Weak Convergence Rate of Symmetrized Euler-Maryama Scheme for CIR Type Process with Poisson Jumps

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Abstract

In this paper, we propose a symmetrized version of the classical Euler scheme for the Cox-Ingersoll-Ross (CIR) type process with Poisson jumps. We focus on the weak error of this scheme and more precisely, we show that the weak error's order is equal to $\min\left(1, \frac{b(0)}{\sigma^2}\right)$, and it does not depend on the value of the jump size a. This result is also confirmed by simulations.

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1. Introduction

This paper analyses the error of a discretization scheme for the Cox-Ingersoll-Ross (CIR) type process with Poisson jumps. This work extends the recent paper of Berkaoui et al. [1]. More precisely, we are interested in the $\mathbb{R}$-value process $\{X_t, t \geq 0\}$ which satisfies the following one-dimensional Itô stochastic differential equation (SDE):

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma\sqrt{X_s}dW_s + a\int_0^t dN_s \tag{1}$$

where $x_0 \geq 0$, $a > 0$ and $\sigma > 0$ are given constants, $(W_t, t \geq 0)$ being a one-dimensional standard Brownian motion and $(N_t,\ t \geq 0)$ an homogeneous Poisson process with intensity $\lambda > 0$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $(W_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ which are supposed to be independent processes, both $(W_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ defined on the complete probability space with a filtration $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$.

For $a = 0$, Equation (1) is well-known (see, [6]) and is widely used in finance especially to model short term interest rates. It is also well known that under such assumptions on $b(.)$, which will be presented later, Equation (1) is non-negative (see, [12]) and the underlying process is also known to be mean-reverting. Due to the characteristics of non-negativity and mean-reverting, several authors focus on the CIR processes and their applications in many fields such as: finance and insurance over the recent years (see for instance [5] and [8]).

The natural generalization of the classical CIR type process taking into account the jumps is given by (1) for $a \neq 0$. The presence of the jumps in practice can be justified since, for instance, interest rates can be affected by some natural disasters, terrorist attacks, etc. These shocks are then taken into account by adding the jump term to the initial process. The process described by Equation (1) can be considered as the simplest one; in which we assume that the jumps occur with the same size and with a constant intensity $\lambda > 0$. However, in real world, the occurence of events may not be time-homogeneous and consequently, they should depend on the course of time. In this case the jump is taken into account by homogeneous Poisson processes. Some illustrative examples can be found in [15], [11] and [20]. The occurence of events may have different sizes that can also be assumed to be random. It is the case of compound Poisson processes well described in [16]. These cases are not addressed in this work.

In this work, we assume that $(\mathcal{H}_1)$: the drift function $b(.)$ satisfies the Lipschitz condition (i.e there exists $K > 0$ s.t. $|b(x)-b(y)| \leq K|x-y|, \quad \forall x, y \in \mathbb{R}$). This assumption is needed to guarantee that there exists a unique stationary process which satisfies the dynamics of Equation (1) (e.g. see [17]), $(\mathcal{H}_2)$: $b(0) > 2\sigma^2$ which is needed to guarantee that $X_t > 0$ with probability 1. We also assume that $(\mathcal{H}_3)$: the drift function $b(.)$ is a $\mathcal{C}^4$ function, and its derivatives up to the order 4 are bounded.

Since it is difficult to obtain the solution of (1) explicitly, many schemes have been proposed in the literature in order to approximate the CIR process without jumps. One of the first scheme was the one of Euler in the form

$$\widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k})(t_{k+1} - t_k) + \sigma\sqrt{\widehat{X}_{t_k}}\big(W_{t_{k+1}} - W_{t_k}\big). \tag{2}$$

For this scheme, we can get a negative value for $\widehat{X}_{t_1}$ due to the Brownian motion. In this case, it is not possible to compute $\widehat{X}_{t_2}$. We can find different ad-hoc discretization schemes in [9]. To preserve the positivity of process with well define scheme (2), Deelstra [7] improved the scheme by adding an indicator function to assure the positivity of the scheme and obtained

$$\widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k}) + \sigma\sqrt{\widehat{X}_{t_k}\mathbb{I}_{\{\widehat{X}_{t_k}\geq 0\}}}\big(W_{t_{k+1}} - W_{t_k}\big). \tag{3}$$

This scheme is well defined and its solves the problem of positivity of the process but unfortunately failed in reducing the negative values of the process to 0. To adress this issue, Berkaoui [1] proposes the 'reflection scheme' is defined as:

$$\widehat{X}_{t_{k+1}} = \left|\widehat{X}_{t_k} + b(\widehat{X}_{t_k})(t_{k+1} - t_k) + \sigma\sqrt{\widehat{X}_{t_k}}\big(W_{t_{k+1}} - W_{t_k}\big)\right|. \tag{4}$$

Unfortunately, this 'reflection scheme' does not take into account the occurence of jump(s). In this paper, we extend scheme (4) based on the process defined in Equation (1). The main objective of this paper is to develop a new scheme accounting for the occurrence of jumps in the process. Specific objectives are: (i) to derive a new scheme which considers the occurence of jump(s); (ii) to prove the weak convergence of the new scheme under assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$; and (iii) to establish by simulations the validity of the new scheme.

The rest of the paper is structured as follows: In the Section 2, we present the new scheme and derive some preliminary results of the positive moments of the processes $(X_t)_{t\geq 0}$ and $(\widehat{X}_t)_{t\geq 0}$. In Section 3, we stretch the main results, that is the weak convergence of the scheme. Simulation results are reported in Section 4. Section 5 concludes the paper.

2. Some Preliminary Results

2.1. The scheme and notations

The proposed symmetrized Euler-Maruyama scheme considered here for (1) is defined as follows. For $x_0 \geq 0$, let $(X_t, t \geq 0)$ be given by (1). For a fixed time $T > 0$, the discretisation scheme $(\widehat{X}_{t_k}, k=0,...,N)$ is given by

$$\begin{cases} \widehat{X}_0 & = & x_0 \geq 0 \\ \widehat{X}_{t^-_{k+1}} & = & \left| \widehat{X}_{t_k} + b(\widehat{X}_{t_k})(t_{k+1}-t_k) + \sigma\sqrt{\widehat{X}_{t_k}}(W_{t_{k+1}} - W_{t_k}) \right| \text{ (whithout jump)} \\ \widehat{X}_{t_{k+1}} & = & \left| \widehat{X}_{t_k} + b(\widehat{X}_{t_k})(t_{k+1}-t_k) + \sigma\sqrt{\widehat{X}_{t_k}}(W_{t_{k+1}} - W_{t_k}) + a(N_{t_{k+1}} - N_{t_k}) \right| \text{(with jumps)} \end{cases}, \tag{5}$$

$k=0,...,N-1$, where N denotes the number of discretisation times $t_k = k\Delta t$ and $\Delta t > 0$ is a constant time step such that $N\Delta t = T$. In the sequel we use the time continuous version $(\widehat{X}_t, 0 \leq t \leq T)$ of the discrete time process, which consists of freezing the coefficients on each interval $[t_k, t_{k+1})$:

$$\begin{cases} \widehat{X}_0 & = & x_0 \geq 0 \\ \widehat{X}_{t^-} & = & \left| \widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(W_t - W_{\eta(t)}) \right| \text{ (without jump)} \\ \widehat{X}_t & = & \left| \widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(W_t - W_{\eta(t)}) + a(N_t - N_{\eta(t)}) \right| \text{(with jumps)} \end{cases}, \tag{6}$$

where $\eta(t) = \sup_{1\leq k\leq N}\{t_k, t_k \leq t\}$. The process $(\widehat{X}_t, 1 \leq t \leq T)$ is valued in $[0, +\infty)$.

We first notice that, if we set for any $t \in [0,T]$

$$\widehat{Y}_t = \widehat{X}_{\eta(t)} + b\left(\widehat{X}_{\eta(t)}\right)(t-\eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(W_t - W_{\eta(t)}) + a(N_t - N_{\eta(t)}), \tag{7}$$

then $\widehat{X}_t = |\widehat{Y}_t|$ is a semi-martingale with a local time $(L^0_t(\hat{X}))$ at point 0. In fact by the Mayer-Tanaka formula (See corollary 3 on Page 217 of [21]), we have

$$d\widehat{X}_t = \operatorname{sgn}(\widehat{Y}_{t^-})b(\widehat{X}_{\eta(t)})dt + \operatorname{sgn}(\widehat{Y}_{t^-})\sqrt{\widehat{X}_{\eta(t)}}dW_t + dL_t(\widehat{X}) + a.\operatorname{sgn}(\widehat{Y}_{t^-})dN_t \tag{8}$$

where $\operatorname{sgn}(x) = 1 - 2.\mathbb{I}_{\{x\leq 0\}}$. Equation (8) can be written in the form $d\widehat{X}_t = d\widehat{X}^c_t + a.\operatorname{sgn}(\widehat{Y}_{t^-})dN_t$ where $d\widehat{X}^c_t$ is the continous part of $d\widehat{X}_t$.

2.2. Positive Moments of the Processes $(X_t)_{t\geq 0}$ and $(\widehat{X}_t)_{t\geq 0}$

The following propositions ensure the existence of positive moments of $(X_t)_{t\geq 0}$, starting at x_0 at time 0, and of $(\widehat{X}_t)_{t\geq 0}$, its associated discrete time process:

Proposition 1. *Assume $(\mathcal{H}_1)$-$(\mathcal{H}_3)$. For any $p \geq 1$, there exists a positive constant C_p depending on p, but also on $b(0), K, \sigma, \alpha, a, \lambda$ and T, such that*

$$\max\left(\mathbb{E}\left(\sup_{0\leq t\leq T} X^{2p}_t\right), \mathbb{E}\left(\sup_{0\leq t\leq T} \widehat{X}^{2p}_t\right)\right) \leq C_p(1 + x^{2p}_0)\ \forall p \geq 1. \tag{9}$$

In the following proof, as well as in the rest of the work, we admit the following remark.

Remark 1. *C_p denotes a constant that can change from line to line and C_p could depend on the parameters of the model, but it is always independent of Δt. This observation is made to avoid having several constants in the work.*

Proof of Proposition 1. Equation (9) is proved in two steps.

- **Step** 1 : Let's prove that $\mathbb{E}\left(\sup_{0\le t\le T} X_t^{2p}\right) \le C_p(1+x_0^{2p})$, for $p\ge 1$.

  Since we don't know whether the expectation of X_t^{2p} for $p \ge 1$ is finite or not, we define the stopping time $\tau_m = \inf\{t \in (0,T), X_t \ge m\}$ with the convention $\inf\{\emptyset\} = 0$. For $t\in(0,T)$, we have

$$X_{t\wedge\tau_m} = x_0 + \int_0^{t\wedge\tau_m} b(X_s)ds + \sigma\int_0^{t\wedge\tau_m}\sqrt{X_s}dW_s + aN_{t\wedge\tau_m}.$$

  By applying the Itô formula, we have

$$\begin{aligned} X_{t\wedge\tau_m}^{2p} &= x_0^{2p} + 2p\int_0^{t\wedge\tau_m} X_s^{2p-1}dX_s^c + p(2p-1)\int_0^{t\wedge\tau_m} X_s^{2p-2}d\left[X,X\right]_s^c \\ &+ \int_0^{t\wedge\tau_m}\left((X_{s^-}+a)^{2p} - X_{s^-}^{2p}\right)dN_s \end{aligned}$$

  where $dX_s^c = b(X_s)ds + \sigma\sqrt{X_s}dW_s$ and $d\left[X,X\right]_s^c = \sigma^2 X_s ds$.
  Based on the expectation, and using the fact that X_{s^-} is independent of N_s we have

$$\begin{aligned} \mathbb{E}\left(X_{t\wedge\tau_m}^{2p}\right) &= x_0^{2p} + 2p\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p-1}b(X_s)\right)ds + p(2p-1)\sigma^2\int_0^{t\wedge\tau_m}\mathbb{E}\left(X_s^{2p-1}\right)ds \\ &+\lambda\int_0^{t\wedge\tau_m}\mathbb{E}\left((X_{s^-}+a)^{2p} - X_{s^-}^{2p}\right)ds. \end{aligned}$$

  We can notice from the Lipschitz condition on b and from the fact that $b(0)>0$, that $|b(X_s)| \le b(0) + KX_s$. We also notice from the convexity inequality that

$$(X_s+a)^{2p} - X_s^{2p} \le (2^{2p-1}-1)X_s^{2p} + 2^{2p-1}a^{2p}. \tag{10}$$

  Using these inequalities, we obtain

$$\begin{aligned} \mathbb{E}\left(X_{t\wedge\tau_m}^{2p}\right) &\le x_0^{2p} + 2p\int_0^{t\wedge\tau_m}\mathbb{E}\left(X_s^{2p-1}b(0)\right)ds + 2pK\int_0^{t\wedge\tau_m}\mathbb{E}\left(X_s^{2p}\right)ds \\ &+ p(2p-1)\sigma^2\int_0^{t\wedge\tau_m}\mathbb{E}\left(X_s^{2p-1}\right)ds + \lambda(2^{2p-1}-1)\int_0^{t\wedge\tau_m}\mathbb{E}\left(X_s^{2p}\right)ds \\ &+ 2^{2p-1}a^{2p}\lambda(t\wedge\tau_m). \end{aligned}$$

  We now point out that $x^p(1-x^{q-p}) \le 1$ (or $x^p \le x^q + 1$) which is true for $x\ge 0$ and $\forall p,q\in\mathbb{N}$ such that $q-p\ge 0$. We can write,

$$\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p-1}\right)ds \le (t\wedge\tau_n) + \int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p}\right)ds \le T + \int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p}\right)ds.$$

  Thus,

$$\begin{aligned} \mathbb{E}\left(X_{t\wedge\tau_m}^{2p}\right) &\le \left(x_0^{2p} + 2^{2p-1}a^{2p}\lambda(t\wedge\tau_m) + (2pb(0) + \sigma^2p(2p-1)))(t\wedge\tau_m)\right) \\ &+ \left(2pK + (2pb(0)+\sigma^2p(2p-1))) + (2^{2p-1}-1)\lambda\right)\int_0^{s\wedge\tau_m}\mathbb{E}\left(X_s^{2p}\right)ds \\ &\le C_p(1+x_0^{2p})\int_0^{t\wedge\tau_m}\mathbb{E}\left(X_s^{2p}\right)ds \end{aligned}$$

where C_p depends on $p, b(0), K, \sigma, T, a$. The right hand side (RHS) term can be writen as

$$\int_0^{t\wedge\tau_m} \mathbb{E}\left(X_s^{2p}\right) ds = \int_0^{t\wedge\tau_m} \mathbb{E}\left(X_{s\wedge(t\wedge\tau_m)}^{2p}\right) ds \leq \int_0^t \mathbb{E}\left(X_{s\wedge(t\wedge\tau_m)}^{2p}\right) ds = \int_0^t \mathbb{E}\left(X_{s\wedge\tau_m}^{2p}\right) ds. \tag{11}$$

