UNIVERSITY OF YAOUNDE I.

MASTER THESIS

Convergence Analysis of symmetric Euler–Maruyama Scheme for CIR type Process with Poissonian jump

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by

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Declaration

I, Ramiyou Karim MACHE, declare that this thesis titled, “Convergence Analysis of symmetric Euler–Maruyama Scheme for CIR type Process with Poissonian jump”, which is submitted in fulfillment of the requirements for the Degree of Master of Mathematics, represents my own work except where due acknowledgement have been made. I further declared that it has not been previously included in a thesis, dissertation, or report submitted to this University or to any other institution for a degree, diploma or other qualifications.

Signed:

Date: December 02, 2018

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Abstract

This work aims to approximate the solutions of one-dimension stochastic differential equations (SDEs) with Poisson jump and whose diffusion coefficient is a non-Lipschitz function. We focus on a class of equations that is frequently used in Finance. We considered a generalization of the Cox-Ingersoll-Ross model (CIR). In this class of SDEs, the drift coefficient is a bounded derivative and Lipschitz function, while the diffusion coefficient is of the type $\mapsto \sigma\sqrt{x}$ where $\sigma > 0$ (volatility). we studied some properties of such process, established the link with PDE. We proposed a so-called symmetric Euler-Maruyama scheme to approximate the solution of such SDE, established the weak convergence of such schemes for a class of regular test functions, more precisely, we show that the weak error's order is equal to $\min(1, b(0)/\sigma^2)$, where $b(x)$ is the drift coefficient , and it does not depend on the value of the jump size. Finally, we illustrate our theoretical results by numerical simulations.

keywords: SDEs, CIR type process with Poisson jump, symmetric Euler-Maruyama scheme, Kolmogorov's PDE.

Resumé

Ce travail traite l'approximation de solution d'Equations Différentielle Stochastique (EDSs) à saut, unidimensionnelle dont le coefficient de diffusion est une fonction non Lipschitzienne. Dans ce travail, nous nous sommes focalisé sur une classe d'équations très utilisée en finance. Nous avons considéré une généralisation du modèle de Cox-Ingersoll-Ross (CIR en abrégé). Dans cette classe d'équations, le coefficient de dérivée est une fonction lipschitzienne à dérivées bornées, alors que le coefficient de diffusion est du type $x \mapsto \sigma\sqrt{x}$ où $\sigma > 0$ (volatilité). Nous avons considéré le coefficient de saut constant positif. Nous nous sommes placé sous les hypothèses qui assurent l'existence et l'unicité de solutions à trajectoires strictement positive presque sûrement (Condition de Feller). Nous avons proposé un schéma de type Euler-Maruyama symmetrique pour approximer la solution de ces type d'EDS . Nous avons établi la convergence faible de ce schéma pour une classe de fonctions tests régulières et nous avons illustré les résultats théoriques obtenus par les simulations numériques.

Mots-clés: Equation differentielle stochastique, processus de type CIR à saut Poissonien, schéma d'Euler-Maruyama symmetrique, EPD de Kolmogorov.

Introduction

Stochastic Differential Equations (SDEs) is one of the most important fields of stochastic analysis, that is widely used in finance (to model unstable stock prices, interest ration, etc.), economies, artificial intelligent, physics, etc [12]. In 1985, John. C. Cox, Jonathan J. Ingersoll and Stephen A. Ross introduced the so-called Cox-Ingersoll-Ross (CIR for short) process [3] to model the short time interest rate in financial market. That is described by the following stochastic differential equation(SDE).

$$X_t = x_0 + \int_0^t (\alpha - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s, \;\; X_0 = x_0 \geq 0.$$

where $(B_t, t \geq 0)$ is a standard Brownian motion define on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and α, b, σ are positive constants.

As we can observe in the above equation, the classical CIR model does not take into account some rare event, for, instance, natural disasters, terrorist attacks, market crashes, interventions by the Federal Reserve, economic surprises, shocks in the foreign exchange market, etc. that can affect the interest rate. In order to take into account such rare events, one consider the so-called CIR process with Poisson jump by adding the jump term on the above equation [20], that satisfies the following SDE.

$$X_t = x_0 + \int_0^t (\alpha - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t a\mathrm{d}N_s, \;\; X_0 = x_0 \geq 0.$$

where $(N_t, t \geq 0)$ is a Poisson process and $a > 0$.

From this SDE, we consider in this work the most general class of SDEs, the so-called CIR type process with Poisson jump, where we consider the drift function not necessary linear but some other smooth function with the 'good properties', that is

$$X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t a\mathrm{d}N_s, \;\; X_0 = x_0 \geq 0.$$

where $b(x)$ is some smooth function.

Using Yamada-Watanabe's theorem, one that this SDE has an unique strong solution. Unfortunately it is very difficult to find such solution in the closed form. Then one seek to numerical methods to approximate this solution. The most useful numerical scheme to approximate the solution of SDE is the so-called Euler-Maruyama scheme, that is defined for the equation by

$$\begin{cases} \widehat{X}_0 = x_0 \\ \widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k})\Delta t + \sigma\sqrt{\widehat{X}_{t_k}}\Delta B_k + a\Delta N_k \end{cases} \tag{0.0.1}$$

where $\Delta t = t_{k+1} - t_k$ is the step size, $t_k = k\frac{T}{N}$, $N \in \mathbb{N}^*$, $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and $\Delta N_k = N_{t_{k+1}} - N_{t_k}$.

Notice that the random variable ΔB_k in this scheme is normally distributed, that means it can take negative value. In this case the right side of this equation (3.0.2) take the negative value and then the iteration will not be defined because of the square root. To fix this problem, we use the so-called symmetric Euler-Maruyama scheme as in [4], that is

$$\begin{cases} \widehat{X}_0 = x_0 \\ \widehat{X}_{t_{k+1}} = |\widehat{X}_{t_k} + b(\widehat{X}_{t_k})\Delta t + \sigma\sqrt{\widehat{X}_{t_k}}\Delta B_k + a\Delta N_k| \end{cases} \tag{0.0.2}$$

where $\Delta t = t_{k+1} - t_k$ is the step size, $t_k = k\frac{T}{N}$, $N \in \mathbb{N}^*$, $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and $\Delta N_k = N_{t_{k+1}} - N_{t_k}$.

In this work, we study some properties of such CIR type process with Poisson jump, establish the link with PDE, study the regularity of the solution of such PDE. We establish the weak convergence of so-called symmetric Euler-Maruyama scheme for CIR type process with Poisson jump and finally, illustrate our theoretical results by numerical simulations.

The remainder of this thesis is organized as follow. Chapter 1 is devoted to basic notion of stochastic analysis and numerical methods for SDEs. Chapter 2 studies the CIR type process with Poisson jump, establish the link with PDE and studies the regularity of the solution of such PDE. Chapter 3 defines the symmetric Euler-Maruyama scheme, studies some properties of of approximated process and establishes the weak convergence of such scheme. Chapter 4 is devoted to numerical simulations. A conclusion ends this work.

Chapter 1

Basic notion of stochastic analysis and numerical method for SDEs.

In this chapter we give some necessaries notion for our work. We resume basic notion of probability theory, stochastic analysis. We introduce the notion of Stochastic Differential Equations (SDEs) and explore some numerical method to approximate the solution (if exist) of theses SDEs. The main references for this chapter are [1], [8], [18], [5], [9] and [15], and all the detailed proof of theorem give in this can be found in these references.

1.1 Background on the concept of probability theory

Probability theory deals with mathematical models of trials whose outcomes depend on chance. In this section we resume some notion of probability theory which are useful in our work.

1.1.1 σ-algebra and measurable space

Definition 1.1.1 ***(σ-algebra and measurable space)***
Let Ω be a non empty set. A σ-algebra on Ω is a class $\mathcal{F}$ of subsets of Ω ($\mathcal{F} \subseteq \mathcal{P}(\Omega)$) such that

- $\emptyset \in \mathcal{F}$,
- *If $A \in \mathcal{F}$, then $A^c := \Omega \setminus A \in \mathcal{F}$,*
- *If $(A_n)_n$ is a countable family of subsets of Ω such that $A_n \in \mathcal{F}$ for all n, then $\cup_n A_n \in \mathcal{F}$.*

If $\mathcal{F}$ is a σ-algebra on Ω, the pair $(\Omega, \mathcal{F})$ is called a measurable space, and the elements of $\mathcal{F}$ is henceforth called measurable sets.

Let $\mathcal{F}$ a σ-algebra on Ω, a sub-sigma-algebra of $\mathcal{F}$ is a subset of $\mathcal{F}$ that is also a sigma-algebra on Ω.

Definition 1.1.2 *If $\mathcal{A}$ is a family of subsets of Ω, then there exists a smallest σ-algebra $\sigma(\mathcal{A})$ on Ω which contains $\mathcal{A}$. This $\sigma(\mathcal{A})$ is celled the σ-algebra generated by $\mathcal{A}$. If $\Omega = \mathbb{R}^d$ and $\mathcal{A}$ is the family of all open sets in $\mathbb{R}^d$, then $\beta^d = \sigma(\mathcal{A})$ is called the Borel σ-algebra and the elements of β are called the Borel sets.*

1.1.2 Probability measure and probability space.

Definition 1.1.3 *Let $(\Omega, \mathcal{F})$ be a measurable space. A probability measure on $(\Omega, \mathcal{F})$ is a mapping*

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

such that

- $\mathbb{P}(\Omega) = 1$;
- *If* $(A_n)_n$ *is a family of events of* $\mathcal{F}$ *that are pairwise disjoint* ($A_n \cap A_m = \emptyset$, *for* $n \neq m$), *then*

$$\mathbb{P}\left(\bigcup_n A_n\right) = \sum_n \mathbb{P}(A_n).$$

If $\mathbb{P}$ is a probability measure on a measurable space $(\Omega, \mathcal{F})$, then the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called probability space.

Definition 1.1.4 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, a subset* $A \subseteq \Omega$ *is said to be* $\mathbb{P}$*-negligible if there is a set* $B \in \mathcal{F}$ *with* $A \subseteq B$ *and* $\mathbb{P}(B) = 0$.

A property is true almost surely (a.s) if it is true outside of all negligible set.

1.1.3 Measurability and random variable.

Definition 1.1.5 ***(Measurable function)***
Let $(\Omega, \mathcal{F})$ *and* $(E, \mathcal{E})$ *two measurable space, a function* $f\colon \Omega \to \mathcal{E}$ *is* $(\mathcal{F}, \mathcal{E})$*-measurable (or measurable for short) if the pre-image of every measurable set is a measurable set, i.e:*

$$f^{-1}(A) \in \mathcal{F}, \forall A \in \mathcal{E}, \ \text{ where } \ f^{-1}(A) = \{\omega \in \Omega |\ f(\omega) \in A\}.$$

Definition 1.1.6 ***(Random variable)***

- *A random variable is a measurable map from a probability space to any measurable space.*
- *A real-valued random variable is a measurable function from a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *into* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition 1.1.7 *Let X be a random variable on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *The distribution measure of X is the probability measure* $\mathbb{P}_X$ *that assigns to each Borel subset B of* $\mathbb{R}$ *the mass* $\mathbb{P}_X(B) = \mathbb{P}(X \in B)$.

For the rest of this chapter we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1.4 Expectation

Definition 1.1.8 *random variable* $X\colon \Omega \to \mathbb{R}$ *is called integrable if*

$$\int_\Omega |X(\omega)| d\mathbb{P}(\omega) = \int_\mathbb{R} |x| f_X(x) dx < +\infty,$$

where f_X *is the probability density function of X. The previous identity is based on changing the domain of integration from* Ω *to* $\mathbb{R}$ *and know as transfer theorem.*
The expectation of an integrable real-valued random variable X is defined by

$$\mathbb{E}(X) = \int_\Omega X(\omega) d\mathbb{P}(\omega) = \int_\mathbb{R} x f_X(x) dx.$$

Customarily, the expectation of X is denoted by μ and is called the mean. In general, for any measurable function $h\colon \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}(h(X)) = \int_\Omega h(X(\omega)) d\mathbb{P}(\omega) = \int_\mathbb{R} h(x) f_X(x) dx$$

In the case of a discrete random variable X the expectation is defined as

$$\mathbb{E}(X) = \sum_{k\geq 1} x_k \mathbb{P}(X = x_k)$$

We note also that, the expectation is a linear operator.
For $p \in (0, \infty)$, let $L^p = L^P(\Omega, \mathbb{R}^d)$ be the family of $\mathbb{R}^d$-valued random variables X with $\mathbb{E}(|X|^p) < \infty$. The following inequality are useful for our work.

Jensen's inequality

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a convex function and let X be an integrable random variable. If $\varphi(X)$ is integrable, then

$$\varphi(\mathbb{E}(X)) \leq \mathbb{E}(\varphi(X))$$

Hölder's inequality

$$|\mathbb{E}(X^T Y)| \leq \mathbb{E}(|X|^p)^{1/p}\mathbb{E}(|X|^q)^{1/q}$$

if $p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1, X \in L^p, Y \in L^q$.

Minkowski's inequality

$$(\mathbb{E}|X+Y|^p)^{1/p} \leq \mathbb{E}(|X|^p)^{1/p} + \mathbb{E}(|Y|^p)^{1/p}$$

if $p > 1, X, Y \in L^p$.

Markov's inequality

For any $\lambda, p > 0$, we have the following inequality:

$$\mathbb{P}(\omega; |X(\omega)| \geq \lambda) \leq \frac{\mathbb{E}(|X|^p)}{\lambda^p}$$

1.1.5 Independence

Roughly speaking, two real-valued random variables X and Y are independent if the occurrence of one of them does not change the probability density of the other. More precisely, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two sub-σ-algebra of $\mathcal{F}$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B), \forall A \in \mathcal{F}_1, \forall B \in \mathcal{F}_2$$

Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We say these two random variables are independent if the σ-algebras they generate, $\sigma(X)$ and $\sigma(Y)$, are independent. We say that the random variable X is independent of the a-algebra $\mathcal{G}$ if $\sigma(X)$ and $\mathcal{G}$ are independent.

Proposition 1.1.9 *Two real-valued random variables X and Y are independent if and only if, for all measurable and bounded functions h and g,*

$$\mathbb{E}(h(X)g(Y)) = \mathbb{E}(h(X))\mathbb{E}(g(Y)).$$

1.1.6 Conditional Expectations.

Definition 1.1.10 *let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a random variable that is either non-negative or integrable. The conditional expectation of X given $\mathcal{G}$, denoted $\mathbb{E}(X|\mathcal{G})$, is any random variable that satisfies*

- ***(Measurability)*** *$\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable,*
- ***(Partial averaging)*** *$\int_A \mathbb{E}(X|\mathcal{G})(\omega)d\mathbb{P}(\omega) = \int_A X(\omega)d\mathbb{P}(\omega)$ for all $A \in \mathcal{G}$.*

If $\mathcal{G}$ is the σ-algebra generated by some other random variable Y (i. e., $\mathcal{G} = \sigma(Y)$) , we generally write $\mathbb{E}(X|Y)$ rather than $\mathbb{E}(X|\sigma(Y))$.

Proposition 1.1.11 *Let X, Y be two real-valued random variables, then there exist a measurable function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}(X|Y) = \varphi(Y)$.*

Some properties of conditional expectation

Let's give in the following section, the most useful properties of conditional expectation.

(a) **(Linearity):** If X and Y are integrable random variables and a and b are constants, then

$$\mathbb{E}\left(aX + bY\,|\mathcal{G}\,\right) = a\mathbb{E}\left(X\,|\mathcal{G}\,\right) + b\mathbb{E}\left(Y\,|\mathcal{G}\,\right)$$

(b) **(Increasing)** If X and Y are integrable random variables such that $X \leq Y$ a.s, then

$$\mathbb{E}\left(X\,|\mathcal{G}\,\right) \leq \mathbb{E}\left(Y\,|\mathcal{G}\,\right)\ a.s$$

(c) If X is $\mathcal{G}$-measurable, then $\mathbb{E}\left(X\,|\mathcal{G}\,\right) = \mathbb{E}\left(X\right)$.

(d) $\mathbb{E}\left[\mathbb{E}\left(X\,|\mathcal{G}\,\right)\right] = \mathbb{E}\left(X\right)$.

(e) If Y est $\mathcal{G}$-measurable, then $\mathbb{E}\left(XY\,|\mathcal{G}\,\right) = Y\mathbb{E}\left(X\,|\mathcal{G}\,\right)$.

(f) $\mathcal{G} = \{\emptyset, \Omega\}$, then $\mathbb{E}\left(X\,|\mathcal{G}\,\right) = \mathbb{E}\left(X\right)$.

(g) If X and $\mathcal{G}$ are independent, then $\mathbb{E}\left(X\,|\mathcal{G}\,\right) = \mathbb{E}\left(X\right)$.

(h) If $\mathcal{G}$ and $\mathcal{H}$ are two sub-σ-algebra such that $\mathcal{H} \subset \mathcal{G}$, then

$$\mathbb{E}\left(\mathbb{E}\left(X\,|\mathcal{G}\,\right)|\mathcal{H}\,\right) = \mathbb{E}\left(\mathbb{E}\left(X\,|\mathcal{H}\,\right)|\mathcal{G}\,\right) = \mathbb{E}\left(X\,|\mathcal{H}\,\right).$$

(i) If X and Y are independent, and φ a measurable and bounded function, then

$$E(\varphi(X,Y)\,|Y\,) = \left[E(\varphi(X,y))\right]_{y=Y}$$

.

1.1.7 Filtration and stopping time

Definition 1.1.12 ***(Filtration and filtered probability space)***
A filtration is a family $\{\mathcal{F}_t, 0 \leq t\}$ of sub-σ-algebras of $\mathcal{F}$ such that whenever $s \leq t, \mathcal{F}_s \subseteq \mathcal{F}_t$
If $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ is a filtration, then the quadruplet $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space.

We say that a filtered probability space $(,\mathcal{F},\mathbb{F},\mathbb{P})$ satisfies the usual conditions if

- $\mathcal{F}_0$ contains all sets of zero probability;
- $\{\mathcal{F}_t, t \geq 0\}$ is right continuous, that is

$$\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u.$$

Definition 1.1.13 ***(Stopping time)***
A random variable T, with values in $\mathbb{R}_+$ is a stopping time with respect to a filtration $\{\mathcal{F}_t, t \geq 0\}$, if

$$\{T \leq t\} \in \mathcal{F}_t, \quad \textit{for all} \quad t \geq 0.$$

1.2 Stochastic Processes

1.2.1 Definition of stochastic process

Definition 1.2.1 *A stochastic process on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is a family of random variables $\{X_t\}$ parameterized by $t \in \mathrm{T}$, where $\mathrm{T} \subseteq \mathbb{R}$. If T is an interval we say that $\{X_t\}$ is a stochastic process in continuous time. If $\mathrm{T} = \{1,2,3,\dots\}$ we shall say that $\{X_t\}$ is a stochastic process in discrete time. The latter case describes a sequence of random variables.*

The evolution in time of a given state of the world $\omega \in \Omega$ given by the function $t \mapsto X_t(\omega)$ is called a path or realization of X_t . The study of stochastic processes using computer simulations is based on retrieving information about the process X_t given a large number of its realizations.

Definition 1.2.2 *A stochastic process X_t is said to be adapted to the filtration $\{\mathcal{F}_t, 0 \leq t\}$ if X_t is $\mathcal{F}_t$-measurable, for any $t \in T$. This means that the information at time t determines the value of the random variable X_t.*

Definition 1.2.3 *A stochastic process $X = \{X_t, t \in \mathrm{T}\}$ is said to be càdlàg if it a.s. has sample paths which are right continuous, with left limits. Similarly, a stochastic process $X = \{X_t, t \in \mathrm{T}\}$ is said to be càglàd if it a.s. has sample paths which are left continuous, with right limits. (The nonsensical words càdlàg and càglàd are acronyms from the French for continu à droite, limites a gauche and continu à gauche, limites à droite, respectively.)*

If a stochastic process $X = \{X_t, t \in \mathrm{T}\}$ is càdlàg, denoting by ΔX the process define by $\Delta X_t = X_t - X_{t^-}$. Then ΔX is not càdlàg, though it is adapted and for a.s.ω , $t \mapsto \Delta X_t(\omega) = 0$ except for at most countably many t.

Definition 1.2.4 *Two stochastic processes $X = \{X_t, t \in \mathrm{T}\}$ and $Y = \{Y_t, t \in \mathrm{T}\}$ are modifications if $X_t = Y_t$ a.s., each t. Two processes $X = \{X_t, t \in \mathrm{T}\}$ and $Y = \{Y_t, t \in \mathrm{T}\}$ are indistinguishable if a.s., for all t,$X_t = Y_t$.*

Theorem 1.2.5 *Let $X = \{X_t, t \in \mathrm{T}\}$ and $Y = \{Y_t, t \in \mathrm{T}\}$ be two stochastic processes, with X a modification of Y. If X and Y have right continuous paths a.s., then X and Y are indistinguishable.*

1.2.2 Brownian motion and Poisson process.

In this section, we consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ satisfying the usual hypotheses.

Definition 1.2.6 *A (standard) one-dimensional Brownian motion is a real-valued continuous $(\mathcal{F}_t)_{t\geq 0}$- adapted process $(B_t : t \geq 0)$ with the following properties:*

- $B_0 = 0$ $\mathbb{P}$*-a.s,*
- *for* $0 \leq s < t < \infty$*, the increment* $B_t - B_s$ *is normally distributed with mean zero and variance* $t - s$*.*
- *for* $0 \leq s < t < \infty$*, the increment* $B_t - B_s$ *is independent of* $\mathcal{F}_s$*.*

The filtration $(\mathcal{F}_t)_{t\geq 0}$ is a part of the definition of Brownian motion. However, we sometimes speak of a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ without filtration, in this case define $\mathcal{F}_t^B = \sigma\{B_s : 0 \leq s < t\}$ for $t \geq 0$, i.e. $(\mathcal{F}^B : t \geq 0)$ is the σ-algebra generated by $\{B_s : 0 \leq s < t\}$. We call $(\mathcal{F}_t^B : t \geq 0)$ the natural filtration generated by $(B_t : t \geq 0)$. Clearly, $(B_t : t \geq 0)$ is a Brownian motion with respect to the natural filtration $(\mathcal{F}_t^B : t \geq 0)$. In this case, once define a d-dimensional Brownian motion as follow.

Definition 1.2.7 *A stochastic process* $(B_t : t \geq 0)$ *with values in* $\mathbb{R}^d$ *is called a d-dimensional standard Brownian motion if the following conditions hold.*

- $B_0 = 0$ $\mathbb{P}$*-a.s*
- *B has independent increments, i.e.* $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent, for all* $0 < t_0 < t_1 < \cdots < t_n$ *with* $n \in \mathbb{N}$
- *The increments are normally distributed, i. e.* $B_s - B_t \sim \mathcal{N}(0, (t-s)I_d),\ 0 \leq s \leq t$*.*
- $\forall\, \omega \in \Omega,\ t \mapsto B_t(\omega)$ *is continue.*

Proposition 1.2.8 *If* $(B_t : t \geq 0)$ *is a standard one-dimensional Brownian motion,for any integer* $k \geq 0$ *we have*

$$\mathbb{E}(B_t^{2k}) = \frac{(2k)!}{2^k k!} t^k, \qquad \mathbb{E}(B_t^{2k+1}) = 0$$

$(T_n)_{n\geq 0}$ be a strictly increasing sequence of positive random variables. We always take $T_0 = 0$ a.s. Recall that the indicator function $\mathbb{1}_{\{t\geq T_n\}}$ is defined as:

$$\mathbb{1}_{\{t\geq T_n\}} = \begin{cases} 1 \text{ if } \ t \geq T_n \\ 0 \text{ if } \ t < T_n \end{cases}$$

Definition 1.2.9 *The process* $(N_t)_{t\geq 0}$ *define by* $N_t = \sum\limits_{n\geq 1} \mathbb{1}_{\{t\geq T_n\}}$ *with values in* $\mathbb{N} \cup \{\infty\}$*, is is called the counting process associated to the sequence* $(T_n)_{n\geq 0}$*.* $\forall n \geq 0$, $[T_n, +\infty[= \{N \geq n\} = \{(t,\omega), N_t(\omega) \geq n\}$*, as well as* $[T_n, T_{n+1}] =: \{N = n\}$*. Setting* $T = \sup\limits_n T_n$*, then* $[T, \infty[= \{N = \infty\}$*.*

The random variable T is the explosion time of N. If $T = \infty$ a.s., then N is a counting process without explosions. For $T = \infty$, let's note that for $0 \leq s < t < \infty$ we have

$$N_t - N_s = \sum_{n\geq 1} \mathbb{1}_{\{s<T_n\leq t\}}$$

As we have defined a counting process it is not necessarily adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, Indeed, we have the following:

Theorem 1.2.10 *A counting process N is adapted if and only if the associated random variables* $(T_n)_{n\geq 0}$ *are stopping times.*

Definition 1.2.11 *An adapted counting process $N = (N_t)_{t\geq 0}$ without explosions is a Poisson process if*

- *N has increments independent of the past i.e for any $s, t,\ 0 \leq s < t <, N_t - N_s$ is independent of $\mathcal{F}_t$.*
- *N has stationary increments i.e for any , $s, t,\ t \leq 0$ and for any $h > 0$, then the distribution of $N_{t+h} - N_t$ is the same as that of N_h.*

Theorem 1.2.12 *Let $(N_t)_{t\geq 0}$ be a Poisson process. Then $\forall t \geq 0,\ N_t \sim \mathcal{P}(\lambda t)$ for some $\lambda > 0$, i.e. N_t has value in $\mathbb{N}$ and*

$$\forall n \in \mathbb{N},\ \mathbb{P}\left(N_t = n\right) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Moreover, $(N_t)_{t\geq 0}$ is continuous in probability.

