

Implied Volatility and Volatility Surfaces

A Comprehensive Course Note (English)

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Focus: equity/FX-style volatility quoting, smiles/skews, surface construction, no-arbitrage constraints, and links to local volatility and risk-neutral density.

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1 Notation and Setup

We work under a frictionless market with deterministic interest rate r (for simplicity) and a non-dividend-paying underlying S_t unless stated otherwise. Let T be maturity, K strike, and $\tau = T - t$ time-to-maturity.

Forward price. For maturity T , the (time- t) forward is

$$F_{t,T} = S_t e^{(r-q)\tau},$$

where q is the continuous dividend yield (or foreign rate in FX). In what follows, we often use F as the natural reference level for moneyness.

Discount factor. With deterministic r , the discount factor is

$$D(t, T) = e^{-r\tau}.$$

2 Black–Scholes Price and the Definition of Implied Volatility

2.1 Black–Scholes call/put formulas

Under the Black–Scholes model with constant volatility σ , the time- t price of a European call is

$$C^{BS}(t; S_t, K, T, \sigma) = D(t, T) \left(F_{t,T} \Phi(d_1) - K \Phi(d_2) \right),$$

and the put is

$$P^{BS}(t; S_t, K, T, \sigma) = D(t, T) \left(K \Phi(-d_2) - F_{t,T} \Phi(-d_1) \right),$$

where

$$d_1 = \frac{\ln(F_{t,T}/K) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

and Φ is the standard normal CDF.

Put–call parity (forward form).

$$C - P = D(t, T) (F_{t,T} - K).$$

This relation is model-free (no-arbitrage).

2.2 Implied volatility

Market option prices are rarely consistent with a constant volatility. Instead, practitioners quote options by the volatility σ that makes the Black–Scholes price match the observed market price.

Implied volatility. Given an observed market price C^{mkt} for a European call, the *implied volatility* σ_{imp} is the (unique) solution to

$$C^{BS}(t; S_t, K, T, \sigma_{\text{imp}}) = C^{mkt}.$$

Similarly for a put.

Practical insight

Implied volatility is a *re-parameterization* of option prices. It is not the true future realized volatility. It is the volatility that, plugged into Black–Scholes, reproduces the market price.

3 Existence, Uniqueness, and Numerical Computation

3.1 Why the implied volatility exists (in practice)

For standard European options, the Black–Scholes price is increasing in σ (positive Vega), so inversion is well-posed.

Vega (sensitivity to volatility). For a call or put in Black–Scholes:

$$\text{Vega} = \frac{\partial C^{BS}}{\partial \sigma} = D(t, T) F_{t,T} \varphi(d_1) \sqrt{\tau} > 0,$$

where φ is the standard normal PDF. Therefore $C^{BS}(\sigma)$ is strictly increasing in σ .

In practice, data issues can break inversion:

- stale quotes,
- bid–ask noise,
- violations of static arbitrage (e.g., call price decreasing in K),
- too deep OTM quotes with tiny prices (numerical instability).

Always clean quotes before building a surface.

3.2 How we compute implied volatility

Closed-form inversion does not exist for Black–Scholes. We solve a 1D root-finding problem:

$$f(\sigma) = C^{BS}(\sigma) - C^{mkt} = 0.$$

Practical insight

Standard methods.

- **Newton–Raphson:** fast, uses Vega.
- **Bisection:** slow but robust (guaranteed convergence if bracketing).
- **Hybrid methods:** bisection + Newton for stability.

A good initial guess matters, especially for short maturities or extreme strikes.

4 Moneyness, Log-Moneyness, and Volatility Quotes

4.1 Choice of x-axis: strike vs moneyness

Option smiles are clearer when plotted against moneyness.

Log-moneyness (forward).

$$k = \ln \left(\frac{K}{F_{t,T}} \right).$$

ATM-forward corresponds to $k = 0$.

Practical insight

Why forward-moneyness is preferred:

- it removes the drift/discounting effect,
- it makes smiles more comparable across maturities,
- it matches the risk-neutral measure naturally used in pricing.

4.2 Delta conventions (especially FX)

In FX, quotes often come in *delta space* rather than strike space. You convert between delta and strike using the Black–Scholes formulas plus the chosen delta convention (spot delta, forward delta, premium-adjusted delta, etc.).

Delta-to-strike conversion depends on:

- call/put,
- spot vs forward delta,
- whether delta is premium-adjusted,
- domestic/foreign discounting.

