Step 26: Advanced Rates Modeling

The Heath-Jarrow-Morton (HJM) Framework

Contents

1 The Philosophy of Forward Rates

Short-rate models (Vasicek, CIR) start with r_t and try to fit the yield curve. This often fails or requires time-dependent parameters. **David Heath, Robert Jarrow, and Andrew Morton (1992)** inverted the problem: Why not take the observed current Forward Curve $f(0, T)$ as an input and model its evolution?

2 Mathematical Formulation

Let $f(t, T)$ be the instantaneous forward rate at time t for maturity T.

$$P(t, T) = \exp\left(-\int_t^T f(t, u) du\right)$$

We assume that for every maturity T, $f(t, T)$ follows an Itô process:

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t \tag{1}$$

3 The HJM No-Arbitrage Drift Condition

This is the central theorem. If the market is free of arbitrage, we cannot choose the drift α and volatility σ independently.

Under the Risk-Neutral Measure $\mathbb{Q}$, the discounted bond price $Z(t, T) = D_t P(t, T)$ must be a martingale. Applying Ito's Lemma to the bond price formula leads to the **HJM Drift Condition**:

$$\boxed{\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,u)du} \tag{2}$$

Interpretation: The drift of the forward rate is completely determined by its volatility structure. High volatility implies high drift (convexity correction).

4 Gaussian HJM and Principal Component Analysis

If $\sigma(t,T)$ is deterministic (Gaussian HJM), the forward rates are normally distributed. In practice, the curve does not move in 1 dimension. We typically use a multi-factor model:

$$df(t,T) = \alpha(t,T)dt + \sum_{k=1}^{3} \sigma_k(t,T)dW_t^{(k)}$$

PCA Analysis of yield curve changes reveals 3 factors:

1. **Level** (Shift): 90% of variance.
2. **Slope** (Tilt): 8% of variance.
3. **Curvature** (Twist): 2% of variance.

5 The Brace-Gatarek-Musiela (BGM) / LIBOR Market Model

HJM models instantaneous rates $f(t,T)$ which are abstract. The LMM models **Forward LIBOR rates** $L(t,T_i,T_{i+1})$, which are the underlying assets of Caps and Swaptions.

$$\frac{dL_k(t)}{L_k(t)} = \mu_k(t)dt + \sigma_k(t)dW_t$$

The LMM is essentially the "discrete, log-normal" version of HJM. It is the gold standard for pricing complex interest rate derivatives (e.g., Bermudan Swaptions, Snowballs).

6 Monte Carlo Implementation

To simulate the HJM framework:

1. Discretize time into steps Δt.
2. Initiate with the current Market Forward Curve $f(0,T)$.
3. At each step, update the entire curve:

$$f(t+\Delta t,T) = f(t,T) + \left(\sigma \int_t^T \sigma\right)\Delta t + \sigma\sqrt{\Delta t}Z$$

4. Calculate Bond Prices $P(t,T)$ from the integral of the new curve.