We then deduce that

$$\mathbb{E}\left(X_{t\wedge\tau_m}^{2p}\right) \leq C_p(1+x_0^{2p}) \int_0^t \mathbb{E}\left(X_{s\wedge\tau_m}^{2p}\right) ds.$$

The Gronwall inequality [24] leads to the evidence that $\mathbb{E}\left(X_{t\wedge\tau_m}^{2p}\right) \leq C_p(1+x_0^{2p})$ where C_p does not depend on m. As m goes to $+\infty$ and taking the supremum, we have

$$\sup_{0\leq t\leq T} \mathbb{E}\left(X_t^{2p}\right) \leq C_p\left(1+x_0^{2p}\right).$$

The Burkhloder-Davis-Gundy [4] gives

$$\mathbb{E}\left(\sup_{0\leq t\leq T} X_t^{2p}\right) \leq \mathbb{E}\left(X_T^{2p}\right)^{1/2} \leq 1 + \sup_{0\leq t\leq T} \mathbb{E}\left(X_t^{2p}\right) \leq C_p\left(1+x_0^{2p}\right) \text{ (since } \sqrt{x} \leq 1+x, \forall x \in \mathbb{R}^+).$$

- **Step** 2: Let's prove that $\mathbb{E}\left(\sup_{0\leq t\leq T} (\widehat{X}_t)^{2p}\right) \leq C_p(1+x_0^{2p})$ for $p \geq 1$.
Using the same argument as before, we define the stopping time

$$\tau_n = \inf\{t \in [0,T], \widehat{X}_t \geq n\}$$

with the convention $\inf\emptyset = T$. By applying again the Itô's formula, for all $t \in [0,T]$, we have

$$\begin{aligned}
\widehat{X}_{t\wedge\tau_n}^{2p} &= x_0^{2p} + 2p\int_0^{t\wedge\tau_n} \widehat{X}_s^{2p-1}\mathrm{sgn}(\widehat{Y}_{s^-})b(\widehat{X}_{\eta(s)})ds \\
&\quad + 2p\sigma\int_0^{t\wedge\tau_n} \widehat{X}_s^{2p-1}\mathrm{sgn}(\widehat{Y}_{s^-})\sqrt{\widehat{X}_{\eta(s)}}dW_s + p(2p-1)\sigma^2\int_0^{t\wedge\tau_n} \widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}ds \\
&\quad + 2p\int_0^{t\wedge\tau_n} \widehat{X}_s^{2p-1}dL_s(\widehat{X}) + \int_0^{t\wedge\tau_n}\left((\widehat{X}_{s^-} + a.\mathrm{sgn}(\widehat{Y}_{s^-}))^{2p} - \widehat{X}_{s^-}^{2p}\right)dN_s.
\end{aligned}$$

Using the fact $\int_0^{t\wedge\tau_n} \widehat{X}_s^{2p-1}dL_s(\widehat{X}) = 0$ and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left(\widehat{X}_{t\wedge\tau_n}^{2p}\right) &= x_0^{2p} + 2p\mathbb{E}\left(\int_0^{t\wedge\tau_n} \widehat{X}_s^{2p-1}\mathrm{sgn}(\widehat{Y}_{s^-})b(\widehat{X}_{\eta(s)})ds\right) \\
&\quad + p(2p-1)\sigma^2\int_0^{t\wedge\tau_n} \widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}ds \\
&\quad + \int_0^{t\wedge\tau_n} \mathbb{E}\left(\left((\widehat{X}_{s^-} + a.\mathrm{sgn}(\widehat{Y}_{s^-}))^{2p} - \widehat{X}_{s^-}^{2p}\right)\right)\lambda ds.
\end{aligned}$$

From the Lipschitz condition of $b(.)$, we have

$$\begin{aligned}\mathbb{E}\left(\widehat{X}_{t\wedge\tau_n}^{2p}\right) \leq x_0^{2p} + 2pb(0)\int_0^{t\wedge\tau_n}\mathbb{E}\left(\widehat{X}_s^{2p-1}\right)ds + 2pK\int_0^{t\wedge\tau_n}\mathbb{E}\left(\widehat{X}_s^{2p-1}\widehat{X}_{\eta(s)}\right)ds \\ + p(2p-1)\sigma^2\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}ds \\ + \int_0^{t\wedge\tau_n}\mathbb{E}\left(\left((\widehat{X}_{s^-} + a.\mathrm{sgn}(\widehat{Y}_{s^-}))^{2p} - \widehat{X}_{s^-}^{2p}\right)\right)\lambda ds.\end{aligned}$$

From the convex inequality (10), we can write that

$$(\widehat{X}_s + a.\mathrm{sgn}(\widehat{Y}_{s^-}))^{2p} - \widehat{X}_s^{2p} \leq (2^{2p-1}-1)\widehat{X}_s^{2p} + 2^{2p-1}a^{2p} \text{ since } \mathrm{sgn}(\widehat{Y}_{s^-})^{2p} = 1.$$

From the Young inequality [10] we have

$$\mathbb{E}\left(\widehat{X}_s^{2p-1}\widehat{X}_{\eta(s)}\right) \leq \frac{(2p-1)}{2p}\mathbb{E}\left(\widehat{X}_s^{2p}\right) + \frac{1}{2p}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right)$$

and

$$\mathbb{E}\left(\widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}\right) \leq \frac{(p-1)}{p}\mathbb{E}\left(\widehat{X}_s^{2p}\right) + \frac{1}{p}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{p}\right) \leq \frac{(p-1)}{p}\mathbb{E}\left(\widehat{X}_s^{2p}\right) + \frac{1}{p} + \frac{1}{p}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right).$$

Collecting terms, we get

$$\begin{aligned}\mathbb{E}\left(\widehat{X}_{t\wedge\tau_n}^{2p}\right) &\leq x_0^{2p} + C_p\int_0^{t\wedge\tau_n}\mathbb{E}\left(\widehat{X}_s^{2p}\right)ds + C_p\int_0^{t\wedge\tau_n}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right)ds \\ &\leq x_0^{2p} + C_p\int_0^{t\wedge\tau_n}\mathbb{E}\left(\widehat{X}_s^{2p}\right)ds + C_p\int_0^{t}\mathbb{E}\left(\widehat{X}_{\eta(s)\wedge\tau_n}^{2p}\right)ds \text{ (from eq.(11))}.\end{aligned}$$

Again using the definition of $\widehat{X}_s$, we have

$$\begin{aligned}\mathbb{E}\left(\widehat{X}_s^{2p}\right) &= \mathbb{E}\left(\widehat{X}_{\eta(s)} + b(\widehat{X}_{\eta(s)})(s-\eta(s)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(W_s - W_{\eta(s)}) + a(N_s - N_{\eta(s)})\right)^{2p} \\ &\leq 2^{2p-1}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p} + b(\widehat{X}_{\eta(s)})^{2p}T^{2p} + \sigma^{2p}\widehat{X}_{\eta(t)}^{p}(W_s - W_{\eta(s)})^{2p} + a^{2p}(N_s - N_{\eta(s)})^{2p}\right).\end{aligned}$$

Since, $N_s - N_{\eta(s)} \sim \mathcal{P}(\lambda(s-\eta(s)))$, from the formula given on page 6 of [19], the moment of $N_s - N_{\eta(s)}$ is given by

$$\mathbb{E}\left(N_s - N_{\eta(s)}\right)^{2p} = \sum_{i=0}^{2p}\lambda^i(s-\eta(s))^i S_{i:2p} \text{ with } S_{n:s} = \frac{1}{n!}\sum_{i=0}^{n}(-1)^{n-i}\binom{i}{n}i^s. \tag{12}$$

Since $W_s - W_{\eta(s)} \sim \mathcal{N}(0, s-\eta(s))$, we have $\mathbb{E}\left((W_s - W_{\eta(s)})^{4p}\right) = \frac{(4p)!}{2^{2p}(2p)!}\Delta t^{2p}$. Young inequality again allows to write that

$$\begin{aligned}\mathbb{E}\left(\widehat{X}_{\eta(t)}^{p}(W_s - W_{\eta(s)})^{2p}\right) &\leq \frac{1}{2}\mathbb{E}\left(\widehat{X}_{\eta(t)}^{2p}\right) + \frac{1}{2}\mathbb{E}\left((W_s - W_{\eta(s)})^{4p}\right) \\ &= \frac{1}{2}\mathbb{E}\left(\widehat{X}_{\eta(t)}^{2p}\right) + \frac{1}{2^{2p+1}}\frac{(4p)!}{(2p)!}(\Delta t)^{2p}.\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left(\widehat{X}_s^{2p}\right) &\leq 2^{2p-1}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right) + 2^{2p-1}b(0)T^{2p} + 2^{2p-1}K^{2p}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right)T^{2p} \\
&+ \frac{\sigma^{2p}}{2}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right) + \frac{\sigma^{2p}}{2}\frac{1}{2^{2p+1}}\frac{(4p)!}{(2p)!} + a^{2p}\sum_{i=0}^{2p}\lambda^i T^i S_{i:2p} \\
&= C_p + C_p\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right).
\end{aligned}$$

We then deduce that

$$\int_0^{t\wedge\tau_n} \mathbb{E}\left(\widehat{X}_s^{2p}\right) ds \leq C_p + C_p \int_0^{t\wedge\tau_n} \mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right) ds \leq C_p + C_p \int_0^t \mathbb{E}\left(\widehat{X}_{\eta(s)\wedge\tau_n}^{2p}\right) ds.$$

We finally obtain

$$\mathbb{E}\left(\widehat{X}_{\eta(t)\wedge\tau_n}^{2p}\right) \leq C_p\left(1 + x_0^{2p} + \int_0^{\eta(t)} \mathbb{E}\left(\widehat{X}_{\eta(s)\wedge\tau_n}^{2p}\right) ds\right).$$

From Grownwall inequality, we have

$$\mathbb{E}\left(\widehat{X}_{\eta(t)\wedge\tau_n}^{2p}\right) \leq C_p\left(1 + x_0^{2p}\right)$$

where C_p depends on $p, b(0), K, \sigma, \lambda, a$ and T. The end of the proof is exactly work out as step 1 of this proof.

□

Corollary 1. *The process X_t^x has moments of any order, that is*

$$\sup_{0\leq t\leq T} \mathbb{E}\left(X_t^n\right) \leq C_n\big(1 + x_0^n\big) \textit{ for all } n \in \mathbb{N}^*.$$

Proof of Corollary 1. We have

$$\sup_{0\leq t\leq T} \mathbb{E}\left(X_t^n\right) \leq \sup_{0\leq t\leq T} \mathbb{E}\left(X_t^{2n}\right)^{1/2} \leq C(p)^{1/2}\big(1 + x_0^{2n}\big)^{1/2} \leq C(p)\big(1 + x_0^n\big)$$

where the second inequality comes from the fact that $x \to x^{1/2}$ is a concave function defined on $[0, +\infty)$. To complete the proof we use the fact that $\sqrt{1+x} \leq 1 + \sqrt{x}$ for $x \geq 0$. Thus,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} X_t^n\right) \leq C_n\big(1 + x_0^n\big).$$

□

In this section (X_t) denotes the solution of (1) starting at the deterministic point x_0 at time 0. When we need to vary the deterministic initial position, we mention it explicitly by using the notation (X_t^x) corresponding to the unique strong solution of the equation

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma\sqrt{X_s^x}dW_s + a\int_0^t dN_s. \tag{13}$$

The following Lemma is used to control the inverse moments and the exponential inverse moment of the process (X_t) and this result serves to justify the use of Girsanov theorem [23].

Lemma 1. *Let us assume* $(\mathcal{H}_1)-(\mathcal{H}_3)$*. We set* $\nu = \frac{2b(0)}{\sigma} - 1 > 1$*. For any* p *such that* $1 < p < \nu$*, for any* $t \in [0,T]$ *and any* $x > 0$*,*

$$\mathbb{E}(X_t^x)^{-1} \leq C(T)x^{-1} \text{ and } \mathbb{E}(X_t^x)^{-p} \leq C(T)t^{-p} \text{ or } \mathbb{E}(X_t^x)^{-p} \leq C(T,p)x^{-p}.$$

Moreover for all $\mu \leq \frac{\nu^2\sigma^2}{8}$*,*

$$\mathbb{E}\left[\exp\left(\mu\int_0^t (X_s^x)^{-1}ds\right)\right] \leq C(T)\left(1 + x^{-\nu/2}\right) \tag{14}$$

where the positive constant $C(T)$ *is a non-decreasing function of* T *and does not depend on* x*.*

Proof of Lemma 1. As $b(x) \geq b(0) - Kx$, the comparison Theorem for SDE without jump (see Theorem 1.1 in [25]); and using the fact that $a > 0$ we have, a.s. $X_t^x \geq Y_t^x$, for all $t \geq 0$, where $(Y_t^x, t \leq T)$ is the CIR process solving

$$Y_t^x = x + \int_0^t (b(0) - KY_s^x)ds + \sigma\int_0^t \sqrt{Y_s^x}dW_s + aN_t > \widetilde{Y}_t^x \tag{15}$$

where $\widetilde{Y}_t^x$ solves a CIR process in the form

$$dr_t^x = (\alpha - \beta r_t^x)dt + \sigma\sqrt{r_t^x}dW_t$$

where $\alpha = b(0), \beta = K$.

