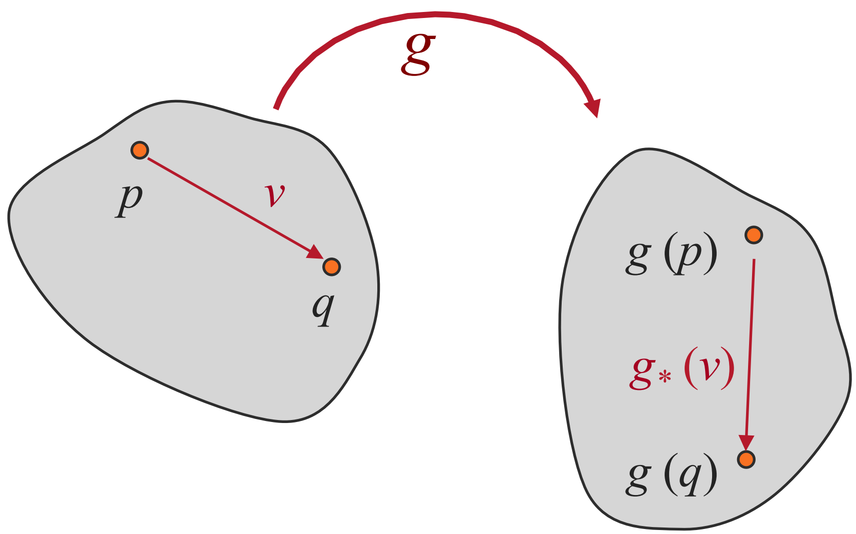
Example



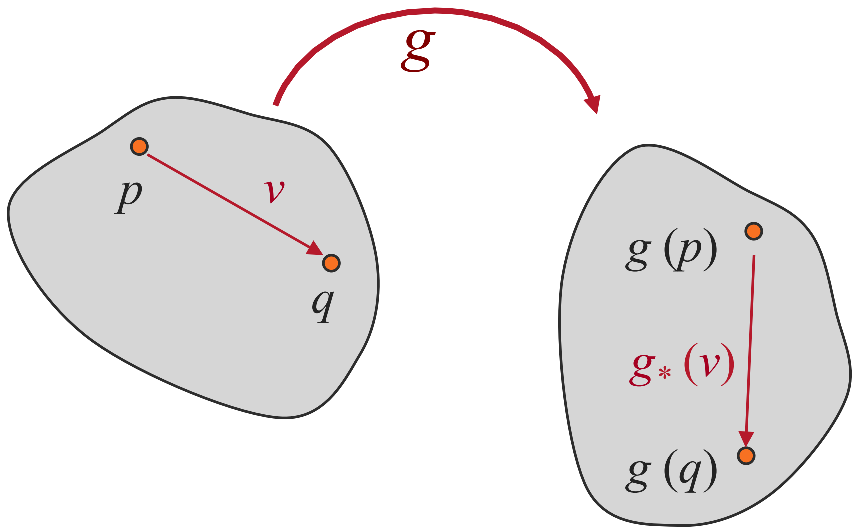
Rigid Body Displacement

A displacement is a transformations of points

- Transformation (g) of points induces an action (g_*) on vectors



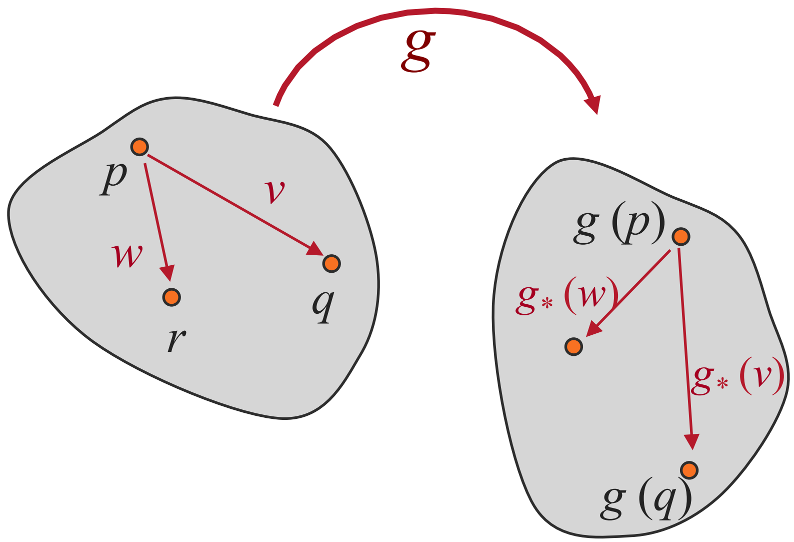
What makes g a *rigid body displacement*?



$$\|g(p) - g(q)\| = \|p - q\|$$

1. Lengths are preserved

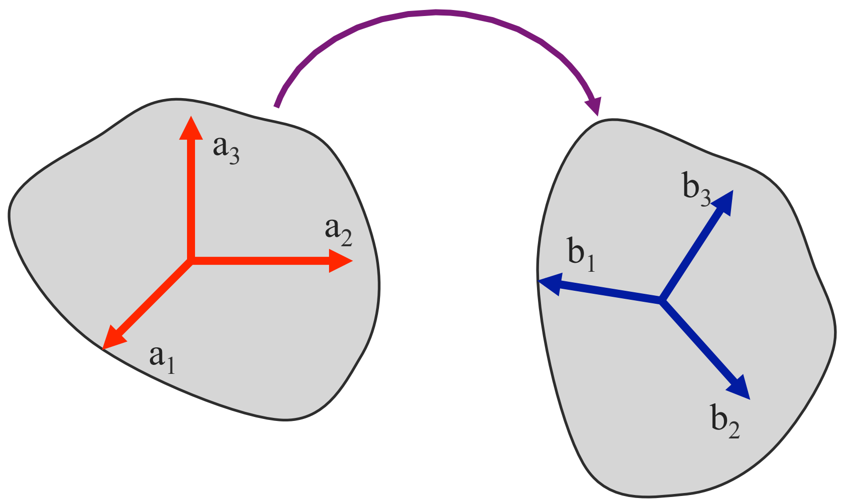
What makes *g* a *rigid body displacement*?



$$g_\star(v) \times g_\star(w) = g_\star(v \times w)$$

2. Cross products are preserved

g is a *rigid body* displacement



mutually orthogonal unit vectors get mapped to mutually orthogonal unit vectors

You should be able to prove

- orthogonal vectors are mapped to orthogonal vectors

- g_* preserves inner products

$$g_\star(v) \cdot g_\star(w) = g_\star(v \cdot w)$$

Summary

Rigid body displacements are transformations (maps) that satisfy two important properties

