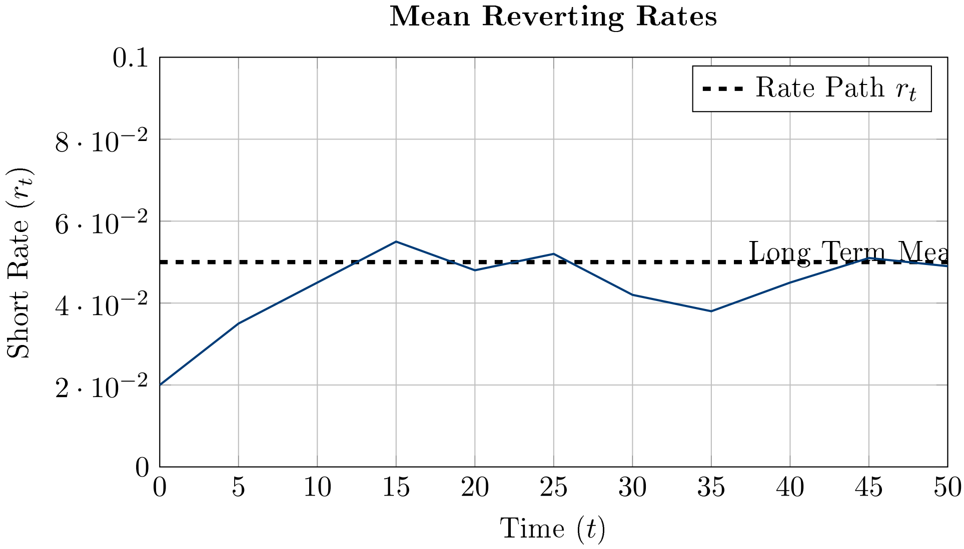
Stochastic Interest Rates

Vasicek, CIR, and Bond Pricing

Lecture 14
M.Sc. Quantitative Finance



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Chapter 1

The Short Rate Framework

1.1 Introduction

So far, we assumed the risk-free rate r was constant. In Fixed Income markets, r is stochastic. We model the instantaneous spot rate, or **Short Rate** r_t. The price of a Zero-Coupon Bond paying \$1 at time T is given by the risk-neutral expectation:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_u du\right) \mid \mathcal{F}_t\right] \tag{1.1}$$

To calculate this, we need an SDE for r_t.

1.2 The Vasicek Model (1977)

Vasicek assumed r_t follows an Ornstein-Uhlenbeck process (Mean-Reverting Gaussian).

Vasicek SDE

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{1.2}$$

- b: Long-term mean level (e.g., 5%).
- a: Speed of mean reversion.
- σ: Volatility of rates.

Properties:

- **Gaussian:** r_T given r_t is normally distributed.
- **Negative Rates:** Since it is Gaussian, r_t can become negative. (Historically considered a flaw, but realistic in post-2008 Europe/Japan).

1.3 The Cox-Ingersoll-Ross (CIR) Model (1985)

To prevent negative interest rates, CIR introduced a square-root diffusion term (similar to Heston).

CIR SDE

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{1.3}$$

Properties:

- **Non-Central Chi-Squared:** The distribution of r_t is no longer Gaussian.
- **Positivity:** If $2ab \geq \sigma^2$ (Feller Condition), rates stay strictly positive.

Chapter 2

Analytical Bond Pricing

Both Vasicek and CIR belong to the class of **Affine Term Structure Models**. This means the bond price has a specific exponential form.

2.1 The Affine Formula

For both models, the price of a Zero-Coupon Bond is:

$$P(t,T) = A(t,T)e^{-B(t,T)r_t} \tag{2.1}$$

This is incredibly powerful: knowing the current rate r_t is enough to generate the entire Yield Curve.

2.2 Vasicek Solution

Let $\tau = T - t$.

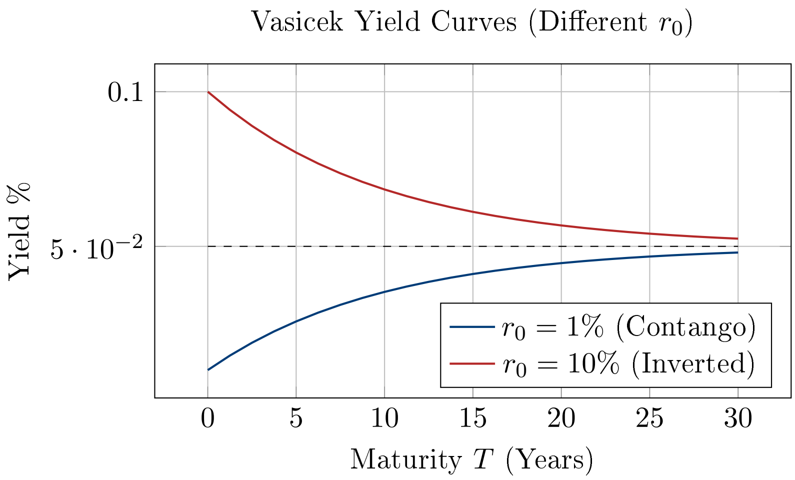
$$B(\tau) = \frac{1 - e^{-a\tau}}{a} \tag{2.2}$$

$$A(\tau) = \exp\left(\left(b - \frac{\sigma^2}{2a^2}\right)(B(\tau) - \tau) - \frac{\sigma^2}{4a}B(\tau)^2\right) \tag{2.3}$$

2.3 The Yield Curve

The Yield $Y(t,T)$ is defined as $-\frac{\ln P(t,T)}{\tau}$.

$$Y(t,T) = \frac{B(\tau)}{\tau}r_t - \frac{\ln A(\tau)}{\tau}$$


Vasicek Yield Curves (Different r_0)
0.1
$5 \cdot 10^{-2}$
Yield %
$r_0 = 1\%$ (Contango)
$r_0 = 10\%$ (Inverted)
0
5
10
15
20
25
30
Maturity T (Years)