In particular, $\mathbb{E}\left[\exp\left(\mu\int_0^t (X_s^x)^{-1}ds\right)\right] \leq \mathbb{E}\left[\exp\left(\mu\int_0^t (\widetilde{Y}_s^x)^{-1}ds\right)\right]$ and it is proven (see Lemma A1 and Lemma A2 in [2]) that

$$\sup_{0\leq t\leq T}\mathbb{E}\exp\left[\left(\frac{\nu^2\sigma^2}{8}\int_0^t \frac{ds}{r_t^x}\right)\right] \leq C(1 + x^{-\nu/2}) \text{ for } \alpha \geq \frac{\sigma^2}{2} \text{ and } \nu = \frac{2\alpha}{\sigma^2} - 1.$$

Similarly, for the upper bounds of the inverse moments of X_t^t, we apply again Lemma A.1 in [2]. □

2.3. On the associated Kolmogorov Partial Differential Equation (PDE)

Let f be a $\mathbb{R}$-valued $\mathcal{C}^4$ bounded funcction, with bounded derivatives up to the order 4. We consider the $\mathbb{R}$-valued function defined on $[0,T] \times [0,+\infty)$ by $u(t,x) = \mathbb{E}(f(X_{T-t}^x)$. We have the following proposition.

Proposition 2. *Under assumptions* $(\mathcal{H}_1)-(\mathcal{H}_3)$*,* u *is in* $\mathcal{C}^{1,4}([0,T]\times[0,+\infty))$*. That is,* u *has a first order derivative with respect to the time variable and derivatives up to order 4 with respect to the space variable. Moreover, there exists a positive constant* C *depending on* f*,* b *and* T *such that, for all* $x \in [0,+\infty)$*,*

$$\sup_{0\leq t\leq T}\left|\frac{\partial u}{\partial t}(t,x)\right| \leq C(1+x) \tag{16}$$

$$||u||_{L^\infty([0,T]\times[0,+\infty))} + \sum_{k=1}^4 \left|\left|\frac{\partial^k u}{\partial x^k}\right|\right|_{L^\infty([0,T]\times[0,+\infty))} \leq C \tag{17}$$

and $u(t,x) = \mathbb{E}\left(f(X_{T-t}^x)\right)$ *solves the following PDE*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + Lu(t,x) = 0, & (t,x) \in [0,T]\times\mathbb{R}_+ \\ u(T,x) = f(x), & x \in \mathbb{R}_+ \end{cases} \tag{18}$$

where $Lu(t,x) = b(x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x\frac{\partial^2 u}{\partial x^2}(t,x) + \lambda u_a(t,x)$ *with* $u_a(t,x) = u(t,x+a) - u(t,x)$ *and* L *is the infinitesimal generator of the process.*

Proof of Proposition 2.

- Proof of $\sup_{0\le t\le T}\left|\frac{\partial u}{\partial t}(t,x)\right| \le C(1+x)$

  Let us prove that $u\in\mathcal{C}^1([0,T])$. Let $f\in C^4(\mathbb{R})$. Applying Itô formula, we have

$$\begin{aligned} f(X^x_{T-t}) &= f(x)+\int_0^{T-t} b(X^x_s)f'(X^x_s)ds+\frac{\sigma}{2}\int_0^{T-t} X^x_s f''(X^x_s)ds \\ &+ \sigma\int_0^{T-t}\sqrt{X^x_s}f'(X^x_s)dW_s+\int_0^{T-t}(f(X^x_{s^-}+a)-f(X^x_{s^-}))dN_s. \end{aligned}$$

  On the basis of the expectation, we have

$$\begin{aligned} u(t,x)=\mathbb{E}f(X^x_{T-t}) &= f(x)+\int_0^{T-t}\mathbb{E}\left[b(X^x_s)f'(X^x_s)\right]ds+\frac{\sigma}{2}\int_0^{T-t}\mathbb{E}\left[X^x_s f''(X^x_s)\right]ds \\ &+\lambda\int_0^{T-t}\mathbb{E}(f(X^x_{s^-}+a)-f(X^x_{s^-}))ds. \end{aligned}$$

  Since f' and f'' are bounded and (X^x_t) has moments of any order, the first order derivative with repect to t gives:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= -\mathbb{E}\left(b(X^x_{T-t})f'(X^x_{T-t})+\frac{\sigma^2}{2}X^x_{T-t}f''(X^x_{T-t})\right) \\ &- \lambda\mathbb{E}(f(X^x_{T-t}+a)-f(X^x_{T-t})). \end{aligned}$$

  Considering the absolute value, we have

$$\begin{aligned} \left|\frac{\partial u}{\partial t}(t,x)\right| &\le C\mathbb{E}\left(|b(X^x_{T-t})|+|X^x_{T-t}|\right)+C \\ &\le C\mathbb{E}\left(C+K|X^x_{T-t}|+|X^x_{T-t}|\right)+C \\ &\le C+C\mathbb{E}\left(|X^x_{T-t}|\right)+C\le C(1+x)(\text{ from corollary 1}). \end{aligned}$$

  It is worth mentioning that $u(t,x)=\mathbb{E}f(X^x_{T-t})$ is a continuous function in x and bounded by $||f||_\infty$.

- Proof of $|u||_{L^\infty([0,T]\times[0,+\infty))}+\sum_{k=1}^{4}\left|\left|\frac{\partial^k u}{\partial x^k}\right|\right|_{L^\infty([0,T]\times[0,+\infty))}\le C$

  We can point out that $u(t,x)=\mathbb{E}f(X^x_{T-t})$ is a continuous function in x and bounded by $||f||_\infty$. For the proof see Appendices **A**1-**A**4.

- Proof of

$$\begin{cases} \frac{\partial u}{\partial t}(t,x)+Lu(t,x)=0, & (t,x)\in[0,T]\times\mathbb{R}_+ \\ u(T,x)=f(x), & x\in\mathbb{R}_+ \end{cases}$$

  We first observe that

$$\mathbb{E}\left(g(X_t)\frac{\partial}{\partial x}f(X_T)\,|X_t\right)=g(X_t)\frac{\partial}{\partial x}\mathbb{E}\left(f(X_T)\mid X_t\right)=g(X_t)\frac{\partial}{\partial x}u(t,X_t)$$

  for any mesurable and bounded function g. From the homogenous property of the process $(X_t)_{t\ge 0}$, we have

$$g(X_t)\frac{\partial}{\partial x}u(t,X_t)=\mathbb{E}\left(g(X_0)\frac{\partial}{\partial x}f(X_{T-t})\,|X_0\right)=g(X_0)\mathbb{E}\left(\frac{\partial}{\partial x}f(X_{T-t})\,|X_0\right).$$

Admitting the conditional expectation, we have

$$\begin{aligned}\mathbb{E}_x\left(g(X_t)\frac{\partial}{\partial x}u(t,X_t)\right) &= \mathbb{E}_x\left(g(X_0)\mathbb{E}\left(\frac{\partial}{\partial x}f(X_{T-t})\,|X_0\right)\right)\\ &= g(x)\frac{\partial}{\partial x}\mathbb{E}_x(f(X_{T-t})) = g(x)\frac{\partial}{\partial x}u(t,x).\end{aligned}$$

Similarly, we prove that $\mathbb{E}_x\left(g(X_t)\frac{\partial^2}{\partial x^2}u(t,X_t)\right) = g(x)\frac{\partial^2}{\partial x^2}u(t,x)$. We also have

$$\mathbb{E}_x\left(\frac{\partial}{\partial t}u(t,X_t)\right) = \mathbb{E}_x\left(\frac{\partial}{\partial t}\mathbb{E}\left(f(X_{T-t})\mid X_0\right)\right) = \frac{\partial}{\partial t}u(t,x)$$

and $\mathbb{E}_x\left(u(t,X_t)\right) = \mathbb{E}_x\left(\mathbb{E}\left(f(X_{T-t})\mid X_0\right)\right) = u(t,x)$. The Itô formula applied to the function u between t and T gives

$$\begin{aligned}u(T,X_T)-u(t,X_t) &= \int_t^T \frac{\partial u}{\partial s}(s,X_s)ds + \int_t^T b(X_s)\frac{\partial u}{\partial x}(s,X_s)ds\\ &\quad+\frac{\sigma^2}{2}\int_t^T X_s\frac{\partial^2 u}{\partial x^2}(s,X_s)ds + \sigma\int_t^T \sqrt{X_s}\frac{\partial u}{\partial x}(s,X_s)dB_s\\ &+ \int_t^T (u(s,X_{s^-}+a)-u(s,X_{s^-}))dN_s.\end{aligned}$$

Taking the conditional expectation and using the previous results, we have

$$\begin{aligned}\mathbb{E}_x\left(u(T,X_T)-u(t,X_t)\right) &= \int_t^T \left(\frac{\partial u}{\partial s}(s,x)+b(x)\frac{\partial u}{\partial x}(s,x)\right.\\ &\quad\left.+\frac{\sigma^2}{2}x\frac{\partial^2 u}{\partial x^2}(s,x)+\lambda u_a(s,x)\right)ds.\end{aligned}$$

Noticing that $\mathbb{E}_x\left(u(T,X_T)\right) = \mathbb{E}_x\big(\mathbb{E}(f(X_T)\mid X_T)\big) = \mathbb{E}_x\big(f(X_T)\big)$ and $\mathbb{E}_x\left(u(0,X_0)\right) = \mathbb{E}_x\big(\mathbb{E}(f(X_T)\mid X_0)\big) = \mathbb{E}_x\big(f(X_T)\big)$, we deduce that u solves the PDE

$$\frac{\partial u}{\partial s}(s,x)+b(x)\frac{\partial u}{\partial x}(s,x)+\frac{\sigma^2}{2}x\frac{\partial^2 u}{\partial x^2}(s,x)+\lambda u_a(s,x)=0$$

with the condition $u(T,x) = \mathbb{E}(f(X_{T-T})\mid X_0=x) = f(x)$.

$\square$

2.4. Analysis of the Local Term $L_t^0(\widehat{X})$

The following proposition is very helpful since the proof of the weak convergence relies on it.

Proposition 3. *Under Assumptions $(\mathcal{H}_1)-(\mathcal{H}_3)$, and for $\triangle t$ sufficiently small, $\left(\triangle t < \frac{1}{(2K\wedge x_0)}\right)$ there exists a positive constant C depending only on the norm of $f, b(0), T, K, a, \lambda, \sigma$ and x_0 and uniformly in $\triangle t$ such that*

$$\mathbb{E}\left[\left(L_t^0(\widehat{X})-L_{\eta(t)}^0(\widehat{X})\right)|\mathcal{F}_{\eta(t)}\right] \le C\triangle t$$

and

$$\mathbb{E}\left(L_T^0(\widehat{X})\right) \le C\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

To prove this Proposition we need the following Lemma.

Lemma 2. *Under Assumptions* $(\mathcal{H}_1) - (\mathcal{H}_3)$*, and for* $\triangle t$ *sufficiently small,* $\left(\triangle t < \frac{1}{(2K \wedge x_0)}\right)$ *there exists a positive constant* C *depending only on* $f, b(0), T, K, a, \lambda, \sigma$ *and* x_0 *and uniformly in* $\triangle t$ *such that*

$$\sup_{k=0,\dots\dots,N} \mathbb{E} \exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2 \triangle t}\right) \leq C(\gamma) \left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}(1-\frac{1}{2\gamma})} \tag{19}$$

Proof of Lemma 2. Let $\gamma \geq 1$. We recall here that $\forall k \in \{1, ..., N+1\}$
$\widehat{X}_{t_{k+1}} = \left| \widehat{X}_{t_k} + b(\widehat{X}_{t_k})\triangle t + \sigma\sqrt{\widehat{X}_{t_k}} \triangle B_k + a\triangle N_k \right|$ with $\triangle B_k = B_{t_k} - B_{t_{k-1}}$ and $\triangle N_k = N_{t_k} - N_{t_k}$.
For all $\mu > 0$,

$$\begin{aligned}
\mathbb{E}\exp\left(-\mu \widehat{X}_{t_{k+1}}\right) &\leq \mathbb{E}\exp\left(-\mu\left(\widehat{X}_{t_k} + b(\widehat{X}_{t_k})\triangle t + \sigma\sqrt{\widehat{X}_{t_k}}\triangle B_k + a\triangle N_k\right)\right) \\
&\leq \mathbb{E}\exp\left(-\mu\left(\widehat{X}_{t_k} + b(0)\triangle t - K\triangle t\widehat{X}_{t_k} + \sigma\sqrt{\widehat{X}_{t_k}}\triangle B_k + a\triangle N_k\right)\right) \\
&\leq \mathbb{E}\exp\left(-\mu\left(\widehat{X}_{t_k} + b(0)\triangle t - K\triangle t\widehat{X}_{t_k} + \sigma\sqrt{\widehat{X}_{t_k}}\triangle B_k\right)\right).
\end{aligned}$$

This last inequality comes from the fact that knowing $\widehat{X}_{t_k}$; $\triangle B_k$ and $\triangle N_k$ are independent; and keeping in mind that $\exp(-\mu a\triangle N_k) \leq 1$. The rest of the proof is similar to that of Lemma 3.6 in [2]. $\square$

We now return to the proof of proposition 3.

