Step 5: Put-Call Parity
(No-Arbitrage Relation)

1 Introduction

Put-Call Parity is a fundamental principle that defines the relationship between the price of European Call (C) and Put (P) options with the same strike (K) and maturity (T). It is derived using a static replication argument, independent of any model (it works for Black-Scholes, but also for any other model, as long as there is no arbitrage).

2 Portfolio Construction

We construct two portfolios, A and B, today (at time $t = 0$).

Portfolio A: Fiduciary Call

- Long one Call option (C).
- Long a zero-coupon bond paying K at maturity (Value today: Ke^{-rT}).

$$\Pi_A = C + Ke^{-rT} \tag{1}$$

Portfolio B: Protective Put

- Long one Put option (P).
- Long one share of the underlying stock (S).

$$\Pi_B = P + S \tag{2}$$

3 Payoff Analysis at Maturity (T)

We analyze the value of both portfolios at time T, depending on the final stock price S_T.

4 Conclusion

As demonstrated in the table, both portfolios deliver exactly the same payoff in all possible states of the world:

$$\Pi_A(T) = \max(S_T, K) = \Pi_B(T)$$

By the **Law of One Price** (No-Arbitrage Principle), if two assets have identical future cash flows, they must have the same price today.

Therefore:

$$\Pi_A(0) = \Pi_B(0) \tag{3}$$

State of Market	$S_T \leq K$	$S_T > K$
Portfolio A		
Call Option $\max(S_T - K, 0)$	0	$S_T - K$
Cash (Bond)	K	K
Total Payoff A	$\mathbf{K}$	$\mathbf{S_T}$
Portfolio B		
Put Option $\max(K - S_T, 0)$	$K - S_T$	0
Stock	S_T	S_T
Total Payoff B	$\mathbf{K}$	$\mathbf{S_T}$

Table 1: Payoff Matrix

This yields the fundamental Put-Call Parity equation:

$$\boxed{C + Ke^{-rT} = P + S} \tag{4}$$

Application: If we know the price of the Call (from Black-Scholes), we can immediately find the price of the Put without evaluating a new integral:

$$P = C - S + Ke^{-rT}$$