Step 7: Implied Volatility (The Inverse Problem)

1 Introduction: Price vs. Volatility

In the previous steps, we treated volatility (σ) as an input parameter to finding the option price (C). This is the theoretical approach.

$$\text{Input: } (S, K, r, T, \sigma) \xrightarrow{\text{Black-Scholes}} \text{Output: Price}$$

However, in professional trading, the logic is reversed.

- **Observable:** The Option Price (C_{mkt}) is determined by supply and demand on the exchange.
- **Unobservable:** Future volatility is unknown.

Therefore, traders use the Black-Scholes formula as a translation tool to convert a market price (in dollars/euros) into a volatility figure (in %). This output is called **Implied Volatility** (σ_{imp}). It represents the market's expectation of the average volatility until maturity.

2 Mathematical Formulation

We seek the value σ_{imp} that solves the following equation:

$$C_{BS}(\sigma_{imp}) - C_{mkt} = 0 \tag{1}$$

Where $C_{BS}(\sigma)$ is the Black-Scholes call price function:

$$C_{BS}(\sigma) = S_0 N(d_1(\sigma)) - Ke^{-rT} N(d_2(\sigma))$$

2.1 The Problem of Inversion

This equation is **non-linear and transcendental**. Because σ appears inside the limits of the normal integral (via d_1 and d_2), it is algebraically impossible to isolate σ.

$$\sigma = \text{Formula}(C_{mkt}) \quad \leftarrow \textbf{Impossible}$$

2.2 Existence and Uniqueness (Bijectivity)

Before trying to solve it numerically, we must ensure a solution exists. Recall from Step 7 that **Vega** is the derivative of price with respect to volatility:

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S\sqrt{T}\varphi(d_1)$$

Since $S > 0$, $T > 0$, and the Gaussian density $\varphi(d_1) > 0$, we have:

$$\mathcal{V} > 0 \quad \text{for all } \sigma > 0$$

Conclusion: The pricing function is strictly monotonic (increasing).

- If $\sigma \to 0$, $C \to \max(S - Ke^{-rT}, 0)$ (Intrinsic Value).
- If $\sigma \to \infty$, $C \to S$ (The option becomes the stock).

As long as the market price C_{mkt} is strictly greater than the intrinsic value (which is always true for traded options due to time value), a unique solution σ_{imp} exists.

3 Numerical Resolution: Newton-Raphson

To find the root of $f(\sigma) = C_{BS}(\sigma) - C_{mkt} = 0$, we use the Newton-Raphson algorithm. This method is extremely fast because we have an analytical formula for the derivative (Vega).

3.1 Geometric Interpretation

Imagine we are at a guess point σ_n. We approximate the complex Black-Scholes curve by its **tangent line** at that point. The slope of this tangent is Vega ($\mathcal{V}(\sigma_n)$). We follow this tangent line down to zero to find our next guess σ_{n+1}.

3.2 The Algorithm

The Taylor expansion around σ_n gives:

$$C_{BS}(\sigma_{imp}) \approx C_{BS}(\sigma_n) + \mathcal{V}(\sigma_n)(\sigma_{imp} - \sigma_n)$$

We set the left side to the target market price C_{mkt}:

$$C_{mkt} \approx C_{BS}(\sigma_n) + \mathcal{V}(\sigma_n)(\sigma_{imp} - \sigma_n)$$

Solving for σ_{imp} (which becomes our next step σ_{n+1}):

$$\boxed{\sigma_{n+1} = \sigma_n - \frac{C_{BS}(\sigma_n) - C_{mkt}}{\mathcal{V}(\sigma_n)}} \tag{2}$$

This iterative process typically converges to machine precision (10^{-8}) in 3 to 5 iterations.

4 Market Reality: The Volatility Surface

If the Black-Scholes assumptions were perfectly true (Log-Normal distribution, constant volatility), calculating σ_{imp} for options with different strikes (K) and maturities (T) should yield the same number.

Reality check: This is not what we observe. When plotting Implied Volatility against Strike Price K, we observe a curve called the **Volatility Smile** or **Skew**.

- **The Smile (Forex):** Implied volatility is higher for both deep ITM and deep OTM options. This suggests the market fears extreme moves (fat tails) more than the Gaussian model predicts.

- **The Skew (Equities):** Low strike Puts have much higher volatility than high strike Calls. This is known as "Crash Phobia." Traders pay a premium for insurance against market drops, inflating the price (and thus the implied vol) of Puts.

This leads to the concept of the **Volatility Surface** $\Sigma(K, T)$, implying that Black-Scholes is used as a quoting convention rather than a perfect physical model.