1. The map preserves lengths

2. Cross products are preserved by the induced map

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Note

Rigid body displacements and rigid body transformations are used interchangeably

1. Transformations generally used to describe relationship between reference frames attached to different rigid bodies.

2. Displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body

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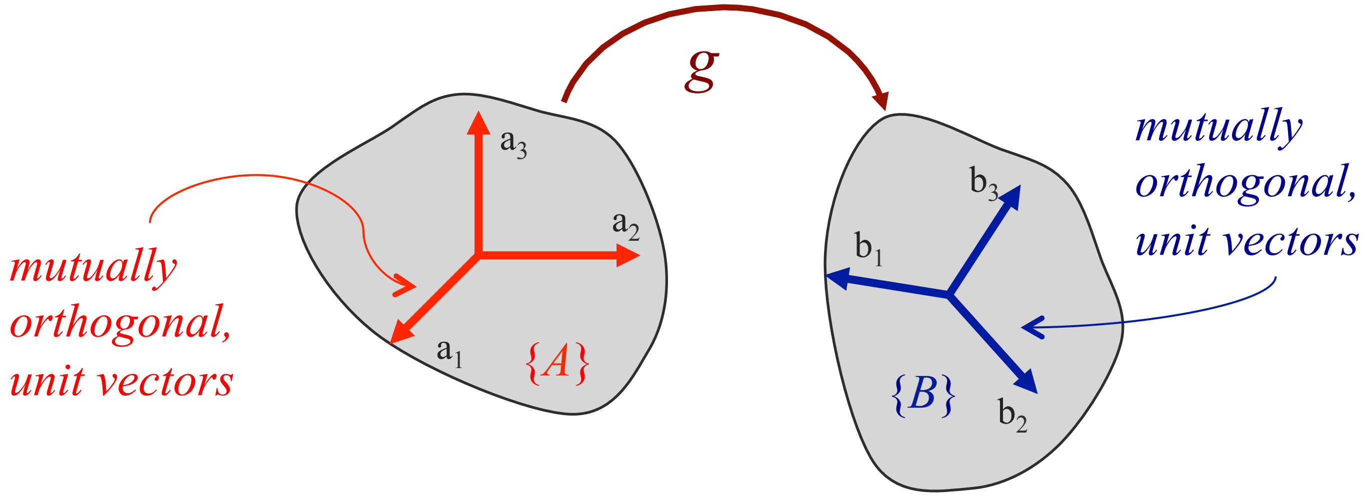
g is a *rigid body* displacement



$$\mathbf{b}_1 = R_{11}\mathbf{a}_1 + R_{12}\mathbf{a}_2 + R_{13}\mathbf{a}_3$$
$$\mathbf{b}_2 = R_{21}\mathbf{a}_1 + R_{22}\mathbf{a}_2 + R_{23}\mathbf{a}_3$$
$$\mathbf{b}_3 = R_{31}\mathbf{a}_1 + R_{32}\mathbf{a}_2 + R_{33}\mathbf{a}_3$$

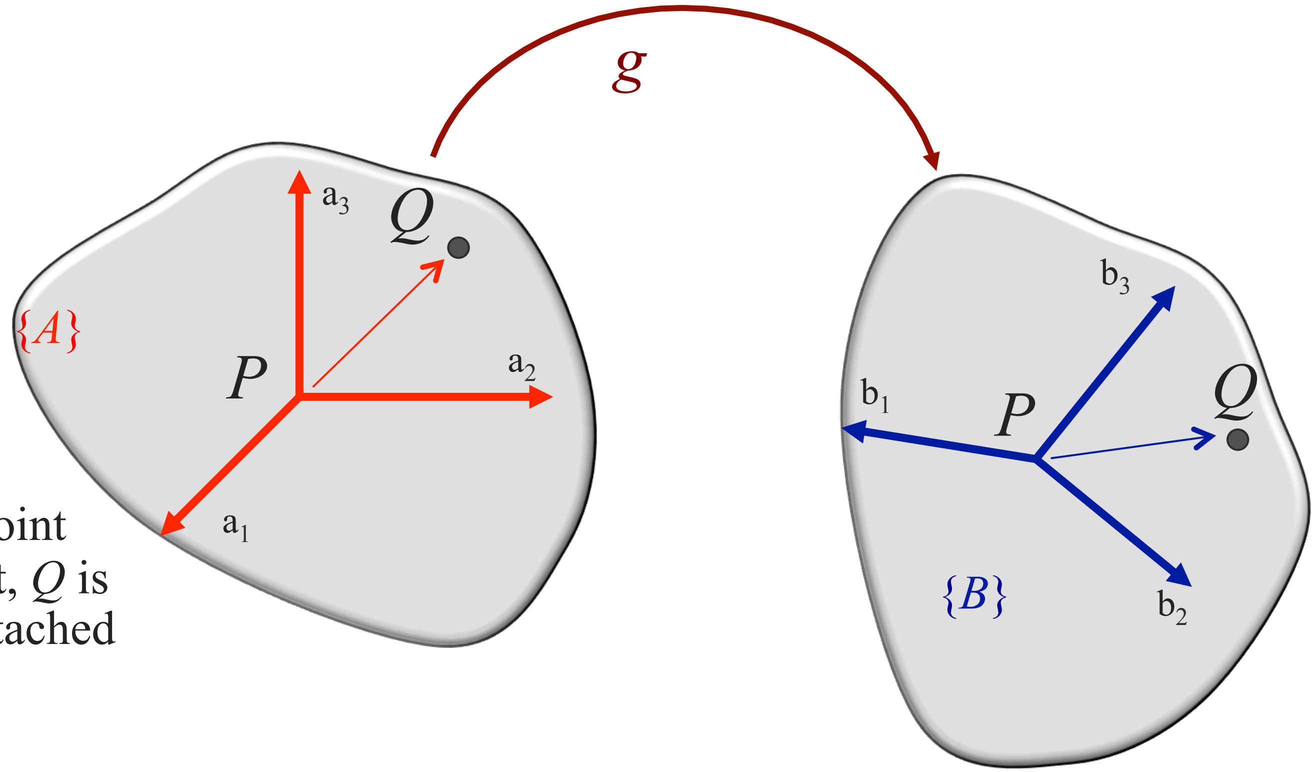
$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

rotation matrix

Properties of a Rotation Matrix

- Orthogonal
  - Matrix times its transpose equals the identity
- Special orthogonal
  - Determinant is +1
- Closed under multiplication
  - The product of any two rotation matrices is another rotation matrix
- The inverse of a rotation matrix is also a rotation matrix

g is a *rigid body* displacement

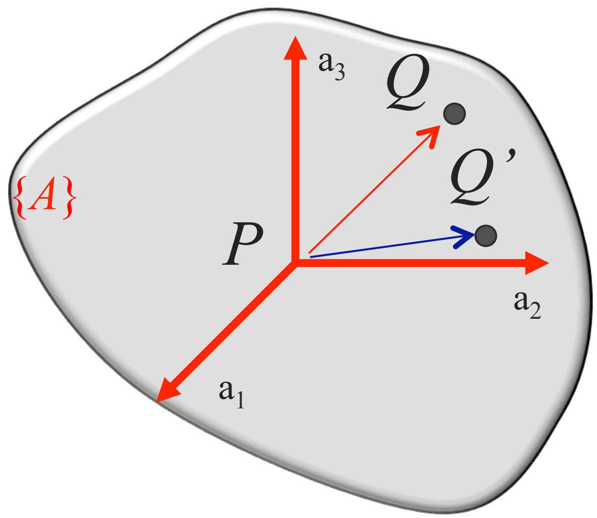


P is a reference point fixed to the object, *Q* is a generic point attached to the object.

