Step 6: The Greeks
(Sensitivity Analysis)

1 Introduction

The "Greeks" measure the sensitivity of the option price to changes in underlying parameters. They are essential for risk management and hedging. We start with the Black-Scholes formula for a Call option:

$$C = SN(d_1) - Ke^{-rT}N(d_2) \tag{1}$$

2 Preliminary Mathematical Identity

Before differentiating, we establish a crucial identity that simplifies calculations. Recall the definitions of d_1 and d_2:

$$d_2 = d_1 - \sigma\sqrt{T}$$

The standard normal density is $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

It can be shown that:

$$S\varphi(d_1) - Ke^{-rT}\varphi(d_2) = 0 \tag{2}$$

Proof:

$$\frac{\varphi(d_1)}{\varphi(d_2)} = \exp\left(-\frac{d_1^2}{2} + \frac{(d_1 - \sigma\sqrt{T})^2}{2}\right) = \exp\left(-\frac{2d_1\sigma\sqrt{T} - \sigma^2 T}{2}\right) = \frac{Ke^{-rT}}{S}$$

Rearranging yields the identity.

3 Derivation of Delta (Δ)

Delta measures the sensitivity to the stock price S:

$$\Delta = \frac{\partial C}{\partial S}$$

Differentiating the pricing formula using the product rule:

$$\frac{\partial C}{\partial S} = N(d_1) + S\frac{\partial N(d_1)}{\partial S} - Ke^{-rT}\frac{\partial N(d_2)}{\partial S}$$

Since $N'(x) = \varphi(x)$:

$$\frac{\partial C}{\partial S} = N(d_1) + S\varphi(d_1)\frac{\partial d_1}{\partial S} - Ke^{-rT}\varphi(d_2)\frac{\partial d_2}{\partial S}$$

Note that $d_2 = d_1 - \sigma\sqrt{T}$, implying $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S}$ (since $\sigma\sqrt{T}$ is constant w.r.t S). We factor out the term $\frac{\partial d_1}{\partial S}$:

$$\frac{\partial C}{\partial S} = N(d_1) + \frac{\partial d_1}{\partial S} \underbrace{\left[S\varphi(d_1) - Ke^{-rT}\varphi(d_2)\right]}_{\text{Zero by Eq. (??)}}$$

Thus, the formula simplifies dramatically to:

$$\boxed{\Delta_{\text{Call}} = N(d_1)} \tag{3}$$

(For a Put option, $\Delta_{\text{Put}} = N(d_1) - 1$).

4 Derivation of Gamma (Γ)

Gamma measures the rate of change of Delta (convexity). It is the same for Calls and Puts.

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial N(d_1)}{\partial S}$$

Applying the chain rule:

$$\Gamma = \varphi(d_1)\frac{\partial d_1}{\partial S}$$

Recall $d_1 = \frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$. The derivative with respect to S is:

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$$

Therefore:

$$\boxed{\Gamma = \frac{\varphi(d_1)}{S\sigma\sqrt{T}}} \tag{4}$$

We rely on the **Magic Identity** derived previously:

$$S\varphi(d_1) = Ke^{-rT}\varphi(d_2) \tag{5}$$

5 Vega ($\mathcal{V}$): Sensitivity to Volatility

Vega measures the sensitivity of the option price to changes in volatility σ.

$$\mathcal{V} = \frac{\partial C}{\partial \sigma}$$

Differentiating the Black-Scholes formula:

$$\frac{\partial C}{\partial \sigma} = S\varphi(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-rT}\varphi(d_2)\frac{\partial d_2}{\partial \sigma}$$

Recall that $d_2 = d_1 - \sigma\sqrt{T}$. Thus:

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T}$$

Substituting this back:

$$\frac{\partial C}{\partial \sigma} = S\varphi(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-rT}\varphi(d_2)\left(\frac{\partial d_1}{\partial \sigma} - \sqrt{T}\right)$$

Grouping terms by $\frac{\partial d_1}{\partial \sigma}$:

$$\frac{\partial C}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} \underbrace{\left(S\varphi(d_1) - Ke^{-rT}\varphi(d_2)\right)}_{\text{Zero by Eq. ??}} + Ke^{-rT}\varphi(d_2)\sqrt{T}$$

Using the Magic Identity on the remaining term ($Ke^{-rT}\varphi(d_2) = S\varphi(d_1)$), we get the final formula:

$$\boxed{\mathcal{V} = S\sqrt{T}\varphi(d_1)} \tag{6}$$

6 Rho (ρ): Sensitivity to Interest Rates

Rho measures sensitivity to the risk-free rate r.

$$\rho = \frac{\partial C}{\partial r}$$

$$\frac{\partial C}{\partial r} = S\varphi(d_1)\frac{\partial d_1}{\partial r} - K\left[-Te^{-rT}N(d_2) + e^{-rT}\varphi(d_2)\frac{\partial d_2}{\partial r}\right]$$

Note that $d_1 - d_2 = \sigma\sqrt{T}$, which does not depend on r. Therefore $\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r}$. The terms involving the partial derivatives cancel out due to the Magic Identity. We are left with the term from differentiating the discount factor:

$$\boxed{\rho = KTe^{-rT}N(d_2)} \tag{7}$$

7 Theta (Θ): Sensitivity to Time Decay

Theta measures the change in price as time to maturity T decreases. By convention, Theta is negative.

$$\Theta = \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial T}$$

Differentiating with respect to T:

$$\frac{\partial C}{\partial T} = S\varphi(d_1)\frac{\partial d_1}{\partial T} - K\left[-re^{-rT}N(d_2) + e^{-rT}\varphi(d_2)\frac{\partial d_2}{\partial T}\right]$$

Using $d_2 = d_1 - \sigma\sqrt{T}$, we have $\frac{\partial d_2}{\partial T} = \frac{\partial d_1}{\partial T} - \frac{\sigma}{2\sqrt{T}}$. Substitute and simplify using the Magic Identity to eliminate $\frac{\partial d_1}{\partial T}$:

$$\frac{\partial C}{\partial T} = rKe^{-rT}N(d_2) + Ke^{-rT}\varphi(d_2)\frac{\sigma}{2\sqrt{T}}$$

Using $Ke^{-rT}\varphi(d_2) = S\varphi(d_1)$ again:

$$\frac{\partial C}{\partial T} = rKe^{-rT}N(d_2) + \frac{S\varphi(d_1)\sigma}{2\sqrt{T}}$$

Since Theta is traditionally the negative of this derivative:

$$\boxed{\Theta = -\frac{S\varphi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)} \tag{8}$$