Proposition 1.2.13 *Let $(N_t)_{t\geq 0}$ be a Poisson process with density λ, then for any $m \in \mathbb{N}$, the m-th moment of $(N_t)_{t\geq 0}$ is well define and finite and we have*

$$t > 0,\ \mathbb{E}\left(N_t^m\right) = \left[e^{-u}\left(u\frac{d}{du}\right)^m e^u\right]|_{u=\lambda t}$$

Applying this equality with $m = 1, 2$ respectively, we obtain

$$\mathbb{E}(N_t) = \lambda t, \quad \text{and} \qquad \mathbb{V}ar(N_t) = \lambda t$$

Proof. (See [6]) ■

1.3 Martingales

In this section we give, some essential results from the theory of continuous time martingales. we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypotheses.

1.3.1 Definition and examples.

Definition 1.3.1 *A real-valued, adapted process $X = (X_t)_{t\geq 0}$ is called a **martingale** (resp. **super-martingale**, **sub-martingale**) with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if*

- $\mathbb{E}(|X_t|) < \infty$,
- *X_t is $\mathcal{F}_t$-adapted,*
- *if $s \leq t < \infty$, $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$, a.s (resp. $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$, a.s, resp. $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$, a.s)*

Example 1.3.2

- *A Brownian motion process $(B_t : t \geq 0)$ is a martingale with respect to the its natural filtration $\mathcal{F}_t^B = \sigma\{B_s : 0 \leq s < t\}$.*
- *If $N = (N_t)_{t\geq 0}$ is a Poisson process with density λ, then the process $N_t - \lambda t$, $(N_t - \lambda t)^2 - \lambda t$ are martingale.*

Indeed, for $s,\ t,\ 0 \leq s < t < \infty,\ \mathbb{E}(N_t - N_s|\mathcal{F}_s) = \lambda(t - s)$.

1.3.2 Some useful inequality on martingale.

We give here some useful inequality on martingale

Doob's inequality

Let X be a positive sub-martingale. For all $p,\ q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left\| \sup_t |X_t| \right\|_{L^p} \leq q \sup_t \|X_t\|_{L^p}.$$

En particular, denoting $\sup_t |X_t|$ by X^*, Note that if X is a martingale with $X_\infty \in L^2$ then $|X|$ is a positive sub-martingale, and taking $p = 2$ we have

$$\mathbb{E}((X^*)^2) \leq 4\mathbb{E}(X_\infty^2)$$

Burkholder-Davis-Gundy's inequality

Let M be a martingale càdlàg, and let $p > 1$ be fixed. Let $M_t^* = \sup_t |M_t|$. Then there exist constants c_p and C_p such that for any such M

$$c_p \mathbb{E}\left([M,M]_t^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq \mathbb{E}\left((M_t^*)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq C_p \mathbb{E}\left([M,M]_t^{\frac{p}{2}} \right)^{\frac{1}{p}}, \qquad \forall t \geq 0$$

The constants are universal: they do not depend on the choice of M.

1.3.3 Semimartingales and local martingale

Definition 1.3.3 *An adapted, càdlàg process $X = (X_t)_{t\geq 0}$ is a local martingale if there exists a sequence of increasing stopping times, (T_n), with $\lim_{n\to\infty} T_n = \infty$ a.s. such that $t \geq 0$, $X_{t\wedge T_n}\mathbb{1}_{\{T_n>0\}}$ is a uniformly integrable martingale for each n. Such a sequence (T_n) of stopping times is called a fundamental sequence.*

Definition 1.3.4 *An adapted, càdlàg process $X = (X_t)_{t\geq 0}$ is a **semimartingale** if for all $t \geq 0$, $X_t = X_0 + M_t + A_t$, where*

- $M_0 = A_0 = 0$
- *$(M_t)_{t\geq 0}$ is a locally square integrable martingale,*
- *$(A_t)_{t\geq 0}$ is càdlàg, adapted, with paths of finite variation on compacts.*

1.3.4 Change of measure and Girsanov's theorem

Let $P, Q\colon \mathcal{F} \to \mathbb{R}$ be two probability measures on Ω, such that there is an integrable random variable $f\colon \Omega \to \mathbb{R}$, such that $dQ = f dP$. This means

$$Q(A) = \int_A dQ(\omega) = \int_A f(\omega) dP(\omega), \ \ \forall A \in \mathcal{F}.$$

Denote by $\mathbb{E}^P$ and $\mathbb{E}^Q$ the expectations with respect to the measures P and Q, respectively. Then we have

$$\mathbb{E}^Q(X) = \int_\Omega X(\omega) dQ(\omega) = \int_\Omega X(\omega) f(\omega) dP(\omega) = \mathbb{E}^P(fX).$$

The property of martingale is linked to a probability, if we change the probability $\mathbb{P}$ by a probability Q , a martingale under $\mathbb{P}$ has no reason to remain a martingale under Q . Girsanov's theorem allows us to see how a semimartingale is transformed when we change the probability.

Definition 1.3.5 *Two prob ability measure P, Q on $(\Omega, \mathcal{F})$ are said to be equivalent if $P \gg Q$ and $Q \ll P$. (Recall that $P \sim Q$ denotes that $P \ll Q$ is absolutely continuous with respect to Q.) We write $Q \sim P$ to denote equivalence.*

Theorem 1.3.6 *Let (B_t) be a $(\mathcal{F}_t, \mathbb{P})$-Brownian motion, and $f \in L^2([0,T])$*

i) The v.a $Z_T = \exp\left(\int_0^T f(s)dB_s - \frac{1}{2}\int_0^T f^2(s)ds\right)$ is a probability density that allows a probability measure to be defined by $\frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathcal{F}_T} = Z_T$

ii) le processus $B^{\mathbb{Q}}$ define by $B_t^{\mathbb{Q}} = B_t - \int_0^{\min(t,T)} f(s)ds$ is also a Brownian motion under $\mathbb{Q}$.

Remark 1.3.7 *If X is a integrable random variable,*

$$\mathbb{E}_{\mathbb{Q}}(X) := \int_\Omega X d\mathbb{Q} = \int_\Omega X Z_T d\mathbb{P} := \mathbb{E}_{\mathbb{P}}(XZ_T)$$

1.4 Stochastic integral

we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypotheses.

Definition 1.4.1 *A real-valued stochastic process $(H)_{0\leq t\leq T}$ is called a simple (or step) process if there exists a partition $0 = t_1 < .... < t_n = T$ of $[0,T]$ and bounded random variables Φ_k, $0 \leq k \leq n$ such that Φ_k is $\mathcal{F}_{t_{k-1}}$-measurable and*

$$H_t(\omega) = \sum_{k=0}^{n-1} \Phi_k(\omega)\mathbb{1}_{[t_k,\, t_{k+1}[}(t).$$

For a simple process $(H)_{0\leq t\leq T}$, we define

$$I(H_t) := \int_0^t H_s dB_s := \sum_{k\geq 0} \Phi_k(\omega)(B_{t_{k+1}} - B_{t_k})$$

For the reminder of this work we set

$$\mathcal{H} = \left\{(H)_{0\leq t\leq T}\,,\ (\mathcal{F}_t)_{t\geq 0} - \text{adapted process such that } (\mathcal{F}_t)_{t\geq 0}, \mathbb{E}\left(\int_0^T H_s^2 ds\right) < \infty\right\}.$$

Proposition 1.4.2 *For any process $(H_t)_{0\leq t\leq T} \in \mathcal{H}$, there exist a sequence of simple process $(H_t^n)_{0\leq t\leq T}$ such that:*

- $\mathbb{E}(|H_t^n|^2) < +\infty \ \ \forall t \in [0,T], \forall n \in \mathbb{N}$.
- $\lim_{n\longrightarrow\infty} \mathbb{E}\left[\int_0^T |H_t - H_t^n|^2 dt\right] = 0$.

Definition 1.4.3 *Let $H = (H_t)_{0\leq t\leq T} \in \mathcal{H}$, the stochastic integral of H with respect to the Brownian motion $(B_t\,:\ t \geq 0)$ is define by:*

$$\int_0^t H_s dB_s := \lim_{n\longrightarrow\infty} \int_0^t H_s^n dB_s$$

where $(H^n)_{0\leq t\leq T}$ is a simple process that converge to $(H_t)_{0\leq t\leq T}$ in the above proposition sense.

Proposition 1.4.4 *Let $H = (H_t)_{0\leq t\leq T} \in \mathcal{H}$, the following properties hold:*

- $\left(\int_0^t H_s dB_s\right)_{0\leq t\leq T}$ *is a continue $\mathcal{F}_t$-martingale.*
- $\mathbb{E}\left(\int_0^t H_s dB_s\right)^2 = \int_0^t \mathbb{E}\left(H_s^2\right) ds$.

- $\mathbb{E}\left(\sup\limits_{0\le t\le T}|\int_0^t H_s dB_s|^2\right) \le 4\mathbb{E}\left(\int_0^t H_s^2 ds\right)$.

We can also define the stochastic integral with respect to a Poisson process.

Definition 1.4.5 *Let $H = (H_t)_{0\le t\le T} \in \mathcal{H}$ and $N = (N_t)_{t\ge 0}$ a Poisson process. We define a stochastic integral of H with respect to N as:*

$$\int_0^t H_s dN_s := \lim_{n\longrightarrow\infty}\left[\sum_{i=0}^{n} H_{t_i}\triangle N_{t_i}\right]$$

And the following properties hold.

Proposition 1.4.6
- $\mathbb{E}\left(\int_0^T H_s dN_s\right) = \int_0^T \mathbb{E}(H_s)\lambda ds$.
- $\mathbb{E}\left(|\int_0^T H_s dN_s|\right) \le \int_0^T \mathbb{E}(|H_s|)\lambda ds$.
- $\mathbb{E}\left[\left(\int_0^T H_s d\widetilde{N}_s\right)^2\right] = \int_0^T \mathbb{E}(H_s^2)\lambda ds$.

More generally, let $H = (H_t)_{0\le t\le T}$ be an adapted càdlàg process and $(X_t)_{0\le t\le T}$ a continuous local martingale. We define the stochastic integral of H with respect to $(X_t)_{0\le t\le T}$ as follow

$$\int_0^t H_s dX_s := \lim_{n\to\infty}\sum_{t\ge t_k\in\Delta_n} H_{t_k}(\omega)(X_{t_{k+1}}(\omega) - X_{t_k}(\omega)),$$

where $\Delta_n = \left\{s = t_0^n < t_1^n < \cdots < t_{k_n}^n = t\right\}, n \in \mathbb{N}$

Theorem 1.4.7 ***(Gronwall's lemma)***
Let $(A_t)_{t\ge 0}$ be a stochastic and $(X_t)_{t\ge 0}$ an increasing càdlàg process. If

$$0 \le A_t \le \alpha + \int_0^t A_{s^-} dX_s \qquad \forall t\ge 0,\ \textit{then}$$

$$A_t \le \alpha\exp(X_t) \qquad \forall t \ge 0.$$

1.5 Itô's Calculus

Recall that we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ satisfying the usual hypotheses.

Definition 1.5.1 *A one-dimensional Itö's process is a real-valued process $(X_t)_{0\le t\le T}$ such that*

$$\mathbb{P}.p.s, \forall t\le T, X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s$$

where X_0 is $\mathcal{F}_0$-adapted, and $(K_t)_{0\le t\le T}$, $(H_t)_{0\le t\le T}$ are $(\mathcal{F}_t)$-adapted,
$K \in L^1([0,T])$ and $H \in L^2([0,T])$.

Theorem 1.5.2 *Let $(X_t)_{0\le t\le T}$ an Itö's process :*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s$$

If $f:\mathbb{R}\longrightarrow\mathbb{R}$ is class $\mathcal{C}^2$ function, then

$$\begin{aligned} f(X_t) &= f(X_0)+\int_0^t f'(X_s)dX_s+\frac{1}{2}\int_0^t f''(X_s)d\left[X,X\right]_s \\ &= f(X_0)+\int_0^t f'(X_s)K_sds+\int_0^t f'(X_s)H_sdB_s+\frac{1}{2}\int_0^t f''(X_s)H_s^2ds \end{aligned}$$

By the same way, if $f:[0,+\infty[\times\mathbb{R}\longrightarrow\mathbb{R}$ is a class $\mathcal{C}^{1,2}$ then

$$f(t,X_t)=f(0,X_0)+\int_0^t\frac{\partial f}{\partial s}(s,X_s)ds+\int_0^t\frac{\partial f}{\partial x}(s,X_s)dX_s+\frac{1}{2}\int_0^t\frac{\partial^2 f}{\partial x^2}(s,X_s)H_s^2d\left[X,X\right]_s.$$

Or in differential form

$$df(t,X_t)=\frac{\partial f}{\partial t}(t,X_t)dt+\frac{\partial f}{\partial x}(t,X_t)dX_t+\frac{1}{2}\int_0^t\frac{\partial^2 f}{\partial x^2}(t,X_t)\left(dX_t\right)^2$$

with the convention $dtdt=dtdB_t=dB_tdt=0$ and $dB_tdB_t=dt$.

Proof. (See [8]) ■

More generally, the Itö's formula can be extend to any càdlàg semi-martingale and we have

Theorem 1.5.3 *Let X be a cádlág semi-martingale f a class C^2 function ,then $f(X)$ is semi-martingale and* $\forall t\geq 0$

$$\begin{aligned} f(X_t) &= f(X_0)+\int_0^t f'(X_{s^-})dX_s+\frac{1}{2}\int_0^t f''(X_{s^-})d\langle X\rangle_s \\ &+ \sum_{0\leq s\leq t}\left\{f(X)-f(X_{s^-})-f'(X_{s^-})\triangle X_s-\frac{1}{2}f''(X_{s^-})\left(\triangle X_s\right)^2\right\} \\ &= \int_0^t f'(X_{s^-})dX_s+\frac{1}{2}\int_0^t f''(X_{s^-})d\langle X^c\rangle_s+\sum_{0\leq s\leq t}\left\{f(X)-f(X_{s^-})-f'(X_{s^-})\triangle X_s\right\} \end{aligned}$$

Proof. (See [15]) ■

Corollary 1.5.4 *Let (X_t) a Itô's process with Poissonian jump, i.e*

$$X_t=X_0+\int_0^t K_sds+\int_0^t H_sdB_s+\int_0^t F_sdN_s.$$

and f a class C^2 function, then

$$\begin{aligned} f(X_t)=f(X_0)+\int_0^t f'(X_s)K_sds &+ \int_0^t f'(X_s)H_sdB_s+\frac{1}{2}\int_0^t f''(X_s)H_s^2ds \\ &+ \int_0^t\left(f(X_{s^-}+F(X_s))-f(X_{s^-})\right)dN_s. \end{aligned}$$

Proposition 1.5.5 *Let X, Y be two semi-martingales,then for all $t\geq 0$*

$$X_tY_t=X_0Y_0+\int_0^t X_{s^-}dY_s+\int_0^t Y_{s^-}dX_s+[X,Y]_t$$

where $[X,Y]_t=[X,Y]_t^c+\sum_{0\leq s\leq t}\triangle X_s\triangle Y_s$.

We design by X^c, the continue part of the process X.

1.6 Local time of a semi-martingale

Theorem 1.6.1 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be convex and let X be a semimartingale. Then $f(X)$ is a semimartingale and one has*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s^-})dX_s + A_t$$

where f' is the left derivative of f and A is an adapted, right continuous, increasing process. Moreover,

$$\triangle A_t = f(X_t) - f(X_{t^-}) - f'(X_{t_-})\triangle X_t$$

Let's define the *sign* function as follow

$$sign(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x \leq 0 \end{cases}$$

Let $a \in \mathbb{R}$, we define a function h_a as $\begin{cases} h_0(x) := |x| \\ h_a(x) := |x-a| \end{cases}$, Then $sign(x)$ is the left derivative of $h_0(x)$, and $sign(x-a)$ is the left derivative of $h_a(x)$. Since $h_a(x)$ is convex, by above theorem we have for a semimartingale X

Proof. (See [15]) ■

$$|X_t - a| = |X_0 - a| + \int_0^t sign(X_{s^-} - a)dX_s + A_t^a$$

Definition 1.6.2 *Let X be a semimartingale, and let h_a and A^a be as defined above. The local time at a of X denoted by $L^a(X)_t$ and define as*

$$L^a(X)_t := A_t^a - \sum_{0<s\leq t} \{|X_s - a| - |X_{s^-} - a| - sign(X_{s^-} - a)\triangle X_s\}$$

Theorem 1.6.3 *Let X be a semimartingale, and let $L^a(X)_t$ be its local time at the level a, for each $a \in \mathbb{R}$. For a.s. ω, the measure in t, $dL_t^a(\omega)$ is carried by the set*

$$\{s : X_s(\omega) = X_{s^-}(\omega) = a\}$$

.

Theorem 1.6.4 *Let f be the difference of two convex functions, let f' be its left derivative, and let μ be the signed measure (when restricted to compacts) which is the second derivative of f in the generalized function sense. Then the following equation holds:*

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s^-})dX_s + \sum_{0<s\leq t} \{f(X_s) - f(X_{s^-}) - f'(X_{s^-})\triangle X_s\} + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^a \mu(da),$$

where X is a semimartingale and $L_t := L^a(X)_t$ is its local time at a.

Proof. *(See [15])* ■

Corollary 1.6.5 *Let X be a semimartingale with local time $(L^a)_{a\in\mathbb{R}}$. Let g be a bounded Borel measurable function. Then a.s.*

$$\int_{\mathbb{R}} L_t^a g(a)da = \int_0^t g(X_{s^-})d\,[X,X]_s^c$$

1.7 Stochastic Differential Equations (SDEs)

1.7.1 Definition and examples of SDEs

Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion, let $(N_t)_{t\geqslant 0}$ be a Poisson process and X_t be a cádlág stochastic process. An equation of the form

$$\begin{cases} \mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t + c(t^-,X_{t^-})\mathrm{d}N_t, \\ X_0 = Z \end{cases} \tag{1.7.1}$$

where $b,\ \sigma, c\ :\ [0,\infty)\times\mathbb{R}\longrightarrow\mathbb{R}$ are given and X_t is the unknown process, is called a stochastic differential equation (SDE) driven by the Brownian motion $(B_t)_{t\geqslant 0}$ and $(N_t)_{t\geqslant 0}$.
The functions b, c and σ are called the coefficients of the stochastic differential equation (1.7.1).
In fact, this differential relation has the following integral meaning:

$$X_t = Z + \int_0^t b(s,X_s)\mathrm{d}s + \int_o^t \sigma(s,X_s)\mathrm{d}B_s + \int_o^t c(s^-,X_{s^-})\mathrm{d}N_s, \tag{1.7.2}$$

where Z is an $\mathcal{F}_0$-measurable random variable and the last two integral is taken in the Itö sense.
The functions $b(t,x)$, $\sigma(t,x)$ and $c(t,x)$ are called drift, volatility and jump coefficient of the process X_t , respectively.

1.7.2 Existence and uniqueness of the solution of a SDE

Given these two functions as input, one may seek for the solution X_t of the stochastic differential equation as an output. The desired outputs X_t are the so-called strong solutions. The precise definition of this concept is given in the following.

Definition 1.7.1 *we consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. Given functions $b\colon\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$, $\sigma\colon\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$, and $c\colon\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$; an $\mathcal{F}_0$-measurable random variable Z, Finally a standard $\mathcal{F}_t$-Brownian motion $(B_t)_{t\geqslant 0}$ and $(N_t)_{t\geqslant 0}$ a $(\mathcal{F}_t)_{t\geq 0}$-adapted Poisson process. A solution to equation 1.7.1 is an $(\mathcal{F}_t$-adapted continuous stochastic process $(X_t)_{t\geqslant 0}$ that satisfies:*

- $\int_0^t |b(s,X_s)|ds + \int_0^t |\sigma(s,Xs)|^2 ds + \int_0^t |c(s,X_{s^-})|ds < +\infty$ $\mathbb{P}$ *p.s.*
- $\forall t\geq 0$, $\mathbb{P}$*-a.s,* $X_t = x + \int_0^t b(s,Xs)ds + \int_0^t \sigma(s,Xs)dB_s + \int_0^t c(s,X_{s^-})dN_s$.

Theorem 1.7.2 *If b, cand σ are continuous functions, and if there exists a constant $K<+\infty$ such that*

- $|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| + |c(t,x)-c(t,y)| \leq K|x-y|$
- $|b(t,x)| + |\sigma(t,x)| + |c(t,x)| \leq K(1+|x|)$
- $\mathbb{E}(Z^2) < \infty$

then, for any $T\geq 0$, (1.7.1) admits a unique solution in the interval $[0,\ T]$. Moreover, this solution $(X_t)_{0\leq t\leq T}$ satisfies

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X_t|^2\right) < \infty$$

The uniqueness of the solution means that if $(X_t)_{0\leq t\leq T}$ and $(Y_t)_{0\leq t\leq T}$ are two solutions of (1.7.1), then $\mathbb{P}$ a.s. $\forall 0 \leq t \leq T,\ X_t = Y_t$.

The following so-called Yamada-Watanabe's theorem give the existence an uniqueness of the solution of (1.7.1) in the case where the volatility coefficient doesn't satisfy the Lipschitz condition.

Theorem 1.7.3 ***Yamada-Watanabe*** *If the functions b and c global Lipschitz condition and there an increasing and continue function $\rho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

- $\sigma(0) = 0$,
- $\int_0^{\varepsilon} \frac{1}{\rho^2}(y)dy = \infty, \forall \varepsilon > 0$,
- $|\sigma(x) - \sigma(y)| \leq \rho\,(|x-y|)$.

Then the pathwise uniqueness of solutions holds for the equation (1.7.1) and hence it has the unique strong solution.

Proof. (See [18]) ■

1.8 Numerical solution for stochastic differential equation

1.8.1 Approach of numerical method

In this section, we find an approach to introduce the numerical method for SDEs.
Let's consider a (deterministic) ODE of the form

$$\begin{cases} X'(t) = f(X(t)) \\ X(0) = X_0 \end{cases}$$

The solution reads

$$X(t) = X_0 + \int_0^t f(X(s))\mathrm{d}s,\ \ t \in [0,T].$$

We first discretize $[0,T]$ into N "small" intervals $[t_n, t_{n+1}]$, where $t_n = n\triangle t$ and $\triangle t = T/N$ is the step-size in time. On the interval $[t_n, t_{n+1}]$ we obtain.

$$\begin{aligned} X(t_{n+1}) &= X(t_n) + \int_{t_n}^{t_{n+1}} fX(s)ds \approx X(t_n) + f(X(t_n)) \int_{t_n}^{t_{n+1}} \mathrm{d}t \\ &\approx X(t_n) + \triangle t f(X(t_n)) \end{aligned}$$

This motivates the Euler method for ordinary differential equation (ODEs).

$$X_{n+1} = X_n + \triangle t f(X_n),\ \ X_n \approx X(t_n).$$

The order of convergence of Euler's method is p = 1 and thus, the global error (in general) satisfies

$$|X_N - X(T)| \leq C\triangle t.$$

We now follow the same approach for SDEs

$$dX_t = b(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t + c(X(s))\mathrm{d}N_t$$

On the interval $[t_n, t_{n+1}]$ the exact solution reads

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{tn+1} b(X(s))\mathrm{d}s + \int_{t_n}^{tn+1} \sigma(X(s))\mathrm{d}B_s + \int_{t_n}^{tn+1} c(X(s))\mathrm{d}B_s.$$

We freeze the integrands at t_n and get

$$\begin{aligned} X(t_{n+1}) &\approx X(t_n) + b(X(t_n)) \int_{t_n}^{t_{n+1}} \mathrm{d}t + \sigma(X(t_n)) \int_{t_n}^{t_{n+1}} \mathrm{d}B_t \\ &= X(t_n) + b(X(t_n))\triangle t + \sigma(X(t_n))(B_{t_{n+1}} - B_{t_n}) + c(X(t_n))(N_{t_{n+1}} - N_{t_n}) \\ &= X(t_n) + b(X(t_n))\triangle t + \sigma(X(t_n))\triangle B_{t_n} + c(X(t_n))\triangle N_{t_n} \end{aligned}$$

where

$$\triangle B_{t_n} = \int_{t_n}^{t_{n+1}} \mathrm{d}B_t = B_{t_{n+1}} - B_{t_n} \sim \mathcal{N}(0, \triangle t) \text{ and } \triangle N_{t_n} = \int_{t_n}^{t_{n+1}} \mathrm{d}N_t = N_{t_{n+1}} - N_{t_n} \sim \mathcal{P}(\lambda \triangle t)$$

are called Wiener increments. This motivates the Euler-Maruyama scheme $X_0 := X(0)$ and

$$X_{n+1} = X(t_n) + b(X(t_n))\triangle t + \sigma(X(t_n))\triangle B_{t_n}, \;\; X_n \approx X(t_n).$$

1.8.2 Convergence of numerical method.

Definition 1.8.1 ***(Weak error)***
Given a function $\varphi\colon \mathbb{R} \to \mathbb{R}$, the weak error

$$err^{Weak}_{\Delta t} := \sup_n |\mathbb{E}[\Phi(X_n)] - \mathbb{E}\Phi[X(t_n)]|$$

measures how well the numerical solution $(X_n)_n$ can approximate the expectation $\mathbb{E}[\Phi(X(t_n))]$ of the exact solution $X(t_n)$.

