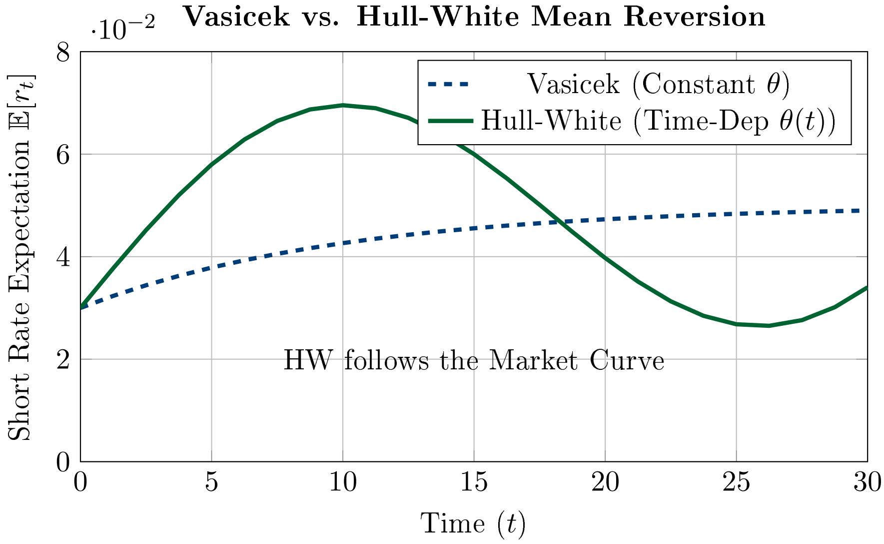
The Hull-White Model

Perfect Calibration to the Term Structure

Lecture 15
M.Sc. Quantitative Finance



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Contents

Chapter 1

The Hull-White Model (1990)

1.1 The Fitting Problem

The Vasicek model ($dr_t = a(b - r_t)dt + \sigma dW_t$) depends on 3 constant parameters: a, b, σ. The market Yield Curve contains dozens of data points (1M, 3M, 6M, 1Y, ... 30Y). It is mathematically impossible for 3 constants to fit 30 points perfectly. **Consequence:** If you use Vasicek to price a bond, your model price P^{Model} will differ from the market price P^{Mkt}. This allows for arbitrage, which is unacceptable.

1.2 The Hull-White Extension

Hull and White (1990) introduced a time-dependent parameter $\theta(t)$ to absorb the error at every maturity.

Hull-White SDE

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t \tag{1.1}$$

Here, a (mean reversion speed) and σ (volatility) are usually kept constant, but $\theta(t)$ changes over time.

1.3 Fitting to the Forward Curve

We want the model to match the initial term structure of interest rates observed in the market. Let $f^M(0,t)$ be the market instantaneous forward rate at time 0 for maturity t.

$$f^M(0,t) = -\frac{\partial \ln P^{Mkt}(0,t)}{\partial t}$$

Hull and White derived the exact functional form required for $\theta(t)$:

Theorem 1.1 (Calibration Formula). *To fit the initial term structure perfectly, $\theta(t)$ must be:*

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{1.2}$$

Remark 1.1. *The term $\frac{\sigma^2}{2a}(1 - e^{-2at})$ is a "convexity adjustment". It arises because bond prices are non-linear functions of rates (Jensen's Inequality).*

Chapter 2

Analytical Pricing

Since Hull-White is still a Gaussian model (just like Vasicek), it retains the beautiful Affine structure.

2.1 Zero-Coupon Bonds

The price of a Zero-Coupon Bond paying \$1 at T is:

$$P(t,T) = A(t,T)e^{-B(t,T)r_t} \tag{2.1}$$

Where $B(t,T)$ is the same as in Vasicek:

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}$$

But $A(t,T)$ is adjusted to ensure the fit:

$$\ln A(t,T) = \ln \frac{P^{Mkt}(0,T)}{P^{Mkt}(0,t)} - B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2$$

2.2 Option on Bonds (Call on ZCB)

A European Call option on a Zero-Coupon Bond (maturity S, strike K, option expiry $T < S$) has a Black-Scholes-like formula:

ZCB Option Price

$$C_t = P(t,S)N(h) - KP(t,T)N(h-\sigma_p) \tag{2.2}$$

Where:

$$\sigma_p = \frac{\sigma}{a}\left(1-e^{-a(S-T)}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}$$

$$h = \frac{1}{\sigma_p}\ln\frac{P(t,S)}{P(t,T)K} + \frac{\sigma_p}{2}$$

Chapter 3

Swaptions & Jamshidian's Decomposition

3.1 The Swaption Problem

A **Swaption** gives the right to enter a Swap. A Swap can be viewed as a portfolio of Zero-Coupon Bonds (a coupon-bearing bond equals par). So a Swaption is an option on a portfolio of bonds:

$$\text{Payoff} = \max\left(\sum c_i P(T, T_i) - K, 0\right)$$

3.2 Jamshidian's Decomposition (1989)

In general, an option on a sum is NOT the sum of options ($\max(\sum x) \neq \sum \max(x)$). However, in 1-factor models like Hull-White, all bond prices $P(T, T_i)$ are driven by the same variable r_T. They move in lockstep (monotonicity).

We can find a critical rate r^* such that the portfolio equals K. Then the Swaption decomposes into a sum of options on individual Zero-Coupon Bonds, each with its own strike $K_i = P(r^*, T, T_i)$.

$$\text{Swaption} = \sum_i c_i \cdot \text{Option}_{ZCB}(K_i) \tag{3.1}$$

This allows for ultra-fast pricing of Swaptions using the formula from Chapter 2.