Step 1: The Mathematical Engine (Brownian Motion & Ito's Lemma)

1 Introduction

Before discussing option pricing, we must define how a stock price moves. Unlike classical functions in physics (smooth and predictable), a stock follows a random and erratic trajectory.

We will cover two fundamental concepts:

1. The Brownian Motion (W_t).

2. Ito's Lemma (the keystone of stochastic calculus).

2 Brownian Motion (W_t)

Imagine a pollen particle trembling in a glass of water. In finance, this represents the source of uncertainty (market "noise").

The 3 properties to memorize:

- $W_0 = 0$ (It starts at zero).

- Its increments are independent (The past does not predict the future).

- The increment dW_t (movement over a very short time dt) follows a Normal distribution:

$$dW_t \sim \mathcal{N}(0, dt)$$

The Golden Rule (Crucial): In classical calculus, $(dx)^2$ is negligible (close to 0). In stochastic calculus, due to infinite volatility in the short term, we have:

$$\boxed{(dW_t)^2 = dt} \tag{1}$$

It is this equality that changes all derivative formulas.

3 The Fundamental Derivation: Solving the Price Equation

This is the "number 1" derivation in quantitative finance.

3.1 The Problem

We assume that the stock price S_t follows the following Stochastic Differential Equation (SDE) (Black-Scholes Model):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{2}$$

Where:

- μ: The average return (Drift).

- σ: The volatility (amplitude of the noise).

- $\frac{dS_t}{S_t}$: The instantaneous return.

Objective: Find the explicit formula for S_t. We cannot simply integrate because S_t appears on both the left and right sides. We must use a transformation function.

The Trick: We use the natural logarithm. Let $f(S_t) = \ln(S_t)$.

3.2 Application of Ito's Lemma

Ito's Lemma states that for a function $f(x)$, the differential is:

$$df = \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2$$

Here $x = S_t$. Let us calculate the partial derivatives of $f(S) = \ln(S)$:

- First derivative: $\frac{\partial f}{\partial S} = \frac{1}{S}$

- Second derivative: $\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$

Let us apply Ito's formula to $d(\ln S_t)$:

$$d(\ln S_t) = \frac{1}{S_t}dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2$$

Replace dS_t with its formula $(\mu S_t dt + \sigma S_t dW_t)$. For the term $(dS_t)^2$, we apply the **Golden Rule** $(dW_t)^2 = dt$ and ignore terms in dt^2 or $dt \cdot dW$ (too small):

$$(dS_t)^2 = (\sigma S_t dW_t)^2 = \sigma^2 S_t^2 (dW_t)^2 = \sigma^2 S_t^2 dt$$

Now, inject everything into the equation for $d(\ln S_t)$:

$$d(\ln S_t) = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2S_t^2}(\sigma^2 S_t^2 dt)$$

We simplify (the S_t terms cancel out miraculously):

$$d(\ln S_t) = (\mu dt + \sigma dW_t) - \frac{1}{2}\sigma^2 dt$$

Group the dt terms:

$$d(\ln S_t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

3.3 Integration

Now that we no longer have S on the right side, we can simply integrate between 0 and T:

$$\int_0^T d(\ln S_t) = \int_0^T \left(\mu - \frac{\sigma^2}{2}\right)dt + \int_0^T \sigma dW_t$$

$$\ln(S_T) - \ln(S_0) = \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma(W_T - W_0)$$

Since $W_0 = 0$ and W_T can be written as $\sqrt{T}Z$ (with $Z \sim \mathcal{N}(0, 1)$):

$$\ln\left(\frac{S_T}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z$$

By taking the exponential of both sides, we obtain the final formula:

$$\boxed{S_T = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right)} \tag{3}$$

4 Key Takeaways

1. We do not differentiate normally in finance; we use **Ito's Lemma**.

2. The term $-\frac{\sigma^2}{2}$ appears mathematically due to convexity. It is the price to pay for volatility (the *volatility drag*).

3. S_T follows a **Log-Normal** distribution.