Derivation of the Black-Scholes Formula

1 Starting Point: Risk-Neutral Valuation

The price of a Call option at time $t = 0$, denoted by C_0, is given by the discounted expectation of the payoff under the risk-neutral measure $\mathbb{Q}$:

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max(S_T - K, 0)\right] \tag{1}$$

Where:

- S_T is the asset price at maturity.
- K is the strike price.
- r is the risk-free rate.

2 Modeling the Underlying Asset

In the Black-Scholes model, the asset price follows a Geometric Brownian Motion. We can express S_T in terms of a standard normal random variable $Z \sim \mathcal{N}(0, 1)$:

$$S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right) \tag{2}$$

3 Option Exercise Condition

The option is exercised if $S_T > K$. Let us find the equivalent condition on Z:

$$\begin{aligned}
S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right) &> K \\
\ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z &> \ln(K) \\
\sigma\sqrt{T}Z &> \ln\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T \\
Z &> \frac{\ln(K/S_0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}
\end{aligned}$$

To simplify notation, we define the lower bound $-d_2$ such that exercise occurs if $Z > -d_2$. We set:

$$-d_2 = \frac{\ln(K/S_0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad \Rightarrow \quad d_2 = \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

4 Calculating the Integral

We can now rewrite the expectation as an integral using the standard normal density function $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$:

$$C_0 = e^{-rT} \int_{-d_2}^{+\infty} (S_T(z) - K)\varphi(z)\, dz \tag{3}$$

By linearity of the integral, we split the calculation into two terms, A and B:

$$C_0 = \underbrace{e^{-rT} \int_{-d_2}^{+\infty} S_T(z)\varphi(z)\, dz}_{\text{Term A}} - \underbrace{e^{-rT} K \int_{-d_2}^{+\infty} \varphi(z)\, dz}_{\text{Term B}}$$

4.1 Calculating Term B (Strike)

This term corresponds to the discounted probability of exercise:

$$B = Ke^{-rT} \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

The integral represents the probability $\mathbb{P}(Z > -d_2)$. By symmetry of the normal distribution, $\mathbb{P}(Z > -x) = \mathbb{P}(Z < x) = N(x)$.

$$B = Ke^{-rT} N(d_2)$$

4.2 Calculating Term A (Asset)

$$A = e^{-rT} \int_{-d_2}^{+\infty} S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T} z\right) \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

The terms e^{-rT} and e^{rT} cancel out. We factor out S_0 and combine the exponentials:

$$A = S_0 \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 T}{2} + \sigma\sqrt{T} z - \frac{z^2}{2}\right) dz$$

We complete the square in the exponent:

$$-\frac{1}{2}(z^2 - 2\sigma\sqrt{T} z + \sigma^2 T) = -\frac{1}{2}(z - \sigma\sqrt{T})^2$$

The integral becomes:

$$A = S_0 \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - \sigma\sqrt{T})^2}\, dz$$

Let us perform the change of variable $u = z - \sigma\sqrt{T}$ (so $dz = du$). The lower bound $-d_2$ becomes $-d_2 - \sigma\sqrt{T}$. Since we define $d_1 = d_2 + \sigma\sqrt{T}$, the new bound is $-d_1$.

$$A = S_0 \int_{-d_1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = S_0 N(d_1)$$

5 Final Result: Black-Scholes Formula

By combining terms A and B, we obtain the Call formula:

$$\boxed{C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)} \tag{4}$$

With:

$$
\begin{aligned}
d_1 &= \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$