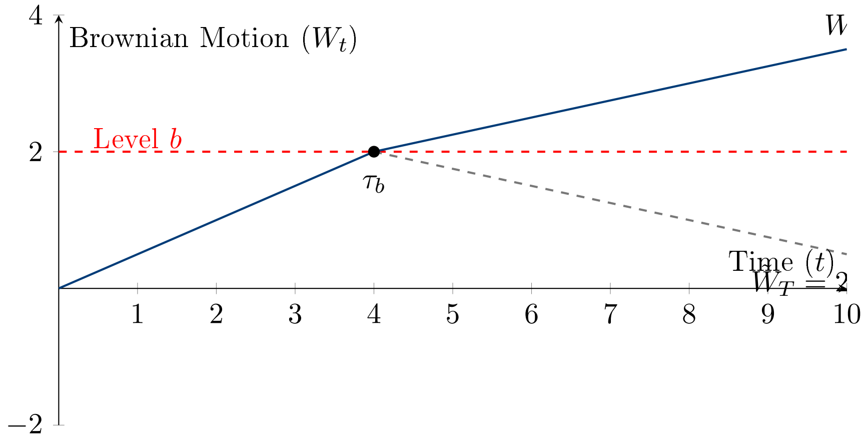
Path-Dependent Derivatives

Barrier Options, Reflection Principle, and Asians

Lecture 12
M.Sc. Quantitative Finance

The Reflection Principle



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Chapter 1

Brownian Motion Extremes

To price Barrier options (which die if S_t hits a level), we need the distribution of the running maximum of a Brownian Motion.

1.1 First Hitting Time

Let W_t be a standard Brownian Motion with $W_0 = 0$. Let $b > 0$. We define the **First Hitting Time** τ_b:

$$\tau_b = \inf\{t > 0 : W_t = b\} \tag{1.1}$$

1.2 The Reflection Principle

This is a fundamental result in stochastic processes. It relies on the symmetry of Brownian motion.

Theorem 1.1 (Reflection Principle). *For any $b > 0$ and $t > 0$:*

$$P(\tau_b \leq t) = 2P(W_t \geq b) = 2\left(1 - N\left(\frac{b}{\sqrt{t}}\right)\right) \tag{1.2}$$

Proof. Consider a path that crosses level b before time t (so $\tau_b \leq t$). At time T, this path is either above b ($W_t \geq b$) or below b ($W_t < b$).

- Case 1: $W_t \geq b$. This clearly implies $\tau_b \leq t$.
- Case 2: $W_t < b$ BUT $\tau_b \leq t$. This means the path went up to b, and then came back down.

By the Strong Markov Property, once the Brownian motion hits b, it "forgets" the past and restarts. Since Brownian motion is symmetric, for every path that continues UP to $x > b$, there is a mirror path that goes DOWN to $2b - x$. Therefore:

$$P(\tau_b \leq t, W_t < b) = P(\tau_b \leq t, W_t > b) = P(W_t > b)$$

Adding the two cases:

$$P(\tau_b \leq t) = P(W_t > b) + P(W_t > b) = 2P(W_t > b)$$

□

1.3 Distribution of the Maximum

Let $M_t = \max_{0\leq s\leq t} W_s$. The event $\{M_t \geq b\}$ is identical to $\{\tau_b \leq t\}$. Thus, the running maximum has the same distribution as $|W_t|$.

Chapter 2

Barrier Option Pricing

2.1 The Down-and-Out Call

Consider a Call option with strike K and barrier $B < S_0$. The option pays $\max(S_T - K, 0)$ if and only if $\min_{0 \le t \le T} S_t > B$.

2.2 Derivation Strategy

We model S_t as a Geometric Brownian Motion:

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}$$

The condition $S_t > B$ is equivalent to a condition on the arithmetic Brownian motion with drift. Let $X_t = \ln(S_t/B)$. The barrier for X_t is 0. The problem reduces to calculating the probability that a **Drifted Brownian Motion** stays positive.

2.3 The Analytical Formula

Using the Girsanov theorem to handle the drift in the reflection principle, we obtain the closed-form solution:

Down-and-Out Call Formula

Let $\lambda = \frac{r-\sigma^2/2}{\sigma^2}$. Let $y = \ln(B^2/(S_0 K))$.

$$C_{DO} = C_{BS}(S_0, K) - \left(\frac{S_0}{B}\right)^{2\lambda+2} C_{BS}\left(\frac{B^2}{S_0}, K\right) \tag{2.1}$$

Interpretation:

- The first term is the price of a standard European Call (ignoring the barrier).
- The second term is the value of the "Knock-Out" clause. It represents the probability-weighted value of the paths that hit B and would have ended ITM.
- $\left(\frac{S_0}{B}\right)^{2\lambda+2}$ is the discount factor related to the distance from the barrier.

Chapter 3

Asian Options

3.1 Geometric Asian Options (Analytic)

The payoff depends on the geometric mean $G_T = \exp\left(\frac{1}{T}\int_0^T \ln S_t dt\right)$.

Proposition 3.1 (Variance Reduction)**.** *The geometric mean of a GBM is also a Log-Normal variable, but with reduced volatility. The effective volatility σ_G is:*

$$\sigma_G = \frac{\sigma}{\sqrt{3}} \tag{3.1}$$

Proof Sketch: $\ln S_t = \ln S_0 + \mu t + \sigma W_t$. The integral $\int_0^T W_t dt$ is a Gaussian variable. The variance of the average is the average of the covariances.

$$\text{Var}\left(\frac{1}{T}\int_0^T W_t dt\right) = \frac{1}{T^2}\int_0^T \int_0^T \min(s,t) ds dt = \frac{T}{3}$$

Thus, we can price Geometric Asians using the Black-Scholes formula with $\sigma_{eff} = \sigma/\sqrt{3}$ and adjusted drift $r_{eff} = \frac{1}{2}(r - \sigma^2/6)$.

3.2 Arithmetic Asian Options (Numerical)

The payoff depends on $A_T = \frac{1}{T}\int_0^T S_t dt$. Since the sum of Log-Normals has no closed-form PDF, we rely on Monte Carlo.

3.3 Control Variates Technique

Standard Monte Carlo is slow. We use the Geometric Asian (whose price V_G is known exactly) to reduce the variance of the Arithmetic estimator.

Algorithm 1 Asian Pricing with Control Variate

Input: S_0, K, r, T, σ, M
Calculate exact price V_{Geom}^{Exact} using BS formula with $\sigma/\sqrt{3}$.
Initialize sums for V_{Arith} and V_{Geom}.
for $i = 1$ to M **do**
  Simulate path $S_0, \dots, S_T$
  Calculate Arithmetic Mean A_T and Payoff $P_A = (A_T - K)^+$
  Calculate Geometric Mean G_T and Payoff $P_G = (G_T - K)^+$
  Store pair (P_A, P_G)
end for
Estimate correlation $\beta = \frac{\mathrm{Cov}(P_A, P_G)}{\mathrm{Var}(P_G)}$
Result: $\hat{V}_{CV} = \hat{V}_{Arith}^{MC} - \beta(\hat{V}_{Geom}^{MC} - V_{Geom}^{Exact})$