

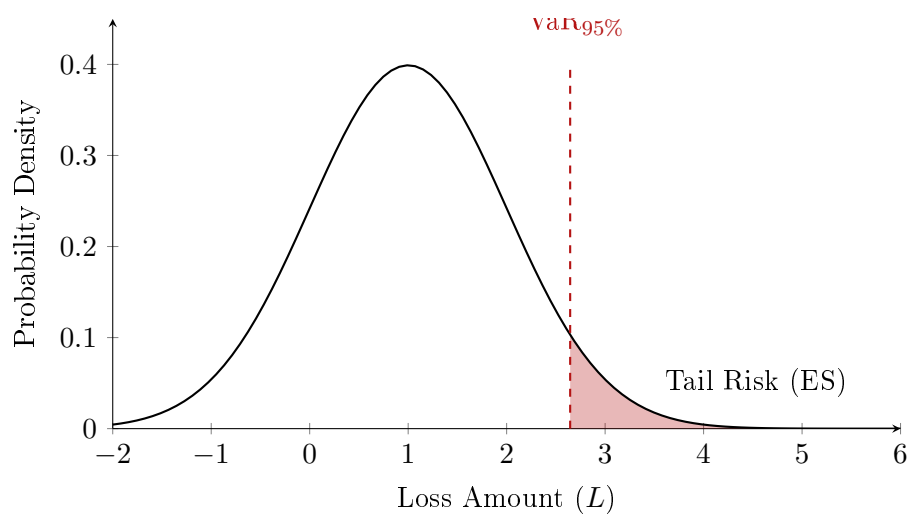
Market Risk Management

Value at Risk (VaR) and Expected Shortfall

Lecture 13

M.Sc. Quantitative Finance

Visualizing Risk Measures (95%)



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Chapter 1

Value at Risk (VaR)

1.1 Definition

Value at Risk is the standard metric for quantifying market risk. It answers the question: *"What is the maximum loss I can expect with $\alpha\%$ confidence over a horizon T ?"*

Let L be the random variable representing the **Loss** of the portfolio over time T . (Note: Positive L means we lost money).

Definition 1.1 (Value at Risk). *Given a confidence level $\alpha \in (0, 1)$ (typically 95% or 99%), the VaR is the smallest number l such that the probability of the loss exceeding l is no larger than $(1 - \alpha)$.*

$$VaR_\alpha(L) = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} \quad (1.1)$$

Mathematically, it is simply the $(1 - \alpha)$ -quantile of the loss distribution.

1.2 Calculation Methods

1.2.1 1. Parametric VaR (Variance-Covariance)

Assume returns follow a Normal distribution $R \sim \mathcal{N}(\mu, \sigma^2)$.

$$VaR_\alpha = \text{Portfolio Value} \times (\sigma \cdot z_\alpha - \mu)$$

where z_α is the normal quantile (e.g., 1.645 for 95%, 2.33 for 99%). *Pros:* Instant calculation. *Cons:* Assumes normality (underestimates fat tails).

1.2.2 2. Historical Simulation

Take the last 500 days of historical returns. Apply them to today's portfolio. The VaR is simply the 5th percentile worst outcome. *Pros:* No assumption on distribution. *Cons:* Assumes the past predicts the future.

1.2.3 3. Monte Carlo VaR

Simulate 10,000 paths using Heston or Jump-Diffusion models. Calculate the portfolio value for each. Sort the losses. *Pros:* Captures non-linearities (Options). *Cons:* Computationally expensive.

Chapter 2

Coherent Risk Measures

2.1 The Failure of VaR

VaR is not a "perfect" risk measure because it fails the property of **Sub-additivity**.

Theorem 2.1 (Sub-additivity). *A risk measure ρ is sub-additive if:*

$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$

Meaning: "Diversification should reduce risk."

For certain non-normal distributions, $\text{VaR}(A + B) > \text{VaR}(A) + \text{VaR}(B)$. This implies that merging two portfolios creates *more* risk, which encourages splitting banks into tiny pieces to hide risk.

2.2 Expected Shortfall (ES)

To fix this, Artzner et al. (1999) proposed Expected Shortfall (also called CVaR or TVaR).

Definition 2.1 (Expected Shortfall). *ES is the expected loss **given that** the loss exceeds the VaR.*

$$ES_{\alpha}(L) = \mathbb{E}[L \mid L \geq \text{VaR}_{\alpha}(L)] \quad (2.1)$$

Why Basel III moved to ES

VaR tells you "We are safe 99% of the time." ES tells you "If the 1% crisis happens, we will lose \$5 Billion." ES captures the "Tail Risk" (the severity of the crash), whereas VaR is blind to anything beyond the threshold.