$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\overrightarrow{PQ} = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$

g is a *rigid body* displacement



$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\overrightarrow{PQ'} = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \text{ or } \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix}$$

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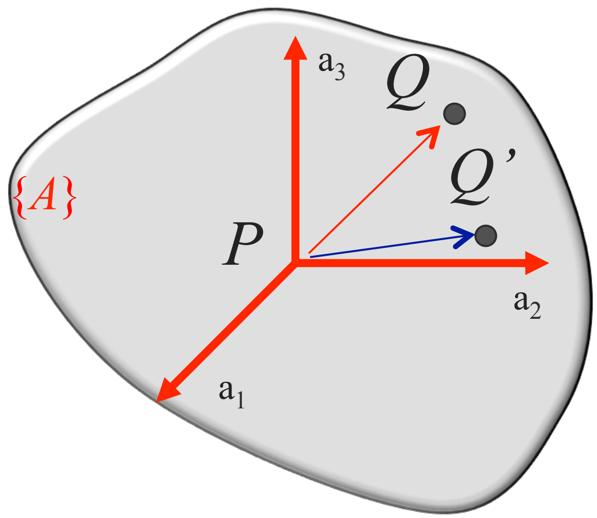
g is a *rigid body* displacement



$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

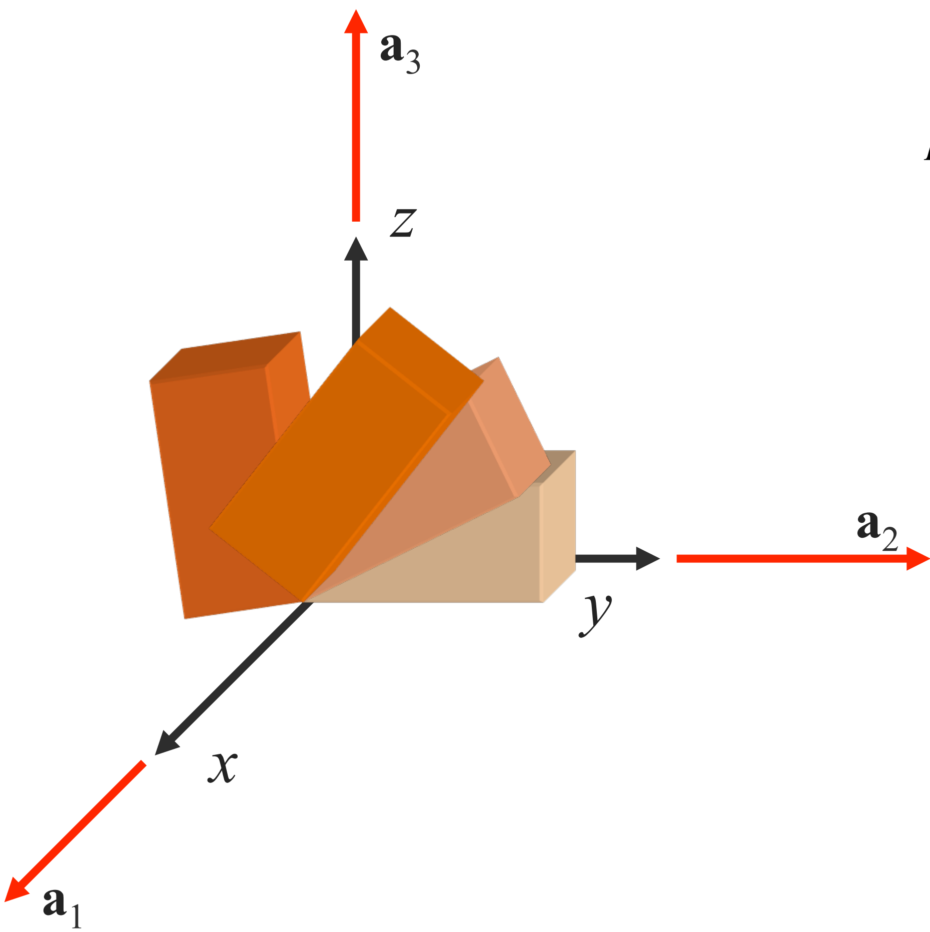
$$\overrightarrow{PQ'} = q'_1\mathbf{a}_1 + q'_2\mathbf{a}_2 + q'_3\mathbf{a}_3$$

Verify

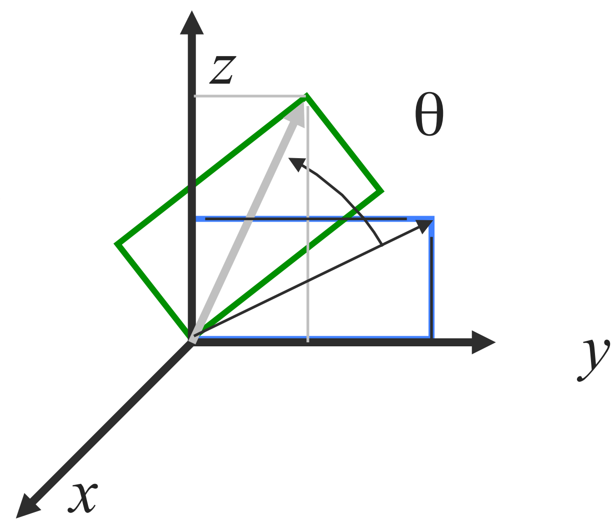
$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}}_{\textit{rotation matrix}} \begin{bmatrix} q'_1 \\ q'_2 \\ q'_3 \end{bmatrix}$$

Example: Rotation

- Rotation about the *x*-axis through θ



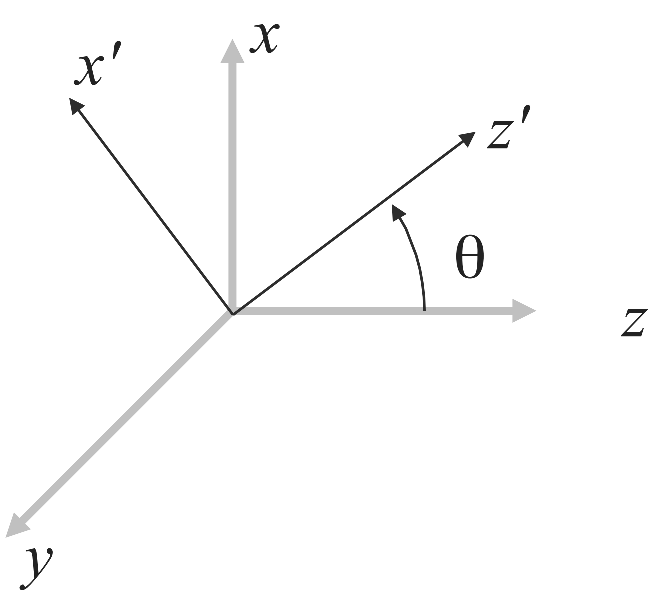
$$Rot(x,\theta)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Example: Rotation

