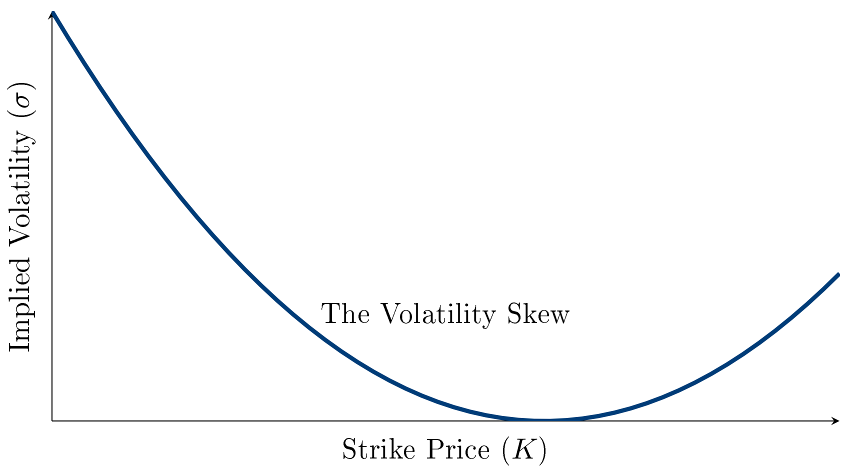
The Heston Model

Stochastic Volatility Pricing & Calibration

Module:
M.Sc. Quantitative Finance



December 25, 2025

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Chapter 1

The Heston Dynamics

1.1 Motivation: Failing of Black-Scholes

The Black-Scholes model assumes constant volatility ($\sigma = \text{const}$). This contradicts market reality, where we observe:

- **Volatility Clustering:** Large moves follow large moves.
- **The Leverage Effect:** Volatility tends to increase when the stock price drops.
- **The Smile/Skew:** OTM Puts are more expensive than ATM Calls.

Steven Heston (1993) proposed modeling the variance as a stochastic process itself.

1.2 The System of SDEs

Under the Risk-Neutral measure $\mathbb{Q}$, the Heston model is defined by two coupled Stochastic Differential Equations (SDEs):

Definition: The Heston Model (1993)

$$\frac{dS_t}{S_t} = r dt + \sqrt{v_t} dW_t^S \qquad \text{(Asset Price Process)} \tag{1.1}$$

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dW_t^v \qquad \text{(Variance Process)} \tag{1.2}$$

With correlation:

$$d\langle W^S, W^v\rangle_t = \rho dt \tag{1.3}$$

1.3 Parameter Interpretation

- $\mathbf{v_t}$: Instantaneous variance.
- θ (Theta): Long-run average variance.
- κ (Kappa): Mean-reversion speed. High κ means volatility returns quickly to θ.
- ξ (Xi): The "Vol-of-Vol". Controls the kurtosis of returns (fat tails).
- ρ (Rho): Correlation. Controls the Skew. Typically $\rho \approx -0.7$ for equities.

1.4 Mathematical Stability: The Feller Condition

The variance process is a CIR (Cox-Ingersoll-Ross) process. Since variance cannot be negative, we must ensure v_t stays strictly positive.

Theorem 1.1 (Feller Condition)**.** *The process v_t remains strictly positive ($v_t > 0$) almost surely if and only if:*

$$2\kappa\theta > \xi^2 \tag{1.4}$$

Proof intuition: The drift $\kappa\theta$ (pushing away from 0) must be strong enough to overcome the diffusion noise ξ near zero.

Chapter 2

Pricing via Fourier Transforms

The beauty of the Heston model is that despite being complex, it admits a semi-analytical solution. We do not need to solve the PDE directly; we use Characteristic Functions.

2.1 The Gil-Pelaez Formula

By analogy with Black-Scholes ($C = SN(d_1) - Ke^{-rT}N(d_2)$), the Heston price is:

$$C(S, K, v_0, T) = SP_1 - Ke^{-rT}P_2 \tag{2.1}$$

Here, P_1 and P_2 are not simple normal CDFs, but probabilities recovered by inverting the characteristic function using the Gil-Pelaez theorem:

Integration in the Complex Plane

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-iu\ln K}\phi_j(u, v_0, T)}{iu}\right] du \tag{2.2}$$

2.2 The Characteristic Function ϕ_j

The functions ϕ_j solve the Affine PDE associated with the Heston model. Let $\tau = T - t$. The solution is of the form:

$$\phi_j(u) = \exp(C_j(\tau) + D_j(\tau)v_0 + iu\ln S)$$

The coefficients C_j and D_j are given by (Heston 1993, Albrecher 2007 "Stable Form"):

$$D_j(\tau) = \frac{b_j - \rho\xi ui + d_j}{\xi^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_je^{d_j\tau}}\right)$$
$$C_j(\tau) = \frac{\kappa\theta}{\xi^2}\left((b_j - \rho\xi ui + d_j)\tau - 2\ln\left(\frac{1 - g_je^{d_j\tau}}{1 - g_j}\right)\right)$$

Where:

$$g_j = \frac{b_j - \rho\xi ui + d_j}{b_j - \rho\xi ui - d_j}, \quad d_j = \sqrt{(\rho\xi ui - b_j)^2 - \xi^2(2u_jui - u^2)}$$

With parameters for P_1 ($u_1 = 0.5, b_1 = \kappa - \lambda - \rho\xi$) and P_2 ($u_2 = -0.5, b_2 = \kappa - \lambda$).