Always document the market convention used.

5 Volatility Smile and Skew: Shapes and Intuition

5.1 What is a smile?

A *volatility smile* means implied volatility varies with strike:

$$\sigma_{\text{imp}} = \sigma_{\text{imp}}(K, T).$$

Typically, equity indices show a *left skew*: high implied vols for low strikes (crash protection is expensive).

Smile vs skew.

- **Smile:** U-shaped curve vs K or k .
- **Skew:** monotone slope, often stronger on one side (equity downside).

5.2 Economic interpretation (qualitative)

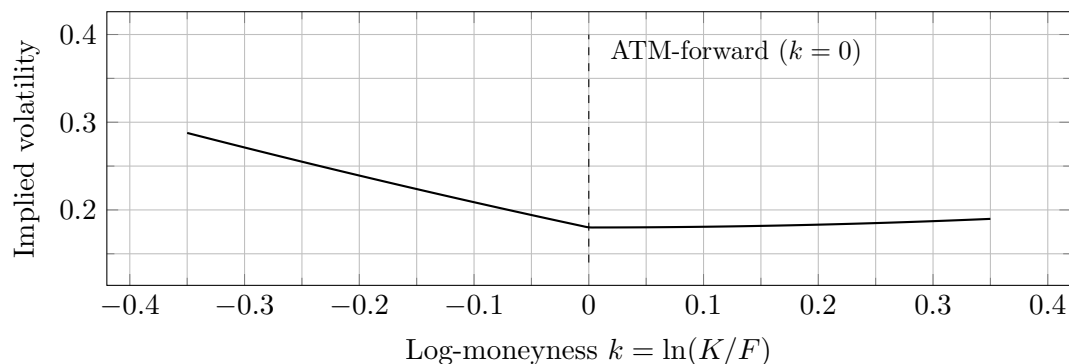
Practical insight

Common drivers behind smile/skew (intuition):

- **Fat tails** and **non-normal returns** under the pricing measure.
- **Leverage effect:** volatility rises when the underlying falls (equities).
- **Jumps:** discontinuous moves produce pronounced smiles.
- **Supply/demand:** persistent demand for puts lifts downside implied vols.

Implied volatility reflects both probabilistic beliefs and risk premia.

5.3 Illustration: a typical equity skew



6 Term Structure of Implied Volatility

Implied volatility changes with maturity. For short maturities, microstructure and jump risk matter more. For long maturities, mean reversion and macro uncertainty dominate.

Term structure. Fix a strike (or moneyness) and view implied vol as a function of maturity:

$$T \mapsto \sigma_{\text{imp}}(K, T).$$

Practical insight

Typical patterns:

- **Event-driven hump:** elevated short maturities around earnings/meetings.
- **Downward short-end:** if near-term uncertainty is low.
- **Upward long-end:** if long-run uncertainty dominates.