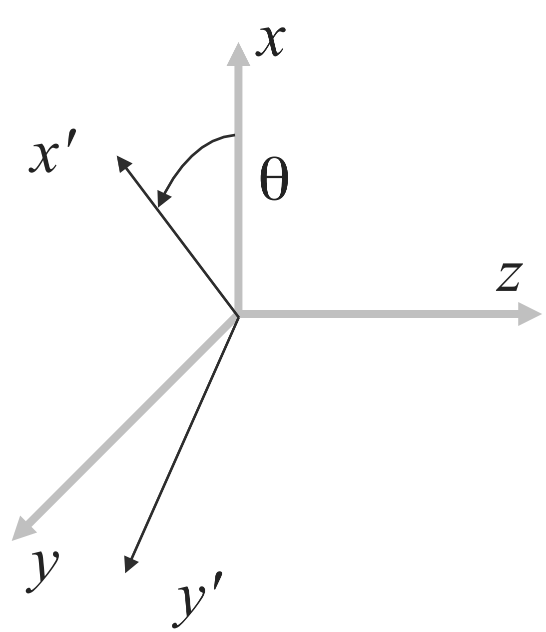
Rotation about the *y*-axis through θ

$$Rot(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$



Rotation about the *z*-axis through θ

$$Rot(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}$$



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Rotations

Special Orthogonal Matrices

$$\{R \in \mathbb{R}^{3\times 3} \mid R^T R = R R^T = I,\ \det R = 1\}$$

Special Orthogonal group in 3 dimensions

- The group of rotations is called *SO*(3)
- Coordinates for *SO*(3)

  1 Rotation matrices

  2 Euler angles

  3 Axis angle parameterization

  4 Exponential coordinates

  5 Quaternions

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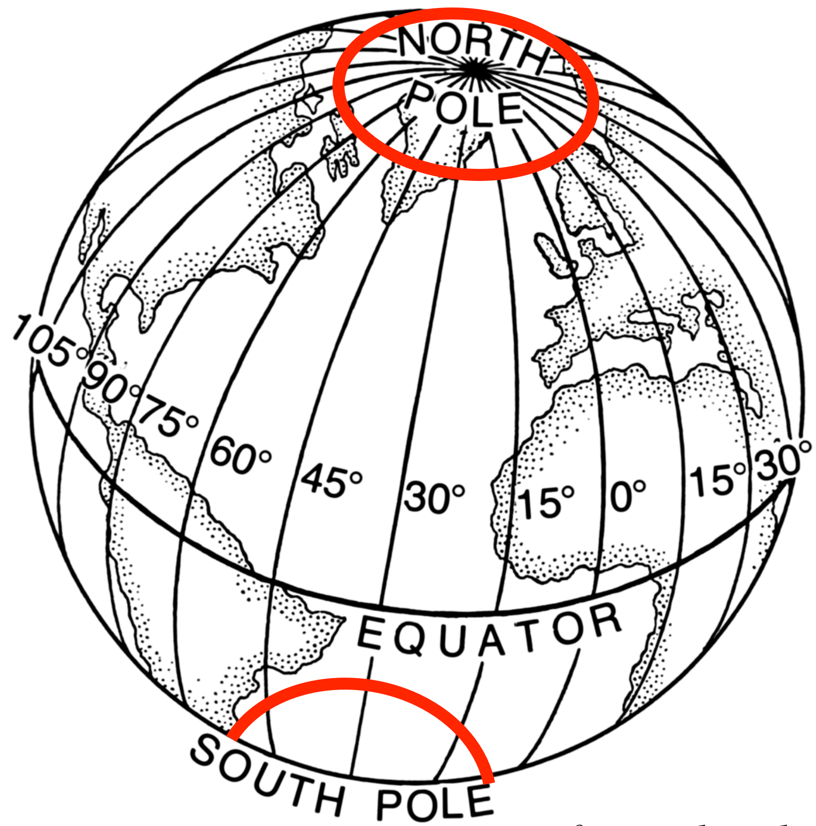
Coordinates for a Sphere

- Parameterize using a set of local coordinate charts (latitude and longitude)
- We want a collection of charts to describe the Earth’s surface



Images from wikipedia

What is the minimum number of charts you need to cover the Earth's surface?

Images from wikipedia

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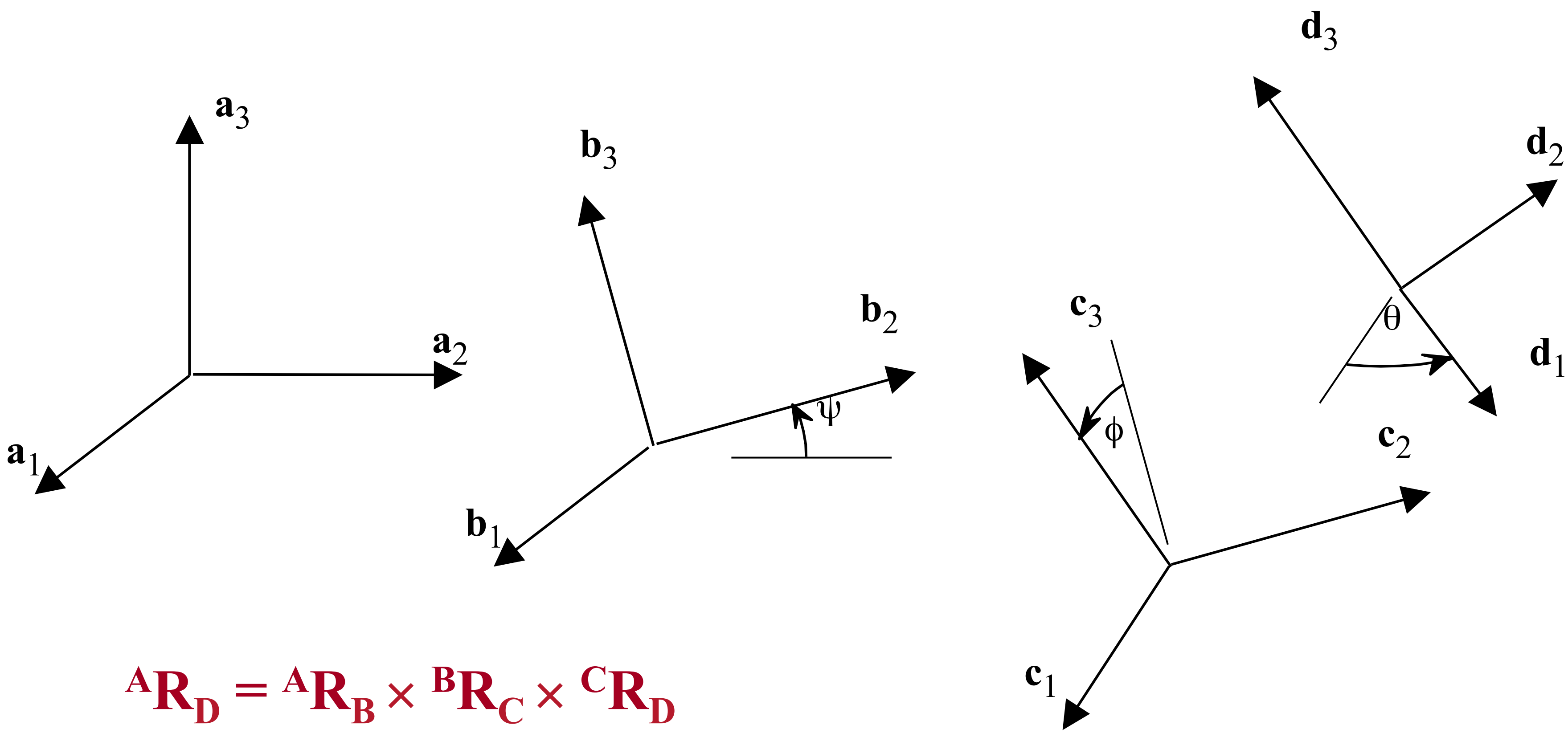
What is the minimum number of charts you need to cover $SO(3)$?

$$SO(3) = \{ R \in \mathbb{R}^{3\times 3} \mid R^T R = R R^T = I,\ \det R = 1 \}$$