Definition 1.8.2 ***(Weak convergence)***

- *A numerical scheme $(X_n)_n$ converges weakly if for any function Φ of a certain class,*

$$err^{Weak}_{\Delta t} \to 0 \text{ as } \Delta t \to 0$$

- *A numerical method $(X_n)_n$ has weak order of convergence p if there exists $K > 0$ and a step size level Δt^* that may depend on Φ, such that*

$$err^{Weak}_{\Delta t} \leq K\Delta t^p, \;\; \text{for } \; 0 \leq \Delta t \leq \Delta t^*.$$

Definition 1.8.3 ***(Strong error)***
The strong error is defined as

$$err^{Strong}_{\Delta t} := \sup_n \mathbb{E}[|X_n - X(t_n)|]$$

where $X(t)$ is the exact solution to (SDE) and $(X_n)_n$ n is the numerical approximation.

Definition 1.8.4 ***(Strong convergence)***

- *A numerical scheme $(X_n)_n$ converges strongly, $err_{\Delta t}^{Strong} \to 0$ as $\Delta t \to 0$*
- *A numerical method $(X_n)_n$ has strong order of convergence q $err_{\Delta t}^{Weak} \leq K\Delta t^q$, where Δt is small enough and $K > 0$ does not depend on Δt.*

In this first chapter, we resumed some useful results of stochastic analysis including the basic notion of probability theory, stochastic process, SDEs and numerical method for approximate solution (if exist) of SDEs. In the next chapter, we will go through CIR type process with Poissonian jump and explore some properties of this kind of process.

Chapter 2

CIR type process with Poissonian jump.

For the reminder of this document, we consider and standard Brownian motion $(B_t)_{0\leq t\leq T}$, a Poisson process $(N_t)_{0\leq t\leq T}$ with density $\lambda > 0$, and a filtration $(\mathcal{F}_t^{B,N})_{0\leq t\leq T}$ generated by that Brownian motion and Poisson process. We consider the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^{B,N})_{t\geq 0}, \mathbb{P}\right)$.

The main references for this chapter are [4], [5], [15], [20] and [11].

2.1 Classical CIR process.

The Cox-Ingersoll-Ross (CIR for short) process is a stochastic process introduce by John. C. Cox, Jonathan. E. Ingersoll and Stephen. A. Ross [3] to modelling the short time interest rate in financial market. That satisfies the following SDE

$$X_t = x_0 + \int_0^t (\alpha - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s, \;\; X_0 = x_0 \geq 0. \tag{2.1.1}$$

where α, b, σ are positive constants.
Note that the classical CIR process satisfies the following

Proposition 2.1.1 *Let $x \in \mathbb{R}_+^*$, and let $(X_t^x)_{0\leq t\leq T}$ the process solution of (2.1.1) started at x and let τ_0^x the stopping time define by*

$$\tau_0^x = \inf\{t \geq 0, X_t^x = 0\}$$

withe the convention $\inf \emptyset = \infty$.

- *If $\alpha \geq \frac{\sigma^2}{2}$, then $\mathbb{P}\left(\tau_0^x = \infty\right) = 1$.*
- *If $0 \leq \alpha \leq \frac{\sigma^2}{2}$ and $b \geq 0$, then $\mathbb{P}\left(\tau_0^x < \infty\right) = 1$.*
- *If $0 \leq \alpha \leq \frac{\sigma^2}{2}$ and $b < 0$, then $\mathbb{P}\left(\tau_0^x < \infty\right) \in]0, 1[$*

Proof. (See [11]) ∎

2.2 Definition of CIR type process with Poissonian jump.

A simple generalisation of CIR process introduce in [20] take account a Poissonian jump with constant size and then modifies as

$$X_t = x_0 + \int_0^t (\alpha - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t a\mathrm{d}N_s, \;\; X_0 = x_0 \geq 0.$$

where $a > 0$.
We define a CIR type process with Poissonian jump as follow

Definition 2.2.1 *A CIR type process with Poissonian jump is a cádlág stochastic process $(X_t)_{0\leq t\leq T}$ satisfying the following stochastic differential equation*

$$X_t = x_0 + \int_0^t b(X_s)ds + \sigma \int_0^t \sqrt{X_s}dB_s + \int_0^t adN_s, \;\; X_0 = x_0 \geq 0. \tag{2.2.1}$$

where $b\colon \mathbb{R} \longrightarrow \mathbb{R}$ designs the drift of $(X_t)_{0\leq t\leq T}$ and $\sigma,\ a$ are positive constant, stand for volatility and jump size of $(X_t)_{0\leq t\leq T}$ respectively.
In order to guarantee the existence and uniqueness of such equation and preserving some properties of classical CIR process, we make the following assumptions.

Assumptions

$$(H)\begin{cases} H_1: \ b(x) \text{ satisfies the Lipschitz condition, i.e there exist, } K > 0 \text{ such that} \\ \qquad\qquad |b(x) - b(y)| \leq K|x-y| \ \forall x, y \in \mathbb{R}. \\ H_2: \ b(x) \text{ is class } C^4\text{, and have all its derivative up to order bounded .} \\ H_3: \ b(0) \geq 2\sigma^2 \\ H_4: \ \text{The process } \ (B_t)_{t\geq 0} \, and \, (N_t)_{t\geq 0} \text{ are independent.} \end{cases}$$

2.3 Existence and uniqueness of the solution (2.5.1).

We start this with the following lemma

Lemma 2.3.1 *The $\varphi : x \mapsto \sqrt{x}$ is Hölderian of order $\frac{1}{2}$, that is, there exist $L > 0$ such that $\forall x, y \in \mathbb{R}_+$,*

$$|\varphi(x) - \varphi(y)| \leq L|x-y|^{\frac{1}{2}}$$

Theorem 2.3.2 *Under the assumption (H), the stochastic differential equation (2.5.1) has an unique solution, moreover, this unique solution has positive path if $x_0 \geq 0$.*

Proof. The existence and uniqueness of the solution of (2.5.1) if given by Yamada-Watanabe's theorem give in (1), indeed, the coefficients of such equation satisfies the so-called Yamada-Watanabe's conditions.
For the positivity of the path, let $(X_t)_{t\geq 0}$ the solution of (2.5.1) with $X_0 = x_0 \geq 0$, since $b(x)$ satisfies the Lipschitz's condition, and $b(0) \geq 0$, then $\forall t \geq 0$, $b(X_t) \geq b(0) - KX_t \geq -KX_t$. Moreover, $x_0 \geq 0$, if (X_t) is the solution of (2.5.1) Then, $X_t \geq Y_t$ $\forall t$ a.s, where Y is the solution of SDE

$$dY_t = -KY_t\mathrm{d}t + \sigma\sqrt{Y_t}\mathrm{d}B_t. \tag{2.3.1}$$

from the Yamada-Watanabe's theorem, (2.3.1) has a unique solution, and $Y \equiv 0$ is a solution of such SDE. Then $Y \equiv 0$ is the unique solution of (2.3.1). Hence $X_t \geq 0, \forall t$ a.s ■

2.4 Moment of CIR type process with Poissonian jump

2.4.1 Existence of the moment

The following theorem show that the CIR type process with Poissonian jump has moment of any order p.

Theorem 2.4.1 *Let $(X_t)_{0\leq t\leq T}$ be a CIR type process with Poissonian jump, with $X_0 = x_0 \geq 0$. under the assumptions (H_1) and (H_3), for all $p \geq 1$, there exist a constants C_p that only depend on $b(0), \sigma, K, a, p$ and T such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} X_t^{2p}\right) \leq C_p(1 + x_0^{2p}).$$

Proof. Let $p \geq 1$, since we are not sure that $\mathbb{E}\left(\sup_{0\leq t\leq T} X_t^{2p}\right)$ is finite or not, let's introduce a control argument, we define the sequence of stopping time (τ_n) by:

$$\tau_n := \inf\{t \in [0, T], |X_t| \geq n\}$$

with the convention $\inf \emptyset = T$.
We have :
$\forall t \in [0, T]$,

$$X_{t\wedge\tau_n} = x_0 + \int_0^{t\wedge\tau_n} b(X_s)\mathrm{d}s + \sigma \int_0^{t\wedge\tau_n} \sqrt{X_s}\mathrm{d}B_s + aN_{t\wedge\tau_n}$$

Applying the Itö's formula with $\varphi : x \mapsto x^{2p}$, we obtain:
$\forall t \in [0, T]$,

$$\begin{aligned} X_{t\wedge\tau_n}^{2p} &= x_0^{2p} + 2p\int_0^{t\wedge\tau_n} X_s^{2p-1} dX_s^c + p(2p-1)\int_0^{t\wedge\tau_n} X_s^{2p-2} d\,[X,X]_s^c + \int_0^{t\wedge\tau_n} \left((X_s+a)^{2p} - X_s^{2p}\right)\mathrm{d}N_s \\ &= x_0^{2p} + 2p\int_0^{t\wedge\tau_n} X_s^{2p-1} b(X_s)\mathrm{d}s + 2p\int_0^{t\wedge\tau_n} X_s^{2p-1}\sqrt{X_s}\mathrm{d}B_s + p(2p-1)\int_0^{t\wedge\tau_n} X_s^{2p-2} X_s \mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau_n} \left((X_s+a)^{2p} - X_s^{2p}\right)\mathrm{d}N_s. \end{aligned}$$

Taking the expectation of the above equation we get

$$\begin{aligned} \mathbb{E}\left(X_{t\wedge\tau_n}^{2p}\right) &= x_0^{2p} + 2p\int_0^{t\wedge\tau_n} \mathbb{E}\left(X_s^{2p-1} b(X_s)\right)\mathrm{d}s + p(2p-1)\sigma^2 \int_0^{t\wedge\tau_n} \mathbb{E}\left(X_s^{2p-2}X_s\right)\mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau_n} \mathbb{E}\left((X_s+a)^{2p} - X_s^{2p}\right)\lambda \mathrm{d}s. \end{aligned}$$

because $\mathbb{E}\left(\int_0^{t\wedge\tau_n} X_s^{2p-1}\sqrt{X_s}\mathrm{d}B_s\right) = 0$ since this process is a martingale.
Since b satisfies the Lipschitz's condition with constant $K > 0$ and $b(0) \geq 0$, then $|b(X_s)| \leq b(0) + KX_s$ using the convexity inequality we havse,

$$(X_s + a)^{2p} - X_s^{2p} \leq (2^{2p-1} - 1)X_s^{2p} + 2^{2p-1}a^{2p}.$$

Introducing this in the above inequality, we obtain

$$\begin{aligned}\mathbb{E}\left(X_{t\wedge\tau_n}^{2p}\right) &\leq x_0^{2p} + 2^{2p-1}a^{2p} + 2p\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p-1}b(0)\right)\mathrm{d}s + 2pK\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p-1}X_s\right)\\ &+ p(2p-1)\sigma^2\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p-1}\right)\mathrm{d}s + \lambda(2^{2p-1}-1)\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p}\right)\mathrm{d}s.\end{aligned}$$

Using the Young's inequality $\left(\forall a,b\geq 0, \forall p,q \in [0,+\infty[\quad \text{such that} \quad \frac{1}{p}+\frac{1}{q}=1, ab\leq \frac{1}{p}a^p+\frac{1}{q}b^q\right)$ we get

$$X_s^{2p-1}b(0) \leq \frac{2p-1}{2p}X_s^{2p} + \frac{1}{2p}b(0)^{2p}$$

Hence

$$\begin{aligned}\mathbb{E}\left(X_{t\wedge\tau_n}^{2p}\right) &\leq x_0^{2p} + 2^{2p-1}a^{2p} + p(2p-1)\sigma^2 + Tb(0)^{2p} + \Big(2p-1+2pK+\lambda(2^{2p-1})\\ &\qquad +p(2p-1)\sigma^2\Big)\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p}\right)\mathrm{d}s\\ &\leq C\left(1+x_0^{2p}+\int_0^{t\wedge\tau_n}\mathbb{E}\left(X_s^{2p}\right)\mathrm{d}s\right)\\ &\leq C\left(1+x_0^{2p}+\int_0^{t}\mathbb{E}\left(X_{s\wedge\tau_n}^{2p}\right)\mathrm{d}s\right)\end{aligned}$$

where

$$C = \max(2^{2p-1}a^{2p} + p(2p-1)\sigma^2 + Tb(0)^{2p},\ 2p-1+2pK+\lambda(2^{2p-1}) + p(2p-1)\sigma^2)$$

Using the Gronwall's lemma, we obtain

$$\mathbb{E}\left(X_{t\wedge\tau_n}^{2p}\right) \leq C(p)(1+x_0^{2p})$$

where $C(p)$ is a positive constant that doesn't depend on n.
Taking the limit as $n\to\infty$ and using the convergence dominate theorem, we get:

$$\sup_{0\leq t\leq T}\mathbb{E}\left(X_t^{2p}\right) \leq C(p)\left(1+x_0^{2p}\right)$$

Using Burkholder-Davis-Gundy's inequality we obtain finally

$$\mathbb{E}\left(\sup_{0\leq t\leq T}X_t^{2p}\right) \leq C(p)\left(1+x_0^{2p}\right)$$

■

2.4.2 Mean and moment of order 2 of classical CIR process with Poison jump.

Lemma 2.4.2 *Let $(X_t)_{0\leq t\leq T}$ be the classical CIR process with Poison jump, that is it satisfies the following SDE*

$$dX_t = (\alpha - bX_t)dt + \sigma\sqrt{X_t}dB_t + adN_t, X_0 = x_0 \geq 0 \tag{2.4.1}$$

Then,

$$X_t = x_0e^{-kt} + \frac{\alpha}{b}\left(1-e^{-bt}\right) + \sigma e^{-bt}\int_0^t e^{bs}\sqrt{X_s}dB_s + ae^{-bt}\int_0^t e^{bs}dN_s.$$

Proof. We have

$$dX_t = (\alpha - bX_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}B_t + a\mathrm{d}N_t, X_0 = x_0 \geq 0.$$

Then

$$\begin{aligned}
e^{bt}dX_t &= \alpha e^{bt}\mathrm{d}t - be^{bt}X_t\mathrm{d}t + \sigma e^{bt}\sqrt{X_t}\mathrm{d}B_t + e^{bt}a\mathrm{d}N_t \\
e^{bt}dX_t + be^{bt}X_t\mathrm{d}t &= \alpha e^{bt}\mathrm{d}t + \sigma e^{bt}\sqrt{X_t}\mathrm{d}B_t + e^{bt}a\mathrm{d}N_t \\
d\left(e^{bt}X_t\right) &= \alpha e^{bt}\mathrm{d}t + \sigma e^{bt}\sqrt{X_t}\mathrm{d}B_t + e^{bt}a\mathrm{d}N_t
\end{aligned}$$

Hence

$$e^{bt}X_t = x_0 + \frac{\alpha}{b}(e^{bt} - 1) + \sigma\int_0^t e^{bs}\sqrt{X_s}\mathrm{d}B_s + a\int_0^t e^{bs}\mathrm{d}N_s$$

Finally

$$X_t = x_0e^{-bt} + \frac{\alpha}{b}\left(1 - e^{-bt}\right) + \sigma e^{-bt}\int_0^t e^{bs}\sqrt{X_s}\mathrm{d}B_s + ae^{-bt}\int_0^t e^{bs}\mathrm{d}N_s.$$

■

Proposition 2.4.3 *Let $(X_t)_{0\leq t\leq T}$ be the classical CIR process with Poissonian jump (**??**), then $\forall t \in [0, T]$,*

$$\mathbb{E}\left(X_t\right) = x_0e^{-bt} + \left(\frac{\alpha}{b} + \frac{a\lambda}{b}\right)\left(1 - e^{-bt}\right).$$

and

$$\begin{aligned}
\mathrm{E}(X_t^2) &= x_0e^{-2bt} + \left(2\alpha + \sigma^2 + 2a\lambda\right)\left(\frac{x_0}{b} - \frac{\alpha + a\lambda}{b^2}\right)\left(e^{-bt} - e^{-2bt}\right) \\
&+ \left[\left(2\alpha + \sigma^2 + 2a\lambda\right)\frac{\alpha + a\lambda}{b^2} + \frac{\lambda a^2}{2b}\right]\left(1 - e^{-2bt}\right)
\end{aligned}$$

Proof.

- For the mean We have

  $$X_t = x_0e^{-bt} + \frac{\alpha}{b}\left(1 - e^{-bt}\right) + \sigma e^{-bt}\int_0^t e^{bs}\sqrt{X_s}\mathrm{d}B_s + ae^{-bt}\int_0^t e^{bs}\mathrm{d}N_s.$$

  Hence

  $$\begin{aligned}
  \mathbb{E}\left(X_t\right) &= x_0e^{-bt} + \frac{\alpha}{b}\left(1 - e^{-bt}\right) + ae^{-bt}\int_0^t e^{bs}\lambda\mathrm{d}s \\
  &= x_0e^{-bt} + \left(\frac{\alpha}{b} + \frac{a\lambda}{k}\right)\left(1 - e^{-bt}\right)
  \end{aligned}$$

- To compute the moment of order of $(X_t)_{0\leq t\leq T}$ let's the Itö's formula with the function $x \mapsto x^2$, we get

  $$X_t^2 = x_0^2 + 2\alpha\int_0^t X_s\mathrm{d}s - 2\int_0^t X_s^2\mathrm{d}s + 2\sigma\int_0^t X_s\sqrt{X_s}\mathrm{d}B_s + \sigma^2\int_0^t X_s\mathrm{d}s + \int_0^t\left(2aX_s + a^2\right)\mathrm{d}N_s.$$

  Hence

  $$\mathbb{E}\left(X_t^2\right) = x_0^2 + \left(2\alpha + \sigma^2 + 2a\lambda\right)\int_0^t \mathbb{E}\left(X_s\right)\mathrm{d}s - 2b\int_0^t \mathbb{E}\left(X_s^2\right)\mathrm{d}s + \lambda a^2 t$$

setting $y(t) = \mathbb{E}\left(X_t^2\right)$, then

$$\begin{aligned} y'(t) &= \left(2\alpha + \sigma^2 + 2a\lambda\right)\mathbb{E}\left(X_t\right) - 2by(t) + \lambda a^2 \\ &= 2by(t) + \left(2\alpha + \sigma^2 + 2a\lambda\right)\left[\left(x_0 - \frac{\alpha + a\lambda}{b}\right)e^{-bt} + \frac{\alpha + a\lambda}{b}\right] + \lambda a^2 \\ &= -2by(t) + A_1 e^{-bt} + A_2 \end{aligned}$$

where $A_1 = \left(2\alpha + \sigma^2 + 2a\lambda\right)\left(x_0 - \frac{\alpha + a\lambda}{b}\right)$ and $A_1 = \left(2\alpha + \sigma^2 + 2a\lambda\right)\frac{\alpha + a\lambda}{b} + \lambda a^2$.
The general solution of the above ordinary differential equation is given by

$$y(t) = ke^{-2bt}$$

then

$$y'(t) + 2by(t) = k'(t)e^{-2bt} = A_1 e^{-bt} + A_2$$

Hence

$$k(t) = \frac{A_1}{b}e^{bt} + \frac{A_2}{2b}e^{2bt} + C$$

Or $k(0) = y(0) = x_0^2$. however $C = x_0^2 - \frac{A_1}{b} - \frac{A_2}{2b}$.
Then

$$k(t) = x_0^2 + \frac{A_1}{b}\left(e^{bt} - 1\right) + \frac{A_2}{2b}\left(e^{2bt} - 1\right)$$

Hence

$$y(t) = x_0^2 e^{-2bt} + \frac{A_1}{b}\left(e^{-bt} - e^{-2bt}\right) + \frac{A_2}{2b}\left(1 - e^{-2bt}\right)$$

replacing A_1 and A_2 by the above expressions, we obtain

$$\begin{aligned} \mathrm{E}(X_t^2) &= x_0 e^{-2bt} + \left(2\alpha + \sigma^2 + 2a\lambda\right)\left(\frac{x_0}{b} - \frac{\alpha + a\lambda}{b^2}\right)\left(e^{-bt} - e^{-2bt}\right) \\ &\quad + \left[\left(2\alpha + \sigma^2 + 2a\lambda\right)\frac{\alpha + a\lambda}{b^2} + \frac{\lambda a^2}{2b}\right]\left(1 - e^{-2bt}\right) \end{aligned}$$

∎

2.4.3 Inverse moment of CIR type Process with Poissonian jump

Proposition 2.4.4 $\forall\, x > 0, p > 0$

$$\mathbb{E}\left[\frac{1}{(X_t^x)^p}\right] \leq \frac{1}{2^p \Gamma(p) L(t)^p}\int_0^1 \theta^{p-1}(1-\theta)^{\frac{2b(0)}{\sigma^2} - p - 1}\exp\left(-\frac{xe^{-Kt}}{2L(t)}\theta\right)d\theta$$

where $L(t) = \frac{\sigma^2}{4K}\left(1 - e^{-Kt}\right)$ *and* $\Gamma(p) = \int_0^{+\infty} u^{p-1}\exp(-u)du$, *designs the Gamma function. In particular, under the assumption* (H_3) $(b(0) \geq \sigma^2)$,

$$\mathbb{E}\left[\frac{1}{X_t^x}\right] \leq \frac{e^{Kt}}{x} \tag{2.4.2}$$

Moreover, for any real number $\mu \leq \frac{\nu\sigma^2}{8}$ *(with* $\nu = \frac{2b(0)}{\sigma^2} - 1$.*),*

$$\mathbb{E}\left[\exp\left(\mu\int_0^T \frac{1}{X_t^x}dt\right)\right] \leq C(T)\left(1 + x^{-\frac{\nu}{2}}\right) \tag{2.4.3}$$

where $C(T)$ is an non deceasing function of T and doesn't depend on x.

Proof. let $x > 0$, $p > 0$,
let's recall that the $(X_t^x)_{0\leq t\leq T}$ satisfies

$$X_t^x = x + \int_0^t b(X_s^x)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s^x}\mathrm{d}B_s + a\int_0^t \mathrm{d}N_s$$

Let $t \in [0,T]$, Since the function $b(x)$ satisfies the Lipschitz's condition with constant K, then $b(X_t^x) \geq b(0) - KX_t^x$ moreover $a > 0$. Hence $X_t^x \geq Z_t^x$ $\mathbb{P} - p.s$ where $(Z_t^x)_{0\leq t\leq T}$ the solution of the stochastic differential equation

$$Z_t^x = x + \int_0^t (b(0) - KZ_s^x)\mathrm{d}s + \sigma \int_0^t \sqrt{Z_s^x}\mathrm{d}B_s.$$

, thus

$$\frac{1}{(X_t^x)^p} \leq \frac{1}{(Z_t^x)^p} \quad \mathbb{P} - p.s$$

Therefore

$$\mathbb{E}\left[\frac{1}{(X_t^x)^p}\right] \leq \mathbb{E}\left[\frac{1}{(Z_t^x)^p}\right].$$

furthermore,

$$\frac{1}{(Z_t^x)^p} = \frac{1}{\Gamma(p)}\int_0^{+\infty} u^{p-1}\exp\left(-uZ_t^x\right)du$$

Taking the expectation, we get

$$\mathbb{E}\left[\frac{1}{(Z_t^x)^p}\right] = \frac{1}{\Gamma(p)}\int_0^{+\infty} u^{p-1}\mathbb{E}\exp\left(-uZ_t^x\right)du.$$

and the Laplace transform of Z_t^x give

$$\mathbb{E}[\exp\left(-uZ_t^x\right)] = \frac{1}{(2uL(t)+1)^{\frac{2b(0)}{\sigma^2}}}\exp\left(-\frac{uL(t)\zeta(t,x)}{2uL(t)+1}\right) [11]$$

with $L(t) = \frac{\sigma^2}{4K}\left(1 - e^{-Kt}\right)$ and $\zeta(t,x) = xe^{-Kt}/L(t)$
Thus

$$\mathbb{E}\left[\frac{1}{(Z_t^x)^p}\right] = \frac{1}{\Gamma(p)}\int_0^{+\infty} \frac{u^{p-1}}{(2uL(t)+1)^{\frac{2b(0)}{\sigma^2}}}\exp\left(-\frac{uL(t)\zeta(t,x)}{2uL(t)+1}\right)du$$

setting $\theta = \frac{2uL(t)}{2uL(t)+1}$, then $u = \frac{\theta}{2L(t)(1-\theta)}$, $2uL(t)+1 = \frac{1}{1-\theta}$ and $du = \frac{1}{2L(t)(1-\theta)^2}d\theta$. Thus

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{(Z_t^x)^p}\right] &= \frac{1}{\Gamma(p)}\int_0^1 \frac{\left(\frac{\theta}{2L(t)(1-\theta)}\right)^{p-1}}{\left(\frac{1}{1-\theta}\right)^{\frac{2b(0)}{\sigma^2}}}\exp\left(-\frac{xe^{-Kt}\theta}{2L(t)}\right)\frac{1}{2L(t)(1\theta)^2}d\theta \\
&= \frac{1}{2^p\Gamma(p)L(t)^p}\int_0^1 \theta^{p-1}(1-\theta)^{\frac{2b(0)}{\sigma^2}-p-1}\exp\left(-\frac{xe^{-Kt}\theta}{2L(t)}\right)d\theta.
\end{aligned}$$

for $p=1$, the assumption (H_3), $\frac{2b(0)}{\sigma^2}-2\geq 0$ implies $(1-\theta)^{\frac{2b(0)}{\sigma^2}-2}\leq 1$
Thus

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{Z_t^x}\right] &= \frac{1}{2L(t)}\int_0^1 (1-\theta)^{\frac{2b(0)}{\sigma^2}} \exp\left(-\frac{xe^{-Kt}\theta}{2L(t)}\right)d\theta \\
&\leq \frac{1}{2L(t)}\int_0^1 \exp\left(-\frac{xe^{-Kt}\theta}{2L(t)}\right)d\theta \\
&= \frac{e^{Kt}}{x}\left(1-\exp\left(-\frac{xe^{-Kt}}{2L(t)}\right)\right) \\
&\leq \frac{e^{Kt}}{x}.
\end{aligned}$$

Hence

$$\mathbb{E}\left[\frac{1}{X_t^x}\right]\leq \frac{e^{Kt}}{x}.$$

For the proof of the inequality (2.4.3), we have, $\forall\mu\leq\frac{\nu\sigma^2}{8}$,

$$\mathbb{E}\left[\exp\left(\mu\int_0^T\frac{1}{Z_t^x}\mathrm{d}t\right)\right]\leq C(T)\left(1+x^{-\frac{\nu}{2}}\right). \quad \text{(see [11])}$$

where $C(T)$ an non decreasing function of T that doesn't x.
since

$$\frac{1}{X_t^x}\leq\frac{1}{Z_t^x}\quad p.s,$$

then

$$\exp\left(\mu\int_0^T\frac{1}{X_t^x}\mathrm{d}t\right)\leq\exp\left(\mu\int_0^T\frac{1}{Z_t^x}\mathrm{d}t\right)\quad p.s$$

Thus

$$\mathbb{E}\left[\exp\left(\mu\int_0^T\frac{1}{X_t^x}\mathrm{d}t\right)\right]\leq\mathbb{E}\left[\exp\left(\mu\int_0^T\frac{1}{Z_t^x}\mathrm{d}t\right)\right]\leq C(T)\left(1+x^{-\frac{\nu}{2}}\right).$$

■

2.5 Kolmogorov's PDE associate with CIR type process with Poissonian jump

2.5.1 Link between PDE and CIR type process with Poissonian jump

The Kolmogorov's partial differential equation (PDE) establishes the link between SDEs and partial differential equation (PDEs). That come from Feynman-Kac formula [19].