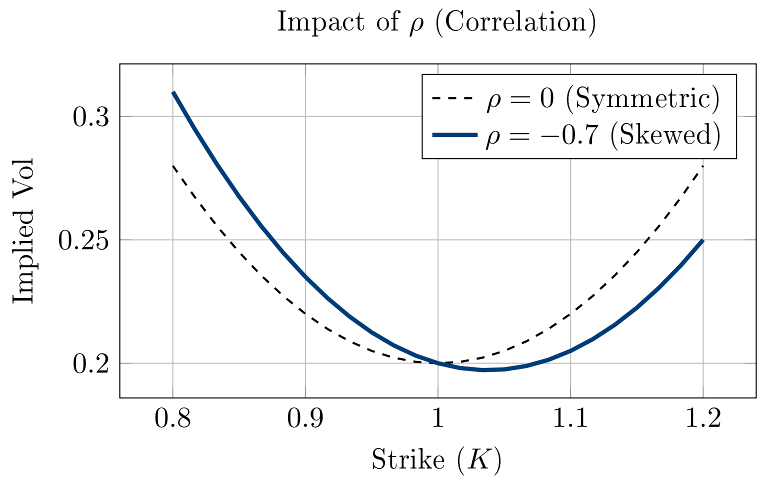
Chapter 3

Analysis of the Smile

This chapter visualizes how Heston parameters shape the Volatility Surface.

3.1 Effect of Correlation ρ (The Skew)

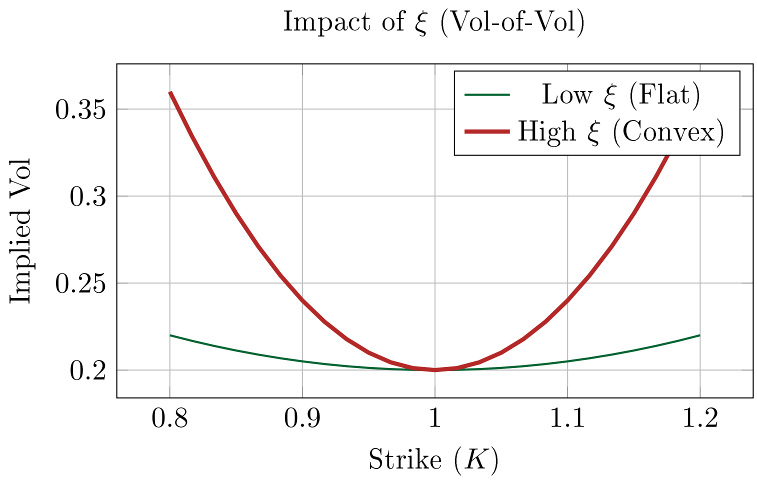
The correlation ρ determines the slope of the smile.



Conclusion: A negative correlation (Equity markets) makes OTM Puts expensive, creating a downward slope.

3.2 Effect of Vol-of-Vol ξ (The Smile)

The parameter ξ determines the curvature (convexity).



Conclusion: High ξ increases the value of both OTM Puts and Calls (Fat Tails).

Chapter 4

Numerical Implementation

4.1 Simulation: Full Truncation Scheme

Standard Euler discretization fails because v_t can become negative. We use the **Full Truncation** scheme (Lord et al.):

Algorithm 1 Heston Monte Carlo Path

$S_0, v_0, \kappa, \theta, \xi, \rho, T, N$
$\Delta t = T/N$
for $i = 0$ to $N - 1$ **do**
    Draw $Z_1, Z_2 \sim \mathcal{N}(0, 1)$
    $Z_S = \rho Z_1 + \sqrt{1 - \rho^2} Z_2$ ▷ Cholesky
    $\tilde{v}_i = \max(v_i, 0)$ ▷ Truncation for drift
    $v_{i+1} = v_i + \kappa(\theta - \tilde{v}_i)\Delta t + \xi\sqrt{\tilde{v}_i}\sqrt{\Delta t} Z_1$
    $S_{i+1} = S_i \exp\left((r - 0.5\tilde{v}_i)\Delta t + \sqrt{\tilde{v}_i}\sqrt{\Delta t} Z_S\right)$
end for

4.2 Calibration Strategy

Calibration involves finding $\Theta = \{v_0, \kappa, \theta, \xi, \rho\}$ to minimize the distance between Heston prices and Market prices.

$$\min_{\Theta} \sum_i w_i \left(C_{Heston}(K_i, T_i, \Theta) - C_{Market}(K_i, T_i)\right)^2$$

This is a non-convex optimization problem, typically solved using **Levenberg-Marquardt** or **Differential Evolution**.