Project 02 — Black–Scholes Pricing via Monte Carlo

A mini-course + report template

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Abstract

This document is a concise mini-course supporting **Project 02: Black–Scholes Monte Carlo**. It introduces the mathematical prerequisites, derives the risk-neutral dynamics, explains why Monte Carlo works, and gives practical guidance (confidence intervals, variance reduction, Greeks). It is designed to compile quickly on Overleaf (free plan friendly) and to live inside your repo as a written report.

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1 What you build in Project 02

In this project you implement a **Monte Carlo (MC)** engine to price European options under the **Black–Scholes (BS)** model. Concretely, you:

- simulate terminal prices S_T under a **risk-neutral** Geometric Brownian Motion (GBM),
- compute discounted payoffs (call/put),
- study convergence as the number of paths increases,
- quantify uncertainty with confidence intervals,
- (optional but impressive) add variance-reduction and estimate Greeks.

2 Prerequisites (math you must know)

2.1 Normal distribution essentials

Let $Z \sim \mathcal{N}(0, 1)$.

- If $X = \mu + \sigma Z$, then $X \sim \mathcal{N}(\mu, \sigma^2)$.
- Tail quantile: z_α such that $\mathbb{P}(Z \leq z_\alpha) = \alpha$.
- CLT (Central Limit Theorem): sample averages of i.i.d. variables become approximately Normal.

2.2 Brownian motion (Wiener process)

A Brownian motion $(W_t)_{t \geq 0}$ satisfies:

- $W_0 = 0$.
- Independent increments: $W_t - W_s$ independent of the past for $t > s$.
- Gaussian increments: $W_t - W_s \sim \mathcal{N}(0, t - s)$.
- Continuous paths (almost surely).

Interpretation: dW_t is the fundamental source of randomness.

2.3 Itô calculus (minimal version)

In stochastic calculus, the GBM SDE is written:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

The key fact you need: the exact solution exists and is lognormal (Section 3.2).

3 Model setup

3.1 Real-world vs risk-neutral measures

Economic point: pricing is done under a measure where discounted asset prices are martingales. Assume a constant risk-free rate r (continuously compounded).

Under the *real-world* (physical) measure $\mathbb{P}$ one may have drift μ:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}}.$$

Under the *risk-neutral* measure $\mathbb{Q}$ the drift becomes r:

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}}.$$

This replacement is the heart of Black–Scholes: **expected growth for pricing is r, not μ.**

3.2 Closed-form GBM solution

Solving the SDE gives:

$$S_T = S_0 \exp\Big(\big(r - \tfrac{1}{2}\sigma^2\big)T + \sigma\sqrt{T}\, Z \Big), \quad Z \sim \mathcal{N}(0,1).$$

So $\log(S_T)$ is Normal and S_T is **lognormal**. This is exactly what you simulate in Monte Carlo.

4 European option payoff and price

4.1 Payoffs

Call and put payoffs at maturity T with strike K:

$$\text{Call payoff } = (S_T - K)^+, \qquad \text{Put payoff } = (K - S_T)^+,$$

where $x^+ = \max(x, 0)$.

4.2 Risk-neutral pricing formula

A European derivative is fully characterized by a payoff function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\text{Payoff at time } T = \Phi(S_T).$$

Under the risk-neutral measure $\mathbb{Q}$, its time-0 price is

$$V_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\big[\Phi(S_T)\big].$$

4.2.1 European call payoff and price

A European call with strike K gives the right (not the obligation) to buy the underlying at price K at time T. At maturity,

$$\Phi_{\text{call}}(S_T) = \begin{cases} 0, & S_T \le K, \\ S_T - K, & S_T > K, \end{cases} \qquad \text{so} \qquad \Phi_{\text{call}}(s) = (s - K)^+.$$

By the risk-neutral pricing formula,

$$C_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\big[(S_T - K)^+\big].$$

4.2.2 European put payoff and price

A European put with strike K gives the right (not the obligation) to sell the underlying at price K at time T. At maturity,

$$\Phi_{\text{put}}(S_T) = \begin{cases} 0, & S_T \geq K, \\ K - S_T, & S_T < K, \end{cases} \qquad \text{so} \qquad \Phi_{\text{put}}(s) = (K - s)^+.$$

By the risk-neutral pricing formula,

$$P_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\big[(K - S_T)^+\big].$$

5 Monte Carlo estimator

5.1 Algorithm

Simulate N independent draws $S_T^{(i)}$ from the GBM formula. Then:

$$\widehat{V}_0^{(N)} = e^{-rT}\,\frac{1}{N}\sum_{i=1}^{N}\Phi(S_T^{(i)}).$$

5.2 Why it converges

- Law of Large Numbers: $\widehat{V}_0^{(N)} \to V_0$ almost surely as $N \to \infty$.
- CLT: the error scales like $1/\sqrt{N}$.

5.3 Confidence interval (CI)

Let $X_i = e^{-rT}\Phi(S_T^{(i)})$ and $\bar{X} = \frac{1}{N}\sum X_i$. Estimate the standard deviation:

$$\widehat{\sigma}_X = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2}.$$

Approx. 95% CI:

$$\bar{X} \ \pm\ 1.96\frac{\widehat{\sigma}_X}{\sqrt{N}}.$$

Practical meaning: **MC returns a price plus an uncertainty band.**

6 Variance reduction (optional but very valuable)

Monte Carlo is slow because it converges at rate $1/\sqrt{N}$. Variance reduction improves speed *without* biasing the estimator.