Euler Angles

Composition of Three Rotations



$$ {}^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D} $$

$$ {}^{A}\mathbf{R}_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta) $$

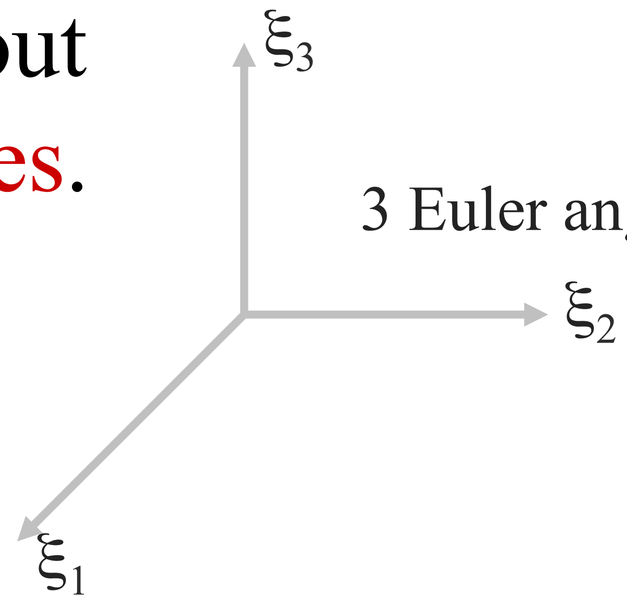
roll *pitch* *yaw*

Euler Angles

Image from wikipedia

Any rotation can be described by three successive rotations about linearly independent axes.

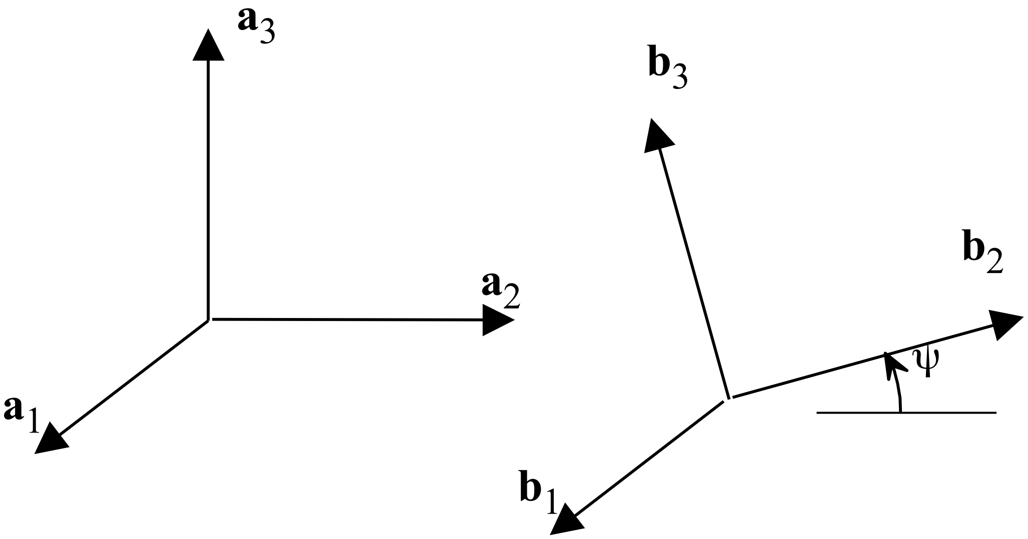
$$\begin{pmatrix} \text{3 × 3 rotation matrix} \end{pmatrix}$$

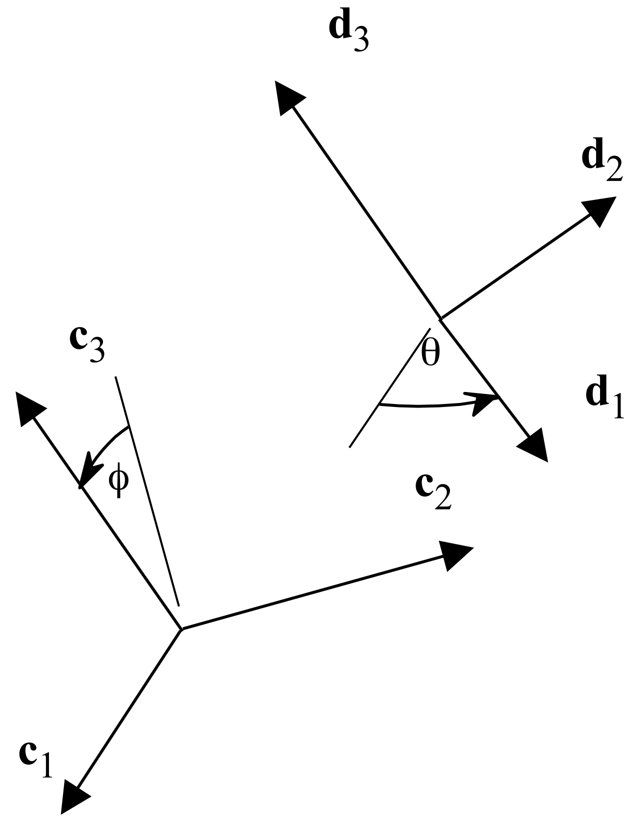


3 Euler angles

Almost 1-1 transformation

X-Y-Z Euler Angles



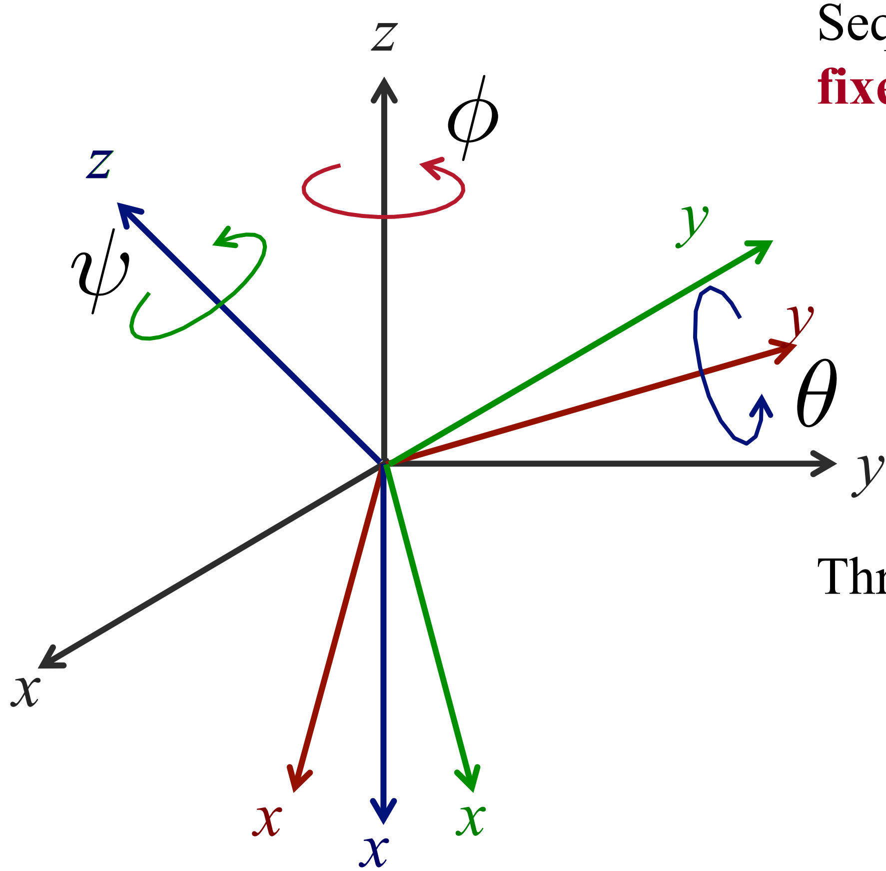


$$ {}^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D} $$

$$ {}^{A}\mathbf{R}_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta) $$

roll *pitch* *yaw*

Z-Y-Z Euler Angles



Sequence of three rotations about **body-fixed** axes

- Rot(z, ϕ)
- Rot(y, θ)
- Rot(z, ψ)

Are these linearly independent?