Proof of Proposition 3. For all $t \in [t_k, t_{k+1}[$, $0 \leq k \leq N-1$ and for all measurable and bounded function ϕ, according to the occupation time formula for semi-martingales (see Corollary 9.7 in [18]), we have $\mathbb{P}$- a.s

$$\int_{\mathbb{R}} \phi(z)\left(L_t^z(\widehat{Y}) - L_{t_k}^z(\widehat{Y})\right)dz = \int_{t_k}^t \phi(\widehat{Y}_{s^-})d\left[\widehat{Y}, \widehat{Y}\right]_s^c = \sigma^2\int_{t_k}^t \phi(\widehat{Y}_{s^-})\widehat{X}_{t_k}ds.$$

Since $\int_{\mathbb{R}} \phi(z)\left(L_t^z(\widehat{X}) - L_{t_k}^z(\widehat{X})\right)dz = \int_{\mathbb{R}} \phi(z)\left(L_t^z(\widehat{Y}) - L_{t_k}^z(\widehat{Y})\right)dz$, we deduce that

$$\int_{\mathbb{R}} \phi(z)\left(L_t^z(\widehat{X}) - L_{t_k}^z(\widehat{X})\right)dz = \sigma^2\int_{t_k}^t \phi(\widehat{Y}_{s^-})\widehat{X}_{t_k}ds$$

For all $x > 0$,

$$\int_{\mathbb{R}} \phi(z)\mathbb{E}\left(L_t^z(\widehat{X}) - L_{t_k}^z(\widehat{X}) \Big| \{\widehat{X}_{t_k} = x\}\right)dz = \sigma^2\int_{t_k}^t x\mathbb{E}\left(\phi(\widehat{Y}_{s^-}) \Big| \{\widehat{X}_{t_k} = x\}\right)ds := J.$$

where ':=' is used as the definition of a given quantity. Substituting $\widehat{Y}_{s^-}$ by its expression given by (7) we obtain:

$$\begin{aligned}
J &= \sigma^2\int_{t_k}^t x\mathbb{E}\Big(\phi(\widehat{X}_{t_k} + b(\widehat{X}_{t_k})(s-t_k) + \sigma\sqrt{\widehat{X}_{t_k}}(B_s - B_{t_k}) \\
&+ a(N_s - N_{t_k}))\Big|\{\widehat{X}_{t_k} = x\}\Big)ds \\
&= \sigma^2\int_{t_k}^t x\mathbb{E}\left(\phi\left(x + b(x)(s-t_k) + \sigma\sqrt{x}(B_s - B_{t_k}) + a\left(N_s - N_{t_k}\right)\right)\right)ds.
\end{aligned}$$

If we set

$$A(n) := \mathbb{E}\left(\phi\left(x + b(x)(s-t_k) + \sigma\sqrt{x}(B_s - B_{t_k}) + a(N_s - N_{t_k})\right) \mid \{N_s - N_{t_k} = n\}\right),$$

we have for all $n \in \mathbb{N}$,

$$\begin{aligned}
A(n) &= \mathbb{E}\left(\phi\left(x + b(x)(s-t_k) + \sigma\sqrt{x}(B_s - B_{t_k}) + a(N_s - N_{t_k})\right) |\{N_s - N_{t_k} = n\}\right) \\
&= \int_{\mathbb{R}} \phi\left(x + b(x)(s-t_k) + \sigma y\sqrt{x} + an\right) \frac{1}{\sqrt{2\pi(s-t_k)}} \exp\left(-\frac{y^2}{2(s-t_k)}\right) dy \\
&= \int_{\mathbb{R}} \phi(z) \frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}} \exp\left(-\frac{(z - x - b(x)(s-t_k) - an)^2}{2\sigma^2 x(s-t_k)}\right) dz.
\end{aligned}$$

We then obtain

$$\begin{aligned}
& \mathbb{E}\left(\phi\left(x + b(x)(s-t_k) + \sigma\sqrt{x}(B_s - B_{t_k}) + a(N_s - N_{t_k})\right)\right) = \mathbb{E}(A(N_s - N_{t_k})) \\
= & \int_{\mathbb{R}} \phi(z) \frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(z - x - b(x)(s-t_k) - an)^2}{2\sigma^2 x(s-t_k)}\right) \\
& \times \mathbb{P}(\{N_s - N_{t_k} = n\}) dz.
\end{aligned}$$

We finally deduce that

$$\begin{aligned}
& \int_{\mathbb{R}} \phi(z) \mathbb{E}\left(L_t^z(\widehat{X}) - L_{t_k}^z(\widehat{X}) \Big| \{\widehat{X}_{t_k} = x\}\right) dz \\
= & \sigma^2 \int_{t_k}^t x \int_{\mathbb{R}} \phi(z) \frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(z - x - b(x)(s-t_k) - an)^2}{2\sigma^2 x(s-t_k)}\right) \\
& \times \mathbb{P}(\{N_s - N_{t_k} = n\}) dz ds \\
= & \int_{\mathbb{R}} \phi(z) \sigma \int_{t_k}^t \frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(z - x - b(x)(s-t_k) - an)^2}{2\sigma^2 x(s-t_k)}\right) \\
& \times \mathbb{P}(\{N_s - N_{t_k} = n\}) ds dz.
\end{aligned}$$

The previous equality being true for all ϕ, by setting $B(x,z) = \mathbb{E}\left(L_t^z(\widehat{X}) - L_{t_k}^z(\widehat{X}) \mid \{\widehat{X}_{t_k} = x\}\right)$, we deduce that

$$\begin{aligned}
B(x) &= \sigma \int_{t_k}^t \frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(z - x - b(x)(s-t_k) - an)^2}{2\sigma^2 x(s-t_k)}\right) \\
& \quad \times \mathbb{P}(\{N_s - N_{t_k} = n\}) ds.
\end{aligned}$$

For $z = 0$

$$\begin{aligned}
B(x,0) &= \mathbb{E}\left(L_t^0(\widehat{X}) - L_{t_k}^0(\widehat{X}) \Big| \{\widehat{X}_{t_k} = x\}\right) \\
&= \sigma \int_{t_k}^t \frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(x + b(x)(s-t_k) + an)^2}{2\sigma^2 x(s-t_k)}\right) \\
& \quad \times \exp(-\lambda(s-t_k)) \frac{(\lambda(s-t_k))^n}{n!} ds \\
&= \sigma \int_0^{\triangle t} \frac{\sqrt{x}}{\sqrt{2\pi u}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(x + b(x)u + an)^2}{2\sigma^2 xu}\right) \exp(-\lambda u) \frac{(\lambda u)^n}{n!} du
\end{aligned}$$

where the integral is obtained by setting $u = s - t_k$. From the Lipshitz condition of b, we have $b(x) \geq -Kx \Longrightarrow x + b(x)u + an \geq x(1 - Ku) + an$. Since $u \in [0, \triangle t]$, we have $x(1 - Ku) + an \geq$

$x(1-Ku)+an$. For $1+K\triangle t \geq 1/2$, we have $x+b(x)u+an \geq 1/2+an$. It then follows that $\frac{1}{2\sigma^2 xu}(x+b(x)u+an)^2 \geq \frac{1}{2\sigma^2 xu}(\frac{1}{4}x^2+xan+a^2n^2)$ and

$$\exp\left[-\frac{1}{2\sigma^2 xu}(x+b(x)x+an)^2\right] \leq \exp\left[-\frac{1}{8\sigma^2 u}\right] \text{ since } xan+a^2n^2 \geq 0.$$

We can now write that

$$\begin{aligned}
B(X_{t_k},0) &= \sigma\int_0^{\triangle t}\frac{\sqrt{X_{t_k}}}{\sqrt{2\pi u}}\sum_{n\in\mathbb{N}}\exp\left(-\frac{X_{t_k}}{8\sigma^2 u}\right)\times\exp(-\lambda u)\frac{(\lambda u)^n}{n!}du. \\
&= \sigma\int_0^{\triangle t}\frac{\sqrt{X_{t_k}}}{\sqrt{2\pi u}}\exp\left(-\frac{X_{t_k}}{8\sigma^2 u}\right)du \leq \sigma\int_0^{\triangle t}\frac{\sqrt{X_{t_k}}}{\sqrt{2\pi u}}\exp\left(-\frac{X_{t_k}}{8\sigma^2\triangle t}\right)du \\
&\leq 2\sigma\sqrt{\triangle t}\frac{\sqrt{X_{t_k}}}{\sqrt{2\pi}}\exp\left(-\frac{X_{t_k}}{8\sigma^2\triangle t}\right).
\end{aligned}$$

Now using the fact that $x\exp(-\frac{x^2}{c}), \leq \sqrt{c/2}\exp(-1/2) \quad \forall x\in\mathbb{R}$ and $c>0$ it comes that

$$\begin{aligned}
\sqrt{x}\exp\left(-\frac{x}{c}\right) &= \sqrt{x}\exp\left(-\frac{x}{2c}\right)\exp\left(-\frac{x}{2c}\right) \\
&= \sqrt{c}.\sqrt{\frac{x}{c}}\exp\left(-\frac{\left(\sqrt{\frac{x}{c}}\right)^2}{2}\right)\exp\left(-\frac{x}{2c}\right) \\
&\leq \sqrt{c}.\exp(-1/2).\exp\left(-\frac{x}{2c}\right) \leq \sqrt{c}\exp\left(-\frac{x}{2c}\right).
\end{aligned}$$

For $c=8\sigma^2\triangle t$, the expression of $B(X_{t_k},0)$ becomes

$$B(X_{t_k},0) \leq \sqrt{\frac{8\sigma^2\triangle t}{2\pi}}\exp\left(-\frac{X_{t_k}}{16\sigma^2\triangle t}\right)2\sigma\sqrt{\triangle t} = \frac{4\sigma^2\triangle t}{\sqrt{\pi}}\exp\left(-\frac{X_{t_k}}{16\sigma^2\triangle t}\right)$$

$$B(X_{t_k},0) \leq C.\triangle t.\exp\left(-\frac{X_{t_k}}{16\sigma^2\triangle t}\right)$$

From Markov hypothesis, we have

$$\mathbb{E}\left(L_t^0(\widehat{X})-L_{t_k}^0(\widehat{X}) \mid \mathcal{F}_{t_k}\right) = B(X_{t_k},0) \leq C.\triangle t.\exp\left(-\frac{X_{t_k}}{16\sigma^2\triangle t}\right)$$

Then we deduce that

$$\begin{aligned}
\mathbb{E}\left(L_t^0(\widehat{X})-L_{t_k}^0(\widehat{X})\right) &= \mathbb{E}(B(X_{t_k},0)) \leq C.\triangle t.\mathbb{E}\exp\left(-\frac{X_{t_k}}{16\sigma^2\triangle t}\right) \\
&\leq C.\triangle t.\sup_{0\leq k\leq N}\mathbb{E}\exp\left(-\frac{X_{t_k}}{16\sigma^2\triangle t}\right)
\end{aligned} \tag{20}$$

Using Lemma 2 with $\gamma=16$ we have:

$$\mathbb{E}\left(L_{t_{k+1}}^0(\widehat{X})-L_{t_k}^0(\widehat{X})\right) \leq C.\triangle t.\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

Taking the summation on all k we obtain:

$$\mathbb{E}\left(L_T^0(\widehat{X})\right) \leq C(T)\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

$\square$

Proposition 4. *Under assumptions $(\mathcal{H}_1)-(\mathcal{H}_3)$, there exists a constant C uniform in $\triangle t$ such that for all $t \in [0,T]$, we have*

$$\mathbb{P}\left(\widehat{Y}_t \le 0\right) \le C\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

Proof of Proposition 4. For all $t \in [0,T]$, if we set $\Gamma(s) := \mathbb{P}\left(\widehat{Y}_t \le 0 \mid \widehat{X}_{\eta(s)}\right)$ we have

$$\begin{aligned}
\Gamma(s) &= \sum_{n\in\mathbb{N}} \mathbb{P}\left(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(B_t - B_{\eta(t)})\right. \\
&\quad \left. + \ an \le 0, \mid \widehat{X}_{\eta(t)}\right) \mathbb{P}\left((N_t - N_{\eta(t)}) = n \mid \widehat{X}_{\eta(t)}\right) \\
&\le \sum_{n\in\mathbb{N}} \mathbb{P}\left(\sigma\sqrt{\widehat{X}_{\eta(t)}}(B_t - B_{\eta(t)}) \le -\widehat{X}_{\eta(t)} - b(\widehat{X}_{\eta(t)})(t-\eta(t)) - an \middle| \widehat{X}_{\eta(t)}\right) \\
&\quad \times \mathbb{P}\left((N_t - N_{\eta(t)}) = n \middle| \widehat{X}_{\eta(t)}\right) \\
&\le \sum_{n\in\mathbb{N}} \mathbb{P}\left((B_t - B_{\eta(t)}) \le \frac{-\widehat{X}_{\eta(t)} - b(\widehat{X}_{\eta(t)})(t-\eta(t)) - an}{\sigma\sqrt{\widehat{X}_{\eta(t)}}} \middle| \widehat{X}_{\eta(t)}\right) \\
&\quad \times \mathbb{P}\left((N_t - N_{\eta(t)}) = n\right).
\end{aligned}$$

Using the Gaussian inequality (if $Z \sim \mathcal{N}(0,1)$ and β a negative real number, then $\mathbb{P}(Z \le \beta) \le \frac{1}{2}\exp\left(-\frac{\beta^2}{2}\right)$) with $\beta = -\frac{\left(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + an\right)^2}{2\sigma^2(t-\eta(t))\widehat{X}_{\eta(t)}}$. We therefore obtain:

$$\begin{aligned}
\Gamma(s) &\le \sum_{n\in\mathbb{N}} \exp\left(-\frac{\left(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + an\right)^2}{2\sigma^2(t-\eta(t))\widehat{X}_{\eta(t)}}\right) \mathbb{P}\left((N_t - N_{\eta(t)}) = n\right) \\
&\le \sum_{n\in\mathbb{N}} \exp\left(-\frac{\left(\widehat{X}_{\eta(t)}(1 - K\triangle t) + an\right)^2}{2\sigma^2.\triangle t.\widehat{X}_{\eta(t)}}\right) \mathbb{P}\left((N_t - N_{\eta(t)}) = n\right) \\
&\le \exp\left(-\frac{\widehat{X}_{\eta(t)}}{8\sigma^2.\triangle t}\right) \ \text{(since } 1 + K\Delta t \ge 1/2).
\end{aligned}$$

We also have

$$\mathbb{P}\left(\widehat{Y}_t \le 0\right) = \mathbb{E}\left(\Gamma(s)\right) \le \frac{1}{2}\mathbb{E}\exp\left(-\frac{\widehat{X}_{\eta(t)}}{8\sigma^2\triangle t}\right).$$

Using Lemma 2 for $\gamma = 8$ we do obtain

$$\mathbb{P}\left(\widehat{Y}_t \le 0\right) \le C(T)\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

where $C(T)$ is a positive constant not depending on $\triangle t$. ☐

3. Weak Congervence

The main result of this paper is given by the following theorem.

Theorem 1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, belongs to $\mathcal{C}^4$, with derivatives up to order 4 which are bounded. Let $(X_t)_{0\leq t\leq T}$, be stochastic process solution of (1), and $\left(\widehat{X}_t\right)_{0\leq t\leq T}$, the scheme $(X_t)_{0\leq t\leq T}$. Under assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$, there exists a positive constant $C(T)$ not depending on $b(0), \sigma, x_0, T, a, \lambda$ and K the lipshitz function $b(.)$, such that*

$$\left|\mathbb{E}f(X_T) - \mathbb{E}(f(\widehat{X}_T))\right| \leq C\left(\triangle t + \left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right). \tag{21}$$

Proof of Theorem 1. To prove this theorem we proceed in three steps:

Step 1: We need to find the expression of $\varepsilon_T = \mathbb{E}f(X_T) - \mathbb{E}(f(\widehat{X}_T))$.