7 From Smile to Volatility Surface

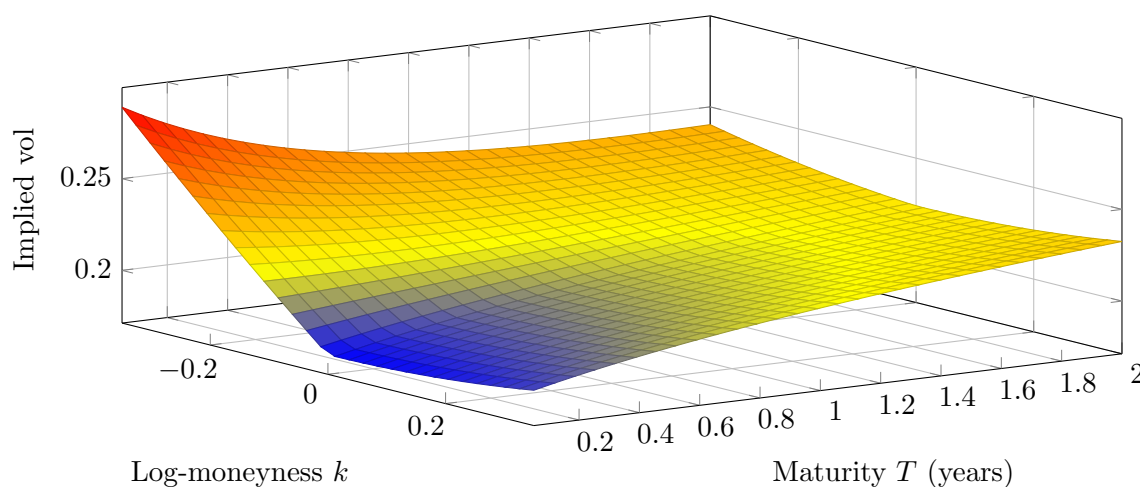
7.1 Definition of a volatility surface

Volatility surface. A volatility surface is a function

$$(K, T) \mapsto \sigma_{\text{imp}}(K, T),$$

(or equivalently $(k, T) \mapsto \sigma_{\text{imp}}(k, T)$) built from market quotes across strikes and maturities.

7.2 Visual illustration: a stylized surface



Practical insight

Reading the surface:

- Fix T : you see a smile/skew slice.
- Fix k : you see the term structure.
- Equity surfaces: downside skew is often strongest at short maturities and gradually flattens.

8 Static No-Arbitrage Constraints for Surfaces

A volatility surface is only useful if it corresponds to a set of option prices that does not violate static arbitrage.

8.1 No-arbitrage in strike (butterfly arbitrage)

For a fixed maturity T , call prices as a function of strike must satisfy:

$$\frac{\partial C}{\partial K} \leq 0, \quad \frac{\partial^2 C}{\partial K^2} \geq 0.$$

The second condition is convexity and prevents butterfly arbitrage.

Risk-neutral density (Breeden–Litzenberger). Under mild regularity,

$$\frac{\partial^2 C}{\partial K^2} = D(t, T) f_{S_T}(K),$$

where $f_{S_T}(K)$ is the risk-neutral density of S_T . Hence convexity ($\partial^2 C / \partial K^2 \geq 0$) is equivalent to a nonnegative density.

8.2 No-arbitrage across maturities (calendar arbitrage)

For a fixed strike K , call prices must be non-decreasing with maturity:

$$C(K, T_2) \geq C(K, T_1) \quad \text{if } T_2 > T_1.$$

A surface may look smooth in implied-vol space but still violate arbitrage in price space. Always check constraints in *price space*, not only in volatility space.

9 Building a Volatility Surface in Practice

9.1 Step-by-step workflow

Practical insight

Pipeline (standard desk approach).

1. Collect quotes (bid/ask) for several maturities and strikes or deltas.
2. Convert everything to a consistent convention:
 - use forward $F_{t,T}$,
 - map delta-quotes to strikes if needed,

- choose call/put consistently and enforce put–call parity.
3. Clean data:
 - remove obvious outliers,
 - ensure monotonicity/convexity (or regularize).
 4. Fit each maturity slice (smile fit):
 - parametric (SVI, polynomial on k with constraints, etc.),
 - or nonparametric (splines with arbitrage control).
 5. Interpolate across maturities:
 - interpolate total variance $w(k, T) = \sigma_{\text{imp}}^2(k, T) T$ rather than σ ,
 - enforce calendar constraints.
 6. Validate:
 - reproduce market quotes within bid/ask,
 - run arbitrage checks,
 - stress-test stability.

9.2 Why interpolate *total variance*?

Total implied variance.

$$w(k, T) = \sigma_{\text{imp}}^2(k, T) T.$$

Practical insight

Interpolating w is often more stable because:

- it behaves more linearly in T ,
- it aligns with diffusion scaling,
- it helps enforce calendar monotonicity (w typically increases with T).

10 A Standard Parametric Smile: SVI (Overview)

10.1 SVI idea

A popular model to fit a single maturity slice is SVI (Stochastic Volatility Inspired). It parameterizes total variance as a function of log-moneyness.

SVI total variance (one common form).

$$w(k) = a + b \left(\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2} \right),$$

with parameters a, b, ρ, m, σ .

Practical insight

SVI is used because it:

- fits many market smiles well,
- is flexible yet compact,
- can be constrained to reduce butterfly arbitrage.

In practice, one fits SVI per maturity and then smooths across maturities.

SVI fitting is not just “least squares”:

- you should weight errors by bid/ask,
- you should fit in price or in implied-vol space consistently,
- you should include arbitrage penalties or constraints.

11 Link with Local Volatility and Dupire’s Formula

Implied volatility surfaces can be transformed into a *local volatility* surface, used to price exotic options via PDE or Monte Carlo.

