Project 01 — Brownian Motion & Geometric Brownian Motion

From continuous-time modeling to a first option-pricing bridge

Quant-Finance Portfolio (Karim)

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1 Why this project

In quantitative finance, most continuous-time models start with *Brownian motion*. This project sets the mathematical foundations (definitions + key properties), then connects them to a first financial model: *Geometric Brownian Motion (GBM)*, which leads naturally to Black–Scholes option pricing.

Notebook link. The accompanying Jupyter notebook contains: Monte Carlo simulations, interactive Plotly charts, and numerical experiments (distribution checks, QQ-plots, convergence). This report is the *written explanation*.

2 Brownian motion (BM): definition and key facts

A standard Brownian motion $(W_t)_{t\geq 0}$ is a stochastic process such that:

- $W_0 = 0$.
- **Independent increments:** for $0 \leq s < t$, $W_t - W_s$ is independent of the past.
- **Gaussian increments:** $W_t - W_s \sim \mathcal{N}(0, t - s)$.
- **Continuous paths:** $t \mapsto W_t$ is almost surely continuous.

2.1 Two immediate consequences

$$\mathbb{E}[W_t] = 0, \tag{1}$$
$$\mathrm{Var}(W_t) = t \quad \text{(variance grows linearly in time)}. \tag{2}$$

2.2 What we test numerically in the notebook

On simulated increments ΔW:

- Histogram vs $\mathcal{N}(0, \Delta t)$,
- QQ-plot vs Normal,
- Scaling: $\mathrm{Var}(W_t) \approx t$,
- Independence (low autocorrelation of increments).

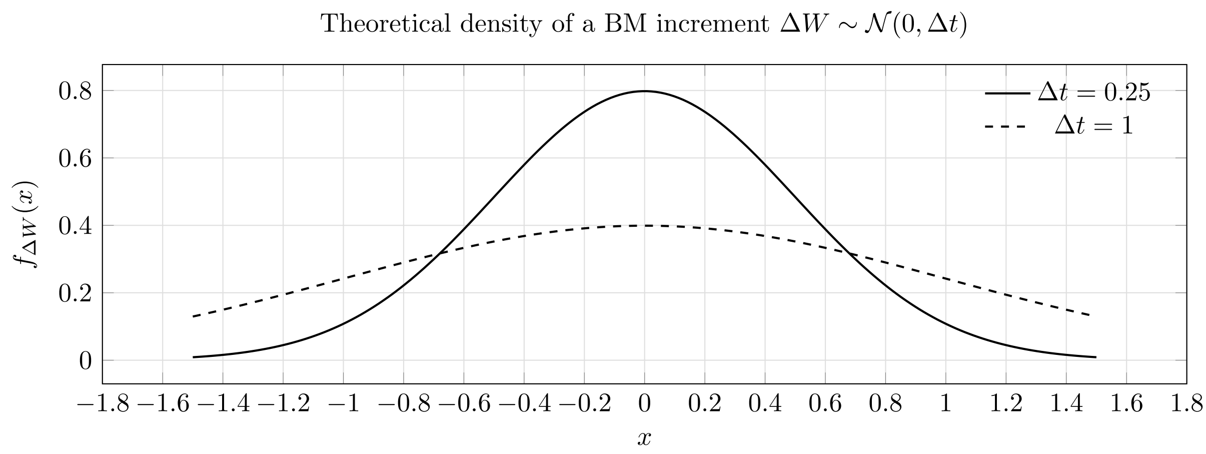

Figure 1: As Δt increases, the distribution spreads: $\mathrm{Var}(\Delta W) = \Delta t$.

3 BM increments: a Normal distribution picture

Below we plot the Normal density of a BM increment $\Delta W \sim \mathcal{N}(0, \Delta t)$. (We draw the *theoretical* density in LaTeX to keep the report light and reproducible on Overleaf.)

4 Geometric Brownian Motion (GBM): the first asset model

In Black–Scholes, the asset price follows:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \tag{3}$$

where μ is the drift and σ the volatility.