By definition, we have $\mathbb{E}\left(f(X_T)\right) = u(0, x_0)$, $f(\widehat{X}_T) = u(T, \widehat{X}_T)$ and $\varepsilon_T = \mathbb{E}f(X_T) - \mathbb{E}(f(\widehat{X}_T)) = u(0, x_0) - u(T, \widehat{X}_T)$. By applying Itô formula on u we have

$$\begin{aligned}
u(T,\widehat{X}_T) &= u(0,x_0) + \int_0^T \frac{\partial u}{\partial t}(s,\widehat{X}_s)ds + \int_0^T \mathrm{sgn}(Y_{s^-})b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds \\
&+ \sigma\int_0^T \mathrm{sgn}(Y_{s^-})\sqrt{\widehat{X}_{\eta(s)}}\frac{\partial u}{\partial x}(s,\widehat{X}_s)dB_s + \frac{\sigma^2}{2}\int_0^T \widehat{X}_s\frac{\partial^2 u}{\partial^2 x}(s,\widehat{X}_s)ds \\
&+ \int_0^T \frac{\partial u}{\partial t}(s,\widehat{X}_s)dL_s(\widehat{X}) + \int_0^T \left(u(s,\widehat{X}_s + \mathrm{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s)\right)dN_s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varepsilon_T &= -\mathbb{E}\left[\int_0^T \frac{\partial u}{\partial t}(s,\widehat{X}_s)ds + \int_0^T \mathrm{sgn}(Y_{s^-})b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds + \frac{\sigma^2}{2}\int_0^T \widehat{X}_s\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)ds\right. \\
&+ \left.\int_0^T \frac{\partial u}{\partial x}(s,\widehat{X}_s)dL_s(\widehat{X}) + \int_0^T \left(u(s,\widehat{X}_s + \mathrm{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s)\right)dN_s\right].
\end{aligned}$$

Since u is the solution of Cauchy problem described in (18), it follows that

$$\frac{\partial u}{\partial t}(T,\widehat{X}_T) = -b(\widehat{X}_s)\frac{\partial u}{\partial x}(s,\widehat{X}_s) - \frac{\sigma^2}{2}\widehat{X}_s\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s) - \lambda\left(u(s,\widehat{X}_s + a) - u(s,\widehat{X}_s)\right).$$

Using the fact that $\mathrm{sgn}(x) = 1 - 2.\mathbb{I}_{\{x\leq 0\}}$, we have

$$\begin{aligned}
\epsilon_T &= \mathbb{E}\int_0^T \left(b(\widehat{X}_s) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds + \frac{\sigma^2}{2}\mathbb{E}\int_0^T \left(\widehat{X}_s - \widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)ds \\
&+ \int_0^T \mathbb{E}\left[u(s,\widehat{X}_s + \mathrm{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s + a)\right]ds \\
&+ 2\mathbb{E}\int_0^T \left(1_{\{\widehat{Y}_{s^-}\leq 0\}}b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\widehat{X}_s)\right)ds - \mathbb{E}\int_0^T \frac{\partial u}{\partial x}(s,\widehat{X}_s)dL_s(\widehat{X}) \\
&= \epsilon_T^1 + \epsilon_T^2 + \epsilon_T^3 + \epsilon_T^4 + \epsilon_T^5.
\end{aligned}$$

where $\epsilon_T^1 = \mathbb{E}\int_0^T \left(b(\widehat{X}_s) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds$, $\epsilon_T^2 = \mathbb{E}\int_0^T \left(\widehat{X}_s - \widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)ds$, $\epsilon_T^3 = \int_0^T \mathbb{E}\left(1_{\{\widehat{Y}_s\leq 0\}}b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right)ds$, $\epsilon_T^4 = \int_0^T \mathbb{E}\left[u(s,\widehat{X}_s + \mathrm{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s + a)\right]ds$, and $\epsilon_T^5 = \int_0^T \mathbb{E}\frac{\partial u}{\partial x}(s,\widehat{X}_s)dL_s(\widehat{X})$.

Step 2: In this step we analyse each term of the previous expression

* **Analysis of the term** $\epsilon_T^1 = \mathbb{E}\int_0^T \left(b(\widehat{X}_s) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds$

Let's consider the function $x \longmapsto \left(b(x) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,x)$. This function belongs to $\mathcal{C}^{1,2}$, and from Itô formula we have

$$\begin{aligned}
\left(b(\widehat{X}_s) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_s) &= \int_{\eta(s)}^s \frac{\partial\left[\left(b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x}(s,\widehat{X}_\theta)d\widehat{X}_\theta^c \\
&+ \frac{1}{2}\int_{\eta(s)}^s \frac{\partial^2\left[\left(b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x^2}(s,\widehat{X}_\theta)d[\ \widehat{X}^c,\widehat{X}^c]_\theta \\
&+ \int_{\eta(s)}^s \left[\left(b(\widehat{X}_\theta + a) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta + a)\right. \\
&- \left.\left(b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)\right]dN_\theta \\
&= A_1 + A_2 + A_3
\end{aligned}$$

where $A_1 = \int_{\eta(s)}^s \frac{\partial\left[\left(b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x}(s,\widehat{X}_\theta)d\widehat{X}_\theta^c$; $A_2 = \int_{\eta(s)}^s \frac{\partial^2\left[\left(b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x^2}(s,\widehat{X}_\theta)d[\ \widehat{X}^c,\widehat{X}^c]_\theta$; $A_3 = \int_{\eta(s)}^s \left[\left(b(\widehat{X}_\theta + a) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta + a) - \left(b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)\right]dN_\theta$.

Replacing $d\widehat{X}_\theta^c$ by its expression, A_1 becomes

$$\begin{aligned}
A_1 &= \int_{\eta(s)}^s \operatorname{sgn}(\widehat{Y}_{\theta^-})b(\widehat{X}_{\eta(s)})\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta) + (b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)d\theta \\
&+ \sigma\int_{\eta(s)}^s \operatorname{sgn}(\widehat{Y}_{\theta^-})\sqrt{\widehat{X}_{\eta(s)}}\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta) + (b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)dB_\theta \\
&+ \int_{\eta(s)}^s \left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta) + (b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)dL_\theta.
\end{aligned}$$

By using the fact that $\operatorname{supp}\left(dL(\widehat{X})\right) = \left\{t \in [0,T]\,/\widehat{X}_t = \widehat{X}_{t^-} = 0\right\}$ (see Theorem 9.12 in [18]) and taking the expectation, we have

$$\begin{aligned}
\mathbb{E}(A_1) &= \mathbb{E}\left[\int_{\eta(s)}^s \operatorname{sgn}(\widehat{Y}_\theta)b(\widehat{X}_{\eta(s)})\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta) + (b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)d\theta\right] \\
&+ \mathbb{E}\left[\int_{\eta(s)}^s \left(b'(0)\frac{\partial u}{\partial x}(s,0) + (b(0) - b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,0)\right)dL_\theta\right].
\end{aligned}$$

From the Lipschitz condition on b, we have, $|b(\widehat{X}_{\eta(s)})| \leq b(0) + K\widehat{X}_{\eta(s)}$. Futhermore all the derivatives of b up to order 4 are bounded and it is the same for u and its own derivatives.

$$\begin{aligned}
\mathbb{E}(A_1) &\leq \int_{\eta(s)}^s \mathbb{E}\left[\left(b(0) + K\widehat{X}_{\eta(s)}\right)\left\{\|b'\|_\infty\left\|\frac{\partial u}{\partial x}\right\|_\infty + K\left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty\left(\widehat{X}_\theta + \widehat{X}_{\eta(s)}\right)\right\}\right]d\theta \\
&+ \int_{\eta(s)}^s \mathbb{E}\left[\left\{\|b'\|\left\|\frac{\partial u}{\partial x}\right\| + K\left\|\frac{\partial^2 u}{\partial x^2}\right\|\widehat{X}_{\eta(s)}\right\}\right]dL_\theta(\widehat{X}) \\
&\leq C\triangle t\left(1 + \mathbb{E}\left(\sup_{0\leq\theta\leq T}\widehat{X}_\theta\right) + \sup_{0\leq\theta\leq T}\mathbb{E}\widehat{X}_\theta^2\right) \\
&+ C\mathbb{E}\left\{\left(1 + \widehat{X}_{\eta(s)}\right)\left(L_s(\widehat{X}) - L_{\eta(s)}(X)\right)\right\} \leq C.\triangle t + B_1
\end{aligned}$$

where the last inequality comes from Proposition 1. The term B_1 can be analysed as follows

$$\begin{aligned}
B_1 &= \mathbb{E}\left(\left(1+\widehat{X}_{\eta(s)}\right)\mathbb{E}\left\{\left(L_s(\widehat{X})-L_{\eta(s)}(X)\right)|\mathcal{F}_{\eta(s)}\right\}\right)\\
&\leq C\mathbb{E}\left(\left(1+\widehat{X}_{\eta(s)}\right)\mathbb{E}\exp\left(\frac{-\widehat{X}_{\eta(s)}}{16\sigma^2\triangle t}\right)\right)\\
&\leq C\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}} \text{(from Proposition 2)}
\end{aligned}$$

We then conclude that

$$|\mathbb{E}(A_1)| \leq C\left(\triangle t+\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{22}$$

where C_p is a positive constant independent of $\triangle t$.

Replacing $d[\ \widehat{X}^c, \widehat{X}^c]_\theta$ by its expression in A_2, yields

$$\begin{aligned}
A_2 &= \frac{1}{2}\int_{\eta(s)}^{s}\left(b''(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right.\\
&\quad\left.+\,(b(\widehat{X}_s)-b(\widehat{X}_{\eta(s)}))\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)d[\ \widehat{X},\widehat{X}]^c_\theta\\
&= \frac{\sigma^2}{2}\int_{\eta(s)}^{s}\widehat{X}_{\eta(s)}\left(b''(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right.\\
&\quad\left.+\,(b(\widehat{X}_s)-b(\widehat{X}_{\eta(s)}))\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)d\theta.
\end{aligned}$$

Based on Proposition 1, we prove that $|\mathbb{E}\,(A_2)\,|\leq C\triangle t$. The term A_3 is also proportional to $\triangle t$ and this can be proved as follows:

$$|\mathbb{E}\,(A_3)\,| \leq \int_{\eta(s)}^{s}\lambda\mathbb{E}\left[\left\|\frac{\partial u}{\partial x}\right\|\left\{2K\left(\widehat{X}_\theta+\widehat{X}_{\eta(s)}\right)+Ka\right\}\right]\leq C_p\triangle t$$

Combining the inequalities of A_1, A_2 and A_3 we have

$$\epsilon_T^1=\left|\int_0^T\mathbb{E}\left((b(\widehat{X}_s)-b(\widehat{X}_{\eta(s)}))\frac{\partial u}{\partial x}(s,\widehat{X}_s)\right)ds\right|\leq C(T)\left(\triangle t+\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{23}$$

* **Analysis of the term** $\epsilon_T^2=\mathbb{E}\int_0^T\left(\widehat{X}_s-\widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)ds$

Following the same approach as before, applying the Itô formula with the function $\phi : x \longmapsto (x-\widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,x)$ between $\eta(s)$ and s, we have:

$$\begin{aligned}
\phi(\widehat{X}_s) &= \phi(\widehat{X}_{\eta(s)})+\int_{\eta(s)}^{s}\phi'(s,\widehat{X}_\theta)d\widehat{X}^c_\theta+\frac{1}{2}\int_{\eta(s)}^{s}\phi''(s,\widehat{X}_\theta)d[\widehat{X},\widehat{X}]^c_\theta\\
&\quad+\int_{\eta(s)}^{s}\left[\phi\left(\widehat{X}_\theta+\operatorname{sgn}(\widehat{Y}_{\theta-})a\right)-\phi\left(\widehat{X}_\theta+\operatorname{sgn}(\widehat{Y}_{\theta-})a\right)\right]dN_\theta.
\end{aligned}$$

Since $\phi(\widehat{X}_{\eta(s)})=0$, we have

$$\begin{aligned}
\phi(\widehat{X}_s) &= \int_{\eta(s)}^{s} \operatorname{sgn}(\widehat{Y}_\theta) b(\widehat{X}_{\eta(s)}) \left(\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta) + (\widehat{X}_\theta - \widehat{X}_{\eta(s)})\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right) d\theta \\
&+ \int_{\eta(s)}^{s} \operatorname{sgn}(\widehat{Y}_\theta)\sqrt{\widehat{X}_{\eta(s)}} \left(\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta) + (\widehat{X}_\theta - \widehat{X}_{\eta(s)})\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right) dB_\theta \\
&+ \int_{\eta(s)}^{s} \left(\frac{\partial^2 u}{\partial x^2}(s,0) - \widehat{X}_{\eta(s)}\frac{\partial^3 u}{\partial x^3}(s,0)\right) dL_\theta(\widehat{X}) \\
&+ \frac{\sigma^2}{2}\int_{\eta(s)}^{s} \widehat{X}_{\eta(s)} \left((\widehat{X}_\theta - \widehat{X}_{\eta(s)})\frac{\partial^4 u}{\partial x^4}(s,\widehat{X}_\theta) + 2\frac{\partial^3 u}{\partial^3 x}(s,\widehat{X}_\theta)\right) d\theta \\
&+ \int_{\eta(s)}^{s} \Bigg[\left((\widehat{X}_\theta + \operatorname{sgn}(\widehat{Y}_{s-})a - \widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta + \operatorname{sgn}(\widehat{Y}_{s-})a)\right) \\
&\qquad - \left((\widehat{X}_\theta - \widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)\Bigg] dN_\theta.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}(\phi(\widehat{X}_s)) &= \int_{\eta(s)}^{s} \mathbb{E}\left[\operatorname{sgn}(\widehat{Y}_\theta) b(\widehat{X}_{\eta(s)}) \left(\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta) + (\widehat{X}_\theta - \widehat{X}_{\eta(s)})\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)\right] d\theta \\
&+ \frac{\sigma^2}{2}\int_{\eta(s)}^{s} \mathbb{E}\left[\widehat{X}_{\eta(s)} \left((\widehat{X}_\theta - \widehat{X}_{\eta(s)})\frac{\partial^4 u}{\partial x^4}(s,\widehat{X}_\theta) + 2\frac{\partial^3 u}{\partial^3 x}(s,\widehat{X}_\theta)\right)\right] d\theta \\
&+ \int_{\eta(s)}^{s} \mathbb{E}\left[\left(\frac{\partial^2 u}{\partial x^2}(s,0) - \widehat{X}_{\eta(s)}\frac{\partial^3 u}{\partial x^3}(s,0)\right)\right] dL_\theta(\widehat{X}) \\
&+ \int_{\eta(s)}^{s} \lambda\mathbb{E}\Bigg[\left((\widehat{X}_\theta + \operatorname{sgn}(\widehat{Y}_{s-})a - \widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta + \operatorname{sgn}(\widehat{Y}_{s-})a)\right) \\
&\qquad - \left((\widehat{X}_\theta - \widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)\Bigg] d\theta.
\end{aligned}$$