Theorem 2.5.1 *Let $x\in\mathbb{R}$, let $(X_t)_{0\leq t\leq T}$ be a CIR type process with Poissonian jump starting from x, that is it satisfies the SDE*

$$X_t = x_0+\int_0^t b(X_s)ds+\sigma\int_0^t\sqrt{X_s}dB_s+\int_0^t a dN_s,\ \ X_0=x_0\geq 0. \tag{2.5.1}$$

For any function $f:\mathbb{R}\longrightarrow\mathbb{R}$ of class $C^4(\mathbb{R})$ with derivative up to order 4 bounded, The function $u(t,x)$ define on $[0,T]\times\mathbb{R}_+$ by

$$u(t,x):=\mathbb{E}[f(X_{T-t}^x)]:=\mathbb{E}[f(X_{T-t})\,|X_0=x]$$

satisfies the following Cauchy's problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + Lu(t,x) = 0 & (t,x) \in [0,T] \times \mathbb{R}_+ \\ u(T,x) = f(x) & x \in \mathbb{R}_+ \end{cases} \tag{2.5.2}$$

where $Lu(t,x) = b(x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x \frac{\partial^2 u}{\partial x^2}(t,x) + \lambda u_a(t,x)$ *with* $u_a(t,x) = u(t,x+a) - u(t,x)$.

The proof of this theorem use the following lemma.

Lemma 2.5.2 *For any measurable function* $g \colon \mathbb{R} \to \mathbb{R}$, $\forall t \in [0,T]$,

$$\mathbb{E}\left(g(X_t)\frac{\partial u}{\partial x}(t,X_t)\,|X_0 = x \right) = g(x)\frac{\partial u}{\partial x}(t,x).$$

Proof. We have

$$\begin{aligned} g(X_t)\frac{\partial}{\partial x}\mathbb{E}\left(f(X_T)\,|X_t\right) &= \mathbb{E}\left(g(X_t)\frac{\partial}{\partial x}f(X_T)\,|X_t \right) \\ &= \mathbb{E}\left(g(X_0)\frac{\partial}{\partial x}f(X_{T-t})\,|X_0 \right) \qquad \text{(homogeneity)} \\ &= h(X_0). \end{aligned}$$

where $h(y) := \mathbb{E}\left(g(X_0)\frac{\partial}{\partial x}f(X_{T-t})\,|X_0 = y \right)$.
Thus

$$\begin{aligned} \mathbb{E}\left(g(X_t)\frac{\partial}{\partial x}\mathbb{E}\left(f(X_T)\,|X_t\right) X_0 = x \right) &= \mathbb{E}\left(h(X_0)\,|X_0 = x\right) \\ &= h(x) \\ &= \mathbb{E}\left(g(X_0)\frac{\partial}{\partial x}f(X_{T-t})\,|X_0 = x \right) \\ &= g(x)\frac{\partial}{\partial x}\mathbb{E}\left(f(X_{T-t})\,|X_0 = x\right) \\ &= g(x)\frac{\partial u}{\partial x}(t,x) \end{aligned}$$

■

Proof. **(Proof of the above theorem)**
Soit $t \in [0,T]$,
Applying the Itö's formula with the function $u(t,x)$ between 0 and $T-t$, we get

$$\begin{aligned} u(T-t, X_{T-t}) - u(0,X_0) &= \int_0^{T-t} \frac{\partial u}{\partial s}(s,X_s)\mathrm{d}s + \int_0^{T-t} b(X_s)\frac{\partial u}{\partial x}(s,X_s)\mathrm{d}s \\ &+ \frac{\sigma^2}{2}\int_0^{T-t} X_s\frac{\partial^2 u}{\partial x^2}(s,X_s)\mathrm{d}s + \sigma\int_0^{T-t}\sqrt{X_s}\frac{\partial u}{\partial x}(s,X_s)\mathrm{d}B_s \\ &+ \int_0^{T-t}(u(s,X_s+a) + u(s,X_s))\mathrm{d}N_s. \end{aligned}$$

Applying the fact that $\mathbb{E}_x(.) := \mathbb{E}(.\,|X_0 = x)$ and $\mathbb{E}\left(\int_0^{T-t} \sqrt{X_s}\frac{\partial u}{\partial s}(s, X_s)\mathrm{d}B_s\right) = 0$, we obtain firstly,

$$\begin{aligned}
\mathbb{E}_x\left[u(T-t, X_{T-t}) - u(0, X_0)\right] &= \int_0^{T-t} \mathbb{E}_x\left[\frac{\partial u}{\partial s}(s, X_s) + b(X_s)\frac{\partial u}{\partial x}(s, X_s) + \frac{\sigma^2}{2}X_s\frac{\partial^2 u}{\partial x^2}(s, X_s)\right]\mathrm{d}s \\
&\quad +\lambda\int_0^{T-t}\mathbb{E}_x\left[(u(s, X_s + a) + u(s, X_s))\right]\mathrm{d}s \qquad (\text{car } X_s \amalg N_s) \\
&= \int_0^{T-t}\mathbb{E}\frac{\partial}{\partial s}\mathbb{E}\left(f(X_T)\,|X_0 = x\right) + \mathbb{E}b(X_s)\frac{\partial}{\partial s}\mathbb{E}\left(f(X_T)\,|X_0 = x\right) \\
&\quad +\frac{\sigma_2}{2}\mathbb{E}X_s\frac{\partial^2}{\partial s^2}\mathbb{E}\left(f(X_T)\,|X_0 = x\right) + \lambda\mathbb{E}\left(f(X_T)\,|X_0 = x\right) \\
&= \int_0^{T-t}\left(\frac{\partial u}{\partial s}(s, x) + b(x)\frac{\partial u}{\partial s}(s, x) + \frac{\sigma^2}{2}x\frac{\partial^2 u}{\partial x^2}(s, x) + \lambda u_a(s, x)\right)\mathrm{d}s
\end{aligned}$$

Secondly

$$\begin{aligned}
\mathbb{E}_x\left[u(T-t, X_{T-t}) - u(0, X_0)\right] &= \mathbb{E}\left[u(T-t, X_{T-t}) - u(0, X_0)\,|X_0 = x\right] \\
&= \mathbb{E}\left(u(T-t, X_{T-t})\,|X_0 = x\right) - \mathbb{E}\left(u(0, X_0)\,|X_0 = x\right) \\
&= \mathbb{E}\left((\mathbb{E}f(X_{T-t})\,|X_{T-t})\,|X_0 = x\right) - \mathbb{E}\left((\mathbb{E}f(X_{T-t})\,|X_0)\,|X_0 = x\right) \\
&= \mathbb{E}\left(f(X_{T-t})\,|X_0 = x\right) - \mathbb{E}\left(f(X_{T-t})\,|X_0 = x\right) \\
&= 0
\end{aligned}$$

Thus

$$\int_0^{T-t}\left(\frac{\partial u}{\partial s}(s, x) + b(x)\frac{\partial u}{\partial s}(s, x) + \frac{\sigma^2}{2}x\frac{\partial^2 u}{\partial x^2}(s, x) + \lambda u_a(s, x)\right)\mathrm{d}s = 0$$

this is true for every tuple (x, t, T) with $t \leq T$ then,

$$\frac{\partial u}{\partial s}u(s, x) + b(x)\frac{\partial u}{\partial s}(s, x) + \frac{\sigma^2}{2}x\frac{\partial^2 u}{\partial x^2}(s, x) + \lambda u_a(s, x) = 0$$

Furthermore

$$u(T, x) = \mathbb{E}\left(f(X_0)\,|X_0 = x\right) = f(x)$$

■

2.5.2 Regularity of the solution of Kolmogorov's PDE.

The following theorem given the regularity of the solution of the function $u(t, x)$ define above.

Theorem 2.5.3 *Under the (H), $u(t, x) \in \mathcal{C}^{1,4}\left([0, T] \times \mathbb{R}_+, \mathbb{R}\right)$ and there exist a constant $C(T)$ depend only on the function f and b, and the constants a, λ, T such that*

$$\sup_{t\in[0,T]}|u(t, x)| + \sup_{t\in[0,T]}\left|\frac{\partial u}{\partial t}(t, x)\right| \leq C(T)(1 + x)$$

and

$$\sum_{k=0}^{4}\left\|\frac{\partial^k u}{\partial x^k}\right\|_{L^\infty([0,T]\times[0,\infty])} \leq C(T)$$

To prove this theorem, we need to first introduce the following notation. For $x \geq 0$, $\xi \geq 0$ let's denote by $(X^x_{t,\xi})_{0\leq t\leq T}$ the stochastic process that satisfies the SDE

$$\begin{cases} dX^x_{t,\xi} = \left(\xi\sigma^2 + b(X^x_{t,\xi})\right)\mathrm{d}t + \sigma\sqrt{X^x_{t,\xi}}\mathrm{d}B_t + a\mathrm{d}N_t \\ X^x_{0,\xi} = x \end{cases} \tag{2.5.3}$$

And prove the following lemmas

Lemma 2.5.4 *Setting* $S^x_{t,\xi} = \frac{dX^x_{t,\xi}}{dx}$ *then the process* $(S^x_{t,\xi})_{0\leq t\leq T}$ *satisfies*

$$S^x_{t,\xi} = \exp\left(\int_0^t b'(X^x_{s,\xi})ds + \frac{\sigma}{2}\int_0^t \frac{dB_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_{s,\xi}}\right).$$

Proof. We have

$$X^x_{t,\xi} = x + \xi\sigma^2 t + \int_0^t b(X^x_{s,\xi})\mathrm{d}s + \sigma\int_0^t \sqrt{X^x_{s,\xi}}\mathrm{d}B_s + \int_0^t a\mathrm{d}N_s$$

Thus

$$\begin{aligned} \frac{dX^x_{t,\xi}}{dx} &= 1 + \int_0^t \frac{dX^x_{s,\xi}}{dx} b'(X^x_{s,\xi})\mathrm{d}s + \sigma\int_0^t \frac{dX^x_{s,\xi}}{dx}\frac{1}{2\sqrt{X^x_{s,\xi}}}\mathrm{d}B_s \\ &= 1 + \int_0^t S^x_{s,\xi} b'(X^x_{s,\xi})\mathrm{d}s + \sigma\int_0^t S^x_{s,\xi}\frac{1}{2\sqrt{X^x_{s,\xi}}}\mathrm{d}B_s. \end{aligned}$$

i.e.

$$S^x_{t,\xi} = 1 + \int_0^t S^x_{s,\xi} b'(X^x_{s,\xi})\mathrm{d}s + \sigma\int_0^t S^x_{s,\xi}\frac{1}{2\sqrt{X^x_{s,\xi}}}\mathrm{d}B_s.$$

Applying the Itö's formula to the process $\left(S^x_{t,\xi}\right)_{0\leq t\leq T}$ with the function $x \mapsto \ln(x)$, we obtain

$$d\ln S^x_{t,\xi} = b'(X^x_{t,\xi})\mathrm{d}t + \frac{\sigma}{2}\frac{\mathrm{d}B_t}{\sqrt{X^x_{t,\xi}}} - \frac{\sigma^2}{8X^x_{t,\xi}}\mathrm{d}t$$

This implies

$$\ln S^x_{t,\xi} = \ln S^x_{0,\xi} + \int_0^t b'(X^x_{s,\xi})\mathrm{d}s + \frac{\sigma}{2}\int_0^t \frac{\mathrm{d}B_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{\mathrm{d}s}{X^x_{s,\xi}}$$

Then

$$S^x_{t,\xi} = \exp\left(\int_0^t b'(X^x_{s,\xi})\mathrm{d}s + \frac{\sigma}{2}\int_0^t \frac{\mathrm{d}B_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{\mathrm{d}s}{X^x_{s,\xi}}\right).$$

■

Remark 2.5.5

$$X^x_{t,0} = X^x_t$$

.

Lemma 2.5.6 *Let $(M^x_{t,\xi})_{0\leq t\leq T}$ be the process define by*

$$M^x_{t,\xi} = \exp\left(\pm\frac{\sigma}{2}\int_0^t \frac{dB_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{ds}{X^x_{s,\xi}}\right).$$

is a martingale and

$$\sup_{0\leq t\leq T} \mathbb{E}\left(S^x_{t,\xi}\right) \leq \exp(CT)$$

Proof. We have

$$\mathbb{E}\left[\exp\left(\mu\int_0^T \frac{1}{X^x_t}\mathrm{d}t\right)\right] < \infty$$

Thus $(M^x_{t,\xi})_{0\leq t\leq T}$ is a martingale.
Let $t \in [0,T]$

$$\begin{aligned} S^x_{t,\xi} &= \exp\left(\int_0^t b'(X^x_{s,\xi})\mathrm{d}s + \frac{\sigma}{2}\int_0^t \frac{\mathrm{d}B_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{\mathrm{d}s}{X^x_{s,\xi}}\right) \\ &= \exp\left(\int_0^t b'(X^x_{s,\xi})\mathrm{d}s\right) M^x_{t,\xi}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}\left(S^x_{t,\xi}\right) &\leq \mathbb{E}\left(\exp\left(\int_0^t \|b'\|_\infty \mathrm{d}s\right) M^x_{t,\xi}\right) \\ &\leq \exp\left(\|b'\|_\infty T\right)\mathbb{E}\left(M^x_{t,\xi}\right). \end{aligned}$$

Since $(M^x_{t,\xi})_{0\leq t\leq T}$ is a martingale, then $\mathbb{E}(M^x_{t,\xi}) = \mathbb{E}(M^x_{0,\xi}) = 1$.
So

$$\mathbb{E}\left(S^x_{t,\xi}\right) \leq \exp\left(\|b'\|_\infty T\right).$$

Thus

$$\sup_{0\leq t\leq T} \mathbb{E}\left(S^x_{t,\xi}\right) \leq \exp(CT).$$

where $C = \|b'\|_\infty$ ∎

Lemma 2.5.7 *There exist a probability measure $\mathbb{Q}^{\xi+\frac{1}{2}}$ under which the process $X^x_{\cdot,\xi+\frac{1}{2}}$ an $X^x_{\cdot,\xi}$ have the same distribution under the probability $\mathbb{P}$ i.e. $\mathcal{L}^{\mathbb{Q}^{\xi+\frac{1}{2}}}\left(X^x_{\cdot,\xi+\frac{1}{2}}\right) = \mathcal{L}^{\mathbb{P}}\left(X^x_{\cdot,\xi+\frac{1}{2}}\right)$.*

Proof. Let's recall that

$$X^x_{t,\xi} = x + \xi\sigma^2 t + \int_0^t b(X^x_{s,\xi})\mathrm{d}s + \sigma\int_0^t \sqrt{X^x_{s,\xi}}\mathrm{d}B_s + \int_0^t a\mathrm{d}N_s$$

Let

$$\mathrm{d}B_t = \frac{dX^x_{t,\xi}}{\sigma\sqrt{X^x_{t,\xi}}} - \frac{b(X^x_{t,\xi}) + \xi\sigma^2}{\sigma\sqrt{X^x_{t,\xi}}}\mathrm{d}t - \frac{a}{\sigma\sqrt{X^x_{t,\xi}}}\mathrm{d}N_t.$$

Thus

$$B_t = \int_0^t \left(\frac{dX^x_{s,\xi}}{\sigma\sqrt{X^x_{s,\xi}}} - \frac{b(X^x_{s,\xi} + \xi\sigma^2}{\sigma\sqrt{X^x_{s,\xi}}} \mathrm{d}s - \frac{a}{\sigma\sqrt{X^x_{s,\xi}}} \mathrm{d}N_s \right).$$

Let's consider the process $(Z_t^{\xi,\xi+\frac{1}{2}})_{0\le t\le T}$ define by

$$Z_t^{\xi,\xi+\frac{1}{2}} = \exp\left(-\frac{\sigma}{2}\int_0^t \frac{\mathrm{d}B_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{\mathrm{d}s}{8X^x_{s,\xi}} \right)$$

Such process is a martingale
Let's consider the probability measure $\mathbb{Q}^{\xi+\frac{1}{2}}$ such that

$$\frac{d\mathbb{Q}^{\xi,\xi+\frac{1}{2}}}{d\mathbb{P}}\,|_{\mathcal{F}_t} = \frac{1}{Z_t^{\xi,\xi+\frac{1}{2}}}.$$

Setting

$$\begin{aligned} B_t^{\xi+\frac{1}{2}} &= B_t - \frac{\sigma}{2}\int_0^t \frac{\mathrm{d}s}{\sqrt{X^x_{s,\xi}}} \\ &= \int_0^t \left(\frac{dX^x_{s,\xi}}{\sigma\sqrt{X^x_{s,\xi}}} - \frac{b(X^x_{s,\xi}) + \xi + \frac{1}{2})\sigma^2}{\sigma\sqrt{X^x_{s,\xi}}} \mathrm{d}s - \frac{a}{\sigma\sqrt{X^x_{s,\xi}}} \mathrm{d}N_s \right) \end{aligned} \tag{2.5.4}$$

from the Girsanov's, the process $(B_t^{\xi+\frac{1}{2}})_{0\le t\le T}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, \mathbb{Q}^{\xi+\frac{1}{2}})$ and, in this probability space,

$$X^x_{t,\xi} = x + (\xi + \frac{1}{2})\sigma^2 t + \int_0^t b(X^x_{s,\xi})\mathrm{d}s + \sigma\int_0^t \sqrt{X^x_{s,\xi}}\mathrm{d}B_s^{\xi+\frac{1}{2}} + \int_0^t a\mathrm{d}N_s,$$

$$Z_t^{\xi,\xi+\frac{1}{2}} = \exp\left(-\frac{\sigma}{2}\int_0^t \frac{\mathrm{d}B_s}{\sqrt{X^x_{s,\xi}}} - \frac{\sigma^2}{8}\int_0^t \frac{\mathrm{d}s}{X^x_{s,\xi}}\mathrm{d}s \right)$$

and

$$\mathcal{L}^{\mathbb{Q}^{\xi+\frac{1}{2}}}(X^x_{.,\xi}) = \mathcal{L}^{\mathbb{P}}(X^x_{.,\xi+\frac{1}{2}}.$$

■

Now we are able to prove the above theorem.

Proof. **(Poof of the theorem)**
We have

$$\begin{aligned} |u(t,x)| = \left|\mathbb{E}(f(X^x_{T-t}))\right| &\le C(1 + \mathbb{E}(\left|X^x_{T-t}\right|)) \\ &\le C(1+x) \end{aligned}$$

from the theorem (**??**)
One the other hand, since $f \in C^4(\mathbb{R})$ we can apply the Itö's formula.

$$f(X^x_{T-t}) = f(x) + \int_0^{T-t} b(X^x_s)f'(X^x_s)\mathrm{d}s + \frac{\sigma}{2}\int_0^{T-t} X^x_s f''(X^x_s)\mathrm{d}s + \sigma\int_0^{T-t}\sqrt{X^x_s}f'(X^x_s)\mathrm{d}B_s + \int_0^{T-t}(f(X^x_s+a)-f(X^x_s))\mathrm{d}N_s.$$

Take the expectation and using the fact that

$$\mathbb{E}\left(\int_0^{T-t}\sqrt{X^x_s}f'(X^x_s)\mathrm{d}B_s\right) = 0,$$

we obtain

$$\begin{aligned} u(t,x) = \mathbb{E}f(X^x_{T-t}) &= f(x) + \int_0^{T-t}\mathbb{E}b(X^x_s)f'(X^x_s)\mathrm{d}s + \frac{\sigma}{2}\int_0^{T-t}\mathbb{E}\left(X^x_s f''(X^x_s)\right)\mathrm{d}s \\ &\quad +\lambda\int_0^{T-t}\mathbb{E}(f(X^x_s+a)-f(X^x_s))\mathrm{d}s. \end{aligned}$$

Take the partial derivative with to t, we obtain

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= -\mathbb{E}\left(b(X^x_{T-t})f'(X^x_{T-t}) + \frac{\sigma^2}{2}X^x_{T-t}f''(X^x_{T-t})\right) \\ &\quad -\lambda\mathbb{E}(f(X^x_{T-t}+a)-f(X^x_{T-t})) \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial u}{\partial t}(t,x)\right| &\leq \mathbb{E}\left(\|b\|\,\|_\infty f'\|_\infty + \frac{\sigma^2}{2}\left\|f''\right\|_\infty\mathbb{E}(X^x_{T-t})\right) + 2\lambda\left\|f\right\|_\infty \\ &\leq C(1+x) \end{aligned}$$

Thus

$$\sup_{t\in[0,T]}|u(t,x)| + \sup_{t\in[0,T]}\left|\frac{\partial u}{\partial t}(t,x)\right| \leq C(T)(1+x)$$

Before grossing up the partial derivatives of u with respect to x, note that $X^x_t = X^x_{t,0}$ is continuously differentiable with respect to x and $\frac{dX^t_{t,0}}{dx} = S^x_{t,0}$ (we set $\xi = 0$) .
Thus

$$\frac{\partial u}{\partial x}(t,x) = \frac{\partial}{\partial x}\mathbb{E}(f(X^x_{T-t,0})) = \mathbb{E}\frac{\partial}{\partial x}f(X^x_{t,0}) = \mathbb{E}(S^x_{T-t,0})f'(X^x_{T-t,0}).$$

Therefore

$$\left|\frac{\partial u}{\partial x}(t,x)\right| \leq \mathbb{E}(S^x_{T-t,0}\left\|f'\right\|_\infty) \leq \left\|f'\right\|\exp(CT) \tag{2.5.5}$$

Analysis of the term $\frac{\partial^2 u}{\partial x^2}$.
Let's introduce the probability measure $\mathbb{Q}^{\frac{1}{2}}$ such that

$$\frac{d\mathbb{Q}^{\frac{1}{2}}}{d\mathbb{P}}\,|_{\mathcal{F}_t} = \frac{1}{Z_t^{(0,\frac{1}{2})}}$$

From the Girsanov's theorem,

$$\mathbb{E}\left(S^x_{T-t,0}f'(X^x_{T-t,0})\right) = \mathbb{E}_{\mathbb{Q}^{\frac{1}{2}}}\left(Z_t^{(0,\frac{1}{2})}S^x_{T-t,0}f'(X^x_{T-t,0})\right).$$

From (2.5.5) $B_t = B_t^{\frac{1}{2}} + \int_0^t \frac{\mathrm{d}s}{\sqrt{X_s^x}}$
Therefore

$$S^x_{T-t,0} = \exp\left(\int_0^{T-t} b'(X_s^x)\mathrm{d}s + \frac{\sigma}{2}\int_0^{T-t}\frac{\mathrm{d}B_s^{\frac{1}{2}}}{\sqrt{X_s^x}} + \frac{\sigma^2}{8}\int_0^{T-t}\frac{\mathrm{d}s}{X_s^x}\right)$$

and

$$Z_t^{(0,\frac{1}{2})} = \exp\left(-\frac{\sigma}{2}\int_0^{T-t}\frac{\mathrm{d}B_s^{\frac{1}{2}}}{\sqrt{X_s^x}} - \frac{\sigma^2}{8}\int_0^{T-t}\frac{\mathrm{d}s}{X_s^x}\right)$$

