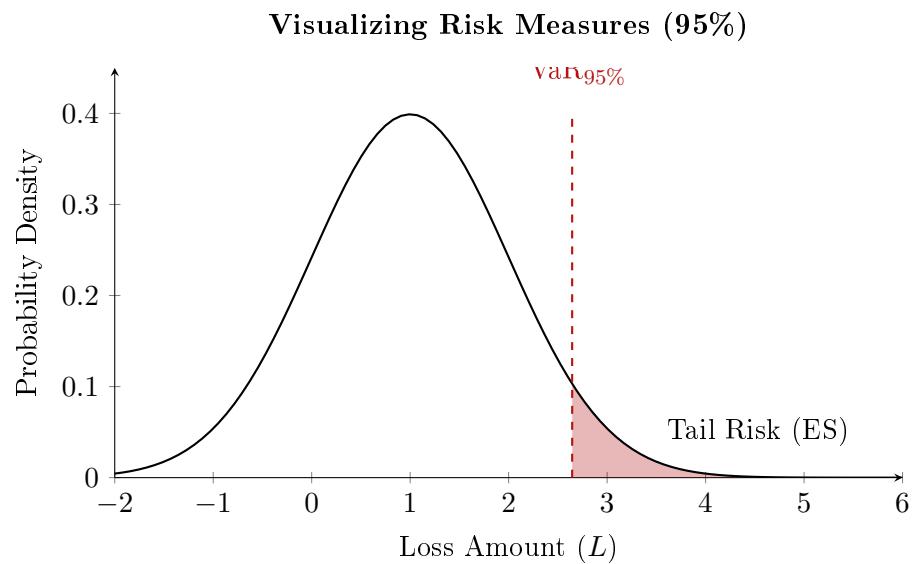


# Market Risk Management

*Value at Risk (VaR) and Expected Shortfall*

**Lecture 13**  
M.Sc. Quantitative Finance



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# Chapter 1

## Value at Risk (VaR)

### 1.1 Definition

Value at Risk is the standard metric for quantifying market risk. It answers the question: "*What is the maximum loss I can expect with  $\alpha\%$  confidence over a horizon  $T$ ?*"

Let  $L$  be the random variable representing the **Loss** of the portfolio over time  $T$ . (Note: Positive  $L$  means we lost money).

**Definition 1.1** (Value at Risk). *Given a confidence level  $\alpha \in (0, 1)$  (typically 95% or 99%), the VaR is the smallest number  $l$  such that the probability of the loss exceeding  $l$  is no larger than  $(1 - \alpha)$ .*

$$VaR_\alpha(L) = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} \quad (1.1)$$

*Mathematically, it is simply the  $(1 - \alpha)$ -quantile of the loss distribution.*

### 1.2 Calculation Methods

#### 1.2.1 1. Parametric VaR (Variance-Covariance)

Assume returns follow a Normal distribution  $R \sim \mathcal{N}(\mu, \sigma^2)$ .

$$VaR_\alpha = \text{Portfolio Value} \times (\sigma \cdot z_\alpha - \mu)$$

where  $z_\alpha$  is the normal quantile (e.g., 1.645 for 95%, 2.33 for 99%). *Pros:* Instant calculation. *Cons:* Assumes normality (underestimates fat tails).

#### 1.2.2 2. Historical Simulation

Take the last 500 days of historical returns. Apply them to today's portfolio. The VaR is simply the 5th percentile worst outcome. *Pros:* No assumption on distribution. *Cons:* Assumes the past predicts the future.

#### 1.2.3 3. Monte Carlo VaR

Simulate 10,000 paths using Heston or Jump-Diffusion models. Calculate the portfolio value for each. Sort the losses. *Pros:* Captures non-linearities (Options). *Cons:* Computationally expensive.

# Chapter 2

## Coherent Risk Measures

### 2.1 The Failure of VaR

VaR is not a "perfect" risk measure because it fails the property of **Sub-additivity**.

**Theorem 2.1** (Sub-additivity). *A risk measure  $\rho$  is sub-additive if:*

$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$

*Meaning: "Diversification should reduce risk."*

For certain non-normal distributions,  $\text{VaR}(A + B) > \text{VaR}(A) + \text{VaR}(B)$ . This implies that merging two portfolios creates *more* risk, which encourages splitting banks into tiny pieces to hide risk.

### 2.2 Expected Shortfall (ES)

To fix this, Artzner et al. (1999) proposed Expected Shortfall (also called CVaR or TVaR).

**Definition 2.1** (Expected Shortfall). *ES is the expected loss given that the loss exceeds the VaR.*

$$ES_\alpha(L) = \mathbb{E}[L \mid L \geq \text{VaR}_\alpha(L)] \quad (2.1)$$

#### Why Basel III moved to ES

VaR tells you "*We are safe 99% of the time.*" ES tells you "*If the 1% crisis happens, we will lose \$5 Billion.*" ES captures the "Tail Risk" (the severity of the crash), whereas VaR is blind to anything beyond the threshold.