4.1 Closed-form solution and distribution

The solution is

$$S_t = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t \Big). \tag{4}$$

Hence $\log(S_t)$ is Normal and S_t is Lognormal:

$$\log(S_t) \sim \mathcal{N}\Big(\log S_0 + (\mu - \tfrac{1}{2}\sigma^2)t, \ \sigma^2 t \Big), \tag{5}$$

$$S_t \sim \mathrm{Lognormal}(\cdot). \tag{6}$$

4.2 Moments

$$\mathbb{E}[S_t] = S_0 e^{\mu t}, \tag{7}$$

$$\mathrm{Var}(S_t) = S_0^2 e^{2\mu t} \big(e^{\sigma^2 t} - 1 \big). \tag{8}$$

5 GBM: fan chart (theoretical quantiles)

Instead of simulating many paths inside LaTeX (slow), we plot a *fan chart* using the analytical quantiles of the Lognormal distribution:

$$Q_p(t) = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma \sqrt{t} \, \Phi^{-1}(p) \Big),$$

where Φ^{-1} is the standard Normal quantile.

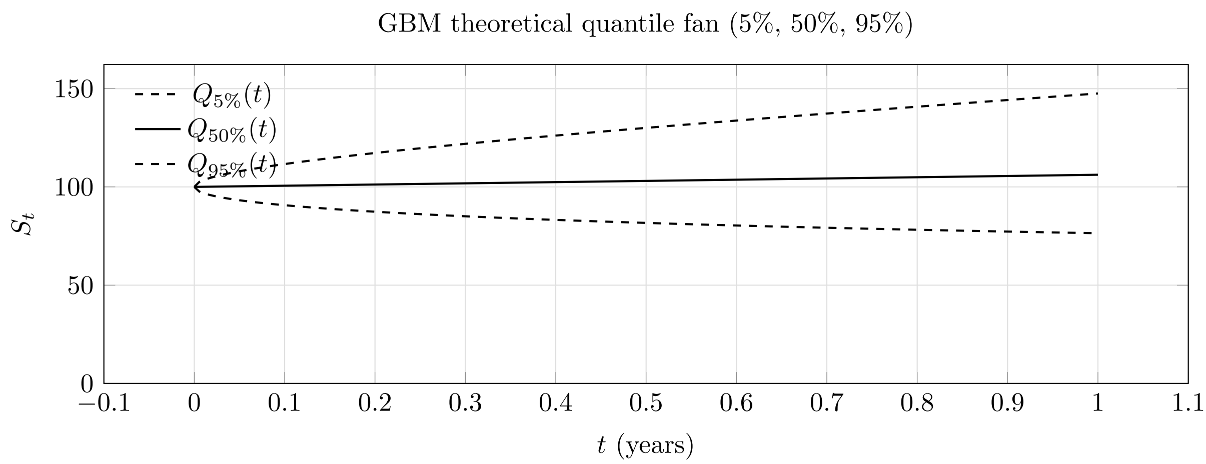

Figure 2: Fast-to-compile alternative to Monte Carlo path plots: the distribution fan is analytical.

6 A first bridge to option pricing (Black–Scholes)

Normal distribution notation. Let $Z \sim \mathcal{N}(0,1)$ be a standard Normal random variable. We denote by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

the *probability density function* (pdf) of the standard Normal distribution, and by

$$\Phi(x) = \mathbb{P}(Z \le x) = \int_{-\infty}^{x} \varphi(t)\, dt$$

its *cumulative distribution function* (cdf). In particular, Φ^{-1} denotes the associated quantile function.

For a European call with strike K and maturity T, Black–Scholes gives

$$C(S_0, K, T, r, \sigma) = S_0\, \Phi(d_1) - K e^{-rT}\, \Phi(d_2), \tag{9}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{10}$$

6.1 Delta (hedging intuition)

The Delta of the call is

$$\Delta = \frac{\partial C}{\partial S_0} = \Phi(d_1).$$

Delta is the first-order hedge ratio: in discrete time, the replication error decreases as we rebalance more often (explored in later projects).

7 What to say in interviews (one paragraph)

I built intuition and numerical checks around Brownian motion (Normal increments, scaling of variance, independence), then used GBM as a first asset model to connect theory to practice: the lognormal distribution, analytical moments, and the Black–Scholes bridge (pricing + delta). This creates a clean foundation for later projects: Monte Carlo pricing, discrete hedging error, VaR/CVaR, stochastic volatility (GARCH/Heston), and volatility surfaces.

8 Appendix: parameters used in this report

Symbol	Meaning	Example value (editable)
S_0	initial price	100
μ	drift	0.08
σ	volatility	0.20
T	maturity (years)	1
r	risk-free rate	0.03

Table 1: These parameters are illustrative. The notebook explores multiple scenarios.