Similarly, we have

$$\left|\int_0^T \mathbb{E}\left(\widehat{X}_s - \widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\widehat{X}_s)\right) ds\right| \le C(T)\left(\triangle t + \left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right). \tag{24}$$

* **Analysis of the term** $\epsilon_T^3 = \int_0^T \mathbb{E}\left(\mathbb{I}_{\{\widehat{Y}_s\le 0\}} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right) ds$. We have

$$\begin{aligned}
\int_0^T \mathbb{E}\left(\mathbb{I}_{\{\widehat{Y}_s\le 0\}} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right) ds &\le \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[\mathbb{E}\left(1_{\{\widehat{Y}_{s^-}\le 0\}} b(\widehat{X}_{\eta(s)}) \,\middle|\, \widehat{X}_{\eta(s)}\right)\right] ds \\
&\le \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[b(\widehat{X}_{\eta(s)})\mathbb{E}\left(\mathbb{I}_{\{\widehat{Y}_{s^-}\le 0\}} \,\middle|\, \widehat{X}_{\eta(s)}\right)\right] ds \\
&\le \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[b(\widehat{X}_{\eta(s)})\mathbb{P}\left(\left\{\widehat{Y}_{s^-}\le 0\right\} \,\middle|\, \widehat{X}_{\eta(s)}\right)\right] ds \\
&\le \frac{1}{2}\left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[b(\widehat{X}_{\eta(s)})\exp\left(-\frac{\widehat{X}_{\eta(s)}}{8\sigma^2\triangle t}\right)\right] ds
\end{aligned}$$

Therefore,

$$\left|\int_0^T \mathbb{E}\left(\mathbb{I}_{\widehat{Y}_s<0} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right) ds\right| \le \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[\left(b(0)+K\widehat{X}_{\eta(s)}\right)\exp\left(-\frac{\widehat{X}_{\eta(s)}}{8\sigma^2\triangle t}\right)\right] ds.$$

Since for all x, we have $x\exp(-x/c) \leq c$, we can write that $\widehat{X}_{\eta(s)} \exp\left(-\frac{\widehat{X}_{\eta(s)}}{8\sigma^2 \triangle t}\right) \leq 8\sigma^2 \triangle t$,

$$\left|\int_0^T \mathbb{E}\left(\mathbb{I}_{\widehat{Y}_s<0} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right) ds\right| \leq C(T)\left(\triangle t + \left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right). \tag{25}$$

* **Analysis of the term** $\epsilon_T^4 = \int_0^T \mathbb{E}\left[u(s,\widehat{X}_s + \text{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s + a)\right] ds = \int_0^T \mathbb{E}(B_2(s))ds$ where $B_2(s) = \left|\mathbb{E}\left[u(s,\widehat{X}_s + \text{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s + a)\right]\right|$. We then have

$$\begin{aligned}
B_2(s) &\leq \left|\mathbb{E}\left[\left(u(s,\widehat{X}_s + \text{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s + a)\right)\mathbb{I}_{\{\widehat{Y}_{s^-}>0\}}\right]\right| \\
&+ \left|\mathbb{E}\left[\left(u(s,\widehat{X}_s + \text{sgn}(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s + a)\right)\mathbb{I}_{\{\widehat{Y}_{s^-}\leq 0\}}\right]\right| \\
&\leq 2\left\|u\right\|_\infty \mathbb{P}\left(\left\{\widehat{Y}_{s^-} \leq 0\right\}\right) \leq C(T)\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.
\end{aligned}$$

Then

$$\epsilon_T^4 = \left|\int_0^T \mathbb{E}\left[u(s,\widehat{X}_s + sgn(\widehat{Y}_{s^-})a) - u(s,\widehat{X}_s + a)\right] ds\right| \leq C(T)\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}} \tag{26}$$

* **Analysis of the term** $\epsilon_T^5 = \int_0^T \mathbb{E}\frac{\partial u}{\partial x}(s,\widehat{X}_s)dL_s(\widehat{X})$
We have

$$\left|\int_0^T \mathbb{E}\frac{\partial u}{\partial x}(s,\widehat{X}_s)dL_s(\widehat{X}_s)\right| \leq T\left\|\frac{\partial u}{\partial x}\right\|_\infty \mathbb{E}\left(L_T(\widehat{X})\right) \leq C(T)\left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}$$

which is obtained from Proposition 3.

Step 3: Hence, from inequalities (24), (25) and (26) we obtain

$$|\mathbb{E}(f(X_t) - f(\widehat{X}_T))| \leq C\left(\triangle t + \left(\frac{\triangle t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right)$$

□

4. A Simulation Study

4.1. The Design

The goal of the simulation is to assess the influence of the jump size on the weak error term since contrary to the weak convergence without jump, the constant $C(T)$ depends on the size of jump a. We evaluate the error term ε_T for $f(x) = x^2$, that is

$$\varepsilon_T = \mathbb{E}(X_T^2) - \mathbb{E}(\widehat{X}_T^2) \approx \mathbb{E}(X_T^2) - \frac{1}{\text{Nsim}}\sum_{i=1}^{\text{Nsim}} \mathbb{E}(\widehat{X}_T^i)^2 \tag{27}$$

where $\mathbb{E}(X_T^2)$ is given in Appendix A1 and $\widehat{X}_T^{i,2}$ is the i^{th} simulated value obtained by the scheme (5) and Nsim represents the size of the simulation. We first describe in Figures ((a) to (d)) the evolution of the process for some values of a.

4.2. Results

We simulate the process using the numerical scheme (5) with $x_0 = 0.05$ and $a \in \{0, 0.5, 1, 1.5, 5, 10\}$. We generated the Poisson process with the parameter $\lambda = 2$ and produce the jump term. We then evaluate the weak error simulation techniques based on equation (27). Results can be summarized as follows:

(i) Figure (a) to (d) describe the evolution of various time series without jump (*Nb_jump*=0), with jump(s) (*Nb_jump*=1,2,4) and for different jump sizes.

(ii) Figure (e) to (f) shows the error with respect to time step, where we consider $\frac{b(0)}{\sigma^2} > 1$. When we compare the error line with the reference line $y = x$, we observe that, the two lines are parallel, meaning that, the weak error order is almost 1 just like the classical Euler Scheme. We should notice thatthe weak error order does not depend on the jump size a.

(iii) Figure (k) to (p) present the error with respect to time step where $\frac{b(0)}{\sigma^2} \leq 1$. In this case, the error curve is close to the curve of the function $x \mapsto x^\beta, \quad \beta < 1$ and immediat interpretation is that the weak error order is less than 1.

5. Final remarks

In this paper, we propose a new and efficient scheme to approximate the CIR prose with jump(s). The new scheme contributes to solving the following: (i) the positivity issue; (ii) the problem of negative values of the underlying process to 0; and (iii) most importantly the weak convergence related issue.

Simulation study indicate that the speed of convergence depends on the position of the ratio $\frac{b(0)}{\sigma^2}$ compared to 1. In particular, when $\frac{b(0)}{\sigma^2} \leq 1$, the convergence rate is of order as in the classical Euler scheme (e.g., see [8], [13] and [14]). When $\frac{b(0)}{\sigma^2} > 1$, the underlying process degenerates a result also obtained by (e.g., see [1]).

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Figure (a)

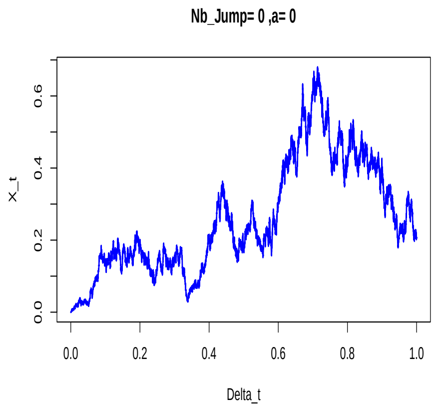
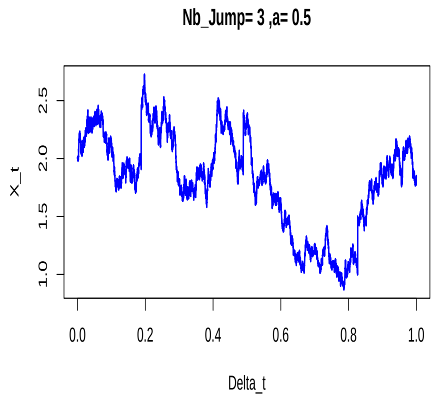


Figure (b)



Notes: $x_0 = 0$, $\alpha = 1$, $b = 1$, $\sigma = 1.5$
$\lambda = 2$, $a = 0$, $T = 1$

Notes: $x_0 = 2$, $\alpha = 2$, $b = 3$, $\sigma = 1$
$\lambda = 2$, $a = 0.5$, $T = 1$

Figure (c)

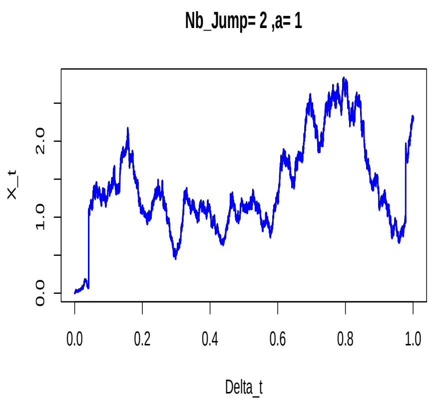
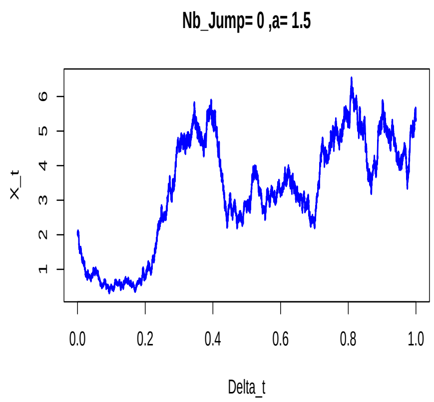


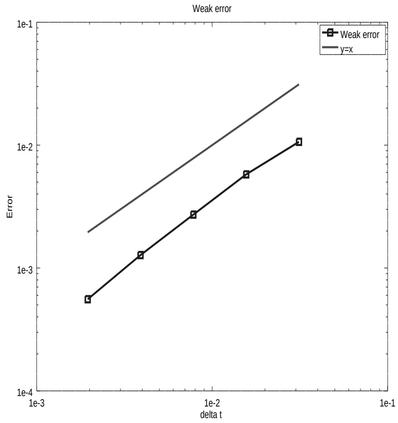
Figure (d)



Notes: $x_0 = 0$,$\alpha = 4$, $b = 3$, $\sigma = 1.5$
$\lambda = 2$, $a = 1$, $T = 1$

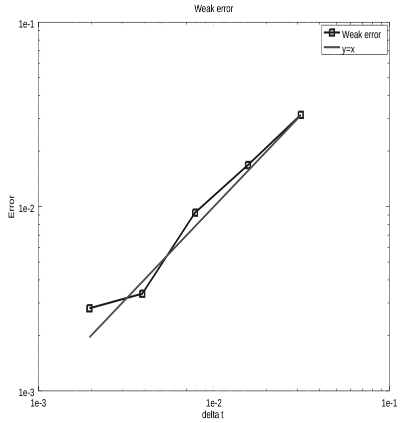
Notes: $x_0 = 5$, $\alpha = 4$, $b = 3$, $\sigma = 1.5$
$\lambda = 2$, $a = 1.5$, $T = 1$

Figure (e)



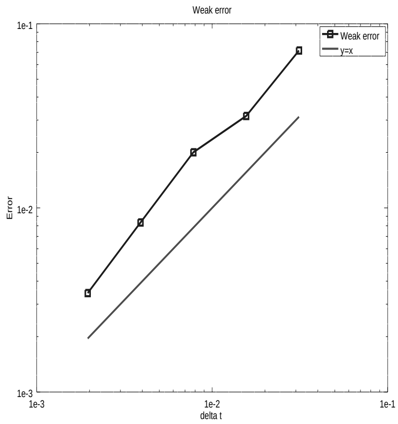
Notes: $x_0 = 0.05$,$\alpha = 1.5$, $\sigma = 0.75$ $\lambda = 2$, $a = 0$, $T = 1$

Figure (f)



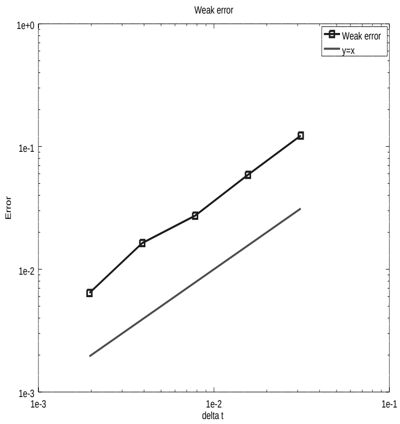
Notes: $x_0 = 0.05$,$\alpha = 1.5$, $\sigma = 0.75$ $\lambda = 2$, $a = 0.5$, $T = 1$

Figure (g)



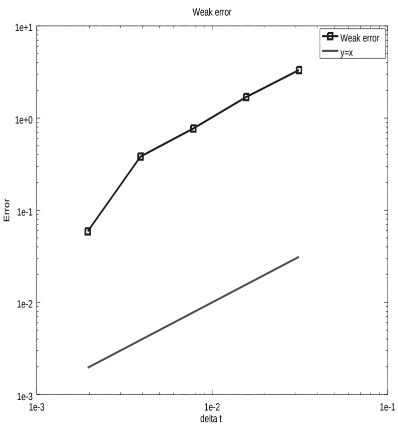
Notes: $x_0 = 0.05$,$\alpha = 1.5$, $\sigma = 0.75$ $\lambda = 2$, $a = 1$, $T = 1$