Thus

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}_{\mathbb{Q}^{\frac{1}{2}}}\left(f'(X^x_{T-t,0})\exp\left(\int_0^{T-t} b'(X^x_{s,0})\right)\right)$$

Since $\mathcal{L}^{\mathbb{Q}^{\frac{1}{2}}}(X^x) = \mathcal{L}^{\mathbb{P}}(X^x)$, then

$$\frac{\partial u}{\partial x}(t,x) = \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\mathrm{d}s\right)\right)$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left(S^x_{T-t,\frac{1}{2}}f''(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\mathrm{d}s\right)\right)\\ &+ \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\mathrm{d}s\right)\left(\int_0^{T-t} b''(X^x_{s,\frac{1}{2}}S^x_{s,\frac{1}{2}})\mathrm{d}s\right)\right)\end{aligned}$$

Therefore

$$\begin{aligned}\left|\frac{\partial^2 u}{\partial x^2}(t,x)\right| &\leq \mathbb{E}\left(S^x_{s,\frac{1}{2}}\|f''\|_\infty \exp\left(\|b'\|_\infty T\right)\right) + \mathbb{E}\left(\|f'\|_\infty \exp\left(\|b'\|_\infty\right)\|b''\|_\infty\int_0^{T-t} S^x_{s,\frac{1}{2}}\mathrm{d}s\right)\\ &\leq \|f''\|_\infty \exp\left(\|b'\|_\infty T\right)\mathbb{E}\left(S^x_{s,\frac{1}{2}}\right) + \|f'\|_\infty\exp\left(\|b'\|_\infty\right)\|b''\|_\infty\int_0^{T-t}\mathbb{E}\left(S^x_{s,\frac{1}{2}}\right)\mathrm{d}s\\ &\leq C(T) \qquad (2.5.6)\end{aligned}$$

Analysis of the term $\frac{\partial^3 u}{\partial x^3}$

Let $(t,x) \in [0,T]\times\mathbb{R}$,
We start by transforming the expression (2.5.6) of $\frac{\partial^2 u}{\partial x^2}(t,x)$. We have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left(S^x_{T-t,\frac{1}{2}}f''(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\right)\\ &+ \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\mathrm{d}s\right)\left(\int_0^{T-t} b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\mathrm{d}s\right)\right)\end{aligned}$$

Using the Markov property of the process $\left(X^x_{t,\frac{1}{2}}\right)$, we get:
Pour tout $s \in [0, T-t]$,

$$\begin{aligned}
\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_s^{T-t} b'(X^x_{u,\frac{1}{2}})du\right)|\mathcal{F}_s\right) &= \mathbb{E}\left(f'(X^y_{T-t,\frac{1}{2}})\exp\left(\int_s^{T-t} b'(X^y_{u,\frac{1}{2}})du\right)\right)\Big|_{y=X^x_{s\frac{1}{2}}} \\
&= \mathbb{E}\left(f'(X^y_{T-t-s,\frac{1}{2}})\exp\left(\int_0^{T-t-s} b'(X^y_{u,\frac{1}{2}})du\right)\right)\Big|_{y=X^x_{s,\frac{1}{2}}}
\end{aligned}$$

Thus

$$\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_s^{T-t} b'(X^x_{u,\frac{1}{2}})du\right)|\mathcal{F}_s\right) = \frac{\partial u}{\partial x}(t+s,x).$$

Setting $A = \mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\left(\int_0^{T-t} b''(X^x_{s,\frac{1}{2}}S^x_{s,\frac{1}{2}}\right)\right)$
Then,

$$\begin{aligned}
A &= \mathbb{E}\int_0^{T-t}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\mathrm{d}s\right)\left(b''(X^x_{s,\frac{1}{2}}S^x_{s,\frac{1}{2}})\right)\right)\mathrm{d}s \\
&= \mathbb{E}\left[\int_0^{T-t}\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_s(\frac{1}{2})\right)|\mathcal{F}_s\ \mathrm{d}s\right)\right] \\
&= \mathbb{E}\left[\int_0^{T-t}\mathbb{E}\left(f'(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)|\mathcal{F}_s\right)\ \exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)\mathrm{d}s\right] \\
&= \int_0^{T-t}\mathbb{E}\left(\frac{\partial u}{\partial x}(t+s,x)\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)\right)\mathrm{d}s.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left(S^x_{T-t,\frac{1}{2}}f''(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\right) \\
&+ \int_0^{T-t}\mathbb{E}\left(\frac{\partial u}{\partial x}(t+s,x).\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)\right)\mathrm{d}s
\end{aligned}$$

Introducing the probability measure $\mathbb{Q}^1$ such that $\frac{d\mathbb{Q}^1}{d\mathbb{P}}|_{\mathcal{F}_t} = \frac{1}{Z_t^{(\frac{1}{2},1)}}$, we obtain

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}_{\mathbb{Q}^1}\left(Z^{(\frac{1}{2},1)}_{T-t}S^x_{T-t,\frac{1}{2}}f''(X^x_{T-t,\frac{1}{2}})\exp\left(\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\right) \\
&\quad+\int_0^{T-t}\mathbb{E}_{\mathbb{Q}^1}\left(Z^{(\frac{1}{2},1)}_{T-t}\frac{\partial u}{\partial x}(t+s,x)\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})S^x_{s,\frac{1}{2}}\right)\right)\mathrm{d}s \\
&= \mathbb{E}_{\mathbb{Q}^1}\left(f''(X^x_{T-t,\frac{1}{2}})\exp\left(2\int_0^{T-t} b'(X^x_{s,\frac{1}{2}})\right)\right) \\
&\quad+\int_0^{T-t}\mathbb{E}_{\mathbb{Q}^1}\left(\frac{\partial u}{\partial x}(t+s,x).\exp\left(\int_0^s b'(X^x_{u,\frac{1}{2}}du)\right)\left(b''(X^x_{s,\frac{1}{2}})\right)\right)\mathrm{d}s
\end{aligned}$$

Since $\mathcal{L}^{\mathbb{Q}^{\frac{1}{2}}}(X^x_{\cdot,\frac{1}{2}}) = \mathcal{L}^{\mathbb{P}}(X^x_{\cdot,1})$, we obtain finally

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(t,x) &= \mathbb{E}\left(f''(X^x_{T-t,1})\exp\left(2\int_0^{T-t} b'(X^x_{s,1})\mathrm{d}s\right)\right) \\
&+ \int_0^{T-t}\mathbb{E}\left(b''(X^x_{s,1})\frac{\partial u}{\partial x}(t+s,x).\exp\left(\int_0^s b'(X^x_{u,1})du\right)\right)\mathrm{d}s.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^3 u}{\partial x^3}(t,x) &= \mathbb{E}\left\{\exp\left(2\int_0^{T-t} b'(X^x_{s,1})\mathrm{d}s\right)\left[f^{(3)}(X^x_{T-t,1})S^x_{T-t,1} + 2f''(X^x_{s,1})\int_0^{T-t} b''(X^x_{s,1})S^x_{s,1}\mathrm{d}s\right]\right\}\\
&+ \int_0^{T-t}\left\{\mathbb{E}\exp\left(2\int_0^s b'(X^x_{u,1})du\right)\left[S^x_{s,1}\left(\frac{\partial^2 u}{\partial x^2}(t+s,X^x_{s,1})b''(X^x_{s,1})\right.\right.\right.\\
&+ \left.\left.\left. b^{(3)}(X^x_{s,1})\frac{\partial u}{\partial x}(t+s,X^x_{s,1})\right) + 2\frac{\partial u}{\partial x}(t+s,X^x_s)b''(X^x_{s,1})\int_0^s b''(X^x_{u,1})S^x_{u,1}du\right]\right\}\mathrm{d}s.
\end{aligned}$$

Using the same argument as previously , we get

$$\left|\frac{\partial^3 u}{\partial x^3}(t,x)\right| \leq C(T). \tag{2.5.7}$$

Analysis of the term $\frac{\partial^4 u}{\partial x^4}$
Let $(t,x) \in [0,T] \times \mathbb{R}$,
Proceeding as before, transforming the expression $\frac{\partial^3 u}{\partial x^3}(t,x)$ and introducing the probability measure $\mathbb{Q}^{\frac{3}{2}}$ such that $\frac{d\mathbb{Q}^{\frac{3}{2}}}{d\mathbb{P}} = \frac{1}{Z_t^{1,\frac{3}{2}}}$
where

$$\frac{1}{Z_t^{1,\frac{3}{2}}} = \exp\left(-\frac{\sigma}{2}\int_0^t \frac{\mathrm{d}B_t}{\sqrt{X^x_{s,1}}} - \frac{\sigma^2}{8}\int_0^t \frac{\mathrm{d}s}{X^x_{s,1}}\right)$$

We get

$$\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{L^\infty([0,T]\times[0,\infty])} \leq C(T) \tag{2.5.8}$$

Using the inequality (2.5.5) , (2.5.6), (2.5.7) and (2.5.8) we obtain finally

$$\sum_{k=0}^{4}\left\|\frac{\partial^k u}{\partial x^k}\right\|_{L^\infty([0,T]\times[0,\infty])} \leq C(T)$$

■

In this Chapter we introduced the classical CIR process with Poissonian jump, define the so-called CIR type process with Poissonian jump, explore some properties of such process and established the link with partial differential equation via Kolmogorov's PDE. In the next chapter we will approximate numerically the of such process.

Chapter 3

Numerical approximation of CIR type process with Poissonian jump.

In this chapter, we propose a numerical scheme to approximate CIR type process with Poissonian jump, and analyze the convergence of such scheme. The main references for this chapter are [4], [7], [2], [17] and [16].

Recall that we consider and standard Brownian motion $(B_t)_{0\leq t\leq T}$, a Poisson process $(N_t)_{0\leq t\leq T}$ with density $\lambda > 0$, and the filtration $(\mathcal{F}_t^{B,N})_{0\leq t\leq T}$ generated by that Brownian motion and Poisson process. We consider the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^{B,N})_{t\geq 0}, \mathbb{P}\right)$. We recall also that, the CIR type process with Poissonian jump is the solution of the stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s)ds + \sigma \int_0^t \sqrt{X_s} dB_s + \int_0^t a dN_s, \quad X_0 := x_0 \geq 0. \tag{3.0.1}$$

where $b\colon \mathbb{R} \longrightarrow \mathbb{R}$ designs the drift of $(X_t)_{0\leq t\leq T}$ and $\sigma,\ a$ are positive constant, stand for volatility and jump size of $(X_t)_{0\leq t\leq T}$ respectively. And we are working under the following assumptions

$$(H)\begin{cases} H_1: \ b(x) \text{ satisfies the Lipschitz condition, i.e there exist, } K>0 \text{ such that} \\ \qquad\qquad |b(x)-b(y)| \leq K|x-y| \ \forall x,y \in \mathbb{R}. \\ H_2: \ b(x) \text{ is class } C^4\text{, and have all its derivative up to order bounded .} \\ H_3: \ b(0) \geq 2\sigma^2 \\ H_4: \ \text{The process } \ (B_t)_{t\geq 0} \ and \ (N_t)_{t\geq 0} \text{ are independent.} \end{cases}$$

3.1 Numerical scheme for approximate CIR type process with Poissonian jump

In this section, we define the numerical method to approximate a CIR type process with Poissonian jump.

Firstly, given a SDE of the form (3.0.1), after proving the existence and uniqueness of the solution, one attempts to approximate such solution on an interval $[0, T]$ with Euler-Maruyama scheme [13], [14], that is

$$\begin{cases} \widehat{X}_0 = x_0 \\ \widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k})\Delta t + \sigma\sqrt{\widehat{X}_{t_k}}\Delta B_k + a\Delta N_k \end{cases} \tag{3.1.1}$$

where $\Delta t = t_{k+1} - t_k$ is the step size, $t_k = k\frac{T}{N}$, $N \in \mathbb{N}^*$, $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and $\Delta N_k = N_{t_{k+1}} - N_{t_k}$.

Notice that the random variable ΔB_k in this scheme is normally distributed, that means it can take negative value. In this case the right side of this equation (3.0.2) take the negative value and then the iteration will not be defined because of the square root. To fix this problem, we take the absolute value of this classical Euler-Maruyama scheme , and defined the so-called symmetric Euler-Maruyama $(\widehat{X}_{t_k}$, $k = 0, ...N)$ by

$$\begin{cases} \widehat{X}_0 = x_0 \\ \widehat{X}_{t_{k+1}} = |\widehat{X}_{t_k} + b(\widehat{X}_{t_k})\Delta t + \sigma\sqrt{\widehat{X}_{t_k}}\Delta B_k + a\Delta N_k| \end{cases} \tag{3.1.2}$$

where $\Delta t = t_{k+1} - t_k$ is the step size, $t_k = k\frac{T}{N}$, $N \in \mathbb{N}^*$, $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and $\Delta N_k = N_{t_{k+1}} - N_{t_k}$.

3.2 Continue version of symmetric Euler-Maruyama scheme.

we have

$$[0, T[= \bigcup_{k=0}^{N-1} [t_k, t_{k+1}[.$$

setting

$$\widehat{X}_t := \sum_{k=0}^{N-1} \widehat{X}_{t_k} \mathbb{1}_{[t_k, t_{k+1}[}(t) \ t \in [0, T],$$

the process $(\widehat{X}_t, 0 \leq t \leq T)$ satisfies

$$\widehat{X}_t = |\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t - \eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(B_t - B_{\eta(t)}) + a(N_t - N_{\eta(t)})| \tag{3.2.1}$$

where $\quad \eta(t) = \sup_{k \in \{1,....,N\}} \{t_k, t_k \leq t\}$, and the function $sign(x)$ is defined by

$$sign(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Setting

$$\widehat{Y}_t = \widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t - \eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(B_t - B_{\eta(t)}) + a(N_t - N_{\eta(t)}). \tag{3.2.2}$$

Then the process $\widehat{Y} = \left(\widehat{Y}_t\right)_{0 \leq t \leq T}$ is a semi-martingale, Hence $|Y|$ is a semi-martingale.
From the Itö's formula, we obtain

$$\left|\widehat{Y}_t\right| = |\widehat{Y}_0| + \int_0^t sign(\widehat{Y}_{s^-})d\widehat{Y}_s + \sum_{0<s\leq t} \left\{|\widehat{Y}_s| - |\widehat{Y}_{s^-}| - sign(\widehat{Y}_{s^-})\Delta\widehat{Y}_s\right\} + L_t^0(\widehat{Y}) \tag{3.2.3}$$

Since $\widehat{X}_t = |\widehat{Y}_t|$ then $sign(\widehat{Y}_{s^-})\Delta\widehat{Y}_s = \Delta X_s$. Therefore

$$\widehat{X}_t = x_0 + \int_0^t sign(\widehat{Y}_{s^-})b(\widehat{X}_{\eta(t)})ds + \sigma\int_0^t sign(\widehat{Y}_s)\sqrt{\widehat{X}_{\eta(t)}}dB_s + \int_0^t \left(sign(\widehat{Y}_{s^-})a\right) dN_s + L_t^0(\widehat{X}). \tag{3.2.4}$$

where $L_t^0(\widehat{X})$ designs the local time of X at 0.

3.3 Some properties of the process $\left(\widehat{X}\right)$

In this section, we provide some results about the process $(\widehat{X})$.

Theorem 3.3.1 *Under the assumption* $\mathbf{H_1}$ *and* $\mathbf{H_2}$, $\forall p \geq 1$ *there exist a positive constant* C_p *that only depend on* $b(0), \sigma, K, a, \lambda, p$ *and* T *such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \widehat{X}_t^{2p}\right) \leq C_p(1+x_0^{2p}) \tag{3.3.1}$$

Proof. Let $p \geq 1$, since we are not sure that $\mathbb{E}\left(\sup_{0\leq t\leq T} X_t^{2p}\right)$ is finite or not, let's introduce a control argument, we define the sequence of stopping time (τ_n) by:

$$\tau_n = \inf\{t \in [0,T], \widehat{X}_t \geq n\}$$

with the convention $\inf \emptyset = T$.
We have :
$\forall t \in [0,T]$,

$$\widehat{X}_{t\wedge\tau_n} = x_0 + \int_0^{t\wedge\tau_n} sign(\widehat{Y}_{s^-})b(\widehat{X}_{\eta(t)})ds + \sigma\int_0^{t\wedge\tau_n} sign(\widehat{Y}_s)\sqrt{\widehat{X}_{\eta(t)}}dB_s + \int_0^{t\wedge\tau_n}\left(sign(\widehat{Y}_{s^-})\right)adN_s + L_{t\wedge\tau_n}(\widehat{X}).$$

Let $p \geq 1$, Using the Itö's formula with the function $\varphi : x \longmapsto x^{2p}$ we obtain

$$\begin{aligned}
\widehat{X}^{2p}_{t\wedge\tau_n} &= x_0^{2p} + \int_0^{t\wedge\tau_n} \varphi'(\widehat{X}_s)d\widehat{X}^c_s + \frac{1}{2}\int_0^{t\wedge\tau_n}\varphi''(\widehat{X}_s)d\left[\widehat{X}^c\right]_s + \int_0^{t\wedge\tau_n}\left(\varphi(\widehat{X}_s + asign(\widehat{Y}_{s^-})) - \varphi(\widehat{X}_s)\right)adN_s \\
&= x_0^{2p} + 2p\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}asign(\widehat{Y}_{s^-})b(\widehat{X}_{\eta(s)})ds + 2p\sigma\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}asign(\widehat{Y}_{s^-})\sqrt{\widehat{X}_{\eta(s)}}dB_s \\
&\quad +2p\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}dL_s(\widehat{X}) + p(2p-1)\sigma^2\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}ds \\
&\quad +\int_0^{t\wedge\tau_n}\left((\widehat{X}_s + asign(\widehat{Y}_{s^-}))^{2p} - \widehat{X}^{2p}_{\eta(s)}\right)dN_s
\end{aligned}$$

Since $2p\sigma\mathbb{E}\left[\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}asign(\widehat{Y}_{s^-})\sqrt{\widehat{X}_{\eta(s)}}dB_s\right] = 0$ since this process is a martingale, and using the fact that

$$\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}dL_s(\widehat{X}) = 0$$

then

$$\begin{aligned}
\mathbb{E}\left(\widehat{X}^{2p}_{t\wedge\tau_n}\right) &= x_0^{2p} + 2p\mathbb{E}\left(\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}asign(\widehat{Y}_{s^-})b(\widehat{X}_{\eta(s)})ds\right) + +p(2p-1)\sigma^2\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}ds \\
&\quad +\int_0^{t\wedge\tau_n}\mathbb{E}\left(\left((\widehat{X}_s + asign(\widehat{Y}_{s^-}))^{2p} - \widehat{X}^{2p}_{\eta(s)}\right)\right)\lambda ds
\end{aligned}$$

From the convexity inequality we have

$$(\widehat{X}_s + asign(\widehat{Y}_{s^-}))^{2p} - \widehat{X}^{2p}_{\eta(s)} \leq 2^{2p-1}a^{2p} + (2^{2p-1}-1)X_s^{2p}$$

Since $b(x)$ satisfies the Lipschitz's condition with constant K, and $b(0) \geq 0$, then
$|b(\widehat{X}_{\eta(s)})| \leq b(0) + K\widehat{X}_{\eta(s)}$.

Therefore

$$\begin{aligned}
\mathbb{E}\left(\widehat{X}_{t\wedge\tau_n}^{2p}\right) &\leq x_0^{2p} + 2^{2p-1}\lambda a^{2p}T + 2p\mathbb{E}\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}\left(b(0)+K\widehat{X}_{\eta(s)}\right)ds \\
&\quad + p(2p-1)\sigma^2\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}ds + (2^{2p-1}-1)\lambda\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p}ds \\
&= x_0^{2p} + 2^{2p-1}\lambda a^{2p}T + 2p\mathbb{E}\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}b(0)ds + 2pK\mathbb{E}\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}\widehat{X}_{\eta(s)}ds \\
&\quad + p(2p-1)\sigma^2\mathbb{E}\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}ds + (2^{2p-1}-1)\lambda\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p}ds \qquad (3.3.2)
\end{aligned}$$

From the Young's inequality, we have

$$\begin{aligned}
2p\mathbb{E}\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p-1}b(0)ds &\leq 2p\mathbb{E}\int_0^{t\wedge\tau_n}\left[\frac{2p-1}{2p}\widehat{X}_s^{2p} + \frac{1}{2p}b(0)^{2p}\right]ds \\
&\leq Tb(0)^{2p} + (2p-1)\mathbb{E}\int_0^{t\wedge\tau_n}\widehat{X}_s^{2p}ds
\end{aligned}$$

Replacing $\widehat{X}_s$ its expression (3.2.1) using the inequality $x \geq 0$, $x^{n-1} \leq (1+x^n)$ for $n \in \mathbb{N}^*$, and $x \geq 0$, we have

$$\begin{aligned}
\mathbb{E}\left(\widehat{X}_s^{2p}\right) &= \mathbb{E}\left(\widehat{X}_{\eta(s)} + b(\widehat{X}_{\eta(s)})(s-\eta(s)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(B_s - B_{\eta(s)}) + a(N_s - N_{\eta(s)})\right)^{2p} \\
&\leq C_1\left(1+\mathbb{E}\widehat{X}_{\eta(s)}^{2p}\right) \qquad \text{where } C_1 \text{ is the positive constant.}
\end{aligned}$$

By the same way, we have

$$\mathbb{E}\left(\widehat{X}_s^{2p-1}\widehat{X}_{\eta(s)}\right) \leq C_2\left(1+\mathbb{E}\widehat{X}_{\eta(s)}^{2p}\right) \qquad \text{et} \qquad \mathbb{E}\left(\widehat{X}_s^{2p-2}\widehat{X}_{\eta(s)}\right) \leq C_2'\left(1+\mathbb{E}\widehat{X}_{\eta(s)}^{2p}\right)$$

Introducing in 3.3.2 and replacing t by $\eta(t)$ we obtain

$$\begin{aligned}
\mathbb{E}\left(\widehat{X}_{\eta(t)\wedge\tau_n}^{2p}\right) &\leq C_3\left(1+x_0^{2p}+\int_0^{\eta(t)\wedge\tau_n}\mathbb{E}\left(\widehat{X}_{\eta(s)}^{2p}\right)ds\right) \\
&\leq C_3\left(1+x_0^{2p}+\int_0^{\eta(t)}\mathbb{E}\left(\widehat{X}_{\eta(s)\wedge\tau_n}^{2p}\right)ds\right)
\end{aligned}$$

From the Gronwall's, lemma, we get

$$\mathbb{E}\left(\widehat{X}_{\eta(t)\wedge\tau_n}^{2p}\right) \leq C(p,b(0),K,\sigma,\lambda,a,T)\left(1+x_0^{2p}\right)$$

Since this is true for all $t \in [0,T]$ and $\eta(t) = \sup_{k\in\{0,\dots,N\}}\{t_k, t_k \leq t\}$ then

$$\sup_{k\in\{0,\dots,N\}}\mathbb{E}\left(\widehat{X}_{t_k\wedge\tau_n}^{2p}\right) \leq C(p,b(0),K,\sigma,\lambda,a,T)\left(1+x_0^{2p}\right)$$

Since $C(p,b(0),K,\sigma,\lambda,a,T)$ doesn't depend on n, taking the limit as $n\to\infty$ we obtain

$$\sup_{k\in\{0,\dots,N\}}\mathbb{E}\left(\widehat{X}_{t_k}^{2p}\right) \leq C(p,b(0),K,\sigma,\lambda,a,T)\left(1+x_0^{2p}\right)$$

Using the new expression of $\widehat{X}_t$ giving in (3.2.1) we get

$$\sup_{t\in[0,T]} \mathbb{E}\left(\widehat{X}_t^{2p}\right) \leq C(p, b(0), K, \sigma, \lambda, a, T)\left(1 + x_0^{2p}\right)$$

Using Burkholder-Davis-Gundy we conclude that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \widehat{X}_t^{2p}\right) \leq C(p, T)\left(1 + x_0^{2p}\right)$$

where $C(p, T)$ is an positive constant that only depend on $p, b(0), K, \sigma, \lambda, a$ and T ∎

The following theorem will be useful for proving the weak convergence of scheme.

