Step 25: Fourier Transform Pricing

The Carr-Madan Formula & Heston Calibration

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1 The Power of Characteristic Functions

In advanced quantitative finance, many models (Heston, Bates, Variance Gamma) do not have a closed-form Probability Density Function (PDF), but their **Characteristic Function (CF)** is known analytically.

The characteristic function $\phi(u)$ is the Fourier Transform of the density $f(x)$:

$$\phi(u) = \int_{-\infty}^{\infty} e^{iux} f(x) dx$$

If we know $\phi(u)$, we can recover the option price without ever knowing the density explicitly.

2 Case Study: The Heston Model

The Heston model dynamics are:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^S \\ dv_t &= \kappa(\theta - v_t)dt + \xi \sqrt{v_t} dW_t^v \end{aligned}$$

with correlation $d\langle W^S, W^v \rangle = \rho dt$.

The characteristic function of the log-price $x_T = \ln(S_T)$ is known (Heston, 1993):

$$\phi(u) = \exp\left(C(u,\tau)\theta + D(u,\tau)v_0 + iu\ln(S_0)\right)$$

Where C and D are complex-valued functions involving $\sqrt{(\rho\xi ui - \kappa)^2 + \xi^2(u^2 + ui)}$.

3 The Carr-Madan Formula (1999)

Carr and Madan revolutionized calibration by linking the Call Price $C(K)$ directly to $\phi(u)$ via FFT.

They define a **Dampened Call Price**:

$$c(k) = e^{\alpha k} \times \text{Call}(e^k)$$

Using Parseval's identity or inverse Fourier transforms, they derived:

$$C(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi(v) dv \tag{1}$$

Where $\psi(v)$ is related to the characteristic function of the log-price:

$$\psi(v) = \frac{e^{-rT} \phi(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}$$

4 The FFT Implementation

The goal is to compute the integral for **many strikes** k simultaneously. We approximate the integral as a sum:

$$I(k_u) \approx \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}ju} \psi(v_j) \Delta v$$

This is exactly the format of a Discrete Fourier Transform (DFT).

4.1 Algorithm Steps

1. **Discretize** the integration domain $[0, A]$ into $N = 4096$ points with spacing η.
2. **Compute** the vector of $\psi(v_j)$ values using the Heston analytic simplifications.
3. **Apply FFT**: Use 'fft(vector)' to compute the summation in $O(N \log N)$ time.
4. **Multiply** by dampening factors and weights (Simpson's rule weights).
5. **Extract** Call prices for the grid of Log-Strikes within the strike range.

5 Why FFT? (Speed Analysis)

If we need to calibrate the 5 Heston parameters $(\kappa, \theta, \xi, \rho, v_0)$ to a surface of 100 options:

- **Direct Integration**: 100 integrals $\times$ 1000 steps $= 10^5$ evaluations.
- **FFT**: 1 FFT call $=$ 100 prices instantly.

This speedup factor (often 100x or 1000x) is why FFT is the standard for real-time volatility surface calibration.