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

Note

- θ= 0 is a special (singular) case

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

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Determination of Euler Angles

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \boxed{\cos\phi\sin\theta} \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \boxed{\cos\theta} \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} & \boxed{R_{13}} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & \boxed{R_{33}} \end{bmatrix}$$

$R_{31} = -\sin\theta\cos\psi$

$R_{32} = \sin\theta\sin\psi$

$R_{33} = \cos\theta$

$R_{13} = \sin\theta\cos\phi$

$R_{23} = \sin\theta\sin\phi$

known rotation matrix

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Determination of Euler Angles

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

If $|R_{33}| < 1$,

$$\theta = \sigma \arccos(R_{33}), \quad \sigma = \pm 1$$

$$\psi = a\tan 2\left(\frac{R_{32}}{\sin\theta}, \frac{-R_{31}}{\sin\theta}\right)$$

$$\phi = a\tan 2\left(\frac{R_{23}}{\sin\theta}, \frac{R_{13}}{\sin\theta}\right)$$

Two sets of Euler angles for every **R** for almost all **R**'s!

If $R_{33} = 1$,

$$R = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi + \sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$f(\phi + \psi)$

If $R_{33} = -1$,

$$R = \begin{bmatrix} -\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$f(\phi + \psi)$

Infinite set of Euler Angles!

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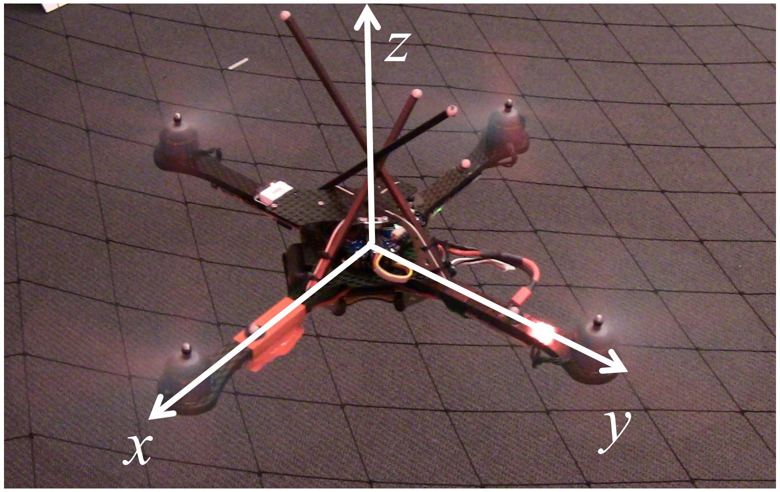
Z-X-Y Euler Angles

Sequence of three rotations about **body-fixed** axes

- Rot(z, ψ)
- Rot(x, ϕ)
- Rot(y, θ)



Verify

$$R = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\theta s\phi c\psi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

N. Michael, D. Mellinger, Q. Lindsey, V. Kumar, *The GRASP Multiple Micro-UAV Testbed*, IEEE Robotics & Automation Magazine, vol.17, no.3, pp.56-65, Sept. 2010

What is the minimum number of sets of Euler angles you need to cover $SO(3)$?

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = RR^T = I\}$$

Axis/Angle Representation

Special Orthogonal Matrices

$$\{R \in \mathbb{R}^{3\times 3} \mid R^T R = RR^T = I,\ \det R = 1\}$$

Special Orthogonal group in 3 dimensions

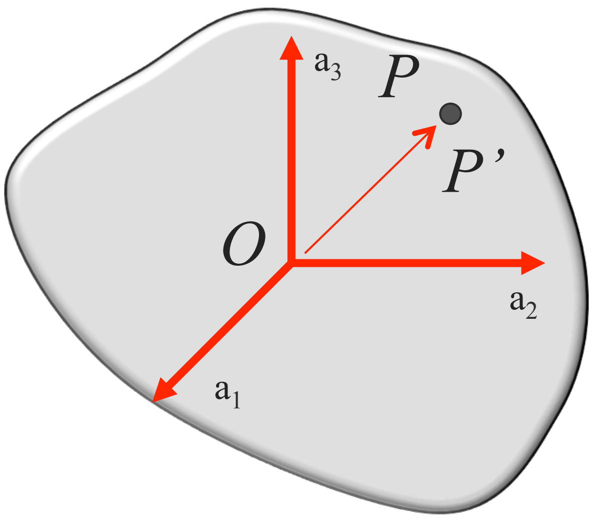
- Coordinates for $SO(3)$
  1 Rotation matrices
  2 Euler angles
  3 Axis angle parameterization
  4 Exponential coordinates
  5 Quaternions

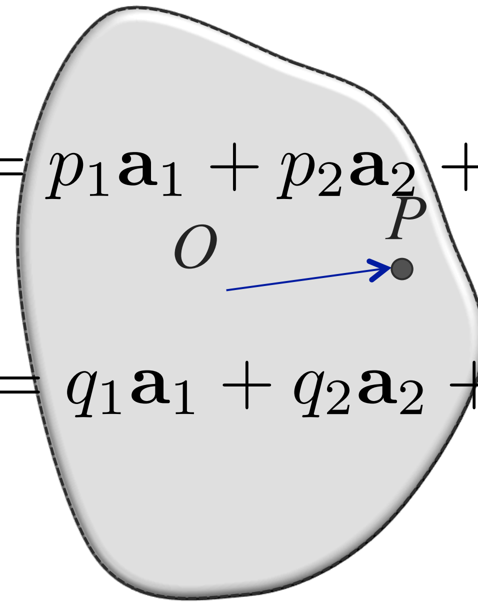
Euler's Theorem

Rotations

Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

Rotation with O fixed





$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$

$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{\mathbf{q}} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{\mathbf{p}}$$

$$\mathbf{q} = R\mathbf{p}$$

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Proof of Euler's Theorem

$$\mathbf{q} = R\mathbf{p}$$

Is there a point **p** that maps onto itself?