Theorem 3.3.2 *Under the above assumption (H), for Δt small enough,($\Delta t < \frac{1}{2K} \wedge x_0$) there exist a positive constant that only depend on $b(0), T, K, a, \lambda, \sigma$, x_0 and uniform with respect to Δt such that*

$$\mathbb{E}\left[\left(L_t^0(\widehat{X}) - L_{\eta(t)}^0(\widehat{X})\right)\Big|\mathcal{F}_{\eta(t)}\right] \leq C\Delta t \exp\left(-\frac{\widehat{X}_{\eta(t)}}{16\sigma^2\Delta t}\right)$$

and

$$\mathbb{E}\left(L_T^0(\widehat{X})\right) \leq C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

The prove of this theorem use the following lemmas

Lemma 3.3.3 *Under the assumption (H), $\forall\gamma \geq 1$, there exist a sequence $(\mu_j, 0 \leq j \leq N)$ such that $\forall k \in \{1, \ldots, N\}$*

$$\mathbb{E}\exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \leq \exp\left(-\sum_{j=0}^{k-1} b(0)\mu_j\Delta t\right)\exp\left(-x_0\mu_k\right)$$

Proof. let $\gamma \geq 1$
remember that $k \in \{1, \ldots, N\}$

$$\widehat{X}_{t_k} = \left|\widehat{X}_{t_{k-1}} + b(\widehat{X}_{t_{k-1}})\Delta t + \sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k + a\Delta N_k\right|.$$

where $\Delta B_k = B_{t_k} - B_{t_{k-1}}$ and $\Delta N_k = N_{t_k} - N_{t_{k-1}}$.
Setting $\mu_0 = \frac{1}{\gamma\sigma^2\Delta t}$
one has

$$\mathbb{E}\exp\left(-\mu_0\widehat{X}_{t_k}\right) = \mathbb{E}\exp\left(-\mu_0\left|\widehat{X}_{t_{k-1}} + b(\widehat{X}_{t_{k-1}})\Delta t + \sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k + a\Delta N_k\right|\right)$$

since $\exp(-\mu|x|) \leq \exp(-\mu x)$, $\forall x \in \mathbb{R}$, $\mu > 0$ and b satisfies the Lipschitz's condition with constant K , then

$$
\begin{aligned}
\mathbb{E}\exp\left(-\mu_0\widehat{X}_{t_k}\right) &\leq \mathbb{E}\left[\exp\left(-\mu_0\left(\widehat{X}_{t_{k-1}}+b(\widehat{X}_{t_{k-1}})\Delta t+\sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k+a\Delta N_k\right)\right)\right]\\
&\leq \mathbb{E}\left[\exp\left(-\mu_0\left(\widehat{X}_{t_{k-1}}+b(0)\Delta t-K\Delta t\widehat{X}_{t_{k-1}}+\sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k+a\Delta N_k\right)\right)\right]\\
&= \mathbb{E}\left[\mathbb{E}\left(\exp\left(-\mu_0\left(\widehat{X}_{t_{k-1}}+b(0)\Delta t-K\Delta t\widehat{X}_{t_{k-1}}+\sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k+a\Delta N_k\right)\right)\Big|\widehat{X}_{t_{k-1}}\right)\right]
\end{aligned}
$$

Since the process $(N_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are independent, then

$$
\begin{aligned}
&\mathbb{E}\left(\exp\left(-\mu_0\left(\widehat{X}_{t_{k-1}}+b(0)\Delta t-K\Delta t\widehat{X}_{t_{k-1}}+\sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k+a\Delta N_k\right)\right)\Big|\widehat{X}_{t_{k-1}}\right)\\
= &\mathbb{E}\left[\exp\left(-\mu_0\left(\widehat{X}_{t_{k-1}}+b(0)\Delta t-K\Delta t\widehat{X}_{t_{k-1}}+\sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k\right)\right)\Big|\widehat{X}_{t_{k-1}}\right]\mathbb{E}\left[\exp\left(-\mu_0 a\Delta N_k\right)\Big|\widehat{X}_{t_{k-1}}\right]
\end{aligned}
$$

On one hand, we have

$$\exp\left(-\mu_0 a\Delta N_k\right)\leq 1 \ a.s$$

Thus

$$\mathbb{E}\left[\exp\left(-\mu_0 a\Delta N_k\right)\Big|\widehat{X}_{t_{k-1}}\right]\leq 1 \ a.s$$

One the other hand, since the random variables ΔB_k and $\widehat{X}_{t_{k-1}}$ are independent, one has

$$
\begin{aligned}
\mathbb{E}\exp\left(-\mu_0\sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k\right) &= \mathbb{E}\left(\mathbb{E}\exp\left(-\mu_0\sigma\sqrt{\widehat{X}_{t_{k-1}}}\Delta B_k\right)\Big|\widehat{X}_{t_{k-1}}\right)\\
&= \mathbb{E}\exp\left(\frac{\sigma^2}{2}\mu_0\Delta t\widehat{X}_{t_{k-1}}\right)
\end{aligned}
$$

Introducing this the inequality (3.3.3), give us

$$\mathbb{E}\exp\left(-\mu_0\widehat{X}_{t_k}\right)\leq\exp\left(-\mu_0 b(0)\Delta t\right)\mathbb{E}\exp\left(-\mu_0\widehat{X}_{t_k}\left(1-K\Delta t-\frac{\sigma^2}{2}\mu_0\Delta t\right)\right)$$

Let's consider the sequence $(\mu_j)_{0\leq j\leq N}$ define by

$$
\begin{cases}
\mu_0=\frac{1}{\gamma\sigma^2\Delta t}\\
\mu_j=\mu_{j-1}\left(1-K\Delta t-\frac{\sigma^2}{2}\Delta t\mu_{j-1}\right), \qquad \forall j\geq 1
\end{cases}
\tag{3.3.3}
$$

Then $\forall j\in\{0,\ldots,k-1\}$, reasoning in the same way as before, we obtain

$$\mathbb{E}\exp\left(-\mu_0\widehat{X}_{t_{k-j}}\right)\leq\exp\left(-\mu_j b(0)\Delta t\right)\mathbb{E}\exp\left(-\mu_{j+1}\widehat{X}_{t_{k-j-1}}\right).$$

Using the induction on j we obtain

$$\mathbb{E}\exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2\Delta t}\right)\leq\exp\left(-\sum_{j=0}^{k-1}b(0)\mu_j\Delta t\right)\exp\left(-x_0\mu_k\right).$$

■

Lemma 3.3.4 *For any $\gamma\geq 1$, the sequence $(\mu_j)_{0\leq j\leq N}$ defined above (3.3.3) is non-negative, decreasing and* *satisfies the following inequality*

$$\mu_j \geq \frac{2\gamma - 1}{\gamma\sigma^2\Delta t\,(2\gamma + j - 1)} - \frac{2K}{\sigma^2},\ \forall j \geq 1 \tag{3.3.4}$$

Proof. To prove that the sequence $(\mu_j)_{0\leq j\leq N}$ is non-negative and decreasing, we will reason by induction on j.
By definition of the sequence $(\mu_j)_{0\leq j\leq N}$, $\mu_0 = \frac{1}{\gamma\sigma^2\Delta t} > 0$.
Under the assumption: $\gamma \geq 1$ and $\Delta t \leq \frac{1}{2K}$.
On a

$$0 \leq 1 - K\Delta t - \frac{\sigma^2}{2}\Delta t\mu_0 \leq 1.$$

Thus

$$0 \leq \mu_0\left(1 - K\Delta t - \frac{\sigma^2}{2}\Delta t\mu_0\right) \leq \mu_0$$

i.e

$$0 \leq \mu_1 \leq \mu_0.$$

Let $j_0 \geq 1$,
assuming that $\quad 0 \leq \mu_{j_0} \leq \mu_{j_0-1} \leq \mu_{j_0-2} \leq .... \leq \mu_0.$
then

$$1 - K\Delta t - \frac{\sigma^2}{2}\Delta t\mu_{j_0} \geq 1 - K\Delta t - \frac{\sigma^2}{2}\Delta t\mu_0 \geq 0$$

that implies

$$\mu_{j_0}\left(1 - K\Delta t - \frac{\sigma^2}{2}\Delta t\mu_{j_0}\right) \geq 0$$

i.e

$$\mu_{j_0+1} \geq 0$$

Furthermore,

$$\mu_{j_0+1} - \mu_{j_0} = -\left(K\Delta t\mu_{j_0} + \frac{\sigma^2}{2}\Delta t\mu_{j_0}^2\right) \leq 0.$$

Before going to the second part of this proof, notice that $\forall j \geq 1$,

$$\mu_j \leq \mu_{j-1} \leq \mu_{j-1} - \frac{\sigma^2}{2}\Delta t\mu_j\mu_{j-1}$$

that implies

$$\mu_j \leq \frac{\mu_{j-1}}{1 + \frac{\sigma^2}{2}\Delta t\mu_{j-1}} = f_{\frac{\sigma^2}{2}\Delta t}(\mu_{j-1}).$$

where $f_{\frac{\sigma^2}{2}\Delta t}$ is a function define by $f_{\frac{\sigma^2}{2}\Delta t}(x) = \frac{x}{1+\frac{\sigma^2}{2}\Delta tx}$.

the function $f_{\frac{\sigma^2}{2}\Delta t}$ is increasing and one has

$$\underbrace{f_{\frac{\sigma^2}{2}\Delta t} \circ f_{\frac{\sigma^2}{2}\Delta t} \circ .... \circ f_{\frac{\sigma^2}{2}\Delta t}}_{jtimes} = f_{\frac{\sigma^2}{2}j\Delta t}$$

Since the sequence $(\mu_j)_{0\leq j\leq N}$ is decreasing, we have

$$\mu_j \leq f_{\frac{\sigma^2}{2}\Delta t}(\mu_{j-1}) \leq f_{\frac{\sigma^2}{2}\Delta t}\left(f_{\frac{\sigma^2}{2}\Delta t}(\mu_{j-2})\right) \leq .... \leq f_{\frac{\sigma^2}{2}j\Delta t}(\mu_0).$$

Now let's prove the inequality (3.3.4) . Firstly, let's start by showing that for any$j \geq 1$

$$\mu_j \geq \mu_1\left(\frac{1}{1+\frac{\sigma^2}{2}(j-1)\Delta t\mu_0}\right) - K\left(\frac{\mu_0\Delta t(j-1)}{1+\frac{\sigma^2}{2}(j-1)\mu_0}\right). \tag{3.3.5}$$

Reasoning by induction j.
It is straightforward to see that the inequality is true for $j = 1$.
Let $j \geq 1$, assume that this inequality is true up to rank j and showing that it also true at the rank $j+1$.
We have

$$\begin{aligned}
\mu_{j+1} &= \mu_j\left(1-\frac{\sigma^2}{2}\Delta t\mu_j\right) - K\Delta t\mu_j \\
&\geq \mu_j\left(1-\frac{\sigma^2}{2}\Delta t f_{j\frac{\sigma^2}{2}\Delta t}(\mu_j)\right) - K\Delta t f_{j\frac{\sigma^2}{2}\Delta t}(\mu_j) \\
&= \mu_j\left(1-\frac{\frac{\sigma^2}{2}\Delta t\mu_0}{1+j\frac{\sigma^2}{2}\Delta t}\right) - K\frac{\mu_0\Delta t}{1+j\frac{\sigma^2}{2}\mu_0\Delta t} \\
&= \mu_j\left(\frac{1+\frac{\sigma^2}{2}\Delta t(j-1)\mu_0}{1+j\frac{\sigma^2}{2}\Delta t\mu_0}\right) - K\frac{\mu_0\Delta t}{1+j\frac{\sigma^2}{2}\mu_0\Delta t}.
\end{aligned}$$

Using the induction hypothesis (3.3.5), we obtain

$$\begin{aligned}
\mu_{j+1} &\geq \left(\frac{\mu_1}{1+\frac{\sigma^2}{2}\mu_0(j-1)\Delta t} - K\frac{\mu_0(j-1)\Delta t}{1+\frac{\sigma^2}{2}\mu_0(j-1)\Delta t}\right)\left(\frac{1+\frac{\sigma^2}{2}\Delta t(j-1)\mu_0}{1+j\frac{\sigma^2}{2}\Delta t\mu_0}\right) - K\frac{\mu_0\Delta t}{1+j\frac{\sigma^2}{2}\mu_0\Delta t} \\
&= \frac{\mu_1}{1+j\frac{\sigma^2}{2}\mu_0\Delta t} - K\frac{\mu_0 j\Delta t}{1+j\frac{\sigma^2}{2}\mu_0\Delta t}
\end{aligned}$$

Therefore the inequality is true at the rank $j+1$, Hence this inequality is true for any $j \geq 1$.
Replacing μ_0 and μ_1 by their expression ($\mu_0 = \frac{1}{\gamma\sigma^2\Delta t}$, $\mu_1 = \mu_0(1-K\Delta t - \frac{\sigma^2}{2}\Delta t\mu_0)$), we get

$$\begin{aligned}
\mu_j &\geq \frac{\mu_0\left(1-K\Delta t-\frac{\sigma^2}{2}\Delta t\mu_0\right)}{1+j\frac{\sigma^2}{2}\mu_0\Delta t} - K\frac{\mu_0 j\Delta t}{1+j\frac{\sigma^2}{2}\mu_0\Delta t} \\
&= \frac{2\gamma-1}{\gamma\sigma^2\Delta t\,(2\gamma+j-1)} - \frac{2Kj}{\sigma^2\,(2\gamma+j-1)} \\
&\geq \frac{2\gamma-1}{\gamma\sigma^2\Delta t\,(2\gamma+j-1)} - \frac{2K}{\sigma^2}
\end{aligned}$$

■

Lemma 3.3.5 *Under the above assumption (H) we have: $\gamma \geq 1$, there exist a non-negative constant $C(\gamma)$ such that*

$$\sup_{k=0,\ldots,N} \mathbb{E}\exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \leq C(\gamma)\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}(1-\frac{1}{2\gamma})} \tag{3.3.6}$$

Proof. Let $\gamma \geq 1$,
From the above lemma 3.3.3, there exist a sequence $(\mu_j, 0 \leq j \leq N)$ such that $\forall k \in \{1, ..., N\}$

$$\mathbb{E}\exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \leq \exp\left(-\sum_{j=0}^{k-1} b(0)\mu_j\Delta t\right)\exp\left(-x_0\mu_k\right). \tag{3.3.7}$$

From the lemma 3.3.4, one has
$\forall j \geq 1$,

$$\mu_j \geq \frac{2\gamma-1}{\gamma\sigma^2\Delta t\left(2\gamma+j-1\right)} - \frac{2K}{\sigma^2}$$

Therefore

$$\begin{aligned}
\sum_{j=0}^{k-1}\Delta t\mu_j &\geq \frac{1}{\gamma\sigma^2}\sum_{j=0}^{k-1}\left(\frac{2\gamma-1}{2\gamma+j-1} - \frac{2K}{\sigma^2}\Delta t\right)\\
&\geq \frac{1}{\gamma\sigma^2}\sum_{j=0}^{k-1}\left(\frac{2\gamma-1}{2\gamma+j-1}\right) - \frac{2K}{\sigma^2}T\\
&\geq \frac{1}{\gamma\sigma^2}\int_0^k \frac{2\gamma-1}{2\gamma+t-1}dt - \frac{2K}{\sigma^2}T
\end{aligned}$$

Because the function $t :\mapsto \frac{2\gamma-1}{2\gamma+t-1}$ is decreasing.
Thus

$$\sum_{j=0}^{k-1}\Delta t\mu_j \geq \frac{2\gamma-1}{\gamma\sigma^2}\ln\left(\frac{2\gamma+k-1}{2\gamma-1}\right) - \frac{2K}{\sigma^2}T.$$

Injecting this inequality in (3.3.7) we obtain

$$\mathbb{E}\exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) \leq \exp\left(\ln\left(\frac{2\gamma-1}{2\gamma+k-1}\right)^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}\right)\exp\left(2b(0)\frac{KT}{\sigma^2}\right)\exp\left(-x_0\mu_k\right).$$

Since

$$\mu_k \geq \frac{2\gamma-1}{\gamma\sigma^2\Delta t\left(2\gamma+k-1\right)} - \frac{2K}{\sigma^2}.$$

Then

$$\exp\left(-x_0\mu_k\right) \leq \exp\left(-\frac{x_0}{\gamma\sigma^2\Delta t}\frac{2\gamma-1}{2\gamma+k-1}\right)\exp\left(2x_0\frac{K}{\sigma^2}\right).$$

Hence

$$\begin{aligned}
\mathbb{E}\exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) &\leq \exp\left(2b(0)\frac{KT}{\sigma^2} + 2x_0\frac{K}{\sigma^2}\right)\left(\frac{2\gamma-1}{2\gamma+k-1}\right)^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}\exp\left(-\frac{x_0}{\gamma\sigma^2\Delta t}\frac{2\gamma-1}{2\gamma+k-1}\right)\\
&= \exp\left(2b(0)\frac{KT}{\sigma^2} + 2x_0\frac{K}{\sigma^2}\right)\left(\frac{\gamma\sigma^2\Delta t}{x_0}\right)^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}\left(\frac{x_0}{\gamma\sigma^2\Delta t}\frac{2\gamma-1}{2\gamma+k-1}\right)^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}\\
&\times \exp\left(-\frac{x_0}{\gamma\sigma^2\Delta t}\frac{2\gamma-1}{2\gamma+k-1}\right).
\end{aligned}$$

For instance, $\forall\alpha \geq 0$ and $\forall x \geq 0$, $x^\alpha\exp(-x) \leq \alpha^\alpha\exp(-\alpha)$, Applying this with $\alpha = \frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)$ and $x = \frac{x_0}{\gamma\sigma^2\Delta t}\frac{2\gamma-1}{2\gamma+k-1}$, we obtain

$$
\begin{aligned}
\mathbb{E}\exp\left(-\frac{\widehat{X}_{t_k}}{\gamma\sigma^2\Delta t}\right) &\leq \exp\left(2b(0)\frac{KT}{\sigma^2}+2x_0\frac{K}{\sigma^2}-\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)\right) \\
&\qquad \times (b(0)(2\gamma-1))^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}\left(\frac{\Delta t}{x_0}\right)^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)} \\
&= C(x_0,b(0),T,K,\sigma,\gamma)\left(\frac{\Delta t}{x_0}\right)^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}.
\end{aligned}
$$

where

$$
C(x_0,b(0),T,K,\sigma,\gamma)=\exp\left(2b(0)\frac{KT}{\sigma^2}+2x_0\frac{K}{\sigma^2}-\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)\right)(b(0)(2\gamma-1))^{\frac{2b(0)}{\sigma^2}\left(1-\frac{1}{2\gamma}\right)}
$$

. ■

Now let's prove the above theorem.

Proof. **Proof of the theorem**
For $t\in[t_k,t_{k+1}[$, $0\leq k\leq N-1$ and for any bounded and measurable function Φ, From the semi-martingale occupation time formula,
one has $\mathbb{P}$-a.s

$$
\begin{aligned}
\int_{\mathbb{R}}\varphi(z)\left(L_t^z(\widehat{Y})-L_{t_k}^z(\widehat{Y})\right)dz &= \int_{t_k}^{t}\varphi(\hat{Y_{s^-}})d\left[\widehat{Y},\widehat{Y}\right]_s^c \\
&= \sigma^2\int_{t_k}^{t}\varphi(\hat{Y_{s^-}})\widehat{X}_{t_k}ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_{\mathbb{R}}\varphi(z)\left(L_t^z(\widehat{X})-L_{t_k}^z(\widehat{X})\right)dz &= \int_{\mathbb{R}}\varphi(z)\left(L_t^z(\widehat{Y})-L_{t_k}^z(\widehat{Y})\right)dz \\
&= \sigma^2\int_{t_k}^{t}\varphi(\hat{Y_{s^-}})\widehat{X}_{t_k}ds.
\end{aligned}
$$

Thus $\forall x>0$,

$$
\begin{aligned}
\int_{\mathbb{R}}\varphi(z)\mathbb{E}\left(L_t^z(\widehat{X})-L_{t_k}^z(\widehat{X})\,\middle|\,\{\widehat{X}_{t_k}=x\}\right)dz &= \sigma^2\int_{t_k}^{t}\mathbb{E}\left(\varphi(\widehat{Y}_{s^-})\widehat{X}_{t_k}\,\middle|\,\{\widehat{X}_{t_k}=x\}\right)ds \\
&= \sigma^2\int_{t_k}^{t}x\mathbb{E}\left(\varphi(\widehat{Y}_{s^-})\,\middle|\,\{\widehat{X}_{t_k}=x\}\right)ds.
\end{aligned}
$$

Replacing $\widehat{Y}_{s^-}$ by its expression given in (3.2.2) we obtain

$$
\begin{aligned}
\sigma^2\int_{t_k}^{t}x\mathbb{E}\left(\varphi(\widehat{Y}_{s^-})\,\middle|\,\{\widehat{X}_{t_k}=x\}\right)ds &= \sigma^2\int_{t_k}^{t}x\mathbb{E}\Big(\varphi(\widehat{X}_{t_k}+b(\widehat{X}_{t_k})(s-t_k)+\sigma\sqrt{\widehat{X}_{t_k}}(B_s-B_{t_k}) \\
&+ a(N_s-N_{t_k}))\,\Big|\,\{\widehat{X}_{t_k}=x\}\Big)ds \\
&= \sigma^2\int_{t_k}^{t}x\mathbb{E}\left(\varphi\left(x+b(x)(s-t_k)+\sigma\sqrt{x}(B_s-B_{t_k})+a\left(N_s-N_{t_k}\right)\right)\right)ds.
\end{aligned}
$$

Furthermore, $\forall n \in \mathbb{N}$,

$$\begin{aligned}
& \mathbb{E}\left(\varphi\left(x+b(x)(s-t_k)+\sigma\sqrt{x}(B_s-B_{t_k})+a(N_s-N_{t_k})\right)\big|\{N_s-N_{t_k}=n\}\right) \\
= \; & \mathbb{E}\left(\varphi\left(x+b(x)(s-t_k)+\sigma\sqrt{x}(B_s-B_{t_k})+an\right)\right) \\
= \; & \int_{\mathbb{R}} \varphi\left(x+b(x)(s-t_k)+\sigma y\sqrt{x}+an\right)\frac{1}{\sqrt{2\pi(s-t_k)}}\exp\left(-\frac{y^2}{2(s-t_k)}\right)dy \\
= \; & \int_{\mathbb{R}} \varphi(z)\frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}}\exp\left(-\frac{(z-x-b(x)(s-t_k)-an)^2}{2\sigma^2 x(s-t_k)}\right)dz.
\end{aligned}$$

Thus

$$\begin{aligned}
& \mathbb{E}\left(\varphi\left(x+b(x)(s-t_k)+\sigma\sqrt{x}(B_s-B_{t_k})+a(N_s-N_{t_k})\right)\big|\{N_s-N_{t_k}\}\right) \\
= \; & \int_{\mathbb{R}} \varphi(z)\frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}}\exp\left(-\frac{(z-x-b(x)(s-t_k)-a(N_s-N_{t_k}))^2}{2\sigma^2 x(s-t_k)}\right)dz.
\end{aligned}$$

Hence

$$\begin{aligned}
& \mathbb{E}\left(\varphi\left(x+b(x)(s-t_k)+\sigma\sqrt{x}(B_s-B_{t_k})+a(N_s-N_{t_k})\right)\right) \\
= \; & \mathbb{E}\left(\mathbb{E}\left(\varphi\left(x+b(x)(s-t_k)+\sigma\sqrt{x}(B_s-B_{t_k})+a(N_s-N_{t_k})\right)\big|N_s-N_{t_k}\right)\right) \\
= \; & \int_{\mathbb{R}} \varphi(z)\frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}}\mathbb{E}\exp\left(-\frac{(z-x-b(x)(s-t_k)-a(N_s-N_{t_k}))^2}{2\sigma^2 x(s-t_k)}\right)dz \\
= \; & \int_{\mathbb{R}} \varphi(z)\frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}}\sum_{n\in\mathbb{N}}\exp\left(-\frac{(z-x-b(x)(s-t_k)-an)^2}{2\sigma^2 x(s-t_k)}\right)\mathbb{P}(\{N_s-N_{t_k}=n\})dz.
\end{aligned}$$

Finally,

$$\begin{aligned}
& \int_{\mathbb{R}} \varphi(z)\mathbb{E}\left(L_t^z(\widehat{X})-L_{t_k}^z(\widehat{X})\Big|\{\widehat{X}_{t_k}=x\}\right)dz \\
= \; & \sigma^2\int_{t_k}^t x\int_{\mathbb{R}}\varphi(z)\frac{1}{\sigma\sqrt{x}\sqrt{2\pi(s-t_k)}}\sum_{n\in\mathbb{N}}\exp\left(-\frac{(z-x-b(x)(s-t_k)-an)^2}{2\sigma^2 x(s-t_k)}\right)\mathbb{P}(\{N_s-N_{t_k}=n\})dzds \\
= \; & \int_{\mathbb{R}}\varphi(z)\,\sigma\int_{t_k}^t\frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}}\sum_{n\in\mathbb{N}}\exp\left(-\frac{(z-x-b(x)(s-t_k)-an)^2}{2\sigma^2 x(s-t_k)}\right)\mathbb{P}(\{N_s-N_{t_k}=n\})dsdz
\end{aligned}$$

Therefore

$$\begin{aligned}
& \mathbb{E}\left(L_t^z(\widehat{X})-L_{t_k}^z(\widehat{X})\Big|\{\widehat{X}_{t_k}=x\}\right) \\
= \; & \sigma\int_{t_k}^t\frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}}\sum_{n\in\mathbb{N}}\exp\left(-\frac{(z-x-b(x)(s-t_k)-an)^2}{2\sigma^2 x(s-t_k)}\right)\mathbb{P}(\{N_s-N_{t_k}=n\})ds.
\end{aligned}$$

