Project 11 — Phoenix Autocall Pricing under Smile

SVI implied volatility → Dupire local volatility → Monte Carlo (Visual Dashboard)

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Abstract

This report is a mini-course accompanying the notebook. We price a Phoenix Autocall structured product with coupons, memory, early redemption (autocall), and a maturity protection barrier. We compare a constant-volatility GBM baseline to a smile-consistent approach built from an SVI implied volatility surface and a pragmatic Dupire local volatility construction. The emphasis is on **visual diagnostics**: interactive 3D volatility sheets, animated smiles, fan charts with barriers, call probabilities, PV distributions, sensitivity heatmaps, and Monte Carlo convergence.

Contents

1 What this project demonstrates (one page)

Main objective. Quantify *smile/model risk* on a path-dependent structured product:

$$\Delta_{\text{smile}} := V_0^{\text{LocalVol}} - V_0^{\text{GBM}}.$$

Pipeline.

1. Build a smooth implied vol surface $\sigma_{\text{imp}}(T, K)$ using **SVI** (synthetic and offline by default).
2. Convert σ_{imp} into call prices $C(T, K)$ via Black–Scholes.
3. Apply a **Dupire** finite-difference recipe to construct a stable local vol grid $\sigma_{\text{loc}}(T, K)$.
4. Simulate under GBM vs LocalVol with **common random numbers** (clean comparison).
5. Price the Phoenix Autocall pathwise; produce a set of **high-signal visuals**.

Deliverables for a GitHub repo. A notebook + an `assets/` folder containing interactive HTML charts (3D vol sheets, smile animation, heatmaps, etc.).

2 Mathematical prerequisites

2.1 Probability and Brownian motion

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. A **Brownian motion** $(W_t)_{t\geq 0}$ satisfies:

- $W_0 = 0$,
- independent increments,
- $W_t - W_s \sim \mathcal{N}(0, t - s)$ for $0 \leq s < t$,
- continuous paths (a.s.).

2.2 Diffusions and Itô calculus (minimal toolkit)

A diffusion model is written:

$$\mathrm{d}S_t = \mu(t, S_t)\,\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}W_t.$$

Itô's formula for $f(t, S_t)$ is:

$$\mathrm{d}f = \left(\partial_t f + \mu\,\partial_S f + \tfrac{1}{2}\sigma^2\partial_{SS}^2 f\right)\mathrm{d}t + \sigma\,\partial_S f\ \mathrm{d}W_t.$$

This is the bridge between stochastic dynamics and PDE-based pricing identities.

2.3 Risk-neutral pricing

Under no-arbitrage, the discounted price is a martingale under $\mathbb{Q}$, and for a payoff H at T:

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT}H],$$

with constant short rate r for simplicity in this mini-project.

3 Baseline: GBM and Black–Scholes

3.1 GBM under $\mathbb{Q}$

The risk-neutral GBM:

$$\mathrm{d}S_t = rS_t\ \mathrm{d}t + \sigma S_t\ \mathrm{d}W_t^{\mathbb{Q}}.$$

Exact time-step (used in the notebook):