If there were such a point **p** ...

$$\mathbf{p} = R\mathbf{p}$$

Solve eigenvalue problem

$$R\mathbf{p} = \lambda\mathbf{p}$$

Verify λ=1 is an eigenvalue for any *R*

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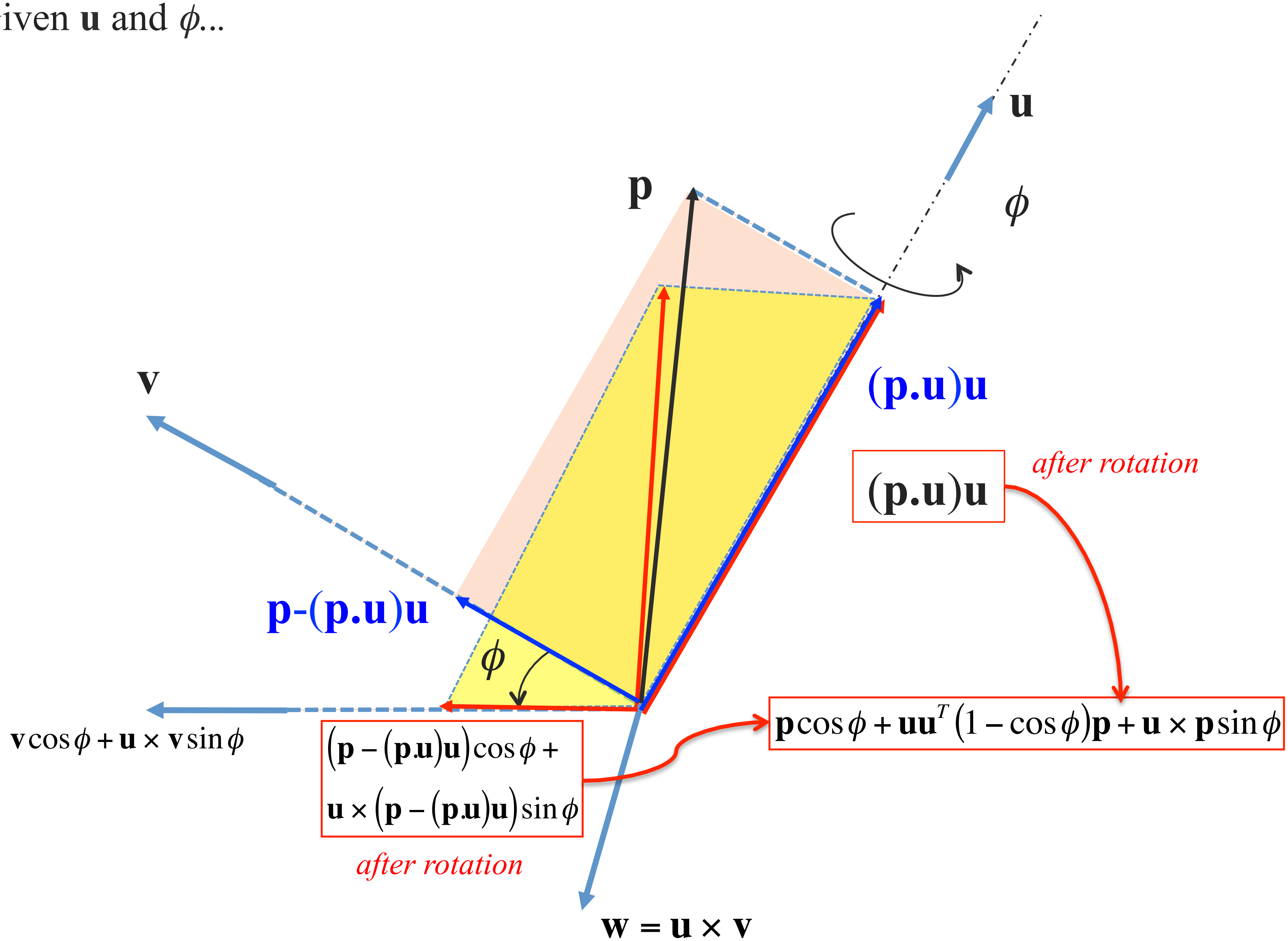
How does one find the rotation matrix for a general axis and angle of rotation?

Note we already know the answer if the axis of rotation is one of the coordinate axes.

Given **u** and ϕ...



1-1 correspondence between any 3×1 vector and a 3×3 skew symmetric matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a

linear operator

For any vector **b**

$$\mathbf{a} \times \mathbf{b} = \mathbf{A}_{3x3}\, \mathbf{b}$$

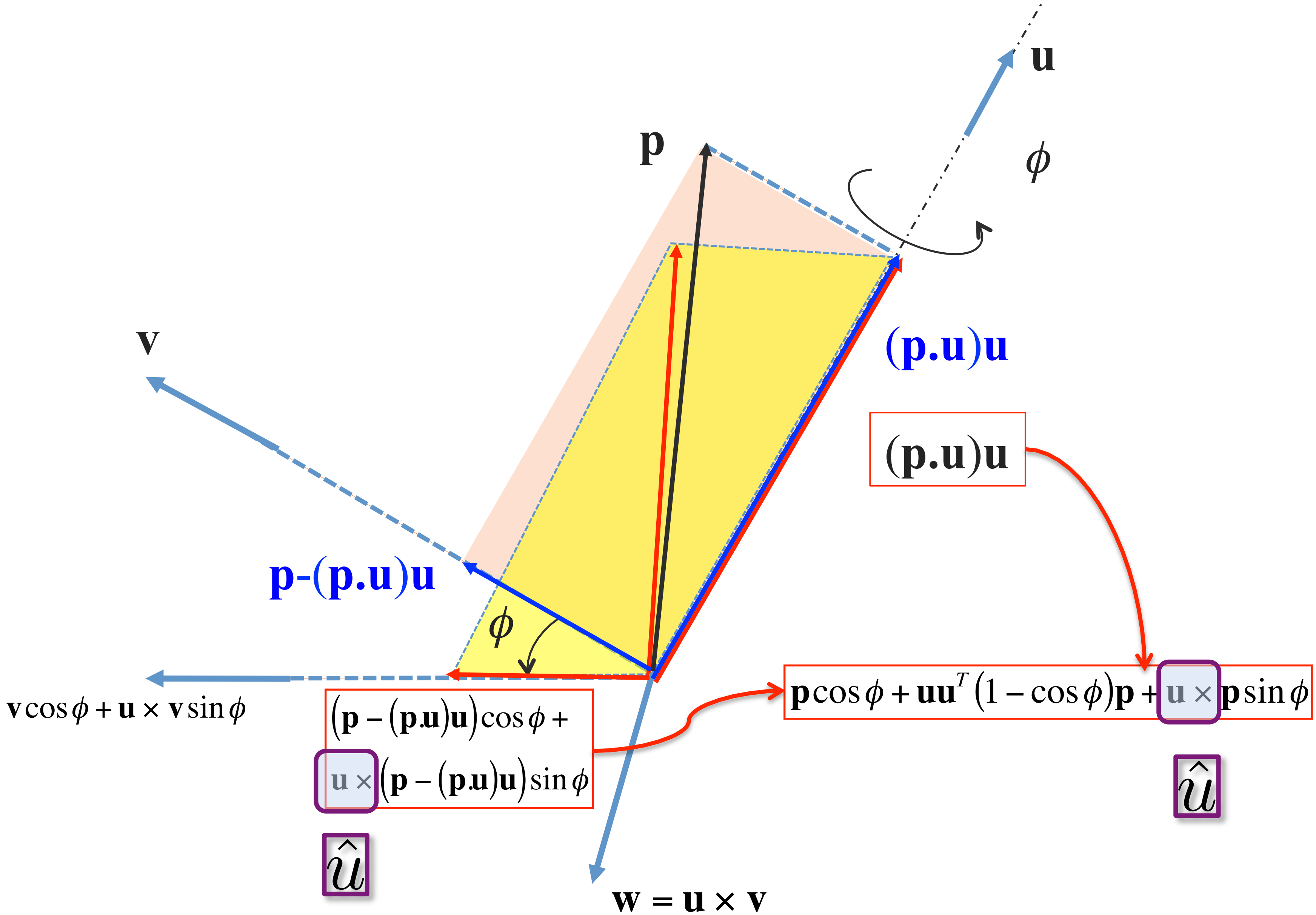
Notation

$\mathbf{A}$

$\mathbf{a}$^

$\hat{\mathbf{a}}$

Given **u** and ϕ...



