Step 23: The Mathematics of Credit Risk

Structural Models, Intensity Models, and Credit Derivatives

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1 Introduction to Credit Risk

Credit risk is the risk of loss resulting from a borrower's failure to repay a loan or meet contractual obligations. Unlike market risk, where prices move continuously, credit risk is characterized by **rare, extreme jumps** (Defaults).

We classify models into two primary frameworks:

1. **Structural Models**: Assume default occurs when firm value falls below a threshold (Endogenous default).
2. **Reduced-Form (Intensity) Models**: Assume default is a random event governed by a hazard rate (Exogenous default).

2 Structural Models: The Merton Model (1974)

Robert Merton revolutionized credit risk by viewing equity as an option on the firm's assets.

2.1 The Setup

Consider a firm with:

- Asset value V_t following a Geometric Brownian Motion (under $\mathbb{Q}$):

$$dV_t = (r - \delta)V_t dt + \sigma_V V_t dW_t$$

- Debt consisting of a single Zero-Coupon Bond with face value D maturing at T.

2.2 Default Mechanism

Default occurs at T if and only if $V_T < D$.

- **Shareholders (Equity E_T)**: They have limited liability. If $V_T < D$, they walk away with 0. If $V_T > D$, they pay debt and keep $V_T - D$.

$$E_T = \max(V_T - D, 0)$$

  This is exactly the payoff of a **European Call Option** on Asset V with strike D.

- **Bondholders (Debt B_T)**: They receive D if solvent, or recover the assets V_T if default.

$$B_T = \min(V_T, D) = D - \max(D - V_T, 0)$$

  This is equivalent to a Risk-Free Bond minus a **Put Option** on the assets.

2.3 Valuation Formulas

Using the Black-Scholes formula:

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2) \tag{1}$$

Where:

$$d_1 = \frac{\ln(V_0/D) + (r + \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}, \quad d_2 = d_1 - \sigma_V \sqrt{T}$$

2.4 The Probability of Default (PD)

The risk-neutral probability of default is the probability that $V_T < D$:

$$\mathbb{Q}(\text{Default}) = N(-d_2) \tag{2}$$

Under the physical measure $\mathbb{P}$ (using real drift μ), we often use "Distance to Default" (DD):

$$DD = \frac{\ln(V_0/D) + (\mu - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}$$

3 Intensity Models (Reduced-Form)

In practice, V_t is not observable. Reduced-form models treat default as a "surprise".

3.1 Poisson Processes and Hazard Rates

Let τ be the default time. We define the **Hazard Rate** (or Intensity) $\lambda(t)$ such that:

$$\mathbb{P}(\tau \in [t, t + dt] \mid \tau > t) = \lambda(t)dt$$

The probability of survival up to time T is:

$$S(T) = \mathbb{P}(\tau > T) = \exp\left(-\int_0^T \lambda(u)du\right) \tag{3}$$

If λ is constant, $S(T) = e^{-\lambda T}$.

3.2 Pricing a Defaultable Zero-Coupon Bond

A bond paying \$1 at T if no default, and Recovery R if default occurs. Using the fundamental asset pricing theorem:

$$B(0,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}\nVdash_{\tau>T} + e^{-\int_0^\tau r_u du}R \cdot \nVdash_{\tau\leq T}\right]$$

Assuming independence between rates and default, and taking $R = 0$ (Zero Recovery):

$$B(0,T) = P_{\text{risk-free}}(0,T) \times S(T) = e^{-rT}e^{-\lambda T} = e^{-(r+\lambda)T}$$

Thus, the yield spread is exactly λ.

4 Credit Default Swaps (CDS)

A CDS is the most liquid credit derivative.

- **Buyer**: Pays spread s (bps per year) on Notional N.
- **Seller**: Pays $(1 - R)$ if default occurs.

4.1 CDS Valuation

We equate the PV of the **Premium Leg** and the **Protection Leg**.

- **Premium Leg (PV)**: $\sum_{i=1}^n s \cdot \Delta t_i \cdot P(0,t_i) \cdot S(t_i)$
- **Protection Leg (PV)**: $\int_0^T (1 - R)P(0,u)\lambda(u)S(u)du$

Solving for the fair spread s (assuming constant λ and continuous premiums):

$$\boxed{s = \lambda(1 - R)} \tag{4}$$

This is the "Credit Triangle". If Spread = 100bps and Recov = 40%, then $\lambda \approx 1\%/0.6 = 1.66\%$.

5 Multi-Name Credit: Copulas

When modeling portfolios (CDOs), correlations matter. A Gaussian Copula imposes a correlation structure on individual default times.

5.1 Li's Model (2000)

We simulate correlated Gaussian variables Z_i.

$$\tau_i = S_i^{-1}(\Phi(Z_i))$$

Despite its flaws (lack of tail dependence), it remains the market standard for correlation mapping.

6 Conclusion

Mastering credit risk requires bridging the gap between balance sheet views (Merton) and market views (CDS/Intensity).