Step 24: Numerical Methods for PDEs

Finite Differences, Crank-Nicolson, and Stability Analysis

Contents

1 The Need for Numerical Methods

While Black-Scholes gives a closed-form solution for European options, most exotic derivatives (American options, Barrier options, Bermudans) obey the same PDE but lack analytical solutions due to complex boundaries.

The Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

To solve this numerically, we transform it into a system of algebraic equations.

2 Discretization

We truncate the domain: $S \in [0, S_{\max}]$ and $t \in [0, T]$.

- Time steps: $\Delta t = T/N$. Points $t_n = n\Delta t$.
- Spot steps: $\Delta S = S_{\max}/M$. Points $S_j = j\Delta S$.

We denote $V_j^n \approx V(t_n, S_j)$.

3 Finite Difference Operators

Using Taylor expansions, we approximate derivatives:

- $\frac{\partial V}{\partial t} \approx \frac{V_j^{n+1} - V_j^n}{\Delta t}$ (Forward Difference)
- $\frac{\partial V}{\partial S} \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S}$ (Central Difference)
- $\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{(\Delta S)^2}$ (Central Second Difference)

4 The Crank-Nicolson Scheme

The Crank-Nicolson method is the industry standard because it is **unconditionally stable** and has **second-order accuracy** in both time and space: $O(\Delta t^2, \Delta S^2)$.

It takes the average of the Explicit (known) and Implicit (unknown) schemes at n and $n+1$.

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} = \frac{1}{2}\mathcal{L}(V^{n+1}) + \frac{1}{2}\mathcal{L}(V^n)$$

After grouping terms, we get a linear system:

$$-\alpha_j V_{j-1}^n + (1 - \beta_j)V_j^n - \gamma_j V_{j+1}^n = \alpha_j V_{j-1}^{n+1} + (1 + \beta_j)V_j^{n+1} + \gamma_j V_{j+1}^{n+1} \tag{1}$$

In matrix form:

$$\mathbf{A}\mathbf{V}^n = \mathbf{B}\mathbf{V}^{n+1}$$

Where $\mathbf{A}$ and $\mathbf{B}$ are **Tridiagonal Matrices**.

5 Boundary Conditions

To solve the system, we need conditions at the edges of the grid ($S = 0$ and $S = S_{\max}$).

5.1 Lower Boundary ($S = 0$)

At $S = 0$, the PDE simplifies (as σS and rS terms vanish):

$$\frac{\partial V}{\partial t} - rV = 0 \implies V_0^n = V_0^{n+1}e^{-r\Delta t}$$

5.2 Upper Boundary ($S = S_{\max}$)

For a Call option, as $S \to \infty$, $V \sim S$. So $\frac{\partial^2 V}{\partial S^2} \approx 0$. We use the linearity condition:

$$V_M^n = 2V_{M-1}^n - V_{M-2}^n$$

(Dirichlet conditions are also possible: $V_M = S_{\max} - Ke^{-r(T-t)}$).

6 Stability Analysis (Von Neumann)

Why not use the simpler Explicit method? Let error evolve as $\epsilon_j^n = \xi^n e^{ikj\Delta S}$. For the Explicit method, the amplification factor ξ can benefit $|\xi| > 1$ (instability) if:

$$\Delta t > \frac{(\Delta S)^2}{\sigma^2 S^2}$$

This puts a severe restriction on time steps. Crank-Nicolson, however, satisfies $|\xi| \leq 1$ for **any** Δt.

7 Dealing with American Options

For American options, we solve the linear system for a temporary candidate $\tilde{V}^n$, then apply the constraint:

$$V_j^n = \max(\tilde{V}_j^n, \text{Payoff}(S_j))$$

This is typically solved using the **PSOR** (Projected Successive Over-Relaxation) algorithm or typically by "operator splitting" (solve then max).