Axis/Angle to Rotation Matrix

Rotation of a generic vector p about u through ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Axis of rotation	u
Rotation angle	ϕ

Rodrigues' formula

$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

Image from wikipedia

1. Set u to be a unit vector along x (or y or z). Verify result is the same as $Rot(x, \phi)$.

2. Is the (axis, angle) to rotation matrix map *onto*? 1-1?

Euler's theorem

$Rot(u,\phi)$ and $Rot(-u, 2\pi-\phi)$?

restrict ϕ to the interval $[0,\pi]$?

Axis/Angle to Rotation Matrix

Rotation of a generic vector p about u through ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Axis of rotation	u
Rotation angle	ϕ

Rodrigues' formula

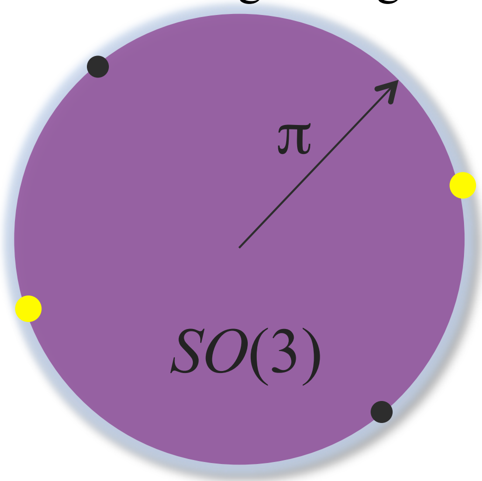
$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

Lets extract the axis and the angle from the rotation matrix, R

Verify

$$\cos\phi = \frac{\tau - 1}{2} \qquad \hat{u} = \frac{1}{2\sin\phi}(R - R^T)$$

(u, without solving for eigenvector)



1. (axis, angle) to rotation matrix map is many to 1

2. restricting angle to the interval [0,π] makes it 1-1 except for

$\tau = 3 \Rightarrow \phi = 0 \Rightarrow$ *no unique axis*

$\tau = -1 \Rightarrow \phi = \pi \Rightarrow$ u or $-u$

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Rotations and Angular Velocities

Time Derivatives of Rotations

Rotation matrix

$$R(t)$$

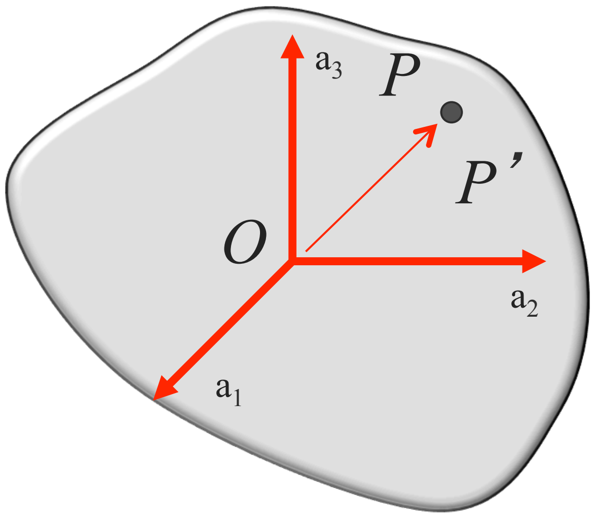
Orthogonality

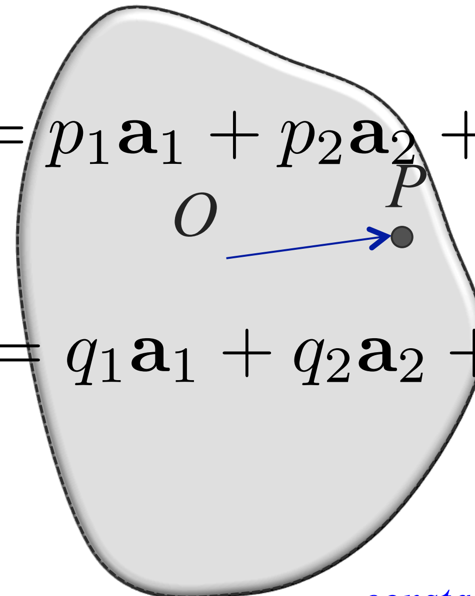
$$R^T(t)R(t) = I \qquad \xrightarrow{\frac{d}{dt}(\,.\,)} \qquad \dot{R}^T R + R^T \dot{R} = 0$$

$$R(t)R^T(t) = I \qquad \qquad R\dot{R}^T + \dot{R}R^T = 0$$

$R^T \dot{R}$ and $\dot{R}R^T$ are skew symmetric

Rotation with O fixed





$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$

$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

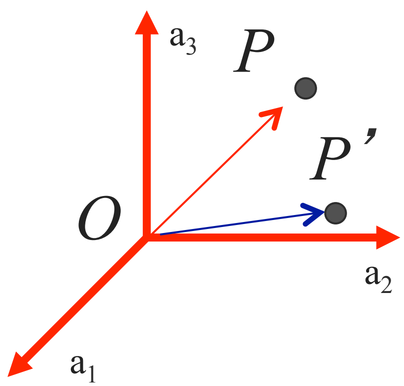
$$q \quad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad p$$

constant coordinates of P in body-fixed frame

$$\boxed{q(t) = R(t)p}$$

changing coordinates of P as the rigid body rotates

Rotation with O fixed



$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\boxed{q(t) = R(t)p}$$

$$R^T \dot{q} = \boxed{R^T \dot{R}} p \qquad \hat{\omega}^b$$

velocity in body-fixed frame

encodes angular velocity in body-fixed frame

$$\dot{q} = \boxed{\dot{R} R^T} q$$

velocity in inertial frame

encodes angular velocity in inertial frame

$$\hat{\omega}^s$$

$$\dot{q} = \dot{R} p$$

velocity in inertial frame

position in body-fixed frame

Exercise

What is the angular velocity for a rotation about the *z* axis?

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dot{R} = \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

Angular velocity for a rotation about the *z*-axis

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T \dot{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$= \dot{R} R^T = \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \boxed{\widehat{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \dot{\theta}}$$

Two Rotations

$$R = R_z(\theta) R_x(\phi)$$

$$\begin{aligned}\hat{\omega}^b = R^T \dot{R} &= (R_z R_x)^T (\dot{R}_z R_x + R_z \dot{R}_x) \\ &= R_x^T R_z^T \dot{R}_z R_x + R_x^T \dot{R}_x\end{aligned}$$

$$\begin{aligned}\hat{\omega}^s = \dot{R} R^T &= (\dot{R}_z R_x + R_z \dot{R}_x)(R_z R_x)^T \\ &= \dot{R}_z R_z^T + R_z \dot{R}_x R_x^T R_z^T\end{aligned}$$

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