Project 04 — VaR & CVaR (Historical + Monte Carlo)

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December 19, 2025

Abstract

This document supports **Project 04: VaR & CVaR**. It defines **Value-at-Risk** and **Conditional Value-at-Risk (Expected Shortfall)** under a consistent loss convention, implements **historical**, **parametric (Normal)** and **Monte Carlo** estimators (including correlated multi-asset portfolios), and illustrates **tail risk**, **model limitations**, and **stress testing**. It is designed to compile quickly on Overleaf and to live in a GitHub repo as a readable report.

Contents

1 What you build in Project 04

You implement an end-to-end risk module for a portfolio:

- a **data pipeline** (prices $\to$ log-returns, with a robust synthetic fallback),
- **portfolio P&L** construction from user-defined weights and horizon,
- **historical VaR/CVaR** based on empirical quantiles,
- **parametric Normal VaR/CVaR** (fast baseline + clear limitations),
- **Monte Carlo VaR/CVaR** for single-asset or multi-asset portfolios (correlated simulation),
- **stress tests** (volatility shock, correlation-to-one),
- interactive visual outputs (distribution, tail zoom, dashboard-style controls).

2 Prerequisites (math you must know)

2.1 Quantiles and tail probabilities

Let X be a real random variable and $\alpha \in (0,1)$. The (lower) α-quantile is

$$q_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge \alpha\}.$$

In risk management, we typically care about **high confidence** levels, e.g. $\alpha = 99\%$.

2.2 Expectation and conditional expectation

We use:

$$\mathbb{E}[X], \qquad \mathbb{E}[X \mid X \le c] \quad \text{(average of outcomes in a tail).}$$

CVaR is essentially a conditional tail expectation (under a loss convention).

2.3 Covariance matrices and correlation

For a vector of returns $R \in \mathbb{R}^d$, the covariance matrix is $\Sigma = \mathrm{Cov}(R)$. To simulate correlated Gaussian vectors, a standard approach is a **Cholesky factorization**

$$\Sigma = LL^\top, \qquad Z \sim \mathcal{N}(0, I_d), \qquad R = \mu + LZ.$$

2.4 Why fat tails matter (Normal vs Student-t)

Empirical returns often exhibit **excess kurtosis** (fat tails). A Student-t model is a simple way to increase tail risk while keeping tractability.

3 Definitions: VaR and CVaR under a consistent loss convention

To avoid sign confusion, we fix a **loss convention**:

Definition (loss). Let L denote the loss over a given horizon:

$$L = -\text{P\&L}.$$

Large positive L means a large loss.

3.1 Value-at-Risk (VaR)

At confidence level α (e.g. 0.99), VaR is the α-quantile of the loss distribution:

$$\mathrm{VaR}_\alpha(L) = q_\alpha(L).$$

Interpretation: **with probability α, losses do not exceed** VaR_α.

3.2 Conditional Value-at-Risk (CVaR / Expected Shortfall)

CVaR measures the **average loss in the tail beyond VaR**:

$$\mathrm{CVaR}_\alpha(L) = \mathbb{E}[L \mid L \geq \mathrm{VaR}_\alpha(L)].$$

CVaR is more sensitive to extreme losses than VaR and is widely used as a tail-risk metric.

Important note. VaR is not always subadditive (may fail diversification in some cases), whereas CVaR is a *coherent* risk measure under standard conditions (convexity/subadditivity).

4 Data pipeline and portfolio P&L

4.1 From prices to log-returns

Given prices (P_t), log-returns are

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right).$$

For a portfolio with weights $w \in \mathbb{R}^d$ (normalized so $\sum_i w_i = 1$), the portfolio return is

$$r_t^{(p)} = w^\top r_t.$$

Over a horizon of h days, you can aggregate returns (e.g. sum of log-returns) and convert to P&L.

4.2 P&L and loss series

Once a return series is built, define P&L (in relative units) and convert to losses:

$$\mathrm{P\&L} = r^{(p)}, \qquad L = -\mathrm{P\&L}.$$

All VaR/CVaR computations below are applied to L.

