Project 06 — Volatility Modeling & Stochastic Vol Pricing

GARCH volatility forecasting + Heston option pricing

Quant Finance Portfolio (Karim)

December 19, 2025

Abstract

This report supports **Project 06**, a volatility-focused workflow built for real-world quant work. It combines (i) **GARCH-type time-series models** to estimate and forecast conditional volatility from returns, and (ii) the **Heston stochastic volatility model** to price European options and study implied-volatility smiles. The emphasis is on correct definitions, clean estimation/calibration steps, and recruiter-relevant validation: out-of-sample volatility forecasts, parameter stability, pricing sanity checks, and sensitivity to key parameters.

Contents

1 What you build in Project 06

This project is organized as a two-part pipeline:

- **Part A — Volatility forecasting (GARCH).**
  - Transform prices into (log) returns and verify basic stylized facts (volatility clustering).
  - Fit a **GARCH(1,1)** model (optionally with Student-t innovations).
  - Produce **one-step and multi-step** volatility forecasts and evaluate them out-of-sample.
- **Part B — Option pricing under stochastic volatility (Heston).**
  - Define the Heston SDE system and risk-neutral dynamics.
  - Price European calls/puts via a **semi-closed-form** approach (Fourier / characteristic function) and/or Monte Carlo.
  - Study the impact of **mean reversion, vol-of-vol, and correlation** on the implied-volatility smile.

Recruiter takeaway. This project demonstrates you can move from *time-series risk modeling* to *derivatives pricing* with consistent volatility concepts, realistic assumptions, and measurable validation.

2 Prerequisites (math and modeling)

2.1 Returns and conditional variance

Given prices (P_t), the log-return is

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right).$$

Volatility models describe the **conditional variance**:

$$\mathrm{Var}(r_t \mid \mathcal{F}_{t-1}) = \sigma_t^2,$$

where $\mathcal{F}_{t-1}$ denotes information available up to time $t-1$.

2.2 Stationarity and basic time-series intuition

A time-series model used for forecasting should be stable:

- **Weak stationarity** (constant mean/variance, autocovariance depends on lag).
- **Mean reversion** in volatility: shocks decay over time.

2.3 Stochastic calculus (minimal Heston toolkit)

In continuous time, asset dynamics are modeled with SDEs:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sqrt{v_t}\, S_t \,\mathrm{d}W_t.$$

Key concepts:

- Itô's formula (how functions of stochastic processes evolve),
- risk-neutral measure $\mathbb{Q}$ for pricing: discounted asset is a $\mathbb{Q}$-martingale,
- correlation between Brownian motions (to generate smiles/skews).

2.4 Implied volatility

Given a market option price, the **implied volatility** IV is the volatility that, plugged into Black–Scholes, reproduces that price. Under stochastic volatility, IV varies with strike and maturity (smile/surface).

3 Part A — GARCH volatility forecasting

3.1 Motivation: volatility clustering

Empirically, returns show weak autocorrelation, while *squared* returns often exhibit persistence:

$$\mathrm{Corr}(r_t, r_{t-k}) \approx 0, \qquad \mathrm{Corr}(r_t^2, r_{t-k}^2) > 0.$$

This motivates conditional heteroskedastic models (time-varying variance).

3.2 GARCH(1,1) definition

A standard specification is:

$$r_t = \mu + \epsilon_t, \qquad \epsilon_t = \sigma_t z_t, \qquad z_t \sim \text{i.i.d. } (0,1),$$
$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2,$$

with constraints $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$.

3.3 Persistence and unconditional variance

The persistence is $\alpha + \beta$. A common stability requirement is $\alpha + \beta < 1$ (finite unconditional variance). Then

$$\mathbb{E}[\sigma_t^2] = \frac{\omega}{1 - \alpha - \beta}.$$

3.4 Forecasting

One-step forecast is directly σ_{t+1}^2 from the recursion. Multi-step forecasts converge to the unconditional variance when $\alpha + \beta < 1$.

3.5 Innovation distribution: Normal vs Student-t

Financial returns often have excess kurtosis; a Student-t innovation can fit tails better. The trade-off is interpretability vs fit.

3.6 Validation (recruiter-facing)

A practical evaluation is:

- split into in-sample vs out-of-sample,
- forecast $\hat{\sigma}_{t+1}$ and compare to a volatility proxy (e.g. realized volatility),
- report errors (MAE/MSE) and qualitative stability across regimes.

4 Part B — Heston stochastic volatility model

4.1 Risk-neutral dynamics

Under $\mathbb{Q}$, the Heston model is:

$$\frac{\mathrm{d}S_t}{S_t} = r\ \mathrm{d}t + \sqrt{v_t}\ \mathrm{d}W_t^{(1)},$$
$$\mathrm{d}v_t = \kappa(\theta - v_t)\ \mathrm{d}t + \xi\sqrt{v_t}\ \mathrm{d}W_t^{(2)}, \qquad \mathrm{Corr}(\mathrm{d}W_t^{(1)}, \mathrm{d}W_t^{(2)}) = \rho.$$

Parameters:

- κ mean-reversion speed,
- θ long-run variance,
- ξ volatility of volatility,
- ρ correlation (drives skew),
- v_0 initial variance.

4.2 Feller condition (positivity intuition)

A sufficient condition for v_t to stay strictly positive is the Feller condition:

$$2\kappa\theta \geq \xi^2.$$

In practice, discretization may still produce negative values; implementations handle this with full truncation / reflection / exact simulation when needed.