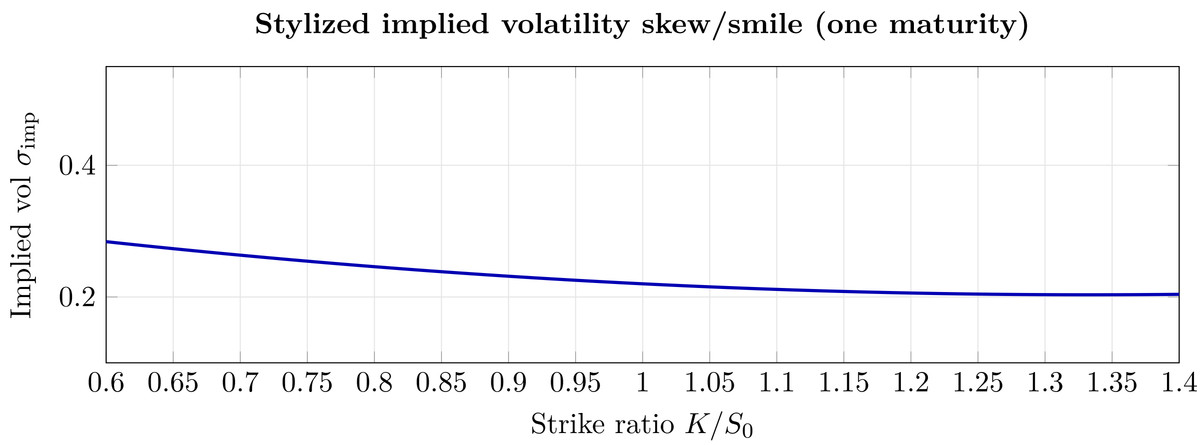
$$S_{t+\Delta t} = S_t \exp\Big((r - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\,Z\Big), \quad Z \sim \mathcal{N}(0,1).$$

3.2 Implied volatility: why the smile matters

Implied volatility $\sigma_{\text{imp}}(T,K)$ is defined by:

$$C^{mkt}(T,K) = C^{BS}(S_0, K, T, r, \sigma_{\text{imp}}(T,K)).$$

Empirically, σ_{imp} depends on both K and T, creating a **surface** (smile + term structure). Path-dependent products (barriers, autocalls, coupons) are highly sensitive to the *shape* of the distribution, hence to the smile.



Interpretation. Left-wing IV often higher (equity crash risk). Skew/curvature significantly impacts barrier-hit probabilities and autocall dynamics.

4 Structured product: Phoenix Autocall (cashflow map)

4.1 Observation dates and barriers

Let $0 < t_1 < \cdots < t_m = T$ be observation dates (typically monthly). Define:

$$B_{\text{cpn}},\ B_{\text{call}},\ B_{\text{prot}} \quad \text{(barriers as fractions of } S_0\text{)}.$$

4.2 Coupon and memory feature

At each t_k:

- Coupon condition: if $S_{t_k} \geq B_{\text{cpn}} S_0$, pay coupon cN.
- Memory: missed coupons accumulate; when the condition is met later, pay $(\text{missed} + 1)cN$.

4.3 Autocall event

If $S_{t_k} \geq B_{\text{call}} S_0$, the product redeems early at t_k and pays:

$$\text{Notional } N \quad \text{(and usually the coupon at call date).}$$

The path stops at the first call.

4.4 Maturity protection

If never called, maturity redemption is:

$$\text{Redemption} = \begin{cases} N, & S_T \geq B_{\text{prot}} S_0, \\ N \cdot \dfrac{S_T}{S_0}, & S_T < B_{\text{prot}} S_0. \end{cases}$$

4.5 Pricing identity

Let τ be the call time (first call date) or $\tau = T$ if never called. Then:

$$V_0 = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{k=1}^{m} e^{-rt_k}\,\text{Coupon}_k + e^{-r\tau}\,\text{Redemption at } \tau\Big].$$

The notebook evaluates this expectation by Monte Carlo with vectorized pathwise logic.

5 SVI surface (offline-friendly smile)

5.1 SVI for total variance

Use log-moneyness:

$$k = \ln\left(\frac{K}{F_T}\right), \qquad F_T = S_0 e^{rT}.$$

SVI models total variance $w(T,k) = \sigma^2_{\text{imp}}(T,K)\,T$:

$$w(T,k) = a + b\Big(\rho(k-m) + \sqrt{(k-m)^2 + \eta^2}\Big).$$

Parameter meaning.

- ρ: skew direction/intensity,
- η: curvature (smile strength),
- a: overall level (ATM-ish),
- b: scale (width/slope),
- m: horizontal shift.

5.2 Visual deliverables

The notebook produces:

- **3D implied vol sheet** $\sigma_{\text{imp}}(T,K)$ (interactive),
- **Smile animation** across maturities (Play/Pause).