5 Historical VaR/CVaR (non-parametric baseline)

Historical risk uses the empirical distribution of observed losses.

5.1 Empirical VaR

Given losses $(L_1, \dots, L_n)$, let $L_{(1)} \leq \dots \leq L_{(n)}$ be the sorted sample. A common estimator is:

$$\widehat{\mathrm{VaR}}_\alpha = L_{(\lceil \alpha n \rceil)}.$$

5.2 Empirical CVaR

A natural estimator averages the tail beyond empirical VaR:

$$\widehat{\mathrm{CVaR}}_\alpha = \frac{1}{\#\{i : L_i \geq \widehat{\mathrm{VaR}}_\alpha\}} \sum_{i : L_i \geq \widehat{\mathrm{VaR}}_\alpha} L_i.$$

Pros/cons.

- **Pros:** model-free, easy to explain and implement.
- **Cons:** depends heavily on the sample window; tail estimates can be noisy; regime changes are not captured unless the window adapts.

6 Parametric Normal VaR/CVaR (fast baseline)

Assume losses are approximately Gaussian:

$$L \sim \mathcal{N}(\mu_L, \sigma_L^2).$$

Let z_α be the standard normal α-quantile: $\mathbb{P}(Z \le z_\alpha) = \alpha$.

6.1 Closed-form VaR (Normal)

$$\mathrm{VaR}_\alpha(L) = \mu_L + \sigma_L z_\alpha.$$

6.2 Closed-form CVaR (Normal)

For a Normal distribution, CVaR has a closed form using the standard normal density φ:

$$\mathrm{CVaR}_\alpha(L) = \mu_L + \sigma_L \frac{\varphi(z_\alpha)}{1-\alpha}.$$

Key limitation. Normal tails are typically too thin for financial returns; this can **underestimate tail risk**, especially during stressed periods.

7 Monte Carlo VaR/CVaR (flexible and portfolio-ready)

Monte Carlo (MC) simulates many scenarios for returns (and thus losses) and computes VaR/CVaR on the simulated distribution.

7.1 Single-asset MC (intuition)

Simulate N outcomes $L^{(1)}, \ldots, L^{(N)}$ and apply the same empirical estimators:

$$\widehat{\mathrm{VaR}}_\alpha^{MC} = \mathrm{quantile}_\alpha(L^{(1:N)}), \qquad \widehat{\mathrm{CVaR}}_\alpha^{MC} = \text{average of tail beyond } \widehat{\mathrm{VaR}}_\alpha^{MC}.$$

7.2 Multi-asset correlated MC (core quantitative point)

Let $R \in \mathbb{R}^d$ be the vector of asset returns over the horizon. Under a Gaussian model:

$$R = \mu + LZ, \qquad Z \sim \mathcal{N}(0, I_d), \qquad \Sigma = LL^\top.$$

Portfolio return is $r^{(p)} = w^\top R$, then loss is $L^{(p)} = -r^{(p)}$. Repeating this generates an empirical loss distribution for the portfolio.

7.3 Student-t MC (fat-tail extension)

A simple fat-tail alternative is to simulate R with a multivariate Student-t structure, increasing extreme outcomes while keeping correlation modeling.

Practical note. MC outputs are only as good as the input model (drift/vol/correlation). This is why **stress tests** are included.

8 Stress testing (simple but effective)

Stress tests answer: *what happens to VaR/CVaR if volatility or correlation suddenly increases?*

8.1 Volatility shock

Scale volatilities by a factor $k > 1$:

$$\sigma \leftarrow k\sigma.$$

This increases dispersion and typically increases both VaR and CVaR.

8.2 Correlation-to-one

A worst-case diversification stress sets correlations to 1 (or near 1), effectively removing diversification benefits. This usually increases portfolio tail risk sharply.