For $z = 0$

$$
\begin{aligned}
&\mathbb{E}\left(L_t^0(\widehat{X}) - L_{t_k}^0(\widehat{X}) \,\Big|\, \{\widehat{X}_{t_k} = x\}\right) \\
&= \sigma \int_{t_k}^{t} \frac{\sqrt{x}}{\sqrt{2\pi(s-t_k)}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(x+b(x)(s-t_k)+an)^2}{2\sigma^2 x(s-t_k)}\right) \mathbb{P}(\{N_s - N_{t_k} = n\}) ds \\
&\le \sigma \int_0^{\Delta t} \frac{\sqrt{x}}{\sqrt{2\pi u}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{(x+b(x)u+an)^2}{2\sigma^2 xu}\right) \exp(-\lambda u)\frac{(\lambda u)^n}{n!} ds \\
&\le \sigma \int_0^{\Delta t} \frac{\sqrt{x}}{\sqrt{2\pi u}} \sum_{n\in\mathbb{N}} \exp\left(-\frac{x}{8\sigma^2\Delta t} - \frac{an}{2\sigma^2\Delta t}\right) \exp(-\lambda u)\frac{(\lambda u)^n}{n!} ds \\
&\le \sigma \int_0^{\Delta t} \sum_{n\in\mathbb{N}} \frac{\sqrt{x}}{\sqrt{2\pi u}} \exp\left(-\frac{x}{16\sigma^2\Delta t} - \frac{an}{2\sigma^2\Delta t}\right) \exp\left(-\frac{x}{16\sigma^2\Delta t}\right) \exp(-\lambda u)\frac{(\lambda u)^n}{n!} ds.
\end{aligned}
$$

Since $\forall x \ge 0$, for any $\alpha, b \in \mathbb{R}_+^*$, $\sqrt{x}\exp(-\alpha x - b) \le \frac{1}{\sqrt{e}}\sqrt{\frac{1}{2a}}\exp(-b)$,
applying this with $\alpha = \frac{1}{16\sigma^2\Delta t}$ and $b = \frac{an}{16\sigma^2\Delta t}$, we obtain

$$
\begin{aligned}
\mathbb{E}\left(L_t^0(\widehat{X}) - L_{t_k}^0(\widehat{X}) \,\Big|\, \{\widehat{X}_{t_k} = x\}\right) &\le \sigma \int_0^{\Delta t} \sum_{n\in\mathbb{N}} \frac{2\sqrt{2}\sigma\sqrt{\Delta t}}{\sqrt{2\pi u}} \exp\left(-\frac{an}{2\sigma^2\Delta t}\right) \exp\left(-\frac{x}{16\sigma^2\Delta t}\right) \\
&\times \quad \exp(-\lambda u)\frac{(\lambda u)^n}{n!} du \\
&= \int_0^{\Delta t} \frac{2\sigma^2\sqrt{\Delta t}}{\sqrt{\pi u}} \exp\left(-\frac{x}{16\sigma^2\Delta t}\right) \\
&\quad \times \sum_{n\in\mathbb{N}} \exp\left(-\frac{an}{2\sigma^2\Delta t}\right) \exp(-\lambda u)\frac{(\lambda u)^n}{n!} du \\
&= \int_0^{\Delta t} \frac{2\sigma^2\sqrt{\Delta t}}{\sqrt{\pi u}} \exp\left(-\frac{x}{16\sigma^2\Delta t}\right) \exp\left(-\lambda u(1 - e^{-\frac{a}{2\sigma^2\Delta t}})\right) du \\
&\le \int_0^{\Delta t} \frac{2\sigma^2\sqrt{\Delta t}}{\sqrt{\pi u}} \exp\left(-\frac{x}{16\sigma^2\Delta t}\right) ds \\
&= \frac{4\sigma^2\Delta t}{\sqrt{\pi}} \exp\left(-\frac{x}{16\sigma^2\Delta t}\right).
\end{aligned}
$$

Thus

$$
\mathbb{E}\left(L_t^0(\widehat{X}) - L_{t_k}^0(\widehat{X}) \,|\mathcal{F}_{t_k}\right) \le \frac{4\sigma^2\Delta t}{\sqrt{\pi}} \exp\left(-\frac{\widehat{X}_{t_k}}{16\sigma^2\Delta t}\right). \tag{3.3.8}
$$

Since this inequality is true for any $t_k, k = 0, \ldots\ldots..N$ in particular for $\eta(t)$.
Therefore

$$
\mathbb{E}\left(L_t^0(\widehat{X}) - L_{\eta(t)}^0(\widehat{X}) \,\Big|\mathcal{F}_{\eta(t)}\right) \le \frac{4\sigma^2\Delta t}{\sqrt{\pi}} \exp\left(-\frac{\widehat{X}_{\eta(t)}}{16\sigma^2\Delta t}\right).
$$

Furthermore, taking $t = t_{k+1}$ in (3.3.8) and using the above lemma 3.3.5 with $\gamma = 16$ we get

$$
\mathbb{E}\left(L_{t_{k+1}}^0(\widehat{X}) - L_{t_k}^0(\widehat{X})\right) \le \frac{4\sigma^2\Delta t}{\sqrt{\pi}} C(x_0, b(0), T, K, \sigma) \left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.
$$

Taking the sum over k we obtain finally

$$\begin{aligned}
\mathbb{E}\left(L_T^0(\widehat{X})\right) &\leq \frac{4\sigma^2 N\Delta t}{\sqrt{\pi}} C(x_0, b(0), T, K, \sigma)\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}} \\
&= \frac{4\sigma^2 T}{\sqrt{\pi}} C(x_0, b(0), T, K, \sigma)\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}} \\
&\leq C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.
\end{aligned}$$

■

Theorem 3.3.6 *Under the assumptions (H), there exist a non-negative constant C that is uniform with respect to Δt such that $\forall t \in [0, T]$*

$$\mathbb{P}\left(\widehat{Y}_t \leq 0\right) \leq C\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}$$

Proof.

For $t \in [0, T]$,

$$\begin{aligned}
\mathbb{P}\left(\widehat{Y}_t \leq 0 \,\middle|\, \widehat{X}_{\eta(s)} > 0\right) &= \mathbb{P}\Bigg(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}\left(B_t - B_{\eta(t)}\right) \\
&\qquad\qquad + a\left(N_t - N_{\eta(t)}\right) \leq 0 \,\Big|\, \widehat{X}_{\eta(t)} > 0\Bigg) \\
&= \sum_{n\in\mathbb{N}} \mathbb{P}\Bigg(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(B_t - B_{\eta(t)}) + an \leq 0, \\
&\qquad\qquad (N_t - N_{\eta(t)}) = n \,\Big|\, \widehat{X}_{\eta(t)} > 0\Bigg) \\
&= \sum_{n\in\mathbb{N}} \mathbb{P}\left(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + \sigma\sqrt{\widehat{X}_{\eta(t)}}(B_t - B_{\eta(t)}) + an \leq 0, \,\middle|\, \widehat{X}_{\eta(t)} > 0\right) \\
&\qquad\qquad \times \mathbb{P}\left((N_t - N_{\eta(t)}) = n \,\middle|\, \widehat{X}_{\eta(t)} > 0\right) \\
&\leq \sum_{n\in\mathbb{N}} \mathbb{P}\left(\sigma\sqrt{\widehat{X}_{\eta(t)}}(B_t - B_{\eta(t)}) \leq -\widehat{X}_{\eta(t)} - b(\widehat{X}_{\eta(t)})(t-\eta(t)) - an \,\middle|\, \widehat{X}_{\eta(t)} > 0\right) \\
&\qquad\qquad \times \mathbb{P}\left((N_t - N_{\eta(t)}) = n \,\middle|\, \widehat{X}_{\eta(t)} > 0\right) \\
&\leq \sum_{n\in\mathbb{N}} \mathbb{P}\left((B_t - B_{\eta(t)}) \leq \frac{-\widehat{X}_{\eta(t)} - b(\widehat{X}_{\eta(t)})(t-\eta(t)) - an}{\sigma\sqrt{\widehat{X}_{\eta(t)}}} \,\middle|\, \widehat{X}_{\eta(t)} > 0\right) \\
&\qquad\qquad \times \mathbb{P}\left((N_t - N_{\eta(t)}) = n\right).
\end{aligned}$$

Using the Gaussian inequality: (If $Z \sim \mathcal{N}(0,1)$ and β be a negative real number , $\mathbb{P}(Z \leq \beta) \leq \frac{1}{2}\exp\left(-\frac{\beta^2}{2}\right)$). With $\beta = -\frac{\left(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + an\right)^2}{2\sigma^2(t-\eta(t))\widehat{X}_{\eta(t)}}$, we obtain

$$\begin{aligned}
\mathbb{P}\left(\widehat{Y}_t \le 0 \middle| \widehat{X}_{\eta(t)} > 0\right) &\le \sum_{n\mathbb{N}} \exp\left(-\frac{\left(\widehat{X}_{\eta(t)} + b(\widehat{X}_{\eta(t)})(t-\eta(t)) + an\right)^2}{2\sigma^2(t-\eta(t))\widehat{X}_{\eta(t)}}\right) \mathbb{P}\left((N_t - N_{\eta(t)}) = n\right) \\
&\le \frac{1}{2}\exp\left(-\frac{\widehat{X}_{\eta(t)}(1-K\Delta t)^2}{2\sigma^2\Delta t}\right) \\
&\le \frac{1}{2}\exp\left(-\frac{\widehat{X}_{\eta(t)}}{8\sigma^2\Delta t}\right).
\end{aligned}$$

Therefore

$$\mathbb{P}\left(\widehat{Y}_t \le 0\right) \le \frac{1}{2}\mathbb{E}\exp\left(-\frac{\widehat{X}_{\eta(t)}}{8\sigma^2\Delta t}\right).$$

Using the lemma 3.3.5 with $\gamma = 8$ we get

$$\mathbb{P}\left(\widehat{Y}_t \le 0\right) \le C(T)\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}.$$

where $C(T)$ is a non-negative constant that doesn't depend on Δt.

3.4 Weak convergence of numerical scheme $(\widehat{X}_{t_k}, k = 0, \dots, N)$

In this section, we will properly prove the weak convergence of the symmetric Euler-Maruyama defined above. This weak convergence is give by the following theorem

Theorem 3.4.1 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, be a C^4 function, with derivative up to order 4 bounded. Let $(X_t)_{0\le t\le T}$ be a CIR type process with Poissonian jump, that is it satisfies the SDE (3.0.1), let $\left(\widehat{X}_t\right)_{0\le t\le T}$, be the symmetric Euler-Maruyama scheme approximating such CIR type process with Poissonian jump. Under the assumptions (H), there exist a non-negative constant $C(T)$ that only depend on $b(0), \sigma, x_0, T, a, \lambda$ and K such that*

$$\left|\mathbb{E}\left(f(X_T)\right) - \mathbb{E}(f(\widehat{X}_T))\right| \le C\left(\Delta t + \left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{3.4.1}$$

Proof. We will prove this theorem in different steps

- **Step 1.** Let's start by computing $\mathbb{E}(f(X_T)) - \mathbb{E}(f(\widehat{X}_T))$
  Set
  $$u(t,x) := \mathbb{E}\left(f(X^x_{T-t})\right)$$
  , then $\mathbb{E}\left(f(X_T)\right) = u(0, x_0)$ and $\mathbb{E}\left(f(\widehat{X}_T)\right) = u(T, \widehat{X}_T)$,
  Therfore
  $$\mathbb{E}(f(X_T)) - \mathbb{E}(f(\widehat{X}_T)) = u(0, x_0) - u(T, \widehat{X}_T)$$

Applying the Itö's formula with the function $u(t,x)$ we obtain

$$\begin{aligned} u(T,\widehat{X}_T) &= u(0,x_0) + \int_0^T \frac{\partial u}{\partial t}(s,\widehat{X}_s)ds + \int_0^T sign(Y_{s-})b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds \\ &+ \sigma\int_0^T sign(Y_{s-})\sqrt{\widehat{X}_{\eta(s)}}\frac{\partial u}{\partial x}(s,\widehat{X}_s)dB_s + \frac{\sigma^2}{2}\int_0^T \widehat{X}_s\frac{\partial^2 u}{\partial^2 x}(s,\widehat{X}_s)ds \\ &+ \int_0^T \frac{\partial u}{\partial t}(s,\widehat{X}_s)dL_s(\widehat{X}) + \int_0^T \left(u(s,\widehat{X}_s + sign(\widehat{Y}_{s-})a) - u(s,\widehat{X}_s)\right)dN_s. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}f(X_T) - \mathbb{E}(f(\widehat{X}_T)) &= -\mathbb{E}\left[\int_0^T \frac{\partial u}{\partial t}(s,\widehat{X}_s)ds + \int_0^T sign(Y_{s-})b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds\right. \\ &+ \sigma\int_0^T sign(Y_{s-})\sqrt{\widehat{X}_{\eta(s)}}\frac{\partial u}{\partial x}(s,\widehat{X}_s)dB_s + \frac{\sigma^2}{2}\int_0^T \widehat{X}_s\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)ds \\ &+ \left.\int_0^T \frac{\partial u}{\partial x}(s,\widehat{X}_s)dL_s(\widehat{X}) + \int_0^T \left(u(s,\widehat{X}_s + sign(\widehat{Y}_{s-})a) - u(s,\widehat{X}_s)\right)dN_s\right]. \end{aligned}$$

Since $u(t,x)$ is the solution of Cauchy's problem (**??**) that we study above.
Thus

$$\frac{\partial u}{\partial t}(T,\widehat{X}_T) = -b(\widehat{X}_s)\frac{\partial u}{\partial x}(s,\widehat{X}_s) - \frac{\sigma^2}{2}\widehat{X}_s\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s) - \lambda\left(u(s,\widehat{X}_s + a) - u(s,\widehat{X}_s)\right).$$

Notice also that $sign(x) = 1 - 2\mathbb{1}_{\{x\leq 0\}}$, we obtain

$$\begin{aligned} \mathbb{E}f(X_T) - \mathbb{E}(f(\widehat{X}_T)) &= \mathbb{E}\int_0^T \left(b(\widehat{X}_s) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds + \frac{\sigma^2}{2}\mathbb{E}\int_0^T \left(\widehat{X}_s - \widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s) \\ &+ \lambda\int_0^T \mathbb{E}\left[u(s,\widehat{X}_s + sign(\widehat{Y}_{s-})a) - u(s,\widehat{X}_s + a)\right]ds \\ &+ 2\mathbb{E}\int_0^T \left(\mathbb{1}_{\{\widehat{Y}_{s-}\leq 0\}}b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s,\widehat{X}_s)\right)ds - \mathbb{E}\int_0^T \frac{\partial u}{\partial x}(s,\widehat{X}_s)dL_s(\widehat{X}) \end{aligned}$$

- **Step 2** In this step, we will analyze each term in the above expression.
  - * **Analysis of the term** $\mathbb{E}\int_0^T \left(b(\widehat{X}_s) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_s)ds$

    Let's consider the function $x \longmapsto \left(b(x) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,x)$. Such function is $\mathcal{C}^{1,2}$, Applying the Itö's formula with this function between s and $\eta(s)$ we have

$$\begin{aligned} \left(b(\widehat{X}_s) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_s) &= \int_{\eta(s)}^s \frac{\partial\left[\left(b - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x}(s,\widehat{X}_\theta)d\widehat{X}^c_\theta \\ &+ \frac{1}{2}\int_{\eta(s)}^s \frac{\partial^2\left[\left(b - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x^2}(s,\widehat{X}_\theta)d[\,\widehat{X}^c,\widehat{X}^c]_\theta \\ &+ \int_{\eta(s)}^s \left[\left(b(\widehat{X}_\theta + a) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta + a)\right. \\ &- \left.\left(b(\widehat{X}_\theta) - b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)\right]dN_\theta \end{aligned}$$

Setting

$$
\begin{aligned}
A_1 &= \int_{\eta(s)}^{s} \frac{\partial\left[\left(b-b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x}(s,\widehat{X}_\theta)d\widehat{X}_\theta^c \\
A_2 &= \frac{1}{2}\int_{\eta(s)}^{s} \frac{\partial^2\left[\left(b-b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x^2}(s,\widehat{X}_\theta)d[\,\widehat{X}^c,\widehat{X}^c]_\theta \\
A_3 &= \int_{\eta(s)}^{s} \left[\left(b(\widehat{X}_\theta+a)-b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta+a)-\left(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)\right]dN_\theta
\end{aligned}
$$

Then,

$$
\begin{aligned}
A_1 &= \int_{\eta(s)}^{s}\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)d\widehat{X}_\theta^c \\
&= \int_{\eta(s)}^{s} sign(\widehat{Y}_\theta)b(\widehat{X}_{\eta(s)})\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)d\theta \\
&+ \sigma\int_{\eta(s)}^{s} sign(\widehat{Y}_\theta)\sqrt{\widehat{X}_{\eta(s)}}\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)dB_\theta \\
&+ \int_{\eta(s)}^{s}\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)dL_\theta
\end{aligned}
$$

Using the fact that $(\widehat{X}) = \left\{t\in[0,T]\,/\,\widehat{X}_t=\widehat{X}_{t^-}=0\right\}$ and taking the expectation, we obtain

$$
\begin{aligned}
\mathbb{E}(A_1) &= \mathbb{E}\left[\int_{\eta(s)}^{s} sign(\widehat{Y}_\theta)b(\widehat{X}_{\eta(s)})\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)d\theta\right] \\
&+ \mathbb{E}\left[\int_{\eta(s)}^{s}\left(b'(0)\frac{\partial u}{\partial x}(s,0)+(b(0)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,0)\right)dL_\theta\right]
\end{aligned}
$$

Because

$$
\sigma\mathbb{E}\left[\int_{\eta(s)}^{s} sign(\widehat{Y}_\theta)\sqrt{\widehat{X}_{\eta(s)}}\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)dB_\theta\right]=0.
$$

Therefore,

$$
\begin{aligned}
|\mathbb{E}(A_1)| &\leq \mathbb{E}\left(\int_{\eta(s)}^{s}\left|b(\widehat{X}_{\eta(s)})\left(b'(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)\right|d\theta\right) \\
&+ \mathbb{E}\left(\int_{\eta(s)}^{s}\left|b'(0)\frac{\partial u}{\partial x}(s,0)+(b(0)-b(\widehat{X}_{\eta(s)}))\frac{\partial^2 u}{\partial x^2}(s,0)\right|dL_\theta\right)
\end{aligned}
$$

Since $b(x)$ satisfies the Lipschitz's condition and $b(0)\geq 0$, then $|b(\widehat{X}_{\eta(s)})|\leq b(0)+K\widehat{X}_{\eta(s)}$. Moreover, the function $b(x)$ has the derivative up to order 4 that are bounded as well as $u(t,x)$.
Thus

$$
\begin{aligned}
\mathbb{E}(A_1) &\leq \int_{\eta(s)}^{s}\mathbb{E}\left[\left(b(0)+K\widehat{X}_{\eta(s)}\right)\left\{\|b'\|_\infty\left\|\frac{\partial u}{\partial x}\right\|_\infty+K\left\|\frac{\partial^2 u}{\partial x^2}\right\|_\infty\left(\widehat{X}_\theta+\widehat{X}_{\eta(s)}\right)\right\}\right]d\theta \\
&+ \int_{\eta(s)}^{s}\mathbb{E}\left[\left\{\|b'\|\left\|\frac{\partial u}{\partial x}\right\|+K\left\|\frac{\partial^2 u}{\partial x^2}\right\|\widehat{X}_{\eta(s)}\right\}\right]dL_\theta(\widehat{X}) \\
&\leq C_1\Delta t\left(1+\mathbb{E}(\sup_{0\leq\theta\leq T}\widehat{X}_\theta)+\sup_{0\leq\theta\leq T}\mathbb{E}\widehat{X}_\theta^2\right)+C_2\mathbb{E}\left\{\left(1+\widehat{X}_{\eta(s)}\right)\left(L_s(\widehat{X})-L_{\eta(s)}(X)\right)\right\}
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\mathbb{E}\left\{\left(1+\widehat{X}_{\eta(s)}\right)\left(L_s(\widehat{X})-L_{\eta(s)}(X)\right)\right\} &= \mathbb{E}\left(\mathbb{E}\left\{\left(1+\widehat{X}_{\eta(s)}\right)\left(L_s(\widehat{X})-L_{\eta(s)}(X)\right)\Big|\mathcal{F}_{\eta(s)}\right\}\right) \\
&= \mathbb{E}\left(\left(1+\widehat{X}_{\eta(s)}\right)\mathbb{E}\left\{\left(L_s(\widehat{X})-L_{\eta(s)}(X)\right)\Big|\mathcal{F}_{\eta(s)}\right\}\right) \\
&\leq \mathbb{E}\left(\left(1+\widehat{X}_{\eta(s)}\right)C_3\mathbb{E}\exp\left(\frac{-\widehat{X}_{\eta(s)}}{16\sigma^2\Delta t}\right)\right)
\end{aligned}
$$

Using the theorems **??** and **??**,we obtain

$$
|\mathbb{E}(A_1)| \leq C_4\left(\Delta t+\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{3.4.2}
$$

where C_4 is a positive constant that doesn't depend on Δt.

Majoration of the term A_2

On a:

$$
\begin{aligned}
A_2 &= \frac{1}{2}\int_{\eta(s)}^{s}\frac{\partial^2\left[\left(b-b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}\right]}{\partial x^2}(s,\widehat{X}_\theta)d[\,\widehat{X}^c,\widehat{X}^c]_\theta \\
&= \frac{1}{2}\int_{\eta(s)}^{s}\left(b''(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right. \\
&\qquad\qquad \left.+\,(b(\widehat{X}_s)-b(\widehat{X}_{\eta(s)}))\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)d[\,\widehat{X},\widehat{X}]^c_\theta \\
&= \frac{\sigma^2}{2}\int_{\eta(s)}^{s}\widehat{X}_{\eta(s)}\left(b''(\widehat{X}_\theta)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)+b'(\widehat{X}_\theta)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right. \\
&\qquad\qquad \left.+\,(b(\widehat{X}_s)-b(\widehat{X}_{\eta(s)}))\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)d\theta
\end{aligned}
$$

Using the same argument as previously, we obtain

$$
|\mathbb{E}\,(A_2)| \leq C_5\left(\Delta t+\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{3.4.3}
$$

where C_5 is a positive constant that doesn't depend on Δt.

Let's major the term A_3.