Figure (h)



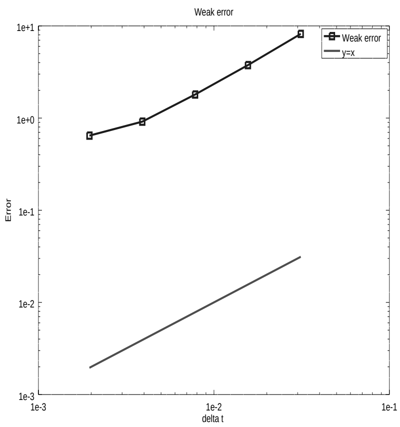
Notes: $x_0 = 0.05$,$\alpha = 1.5$, $\sigma = 0.75$ $\lambda = 2$, $a = 1.5$, $T = 1$

Figure (i)



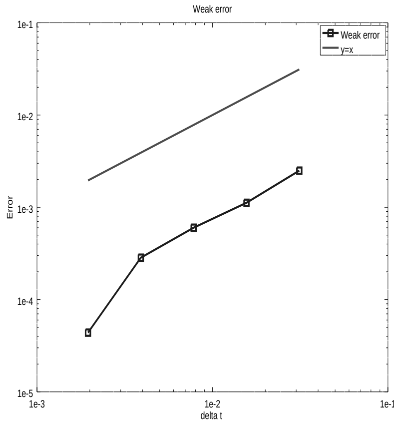
Notes: $x_0 = 0.05$,$\alpha = 1.5$, $\sigma = 0.75$ $\lambda = 2$, $a = 5$, $T = 1$

Figure (j)



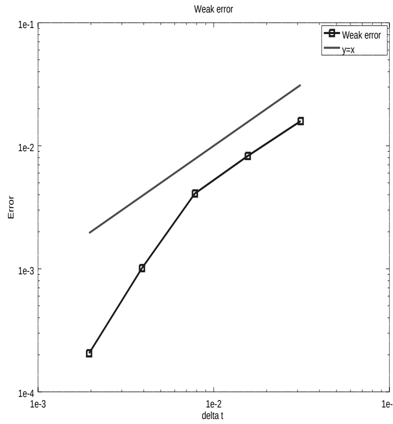
Notes: $x_0 = 0.05$,$\alpha = 1.5$, $\sigma = 0.75$, $\lambda = 2$, $a = 15$, $T = 1$

Figure (k)



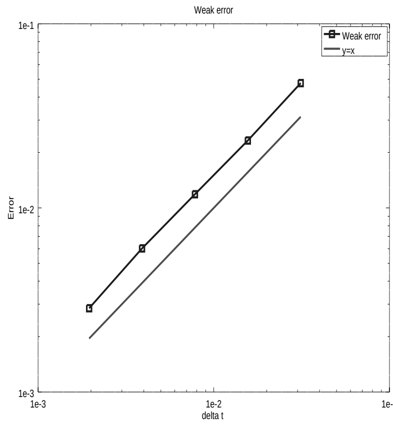
Notes: $x_0 = 5$,$\alpha = 0.5$, $b = 2$, $\sigma = 1$, $\lambda = 2$, $a = 0.5$, $T = 1$

Figure (l)



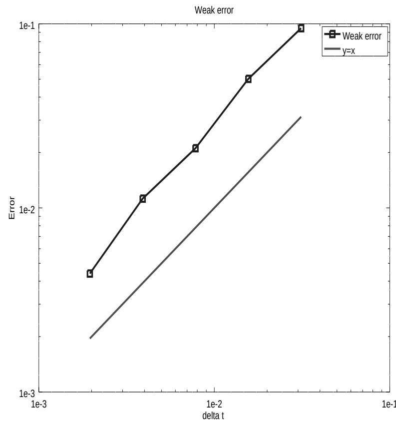
Notes: $x_0 = 5$, $\alpha = 0.5$, $b = 2$, $\sigma = 1$ $\lambda = 2$, $a = 1$, $T = 1$

Figure (m)



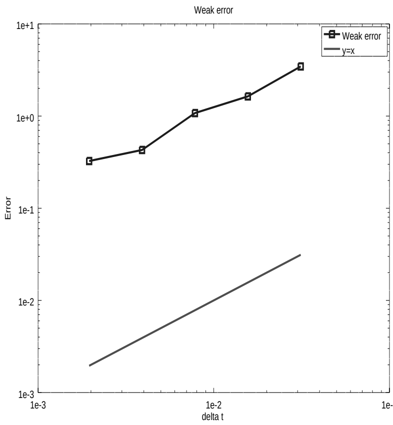
Notes: $x_0 = 5$,$\alpha = 0.5$, $b = 2$, $\sigma = 1$ $\lambda = 2$, $a = 1.5$, $T = 1$

Figure (n)



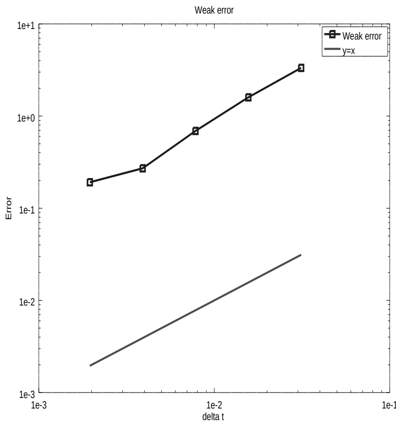
Notes: $x_0 = 5$, $\alpha = 0.5$, $b = 2$, $\sigma = 1$, $\lambda = 2$, $a = 3$, $T = 1$

Figure (o)



Notes: $x_0 = 5$,$\alpha = 0.5$, $b = 2$, $\sigma = 1$ $\lambda = 2$, $a = 1.5$, $T = 1$

Figure (p)



Notes: $x_0 = 5$, $\alpha = 0.5$, $b = 2$, $\sigma = 1$, $\lambda = 2$, $a = 3$, $T = 1$

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Appendices

Appendix 1: Proof of Equation (17)

Let's consider for all $x \geq 0$, $\xi \geq 0$ the process $(X^x_{t,\xi})_{0\leq t\leq T}$ that verifies the SDE

$$dX^x_{t,\xi} = \left(\xi\sigma^2 + b(X^x_{t,\xi})\right)dt + \sigma\sqrt{X^x_{t,\xi}}dB_t + adN_t \tag{A1.1}$$

This SDE has a unique strong solution from Yamada and Watanabe theorem (see Theorem 16 in [25]). Futhermore, $X^x_{t,\xi}$ is continously differentiable respect to x and is cadlag in t. We consider several steps

• Step 1: $S^x_{t,\xi}$ defined by $S^x_{t,\xi} = \frac{dX^x_{t,\xi}}{dx}$ is given by

$$S^x_{t,\xi} = \exp\left(\int_0^t b'(X^x_{s,\xi})ds + \frac{\sigma}{2}\int_0^t \frac{dB_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_{s,\xi}}\right). \tag{A1.2}$$

In fact from the definition of $S^x_{t,\xi}$, we have

$$X^x_{t,\xi} = x + \xi\sigma^2 t + \int_0^t b(X^x_{s,\xi})ds + \sigma\int_0^t \sqrt{X^x_{s,\xi}}dB_s + \int_0^t adN_s \tag{A1.3}$$

and the derivative respect to x gives

$$\begin{aligned} S^x_{t,\xi} = \frac{dX^x_{t,\xi}}{dx} &= 1 + \int_0^t \frac{dX^x_{s,\xi}}{dx}b'(X^x_{s,\xi})ds + \sigma\int_0^t \frac{dX^x_{s,\xi}}{dx}\frac{1}{2\sqrt{X^x_{s,\xi}}}dB_s \\ &= 1 + \int_0^t S^x_{s,\xi}b'(X^x_{s,\xi})ds + \sigma\int_0^t S^x_{s,\xi}\frac{1}{2\sqrt{X^x_{s,\xi}}}dB_s. \end{aligned} \tag{A1.4}$$

From the Itô formula, we have

$$\ln S^x_{t,\xi} = \ln S^x_{0,\xi} + \int_0^t b'(X^x_{s,\xi})ds + \frac{\sigma}{2}\int_0^t \frac{dB_t}{\sqrt{X^x_{t,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_{t,\xi}} \tag{A1.5}$$

and we dedude that

$$S^x_{t,\xi} = \exp\left(\int_0^t b'(X^x_{s,\xi})ds + \frac{\sigma}{2}\int_0^t \frac{dB_t}{\sqrt{X^x_{t,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_{t,\xi}}\right) \text{ with } X^x_{t,0} = X^x_t \tag{A1.6}$$

• Step2: The process $(M^x_{t,\xi})_{0\leq t\leq T}$ defined by

$$M^x_{t,\xi} = \exp\left(\pm\frac{\sigma}{2}\int_0^t \frac{dB_t}{\sqrt{X^x_{t,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_{t,\xi}}\right). \tag{A1.7}$$

is a martingale process and

$$\sup_{0\leq t\leq T} \mathbb{E}\left(S^x_{t,\xi}\right) \leq \exp(CT). \tag{A1.8}$$

In fact by definition, $(M^x_{t,\xi})_{0\le t\le T}$is a martingale process as exponential martingale. For $t\in[0,T]$, we have

$$\begin{aligned} S^x_{t,\xi} &= \exp\left(\int_0^t b'(X^x_{s,\xi})ds + \frac{\sigma}{2}\int_0^t \frac{dB_t}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_{s,\xi}}\right) \\ &= \exp\left(\int_0^t b'(X^x_{s,\xi})ds\right) M^x_{t,\xi}. \end{aligned} \tag{A1.9}$$

and taking expectation, we have

$$\begin{aligned} \mathbb{E}\left(S^x_{t,\xi}\right) &\le \mathbb{E}\left(\exp\left(\int_0^t \left\|b'\right\|_\infty ds\right) M^x_{t,\xi}\right) \\ &\le \exp\left(\left\|b'\right\|_\infty T\right)\mathbb{E}M^x_{t,\xi} = \exp\left(\left\|b'\right\|_\infty T\right)\mathbb{E}M^x_{0,\xi} = \exp\left(\left\|b'\right\|_\infty T\right). \end{aligned} \tag{A1.10}$$

Therefore

$$\sup_{0\le t\le T} \mathbb{E}\left(S^x_{t,\xi}\right) \le \exp(CT). \tag{A1.11}$$

• Step 3: There exists a probability measure $\mathbb{Q}^\xi$ under which the process $X^x_{\cdot,\xi+\frac{1}{2}}$ has the same distribution with the process $X^x_{\cdot,\xi}$ under $\mathbb{P}$. More precisely, $\mathcal{L}^{\mathbb{Q}^{\xi+\frac{1}{2}}}\left(X_{\cdot,\xi}\right) = \mathcal{L}^{\mathbb{P}}\left(X^x_{\cdot,\xi+\frac{1}{2}}\right)$.

From the definition of $X^x_{t,\xi}$, we deduce that

$$B_t = \int_0^t \left(\frac{dX^x_{s,\xi}}{\sigma\sqrt{X^x_{s,\xi}}} - \frac{b(X^x_{s,\xi} + \xi\sigma^2}{\sigma\sqrt{X^x_{s,\xi}}}ds - \frac{a}{\sigma\sqrt{X^x_{s,\xi}}}dN_s\right). \tag{A1.12}$$

We then define the process $(Z^{\xi,\xi+\frac{1}{2}}_t)_{0\le t\le T}$ such that

$$Z^{\xi,\xi+\frac{1}{2}}_t = \exp\left(\frac{\sigma}{2}\int_0^t \frac{dB_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{8X^x_t}\right) \tag{A1.13}$$

which is, by definition, a martingale process. We also define the probability measure $\mathbb{Q}^{\xi+\frac{1}{2}}$ by

$$\frac{d\mathbb{Q}^{\xi,\xi+\frac{1}{2}}}{d\mathbb{P}}|_{\mathcal{F}_t} = \frac{1}{Z^{\xi,\xi+\frac{1}{2}}_t}. \tag{A1.14}$$

From the Girsanov theorem, $\mathbb{Q}^{\xi+\frac{1}{2}}$ is a measure equivalent to $\mathbb{P}$. If we set

$$\begin{aligned} B^{\xi+\frac{1}{2}}_t &= B_t - \frac{\sigma}{2}\int_0^t \frac{ds}{\sqrt{X^x_{s,\xi}}} \qquad\qquad (A1.15)\\ &= \int_0^t \left(\frac{dX^x_{s,\xi}}{\sigma\sqrt{X^x_{s,\xi}}} - \frac{(b(X^x_{s,\xi}) + \xi + \frac{1}{2})\sigma^2}{\sigma\sqrt{X^x_{s,\xi}}}ds - \frac{a}{\sigma\sqrt{X^x_{s,\xi}}}dN_s\right), \end{aligned}$$

from Girsanov theorem, the process $(B^{\xi+\frac{1}{2}}_t)_{0\le t\le T}$ is a Brownian motion on the probability set $(\Omega,\mathcal{F}_T,\mathbb{Q}^{\xi+\frac{1}{2}})$ and on this set, we have

$$X^x_{t,\xi} = x + (\xi + \frac{1}{2})\sigma^2 t + \int_0^t b(X^x_{s,\xi})ds + \sigma \int_0^t \sqrt{X^x_{t,\xi}} dB_s^{\xi+\frac{1}{2}} + aN_t, \tag{A1.16}$$

We can now analyse the different partial derivative of u.

Appendix 2: Analysis of the term $\frac{\partial^2 u}{\partial x^2}$

.