4.3 European option pricing

The Heston model admits a semi-closed-form European price using the characteristic function of $\log S_T$. A common implementation route is:

- compute the characteristic function $\varphi(u)$,
- integrate to get call prices (Fourier inversion / Carr–Madan),
- recover implied volatilities to compare with market smiles.

Monte Carlo provides an alternative pricing method (slower but intuitive and flexible).

4.4 How parameters shape the smile

Qualitative sensitivities:

- higher ξ (vol-of-vol) $\Rightarrow$ stronger smile curvature,
- negative $\rho \Rightarrow$ equity-like negative skew (OTM puts expensive),
- larger $\kappa \Rightarrow$ faster variance mean reversion (short maturities dominated by v_0),
- larger $\theta \Rightarrow$ higher long-term implied vol level.

4.5 Calibration workflow (what a desk expects)

A standard calibration loop:

1. choose a set of option quotes (strikes, maturities),
2. pick an objective function (price RMSE or implied vol RMSE),
3. constrain parameters (e.g. bounds, Feller soft constraint),
4. optimize and validate (in-sample fit + out-of-sample check).

5 Implementation notes (what the notebook is doing)

- **Data:** price series $\rightarrow$ log-returns; diagnostics for volatility clustering.
- **GARCH:** fit GARCH(1,1), extract parameters, forecast conditional volatility.
- **Heston:** define characteristic function / pricing integrals (and/or Monte Carlo simulation), compute call prices, derive implied vols, plot smile curves.
- **Validation:** sanity checks, parameter stability, and sensitivity runs.
- **Outputs:** interactive charts exported as HTML for the GitHub repo, with the PDF as the narrative layer.

6 Sanity checks you should always do

GARCH checks

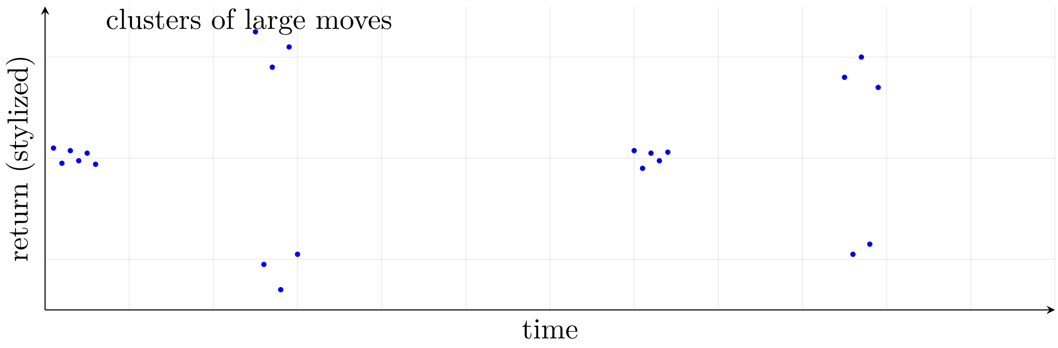
- Verify $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$ and usually $\alpha + \beta < 1$.
- Volatility forecasts should be positive and respond to large shocks in r_t^2.
- Increasing the lookback window should not create look-ahead behavior (alignment check).

Heston checks

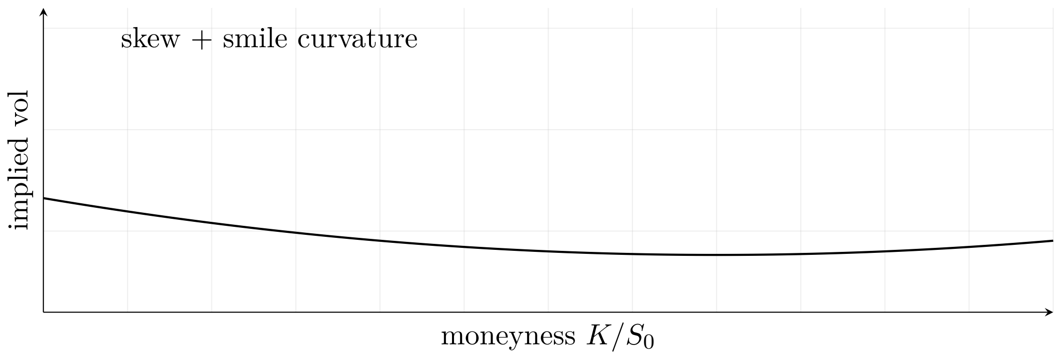
- Option prices must satisfy basic bounds: $0 \le C \le S_0$, and monotonicity in strike.
- As $\xi \to 0$ and $v_t \equiv v_0$, Heston prices should approach Black–Scholes with $\sigma = \sqrt{v_0}$.
- Smile direction: negative ρ should generate negative skew (equity-style).

7 Overleaf plots (conceptual, fast to compile)

7.1 Volatility clustering (stylized)



7.2 Implied-volatility smile intuition (stylized)



8 Interview pitch

I built a volatility modeling pipeline: I fit a GARCH(1,1) model on returns to forecast conditional volatility and validated forecasts out-of-sample. Then I implemented Heston stochastic volatility pricing for European options and analyzed how mean reversion, vol-of-vol and correlation drive the implied-volatility smile. The project includes parameter checks, pricing bounds, and sensitivity analysis to ensure the results are robust.

GARCH captures clustering in discrete time but does not directly model option smiles. Heston generates smiles but requires calibration and careful numerical integration or simulation. The framework is set up to compare assumptions and quantify their impact.