6.1 Antithetic variates

Use pairs $(Z, -Z)$:

$$S_T(Z) = S_0 \exp\Big((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Z\Big), \quad S_T(-Z) = S_0 \exp\Big((r - \tfrac{1}{2}\sigma^2)T - \sigma\sqrt{T}\,Z\Big).$$

Average the two payoffs. This often reduces variance for monotone payoffs (calls/puts).

6.2 Control variate (strong method)

If you know $\mathbb{E}[Y]$ analytically and Y is correlated with your payoff, use:

$$\widehat{V}_{cv} = \bar{X} - b(\bar{Y} - \mathbb{E}[Y]),$$

with b chosen to minimize variance (estimated from data). A classic control variate is $Y = S_T$ because $\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$ is known.

7 Greeks (optional extension)

Greeks measure sensitivities of the price.

7.1 Delta

$\Delta = \partial V_0 / \partial S_0$. Two common MC approaches:

- **Bump-and-revalue**: price at S_0 and $S_0 + \varepsilon$, then finite difference.
- **Pathwise estimator** (when allowed): differentiate payoff along each path.

7.2 Vega

$\nu = \partial V_0 / \partial \sigma$ (sensitivity to volatility). Again bump-and-revalue is easiest and robust.

8 Sanity checks you should always do

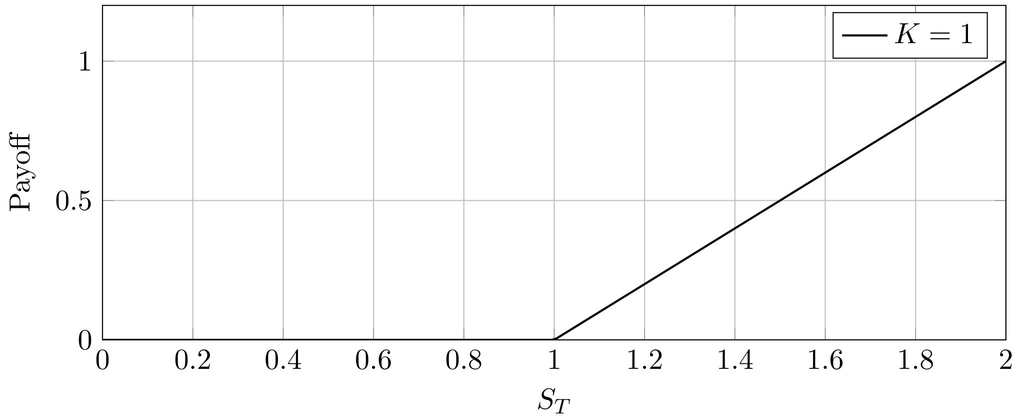
- **No-arbitrage bounds:** $0 \le C \le S_0$ and $0 \le P \le Ke^{-rT}$.
- **Monotonicity:** call price increases with S_0 and σ; put increases with σ.
- **Put-call parity:** $C - P = S_0 - Ke^{-rT}$.
- **Compare MC to closed-form BS price** (if implemented): errors should shrink with N.

9 Overleaf plots (generated directly here)

These figures are intentionally *conceptual* (fast to compile). They are useful to explain ideas in interviews.

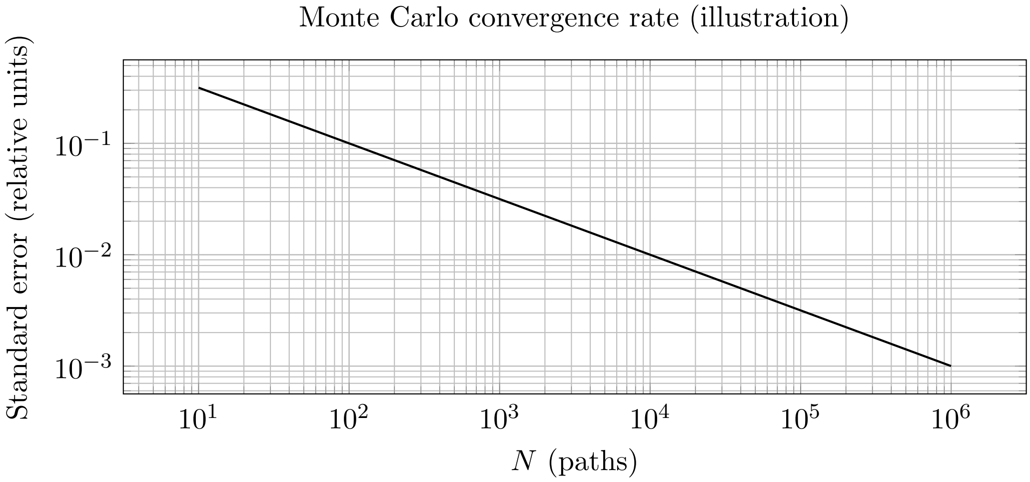
9.1 Call payoff diagram

Call payoff $(S_T - K)^+$ (normalized example)



9.2 Why the error decays like $1/\sqrt{N}$

If the standard error is $\propto N^{-1/2}$, then in log-log scale it is a straight line of slope $-1/2$.



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10 Interview pitch

> "I built a risk-neutral Monte Carlo pricer for Black–Scholes: I simulate lognormal terminal prices, discount payoffs, compute confidence intervals and convergence, and I can improve efficiency with antithetic variates or control variates. I also validated the model with put-call parity and no-arbitrage bounds, and I can estimate Greeks by bump-and-revalue or pathwise methods."