Step 2: The Change of Measure (Girsanov's Theorem)

1 The Problem: Real World Dynamics ($\mathbb{P}$)

In the real world (historical probability measure $\mathbb{P}$), the asset price dynamics depend on the investors' risk aversion. The Stochastic Differential Equation (SDE) is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}} \tag{1}$$

The parameter μ (real drift) is unknown, subjective, and difficult to estimate. To price a derivative objectively, we need a framework that does not depend on μ. We must move to the **Risk-Neutral World** ($\mathbb{Q}$).

2 The Market Price of Risk

First, we define the "Market Price of Risk" λ. It represents the excess return earned per unit of volatility:

$$\lambda = \frac{\mu - r}{\sigma} \tag{2}$$

Rearranging this term gives us a useful expression for μ:

$$\mu = r + \sigma\lambda \tag{3}$$

3 Girsanov's Theorem

Girsanov's Theorem allows us to change the probability measure from $\mathbb{P}$ to $\mathbb{Q}$. It states that we can define a new Brownian motion $W_t^{\mathbb{Q}}$ under the risk-neutral measure such that:

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \lambda dt \tag{4}$$

Intuitively, we are shifting the mean of the noise. We can express the "real" Brownian motion in terms of the "risk-neutral" one:

$$dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} - \lambda dt \tag{5}$$

4 Derivation of Risk-Neutral Dynamics

We now substitute equation (**??**) into the original asset dynamics equation:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}} \\ dS_t &= \mu S_t dt + \sigma S_t \left(dW_t^{\mathbb{Q}} - \lambda dt \right) \end{aligned}$$

We expand the terms:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{Q}} - \sigma\lambda S_t dt$$
$$dS_t = (\mu - \sigma\lambda) S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

Using equation (**??**) ($\mu = r + \sigma\lambda$), we can substitute $\mu - \sigma\lambda$ with r:

$$\mu - \sigma\lambda = r$$

5 Conclusion

The dynamics of the asset under the Risk-Neutral measure $\mathbb{Q}$ become:

$$\boxed{dS_t = r S_t dt + \sigma S_t dW_t^{\mathbb{Q}}} \tag{6}$$

Key Interpretation:

- The unknown parameter μ has disappeared.
- It has been replaced by the risk-free rate r, which is observable.
- In the world $\mathbb{Q}$, on average, the asset grows at the rate of a bank account. This justifies the use of r in the Black-Scholes pricing formula.