9 Implementation notes (what the notebook is doing)

The implementation is organized as small, testable functions:

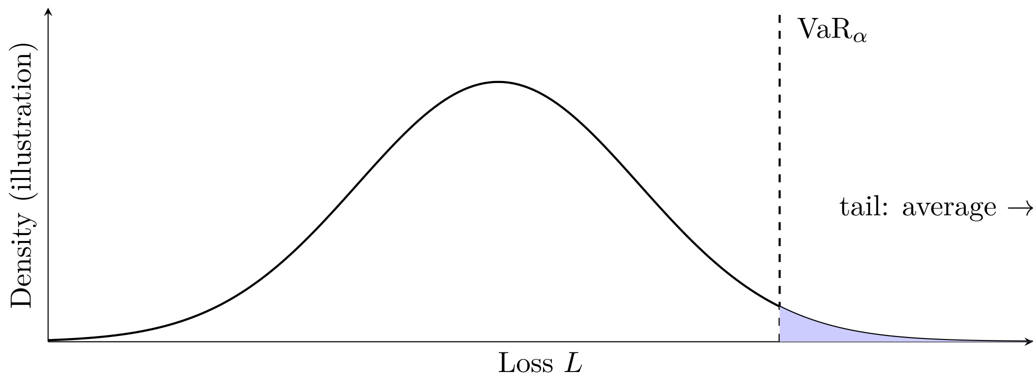
- **Data:** download prices, compute log-returns; fallback to synthetic returns if needed.
- **Portfolio:** normalize weights and aggregate returns to portfolio P&L.
- **Core risk:** a single function computing $(\mathrm{VaR}_\alpha, \mathrm{CVaR}_\alpha)$ from a loss array.
- **MC engine:** simulate correlated scenarios (Normal and Student-t variants), then compute risk on the simulated losses.
- **Stress:** override volatility and/or correlation assumptions and re-run MC.
- **Outputs:** distribution plots, tail zoom, and dashboard-style controls for quick exploration.

10 Sanity checks you should always do

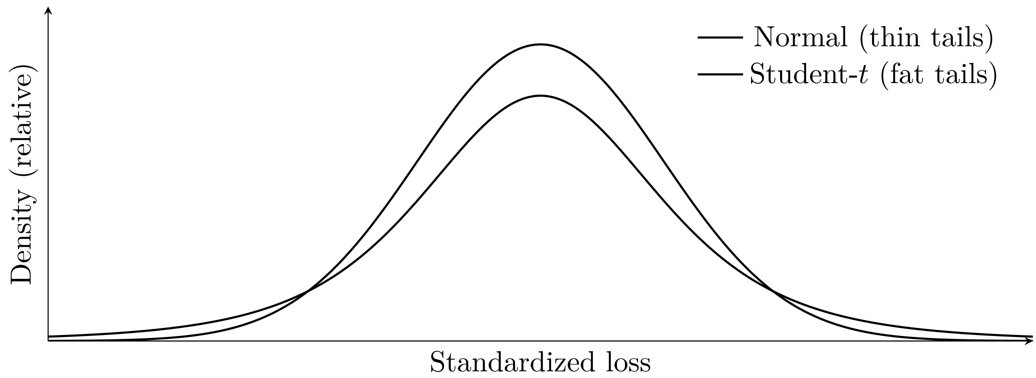
- **Sign convention:** confirm $L = -\mathrm{P\&L}$ everywhere (avoid accidental sign flips).
- **Ordering:** CVaR should be $\geq$ VaR (same α and same loss convention).
- **Confidence level:** VaR/CVaR should increase as α increases (e.g. 95% $\to$ 99%).
- **Window sensitivity:** historical estimates change with the lookback period; report it.
- **Diversification:** increasing correlations should increase risk; correlation-to-one should be a worst case.
- **Fat tails:** Student-t simulations should yield larger tail risk than Normal (all else equal).

11 Overleaf plots

11.1 VaR and CVaR on a loss distribution (illustration)



11.2 Normal vs fat tails (illustration)



12 Interview pitch

I implemented VaR and CVaR for portfolios using three approaches: historical quantiles, parametric Normal formulas, and a Monte Carlo engine. The Monte Carlo version supports correlated multi-asset simulation via covariance/Cholesky and includes a fat-tail Student-t option. I added stress tests (volatility shocks and correlation-to-one) and interactive visuals to inspect the loss distribution and the tail.

VaR gives a threshold, but CVaR tells the average loss beyond that threshold, so it captures tail severity and behaves better under diversification.