Step 3: The Black-Scholes PDE
(The Delta-Hedging Argument)

1 Introduction: The Hedging Argument

Instead of using probabilistic expectations, we can derive the option price by constructing a risk-free portfolio. This approach relies on the assumption of **No-Arbitrage**: If a portfolio has no risk, it must earn the risk-free rate r.

2 Constructing the Portfolio

Consider a portfolio Π consisting of:

- A long position in one option $V(S,t)$.
- A short position in Δ units of the underlying asset S.

The value of the portfolio is:

$$\Pi = V(S,t) - \Delta S \tag{1}$$

The change in the value of the portfolio over a small time interval dt is:

$$d\Pi = dV - \Delta dS \tag{2}$$

3 Applying Ito's Lemma

Since V is a function of S and t, we apply Ito's Lemma to find dV:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \tag{3}$$

Substitute this into equation (**??**):

$$d\Pi = \left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt\right) - \Delta dS$$

Rearranging terms to group dt and dS:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS \tag{4}$$

4 Eliminating Risk (Delta Hedging)

The term dS contains the stochastic component (dW_t). To make the portfolio risk-free, we must eliminate this source of uncertainty. We choose Δ such that the coefficient of dS is zero:

$$\frac{\partial V}{\partial S} - \Delta = 0 \quad \Longrightarrow \quad \Delta = \frac{\partial V}{\partial S} \tag{5}$$

This is the mathematical definition of the **Delta** of the option. The portfolio dynamics are now deterministic (no randomness):

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{6}$$

5 No-Arbitrage Condition

Since the portfolio Π is risk-free, it must grow at the risk-free rate r to prevent arbitrage opportunities:

$$d\Pi = r\Pi dt \tag{7}$$

Substituting $\Pi = V - \Delta S$:

$$d\Pi = r \left(V - \frac{\partial V}{\partial S} S \right) dt \tag{8}$$

6 The Black-Scholes Equation

We equate (**??**) and (**??**):

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left(V - S\frac{\partial V}{\partial S} \right) dt$$

Dividing by dt and rearranging all terms to one side, we obtain the Black-Scholes Partial Differential Equation (PDE):

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0} \tag{9}$$

Note: This is a heat equation (diffusion equation). Solving this PDE with the boundary condition $\max(S_T - K, 0)$ yields the exact same Black-Scholes formula derived in Step 1 and 2.