6 Dupire local volatility: from vanillas to dynamics

6.1 Local vol model

We consider:

$$dS_t = rS_t \, dt + \sigma_{\mathrm{loc}}(t, S_t)S_t \, dW_t^{\mathbb{Q}}.$$

The goal is to choose σ_{loc} so that model European call prices match the target surface $C(T, K)$.

6.2 Dupire formula

If $C(T, K)$ is smooth:

$$\sigma_{\mathrm{loc}}^2(T, K) = \frac{\partial_T C(T, K) + rK \, \partial_K C(T, K)}{\frac{1}{2} K^2 \, \partial_{KK}^2 C(T, K)}.$$

6.3 Numerical recipe (practical implementation)

1. Build a grid (T_i, K_j).
2. Compute $\sigma_{\mathrm{imp}}(T_i, K_j)$ (SVI).
3. Convert to call prices via BS:

$$C(T_i, K_j) = C^{BS}(S_0, K_j, T_i, r, \sigma_{\mathrm{imp}}(T_i, K_j)) \, .$$

4. Approximate derivatives $\partial_T C, \partial_K C, \partial_{KK} C$ using finite differences.
5. Compute $\sigma_{\mathrm{loc}}(T_i, K_j)$, then apply:
   - **clipping** (avoid explosions),
   - **smoothing** (reduce grid noise).
6. Use bilinear interpolation to query $\sigma_{\mathrm{loc}}(t, S)$ during simulation.

Deliverable. The notebook exports `assets/p11_ultimate_localvol_surface_3d.html`.

7 Monte Carlo: ultimate implementation choices

7.1 Common random numbers (CRN)

To compare models cleanly, we reuse the same Gaussian shocks Z in GBM and LocalVol. This reduces the variance of Δ_{smile} and yields a sharp **model-difference histogram**.

7.2 Antithetic variates

Given Z, also simulate $-Z$. This often reduces variance with nearly no extra work.

7.3 Vectorized payoff evaluation

The product is valued by scanning observation dates and applying:

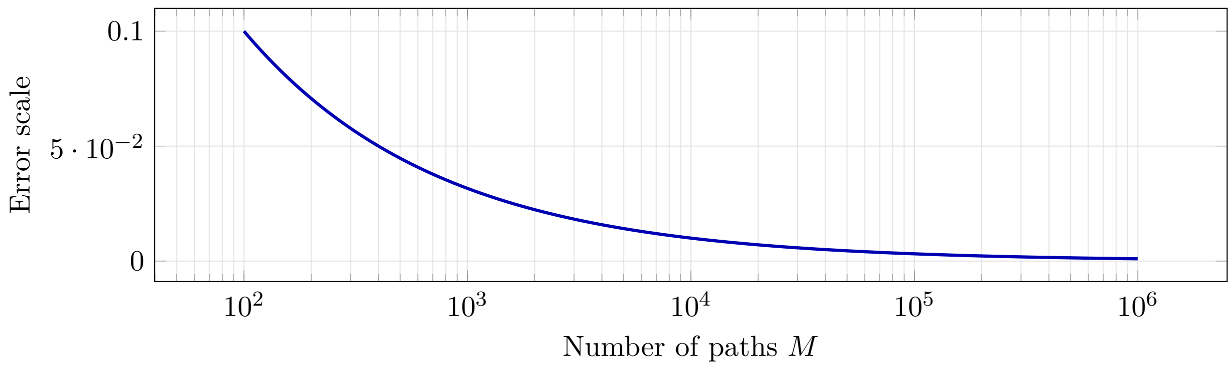
- coupon barrier,
- memory accumulation,
- first-call stopping time,
- maturity barrier redemption.

The notebook implements this **vectorized** across paths, making it fast enough for sensitivity grids.

7.4 Monte Carlo error scaling

Standard error behaves like $1/\sqrt{M}$ where M is the number of paths:

Monte Carlo error $\propto 1/\sqrt{M}$



8 Visual diagnostics and how to read them

8.1 (A) Fan chart with barriers

What it shows. Percentile bands of simulated paths with horizontal lines for:

$$B_{\text{cpn}}S_0, \quad B_{\text{call}}S_0, \quad B_{\text{prot}}S_0.$$

Interpretation.