Local volatility model (concept). Assume under the risk-neutral measure:

$$dS_t = (r - q) S_t dt + \sigma_{\text{loc}}(S_t, t) S_t dW_t.$$

Here $\sigma_{\text{loc}}(S, t)$ depends on both spot and time.

Dupire’s idea (high level). If you know the full surface of European call prices $C(K, T)$ for all strikes and maturities, you can compute $\sigma_{\text{loc}}(K, T)$ from partial derivatives of C (in a suitable form). This is the theoretical bridge: *surface* \Rightarrow *local vol*.

Dupire requires *smooth* $C(K, T)$. Market data is discrete and noisy. If you differentiate noisy surfaces, you amplify noise and can create negative densities or unstable local vol. Regularization is essential.

12 Greeks and the Practical Meaning of Implied Vol Changes

12.1 Vega and “Vol risk”

Implied volatility changes move option prices. For a small change $\Delta\sigma$:

$$\Delta C \approx \text{Vega} \cdot \Delta\sigma.$$

Practical insight

Rule of thumb. For ATM options, Vega is typically large. For deep ITM/OTM options, Vega is smaller (but not always negligible). Short maturities can have low Vega even if implied vol is high.

12.2 Smile risk and higher-order effects

When the whole smile shifts, one needs more than a single Vega. Practitioners talk about:

- **parallel shift** of the surface,
- **skew change** (tilt),
- **curvature change**.

Practical insight

A common approximation is to describe surface moves by a few factors (PCA on implied vols across strikes/maturities). This is widely used in risk systems.

13 Common Market Conventions and Data Issues

13.1 Bid/ask and mid

Market quotes come with bid and ask implied vols (or prices). Using mid is common for marking, but calibration should respect bid/ask.

Never force the surface to hit a bad quote. If one quote is inconsistent with its neighbors, it can create arbitrage or distort the entire fit. Robust fitting and outlier detection are part of real-world surface construction.

13.2 Discrete dividends (equity single names)

With discrete dividends, forward and discounting are more complex. Implied vol depends on the dividend assumption. This can shift the smile and distort the skew if handled incorrectly.

Practical insight

Best practice for equities with dividends:

- build a dividend curve (market implied or forecast),
- compute forwards consistently,
- document the dividend model used for smile construction.

14 Mini-Checklist for a “Good” Volatility Surface

A volatility surface is strong if it is:

- **accurate:** fits liquid quotes within bid/ask,
- **stable:** small quote changes do not cause big surface swings,
- **smooth:** no artificial oscillations,
- **arbitrage-aware:** avoids butterfly and calendar arbitrage,
- **consistent:** respects conventions (forwards, deltas, discounting),
- **useful:** prices and hedges exotics reasonably.

15 Exercises (with guidance)

Exercises

Exercise 1 (Implied vol inversion).

Fix $S_0 = 100$, $r = 2\%$, $q = 0$, $T = 0.5$, $K = 100$, and suppose the market call price is $C^{mkt} = 7.50$.

- Write the equation defining σ_{imp} .
- Explain why the solution is unique.
- Describe a robust algorithm (hybrid bisection/Newton).

Exercise 2 (Skew interpretation).

You observe that $\sigma_{\text{imp}}(k, T)$ increases as k becomes negative (downside). Give two qualitative explanations: one probabilistic and one supply/demand-based.

Exercise 3 (Static arbitrage checks).

For a fixed T , explain the meaning of:

$$\frac{\partial C}{\partial K} \leq 0 \quad \text{and} \quad \frac{\partial^2 C}{\partial K^2} \geq 0.$$

Interpret the second condition via the risk-neutral density.

Exercise 4 (Why total variance).

Explain why one often interpolates $w(k, T) = \sigma_{\text{imp}}^2(k, T) T$ across maturities instead of $\sigma_{\text{imp}}(k, T)$.

16 Summary

Practical insight

Core takeaways.

- Implied volatility is defined by inverting Black–Scholes to match market prices.
- Smiles/skews arise because constant-vol Black–Scholes is inconsistent with observed option prices.
- A volatility surface is the full map $(K, T) \mapsto \sigma_{\text{imp}}(K, T)$.
- Surface construction is a data + modeling problem: clean quotes, fit per slice, interpolate across maturities, and enforce no-arbitrage.
- The surface contains information about the risk-neutral distribution and links to local volatility (Dupire).

End of note.