Let's introduce the probability measure $\mathbb{Q}^{\frac{1}{2}}$ such that

$$\frac{d\mathbb{Q}^{\frac{1}{2}}}{d\mathbb{P}} \|_{F_t} = \frac{1}{Z_t^{(0,\frac{1}{2})}} \tag{A2.1}$$

From the Girsanov theorem,

$$\mathbb{E}\left(S^x_{T-t,0} f'(X^x_{T-t,0})\right) = \mathbb{E}_{\mathbb{Q}^{\frac{1}{2}}}\left(Z_t^{(0,\frac{1}{2})} S^x_{T-t}(0) f'(X^x_{T-t}(0))\right). \tag{A2.2}$$

From the relation $(A1.14)$, $B_t = B_t^{\frac{1}{2}} + \int_0^t \frac{ds}{\sqrt{X^x_s}}$

Then

$$S^x_{T-t,0} \;=\; \exp\left(\int_0^{T-t} b'(X^x_s)ds + \frac{\sigma}{2}\int_0^{T-t} \frac{dB_s^{\frac{1}{2}}}{\sqrt{X^x_s}} + \frac{\sigma^2}{8}\int_0^{T-t} \frac{ds}{X^x_s}\right) \tag{A2.3}$$

and

$$Z_{t-T}^{(0,\frac{1}{2})} \;=\; \exp\left(-\frac{\sigma}{2}\int_0^{T-t} \frac{dB_s^{\frac{1}{2}}}{\sqrt{X^x_s}} - \frac{\sigma^2}{8}\int_0^{T-t} \frac{ds}{X^x_s}\right). \tag{A2.4}$$

The derivative of u respect to u is given by

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}_{\mathbb{Q}^{\frac{1}{2}}}\left(f'(X^x_{T-t}(0)) \exp\left(\int_0^{T-t} b'(X^x_s(0))\right)\right). \tag{A2.5}$$

Since $\mathcal{L}^{\mathbb{Q}^{\frac{1}{2}}}(X^x) = \mathcal{L}^{\mathbb{P}}(X^x)$, then

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})ds\right)\right). \tag{A2.6}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(t,x) \;&=\; \mathbb{E}\left(S^x_{T-t,\frac{1}{2}} f''(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})ds\right)\right) \\
&+\; \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})ds\right)\left(\int_0^{T-t} b''(X^x_{s,\frac{1}{2}} S^x_{s,\frac{1}{2}})ds\right)\right).
\end{aligned} \tag{A2.7}$$

Then

$$\begin{aligned}
\left|\frac{\partial^2 u}{\partial x^2}(t,x)\right| \;&\leq\; \mathbb{E}\left(S^x_{s,\frac{1}{2}} \|f''\|_\infty \exp\left(\|b'\|_\infty T\right)\right) + \mathbb{E}\left(\|f'\|_\infty \exp\left(\|b'\|_\infty T\right)\|b''\|_\infty \int_0^{T-t} S^x_{s,\frac{1}{2}} ds\right) \\
&\leq\; \|f''\|_\infty \exp\left(\|b'\|_\infty T\right) \mathbb{E}_{\mathbb{P}}\left(S^x_{s,\frac{1}{2}}\right) + \|f'\|_\infty \exp\left(\|b'\|_\infty T\right)\|b''\|_\infty \int_0^{T-t} \mathbb{E}\left(S^x_{s,\frac{1}{2}}\right) ds \\
&\leq\; C(T).
\end{aligned} \tag{A2.8}$$

Appendix 3: Analysis of the term $\frac{\partial^3 u}{\partial x^3}$

Let's $(t,x) \in [0,T] \times \mathbb{R}$. We have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left(S^x_{T-t,\frac{1}{2}} f''(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})ds\right)\right) \\ &+ \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})ds\right)\left(\int_0^{T-t} b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}ds\right)\right) \quad \text{(A3.1)}\end{aligned}$$

The Markov property on the process $\left(X^x_{t,\frac{1}{2}}\right)$ allows to write for all $s \in [0, T-t]$,

$$\begin{aligned}\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_s^{T-t} b'(X^x_{u,\frac{1}{2}})du\right)|\mathcal{F}_s\right) &= \mathbb{E}\left(f'(X^y_{T-t,\frac{1}{2}}) \exp\left(\int_s^{T-t} b'(X^y_{u,\frac{1}{2}})du\right)\right)\Big|_{y=X^x_{s\frac{1}{2}}} \\ &= \mathbb{E}\left(f'(X^y_{T-t-s,\frac{1}{2}}) \exp\left(\int_0^{T-t-s} b'(X^y_{u,\frac{1}{2}})du\right)\right)\Big|_{y=X^x_{s,\frac{1}{2}}}.\end{aligned}$$

Then

$$\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_s^{T-t} b'(X^x_{u,\frac{1}{2}})du\right)|\mathcal{F}_s\right) = \frac{\partial u}{\partial x}(t+s,x). \quad \text{(A3.2)}$$

Let's set $A = \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\left(\int_0^{T-t} b''(X^x_{s,\frac{1}{2}}S^x_{s,\frac{1}{2}})\right)\right)$

Then,

$$\begin{aligned}A &= \mathbb{E}\int_0^{T-t}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})ds\right)\left(b''(X^x_{s,\frac{1}{2}}S^x_{s,\frac{1}{2}})\right)\right)ds \\ &= \mathbb{E}\left[\int_0^{T-t}\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_s(\frac{1}{2})\right)|\mathcal{F}_s\,ds\right)\right] \\ &= \mathbb{E}\left[\int_0^{T-t}\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)|\mathcal{F}_s\right)\ \exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)ds\right] \\ &= \int_0^{T-t}\mathbb{E}\left(\frac{\partial u}{\partial x}(t+s,x)\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)\right)ds. \quad \text{(A3.3)}\end{aligned}$$

It comes that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left(S^x_{T-t,\frac{1}{2}} f''(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\right) \quad \text{(A3.4)} \\ &+ \int_0^{T-t}\mathbb{E}\left(\frac{\partial u}{\partial x}(t+s,x).\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)\right)ds\end{aligned}$$

By introducing the measure $\mathbb{Q}^1$ such that $\frac{d\mathbb{Q}^1}{d\mathbb{P}}|_{\mathcal{F}_t} = \frac{1}{Z_t^{(\frac{1}{2},1)}}$, we have:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}_{\mathbb{Q}^1}\left(Z^{(\frac{1}{2},1)}_{T-t} S^x_{T-t,\frac{1}{2}} f''(X^x_{T-t,\frac{1}{2}}) \exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\right) \\ &\quad + \int_0^{T-t}\mathbb{E}_{\mathbb{Q}^1}\left(Z^{(\frac{1}{2},1)}_{T-t}\frac{\partial u}{\partial x}(t+s,x)\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)\right)ds \\ &= \mathbb{E}_{\mathbb{Q}^1}\left(f''(X^x_{T-t,\frac{1}{2}}) \exp\left(2\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\right) \\ &\quad + \int_0^{T-t}\mathbb{E}_{\mathbb{Q}^1}\left(\frac{\partial u}{\partial x}(t+s,x).\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})\right)\right)ds \quad \text{(A3.5)}\end{aligned}$$

$\mathcal{L}^{\mathbb{Q}^{\frac{1}{2}}}(X^x_{\cdot,\frac{1}{2}}) = \mathcal{L}^{\mathbb{P}}(X^x_{\cdot,\frac{1}{2}})$, we finally have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left(f''(X^x_{T-t,1})\exp\left(2\int_0^{T-t} b'(X^x_{s,1})ds\right)\right) \\ &+ \int_0^{T-t}\mathbb{E}\left(b''(X^x_{s,1})\frac{\partial u}{\partial x}(t+s,x).\exp\left(\int_0^s b'(X^x_{u,1})du\right)\right)ds.\end{aligned} \tag{A3.6}$$

and

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}(t,x) &= \mathbb{E}\left\{\exp\left(2\int_0^{T-t} b'(X^x_{s,1})ds\right)\left[f^{(3)}(X^x_{T-t,1})S^x_{T-t,1} + 2f''(X^x_{s,1})\int_0^{T-t} b''(X^x_{s,1})S^x_{s,1}ds\right]\right\} \\ &+ \int_0^{T-t}\left\{\mathbb{E}\exp\left(2\int_0^s b'(X^x_{u,1})du\right)\left[S^x_{s,1}\left(\frac{\partial^2 u}{\partial x^2}(t+s,X^x_{s,1})b''(X^x_{s,1})\right.\right.\right. \\ &+ \left.\left.\left. b^{(3)}(X^x_{s,1})\frac{\partial u}{\partial x}(t+s,X^x_{s,1})\right) + 2\frac{\partial u}{\partial x}(t+s,X^x_s)b''(X^x_{s,1})\int_0^s b''(X^x_{u,1})S^x_{u,1}du\right]\right\}ds. \end{aligned} \tag{A3.7}$$

Similarly, we have :

$$\left|\frac{\partial^3 u}{\partial x^3}(t,x)\right| \le C(T). \tag{A3.8}$$

Appendix 4: Analysis of the term $\frac{\partial^4 u}{\partial x^4}$

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Let $(t,x) \in [0,T]\times\mathbb{R}$, by using the same arguments we proceed as in the previous step by transforming the expression of $\frac{\partial^3 u}{\partial x^3}(t,x)$ and by introducing the probability measure $\mathbb{Q}^{\frac{3}{2}}$ such that $\frac{d\mathbb{Q}^{\frac{3}{2}}}{d\mathbb{P}} = \frac{1}{Z_t^{1,\frac{3}{2}}}$
where

$$\frac{1}{Z_t^{1,\frac{3}{2}}} = \exp\left(-\frac{\sigma}{2}\int_0^t \frac{dB_t}{\sqrt{X^x_{s,1}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{8X^x_{s,1}}\right)$$

We then obtain

$$\left\|\frac{\partial^4 u}{\partial x^4}\right\| \le C(T). \tag{A4.1}$$

By using the inequalities $(A2.8)$, $(A3.7)$ and $(A3.8)$ we finally obtained

$$\sum_{k=0}^4 \left\|\frac{\partial^k u}{\partial x^k}\right\|_{L^\infty([0,T]\times[0,\infty])} \le C(T) \tag{A4.2}$$

Appendix 5: Details of the expectation of X_t^2

We first notice that

$$\begin{aligned} &e^{bt}dX_t = \alpha e^{bt}dt - be^{bt}X_tdt + \sigma e^{bt}\sqrt{X_t}dW_t + e^{bt}adN_t \\ &e^{bt}dX_t + be^{bt}X_tdt = \alpha e^{bt}dt + \sigma e^{bt}\sqrt{X_t}dW_t + e^{bt}adN_t \\ &d\left(e^{bt}X_t\right) = \alpha e^{bt}dt + \sigma e^{bt}\sqrt{X_t}dW_t + e^{bt}adN_t \end{aligned} \tag{A5.1}$$

and it comes that

$$e^{kt}X_t = x_0 + \frac{\alpha}{b}(e^{bt}-1) + \sigma\int_0^t e^{bs}\sqrt{X_s}dW_s + a\int_0^t e^{bs}dN_s. \tag{A5.2}$$

Thus

$$X_t = x_0e^{-bt} + \frac{\alpha}{b}\left(1-e^{-bt}\right) + \sigma e^{-bt}\int_0^t e^{bs}\sqrt{X_s}dW_s + ae^{-bt}\int_0^t e^{bs}dN_s \tag{A5.3}$$

and taking the expectation we have

$$\mathbb{E}\left(X_t\right) = x_0e^{-bt} + \left(\frac{\alpha}{b}+\frac{a\lambda}{b}\right)\left(1-e^{-bt}\right). \tag{A5.4}$$

For the second order moment, we use the Itô formula for the process $(X_t)_{0\le t\le T}$ with the function $x \mapsto x^2$:

$$\begin{aligned} X_t^2 &= x_0^2 + 2\alpha\int_0^t X_sds - 2\int_0^t X_s^2ds + 2\sigma\int_0^t X_s\sqrt{X_s}dW_s \\ &\quad +\sigma^2\int_0^t X_sds + \int_0^t \left(2aX_s + a^2\right)dN_s. \end{aligned} \tag{A5.6}$$

Taking the expectation, again we obtain

$$\mathbb{E}\left(X_t^2\right) = x_0^2 + \left(2\alpha+\sigma^2+2a\lambda\right)\int_0^t \mathbb{E}\left(X_s\right)ds - 2b\int_0^t \mathbb{E}\left(X_s^2\right)ds + \lambda a^2t. \tag{A5.7}$$

Putting $y(t) = \mathbb{E}\left(X_t^2\right)$, then

$$\begin{aligned} y'(t) &= \left(2\alpha+\sigma^2+2a\lambda\right)\mathbb{E}\left(X_t\right) - 2by(t) + \lambda a^2 \\ &= -2by(t) + \left(2\alpha+\sigma^2+2a\lambda\right)\left[\left(x_0 - \frac{\alpha+a\lambda}{b}\right)e^{-bt} + \frac{\alpha+a\lambda}{b}\right] + \lambda a^2 \\ &= -2by(t) + A_1e^{-bt} + A_2 \end{aligned} \tag{A5.8}$$

where $A_1 = \left(2\alpha+\sigma^2+2a\lambda\right)\left(x_0 - \frac{\alpha+a\lambda}{b}\right)$ and $A_2 = \left(2\alpha+\sigma^2+2a\lambda\right)\frac{\alpha+a\lambda}{b} + \lambda a^2$
Using

$$y(t) = k(t)e^{-2bt},$$

we have

$$y'(t) + 2by(t) = k'(t)e^{-2bt} = A_1e^{-bt} + A_2$$

and

$$k(t) = \frac{A_1}{b}e^{bt} + \frac{A_2}{2b}e^{2bt} + C.$$

Since $k(0) = y(0) = x_0^2$ we have $C = x_0^2 - \frac{A_1}{b} - \frac{A_2}{2b}$. It comes that

$$k(t) = x_0^2 + \frac{A_1}{b}\left(e^{bt}-1\right) + \frac{A_2}{2b}\left(e^{2bt}-1\right)$$

and we deduce that

$$y(t) = x_0^2e^{-2bt} + \frac{A_1}{b}\left(e^{-bt}-e^{-2bt}\right) + \frac{A_2}{2b}\left(1-e^{-2bt}\right).$$

By replacing A_1 and A_2 by their expressions, we have:

$$\begin{aligned} \mathrm{E}\left(X_t^2\right) &= x_0e^{-2bt} + \left(2\alpha+\sigma^2+2a\lambda\right)\left(\frac{x_0}{b} - \frac{\alpha+a\lambda}{b^2}\right)\left(e^{-bt}-e^{-2bt}\right) \\ &\quad + \left[\left(2\alpha+\sigma^2+2a\lambda\right)\frac{\alpha+a\lambda}{b^2} + \frac{\lambda a^2}{2b}\right]\left(1-e^{-2bt}\right). \end{aligned} \tag{A5.9}$$