- If the median path spends a lot of time above $B_{\text{call}}S_0$, calls are frequent (short product life).
- If lower percentiles dip below $B_{\text{prot}}S_0$, tail risk at maturity increases.
- Compare GBM vs LocalVol: skewed smiles often increase downside tail probability relative to constant vol.

Exported HTML: `assets/p11_ultimate_fan_gbm.html` and `assets/p11_ultimate_fan_localvol.html`.

8.2 (B) Cumulative call probability by observation date

What it shows. The curve $P(\tau \le t_k)$ across monthly dates.
Interpretation.

- Early call shifts PV earlier (less discounting).
- Call probability is highly sensitive to the smile because tail asymmetry changes barrier-hit odds.

Exported HTML: `assets/p11_ultimate_call_probability.html`.

8.3 (C) PV distribution and model-difference distribution

PV histogram. Shows how discrete coupons and call events create multi-modal outcomes.
Model-difference histogram.

$$\text{PV(LocalVol)} - \text{PV(GBM)}$$

computed pathwise (using CRN) provides a direct visualization of **smile risk**. A wide distribution indicates strong model dependence.
Exported: `assets/p11_ultimate_pv_distribution.html` and `assets/p11_ultimate_pv_difference_dist`

8.4 (D) Sensitivity heatmap: coupon × autocall barrier

Why it is powerful. Structured products are designed by selecting coupon levels and barriers. A heatmap summarizes a *pricing landscape* over these choices.

Interpretation.

- Higher coupon increases PV directly (cashflows), but may also change effective call likelihood (if the issuer chooses barriers accordingly).
- Higher B_{call} reduces call probability, increasing maturity exposure and often increasing tail risk.
- A second heatmap can display Δ_{smile} to show where smile effects are largest.

Exported: `assets/p11_ultimate_sensitivity_heatmap.html` and `assets/p11_ultimate_smile_diff_hea`

8.5 (E) Monte Carlo convergence

What it shows. Price estimates vs number of paths (log-scale).

Interpretation. Stability indicates the estimate is reliable. If LocalVol converges slower, the product may be more sensitive to tail/rare events.

Exported: `assets/p11_ultimate_convergence.html`.

8.6 (F) Interactive dashboard (widgets)

The notebook includes a dashboard to update:

- smile parameters: level / skew / curvature,
- product parameters: coupon and barriers,
- simulation size (paths/steps),

and re-render surfaces + diagnostics on click. This provides a strong story for GitHub and interviews: *how smile and design choices impact structured product pricing and risk.*

9 Limitations and how to extend (honest quant section)

9.1 Limitations

- **Local vol is deterministic.** It matches vanillas but cannot reproduce stochastic smile dynamics (vol-of-vol).
- **Dupire is sensitive.** Finite differences amplify noise; smoothing/clipping is essential.
- **Synthetic smile.** This notebook is offline-first; for production, you calibrate to market chains.

9.2 Extensions (high-value upgrades)

- Calibrate SVI per maturity slice to real option chains (or build a smooth surface with arbitrage checks).
- Compare LocalVol vs Heston (stochastic vol) for the same Phoenix (quantify model risk further).
- Add Greeks: finite-difference Delta/Vega, and discuss hedging challenges for autocalls.
- Add rate/dividend term structures and forward-based moneyness consistently.

10 30-second interview pitch

"I priced a Phoenix Autocall using risk-neutral Monte Carlo and built a smile-consistent pricing pipeline. I generated a smooth SVI implied volatility surface, derived a stable Dupire local volatility grid, and simulated under both GBM and local vol using common random numbers. I produced interactive diagnostics: 3D volatility sheets, smile animation, barrier fan charts, call probability curves, PV and model-diff distributions, sensitivity heatmaps, and convergence plots."