We have

$$
A_3 = \int_{\eta(s)}^{s}\left[\left(b(\widehat{X}_\theta+a)-b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta+a)-\left(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)})\right)\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)\right]dN_\theta
$$

Thus

$$
\begin{aligned}
|\mathbb{E}\,(A_3)| &= |\lambda\int_{\eta(s)}^{s}\mathbb{E}\left[(b(\widehat{X}_\theta+a)-b(\widehat{X}_{\eta(s)}))\frac{\partial u}{\partial x}(s,\widehat{X}_\theta+a)-(b(\widehat{X}_\theta)-b(\widehat{X}_{\eta(s)}))\frac{\partial u}{\partial x}(s,\widehat{X}_\theta)\right]d\theta| \\
&\leq \int_{\eta(s)}^{s}\lambda\mathbb{E}\left[\left\|\frac{\partial u}{\partial x}\right\|\left\{2K\left(\widehat{X}_\theta+\widehat{X}_{\eta(s)}\right)+Ka\right\}\right] \\
&\leq C_6\Delta t
\end{aligned} \tag{3.4.4}
$$

Using the inequalities (3.4.2), (3.4.3) and (3.4.4) we obtain

$$\left|\int_0^T \mathbb{E}\left((b(\widehat{X}_s)-b(\widehat{X}_{\eta(s)}))\frac{\partial u}{\partial x}(s,\widehat{X}_s)\right)ds\right| \leq C(T)\left(\Delta t+\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{3.4.5}$$

* **Analysis of the term $\mathbb{E}\int_0^T \left(\widehat{X}_s-\widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)ds$**
Reasoning in the same way as before, and applying the Itö's formula with the function $\varphi : x \longmapsto (x-\widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,x)$between $\eta(s)$and s we obtain

$$\begin{aligned}\left(\widehat{X}_s-\widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)=\varphi(\widehat{X}_s) &= \varphi(\widehat{X}_{\eta(s)})+\int_{\eta(s)}^s \varphi'(s,\widehat{X}_\theta)d\widehat{X}^c_\theta+\frac{1}{2}\int_{\eta(s)}^s \varphi''(s,\widehat{X}_\theta)d[\widehat{X},\widehat{X}]^c_\theta \\ &+\int_{\eta(s)}^s \left[\varphi\left(\widehat{X}_\theta+sign(\widehat{Y}_{\theta-})a\right)-\varphi\left(\widehat{X}_\theta+sign(\widehat{Y}_{\theta-})a\right)\right]dN_\theta\end{aligned}$$

Or $\varphi(\widehat{X}_{\eta(s)})=0$
Therefore

$$\begin{aligned}\left(\widehat{X}_s-\widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s) &= \int_{\eta(s)}^s sign(\widehat{Y}_\theta)b(\widehat{X}_{\eta(s)})\left(\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)+(\widehat{X}_\theta-\widehat{X}_{\eta(s)})\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)d\theta \\ &+\int_{\eta(s)}^s sign(\widehat{Y}_\theta)\sqrt{\widehat{X}_{\eta(s)}}\left(\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)+(\widehat{X}_\theta-\widehat{X}_{\eta(s)})\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)dB_\theta \\ &+\int_{\eta(s)}^s \left(\frac{\partial^2 u}{\partial x^2}(s,0)-\widehat{X}_{\eta(s)}\frac{\partial^3 u}{\partial x^3}(s,0)\right)dL_\theta(\widehat{X}) \\ &+\frac{\sigma^2}{2}\int_{\eta(s)}^s \widehat{X}_{\eta(s)}\left((\widehat{X}_\theta-\widehat{X}_{\eta(s)})\frac{\partial^4 u}{\partial x^4}(s,\widehat{X}_\theta)+2\frac{\partial^3 u}{\partial^3 x}(s,\widehat{X}_\theta)\right)d\theta \\ &+\int_{\eta(s)}^s \left[\left((\widehat{X}_\theta+sign(\widehat{Y}_{s^-})a-\widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta+sign(\widehat{Y}_{s^-})a)\right)\right. \\ &\left.-\left((\widehat{X}_\theta-\widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)\right]dN_\theta\end{aligned}$$

Taking the expectation we have

$$\begin{aligned}\mathbb{E}\left[\left(\widehat{X}_s-\widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)\right] &= \int_{\eta(s)}^s \mathbb{E}\left[sign(\widehat{Y}_\theta)b(\widehat{X}_{\eta(s)})\left(\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)+(\widehat{X}_\theta-\widehat{X}_{\eta(s)})\frac{\partial^3 u}{\partial x^3}(s,\widehat{X}_\theta)\right)\right]d\theta \\ &+\frac{\sigma^2}{2}\int_{\eta(s)}^s \mathbb{E}\left[\widehat{X}_{\eta(s)}\left((\widehat{X}_\theta-\widehat{X}_{\eta(s)})\frac{\partial^4 u}{\partial x^4}(s,\widehat{X}_\theta)+2\frac{\partial^3 u}{\partial^3 x}(s,\widehat{X}_\theta)\right)\right]d\theta \\ &+\int_{\eta(s)}^s \mathbb{E}\left[\left(\frac{\partial^2 u}{\partial x^2}(s,0)-\widehat{X}_{\eta(s)}\frac{\partial^3 u}{\partial x^3}(s,0)\right)\right]dL_\theta(\widehat{X}) \\ &+\int_{\eta(s)}^s \lambda\mathbb{E}\left[\left((\widehat{X}_\theta+sign(\widehat{Y}_{s^-})a-\widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta+sign(\widehat{Y}_{s^-})a)\right)\right. \\ &\left.-\left((\widehat{X}_\theta-\widehat{X}_{\eta(s)})\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_\theta)\right)\right]d\theta\end{aligned}$$

Proceeding as the same way , we obtain

$$|\mathbb{E}\left[\left(\widehat{X}_s-\widehat{X}_{\eta(s)}\right)\frac{\partial^2 u}{\partial x^2}(s,\widehat{X}_s)\right]| \leq C_7\left(\Delta t+\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right)$$

Therefore

$$\left|\int_0^T \mathbb{E}\left(\left(\widehat{X}_s - \widehat{X}_{\eta(s)}\right)\frac{\partial u}{\partial x}(s, \widehat{X}_s)\right) ds\right| \leq C(T)\left(\Delta t + \left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{3.4.6}$$

* **Analysis of the term** $\int_0^T \mathbb{E}\left(\mathbb{1}_{\{\widehat{Y}_s \leq 0\}} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right) d\theta$
One has

$$\begin{aligned}
\int_0^T \mathbb{E}\left(\mathbb{1}_{\{\widehat{Y}_s \leq 0\}} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right) ds &\leq \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[\mathbb{E}\left(\mathbb{1}_{\{\widehat{Y}_{s^-} \leq 0\}} b(\widehat{X}_{\eta(s)}) \,\Big|\widehat{X}_{\eta(s)}\right)\right] ds \\
&\leq \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[b(\widehat{X}_{\eta(s)})\mathbb{E}\left(\mathbb{1}_{\{\widehat{Y}_{s^-} \leq 0\}} \,\Big|\widehat{X}_{\eta(s)}\right)\right] ds \\
&\leq \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[b(\widehat{X}_{\eta(s)})\mathbb{P}\left(\left\{\widehat{Y}_{s^-} \leq 0\right\} \Big|\widehat{X}_{\eta(s)}\right)\right] ds \\
&\leq \frac{1}{2}\left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[b(\widehat{X}_{\eta(s)}) \exp\left(-\frac{\widehat{X}_{\eta(s)}}{8\sigma^2\Delta t}\right)\right] ds
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\int_0^T \mathbb{E}\left(\mathbb{1}_{\widehat{Y}_s < 0} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}\right) ds\right| &\leq \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left(\mathbb{1}_{\{\widehat{Y} < 0\}}\left(b(0) + K\widehat{X}_{\eta(s)}\right)\right) ds \\
&\leq \left\|\frac{\partial u}{\partial x}\right\|_\infty \int_0^T \mathbb{E}\left[\left(b(0) + K\widehat{X}_{\eta(s)}\right)\exp\left(-\frac{\widehat{X}_{\eta(s)}}{8\sigma^2\Delta t}\right)\right] ds
\end{aligned}$$

Or pour tout $x \geq 0$, $x\exp\left(-\frac{x}{8\sigma^2\Delta t}\right) \leq 8\sigma^2\Delta t$
Using this inequality and the lemma 3.3.5 we obtain

$$\left|\int_0^T \mathbb{E}\left[\mathbb{1}_{\widehat{Y}_s < 0} b(\widehat{X}_{\eta(s)})\frac{\partial u}{\partial x}(s, \widehat{X}_s)\right] ds\right| \leq C\left(\Delta t + \left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}\right) \tag{3.4.7}$$

* **Analysis of the term** $\int_0^T \mathbb{E}\left[u(s, \widehat{X}_s + sign(\widehat{Y}_{s^-})a) - u(s, \widehat{X}_s + a)\right] ds$ we have

$$\begin{aligned}
\left|\mathbb{E}\left[u(s, \widehat{X}_s + sign(\widehat{Y}_{s^-})a) - u(s, \widehat{X}_s + a)\right]\right| &= \left|\mathbb{E}\left[\left(u(s, \widehat{X}_s + sign(\widehat{Y}_{s^-})a) - u(s, \widehat{X}_s + a)\right)\mathbb{1}_{\{\widehat{Y}_{s^-} > 0\}}\right]\right| \\
&+ \left|\mathbb{E}\left[\left(u(s, \widehat{X}_s + sign(\widehat{Y}_{s^-})a) - u(s, \widehat{X}_s + a)\right)\mathbb{1}_{\{\widehat{Y}_{s^-} \leq 0\}}\right]\right| \\
&= \left|\mathbb{E}\left[\left(u(s, \widehat{X}_s + sign(\widehat{Y}_{s^-})a) - u(s, \widehat{X}_s + a)\right)\mathbb{1}_{\{\widehat{Y}_{s^-} \leq 0\}}\right]\right| \\
&\leq \mathbb{E}\left[2\|u\|_\infty \mathbb{1}_{\{\widehat{Y}_{s^-} \leq 0\}}\right] \\
&= 2\|u\|_\infty \mathbb{E}\left(\mathbb{1}_{\{\widehat{Y}_{s^-} \leq 0\}}\right) \\
&\leq 2\|u\|_\infty \mathbb{P}\left(\left\{\widehat{Y}_{s^-} \leq 0\right\}\right)
\end{aligned}$$

Using 3.3.5, we obtain

$$\mathbb{P}\left(\left\{\widehat{Y}_{s^-} \leq 0\right\}\right) \leq C_8\left(\frac{\Delta t}{x_0}\right)^{\frac{b(0)}{\sigma^2}}$$

Thus

$$\left| \int_0^T \mathbb{E}\left[u(s, \widehat{X}_s + sign(\widehat{Y}_{s-})a) - u(s, \widehat{X}_s + a) \right] ds \right| \leq C(T) \left(\frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}} \tag{3.4.8}$$

* **Analysis of the term** $\int_0^T \mathbb{E}\left[\frac{\partial u}{\partial x}(s, \widehat{X}_s) \right] dL_s(\widehat{X})$
  we have

$$\left| \int_0^T \mathbb{E}\left[\frac{\partial u}{\partial x}(s, \widehat{X}_s) \right] dL_s(\widehat{X}_s) \right| \leq T \left\| \frac{\partial u}{\partial x} \right\|_\infty \mathbb{E}\left(L_T(\widehat{X}) \right)$$

  Using the theorem **??** we obtain

$$\left| \int_0^T \mathbb{E}\left[\frac{\partial u}{\partial x}(s, \widehat{X}_s) \right] dL_s(\widehat{X}) \right| \quad \leq \quad C \left(\frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}} \tag{3.4.9}$$

- **Step 3**: Conclusion.
  Using (3.4.6), (3.4.7), (3.4.8) and (3.4.9) we have finally

$$|\mathbb{E}(f(X_T) - f(\widehat{X}_T))| \leq C \left(\Delta t + \left(\frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}} \right)$$

■

We conclude that our symmetric Euler-Maruyama scheme converges weakly to the exact CIR type process with order $\frac{b(0)}{\sigma^2}$.

In this chapter, we defined the symmetric Euler-Maruyama scheme for approximating the CIR type process with Poissonian jump, and prove the weak convergence of such scheme to the true CIR type process with Poissonian jump. In the next chapter we use some numerical simulation to illustrate our result.

Chapter 4

Numerical Simulation.

In this chapter, we present some numerical results in order to illustrate the theoretical results obtained. In order to effectively confront the numerical simulation with our theoretical result, we will use a classical CIR process with Poissonian jump, that is the solution of the following stochastic differential equation

$$X_t = x_0 + \int_0^t (\alpha - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t a\mathrm{d}N_s. \tag{4.0.1}$$

The choice is motivated by the fact that, we can explicitly compute the expectation $\mathbb{E}[f(X_t)]$ for $f\colon \mathbb{R} \to \mathbb{R}$ in certain class. Then can effectively compare the numerical result with the true value and evaluate the weak error. The main references for this chapter are [4], [10], [14]

4.1 Simulation procedures

To simulate our so-called symmetric Euler-Maruyama scheme, $(\widehat{X}_{t_k}, k = 0, ...N)$ defined by

$$\begin{cases} \widehat{X}_0 = x_0 \\ \widehat{X}_{t_{k+1}} = |\widehat{X}_{t_k} + b(\widehat{X}_{t_k})\Delta t + \sigma\sqrt{\widehat{X}_{t_k}}\Delta B_k + a\Delta N_k| \end{cases} \tag{4.1.1}$$

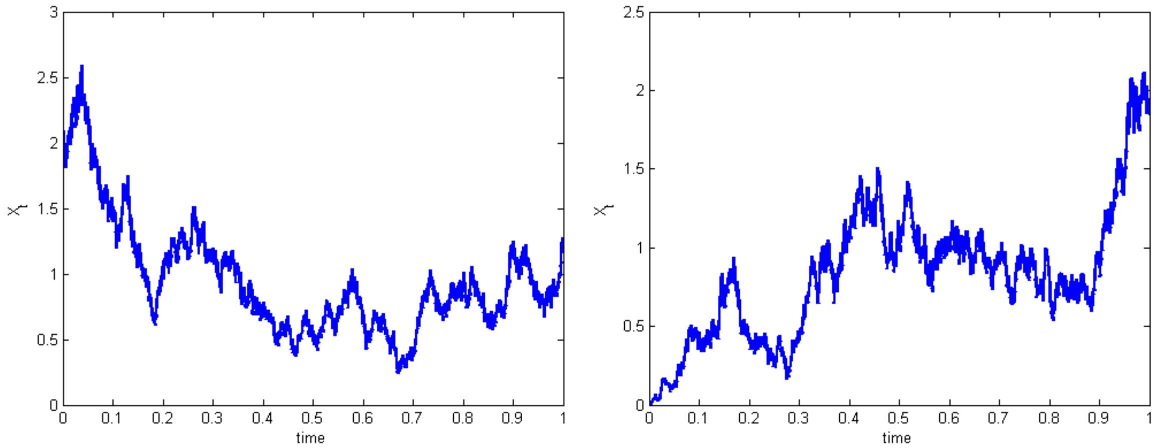
where $\Delta t = t_{k+1} - t_k$ is the step size, $t_k = k\frac{T}{N}$, $N \in \mathbb{N}^*$, $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and $\Delta N_k = N_{t_{k+1}} - N_{t_k}$. Using MATLAB software, we go through the following step.

- We use the function *randn* to generate the ΔB_k, that is a normal distributed random variable with mean zero and standard deviation Δt, $(\Delta B_k \sim \mathcal{N}(0,\ \Delta t))$.
- We take the time-step small enough such that in any interval $[t_k, t_{k+1}]$, there are 0 or 1 jump with the probability $1 - \exp(-\lambda\Delta t)$ and $\exp(-\lambda\Delta t)$ respectively. Since $\exp(-\lambda\Delta t) \approx 1 - \lambda\Delta t$ as $\Delta t \to 0$. The probability to get a jump in the interval $[t_k, t_{k+1}]$ is $\mathbb{P}(\Delta N_k = 1) \approx \lambda\Delta t$, the we use zero-one Poisson-Bernoulli jump law and the the acceptance-rejection method [6].

In the following sections, we will firstly simulate the trajectories of the CIR type process with Poisson jump for some parameters in 2D, that is the evolution of the process with respect to time. Secondly, we will simulate the evolution of CIR type process with Poisson jump in 3D, that is the evolution of the process with respect to time and the volatility parameter σ. Finally, we will the weak error for $\mathbb{E}[X_T^2]$.

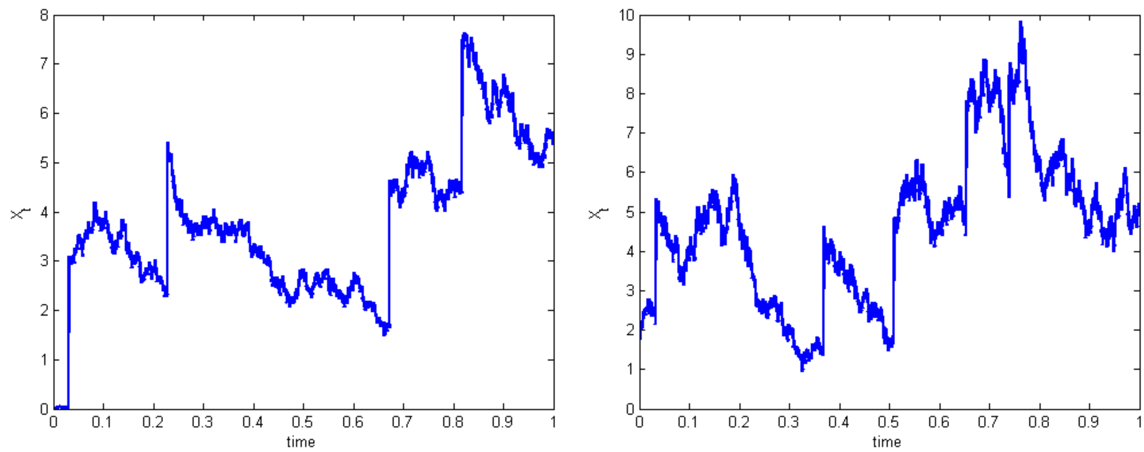
4.2 Trajectories of CIR type process in 1D

Here we simulate the trajectories of CIR type process with Poisson jump in the interval $[0,1]$ for some values of x_0, α, b, σ, λ, a. We use Monte Carlo with samples of size $N = 10000$



(a) $x_0 = 2, \alpha = 4, b = 3, \sigma = 1.5, \lambda = 2, a = 0$ **(b)** $x_0 = 0, \alpha = 4, b = 3, \sigma = 1, \lambda = 2, a = 0$

Figure 4.1: Trajectories of CIR process without jump, here we take the jump coefficient $a = 0$ for some parameters.

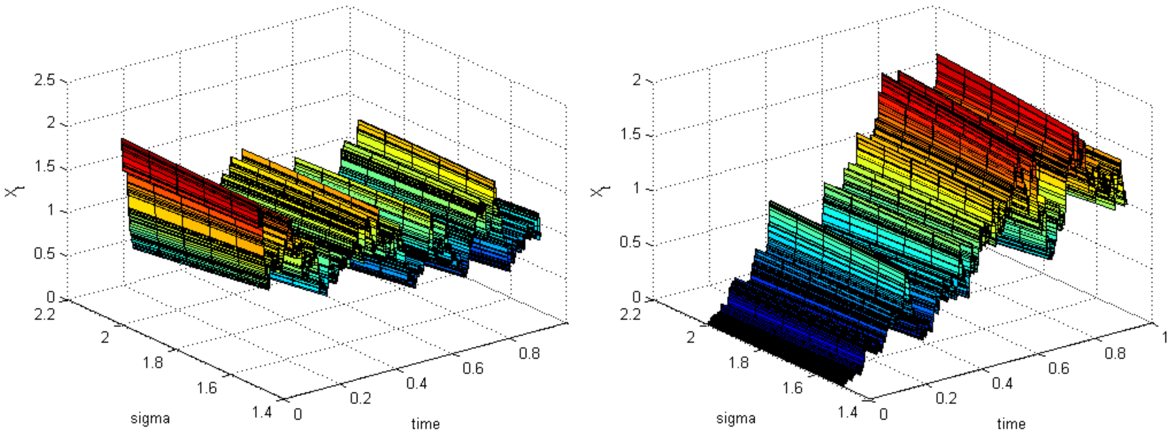


(a) $x_0 = 0$,$\alpha = 4$, $b = 3$, $\sigma = 1.5$, $\lambda = 2$, $a = 3$ **(b)** $x_0 = 2$, $\alpha = 4$, $b = 3$, $\sigma = 2.5$, $\lambda = 2$, $a = 3$

Figure 4.2: Trajectories or CIR process with Poisson jump, for different starting point x_0, here we take a non null jump coefficient .

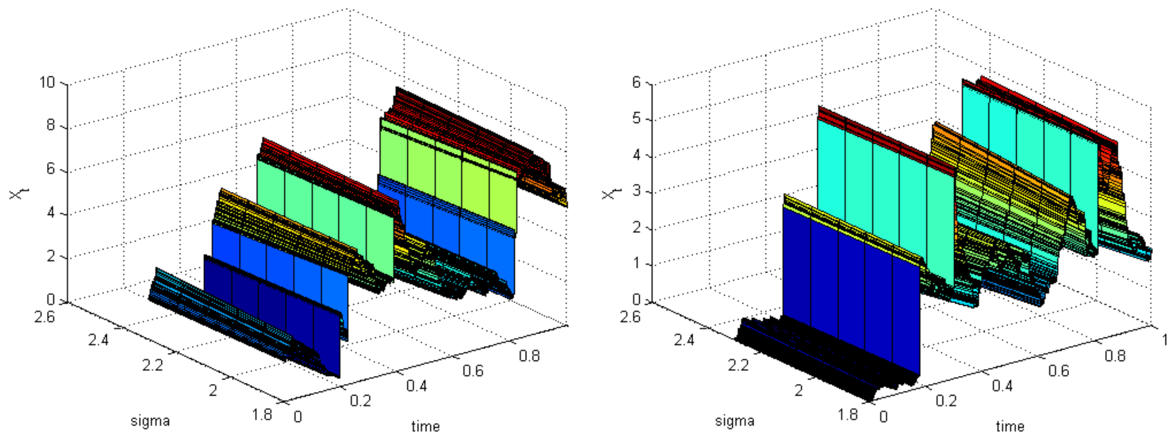
4.3 Evolution of CIR type process with respect to time and volatility.

Here we simulate the evolution of CIR type process with respect to time en volatility in the interval $[0, 1]$ with $N = 20000$



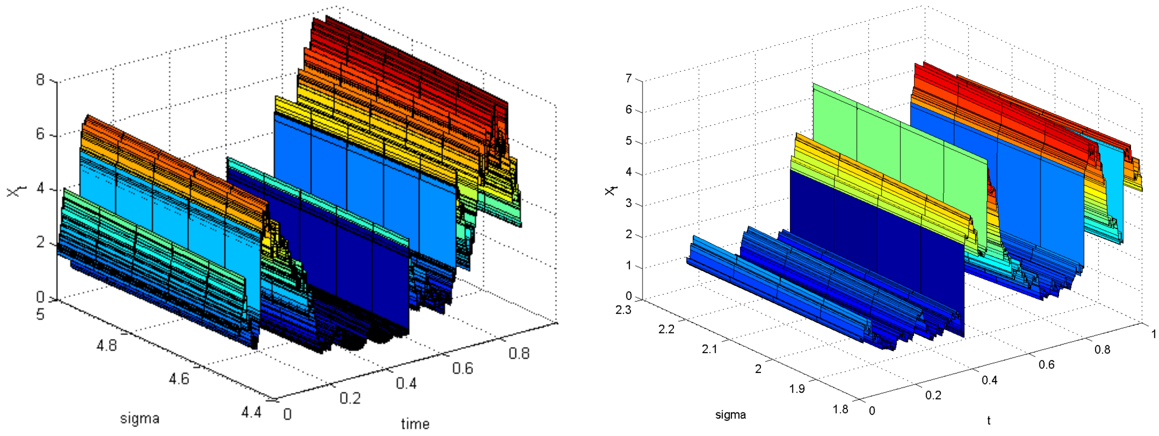
(a) $x_0 = 2$,$\alpha = 4$, $b = 3$, $\sigma = 1.5$, $\lambda = 2$, $a = 0$ **(b)** $x_0 = 0$, $\alpha = 4$, $b = 3$, $\sigma = 1.5$, $\lambda = 2$, $a = 0$

Figure 4.3: Evolution of CIR process without jump, with respect to time and volatility.



(a) $x_0 = 2$,$\alpha = 4$, $b = 3$, $\sigma = 1.8$, $\lambda = 2$, $a = 3$ **(b)** $x_0 = 0$, $\alpha = 4$, $b = 3$, $\sigma = 1.8$, $\lambda = 2$, $a = 3$

Figure 4.4: Evolution of CIR process with Poisson jump, with respect to time and the volatility parameter.



(a) $x_0 = 5$,$\alpha = 1.5$, $b = 3$, $\sigma = 4.5$, $\lambda = 2$, $a = 3$ **(b)** $x_0 = 5$, $\alpha = 1.5$, $b = 3$, $\sigma = 4.5$, $\lambda = 2$, $a = 3$

Figure 4.5: Evolution of CIR process with Poisson jump, with respect to time and the volatility parameter for some other values of x_0, α, b, a

4.4 Weak error for the moment of order 2

Here we simulate the weak error for $\mathbb{E}\left(X_T^2\right)$ in the interval $[0,1]$ and $N = 1000000$.

Setting $Err_T^{weak} := \left|\mathbb{E}(X_T^2) - \mathbb{E}(\widehat{X}_T^2)\right|$, recall that in 2 we find

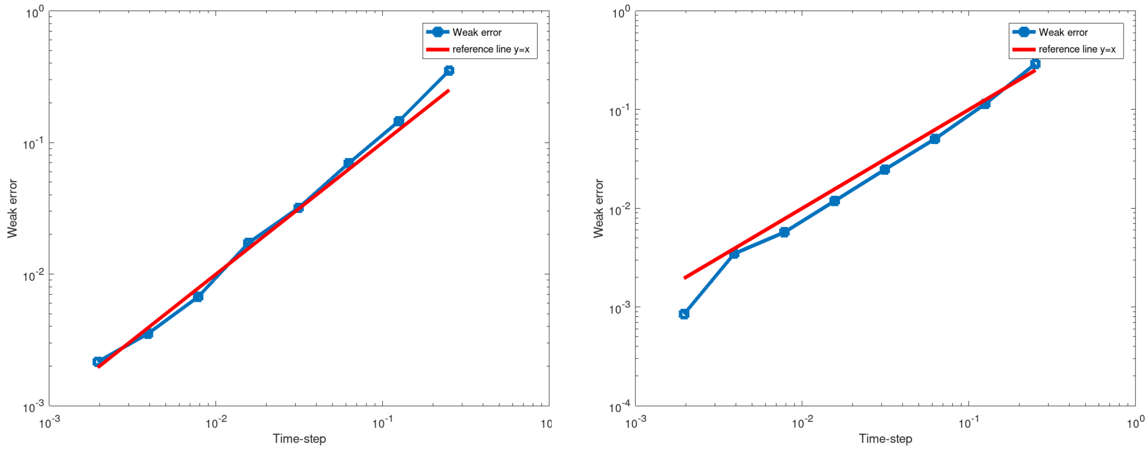
$$\begin{aligned}\mathbb{E}\left(X_T^2\right) = x_0^2 \exp\left(-2bT\right) \quad &+ \quad \left[\left(2\alpha + \sigma^2 + 2a\lambda\right)\frac{\alpha + a\lambda}{2b^2} + \frac{a^2\lambda}{2b}\right]\left[1 - \exp\left(-2bT\right)\right] \\ &+ \quad \left[\left(2\alpha + \sigma^2 + 2a\lambda\right)\left(\frac{x_0}{b} - \frac{\alpha + a\lambda}{b^2}\right)\right]\left[\exp\left(-bT\right) - \exp\left(-2bT\right)\right].\end{aligned} \tag{4.4.1}$$

For $\mathbb{E}(\widehat{X}_T^2)$, we use Monte Carlo approximation i.e

$$\mathbb{E}\left(\widehat{X}_T^2\right) \approx \frac{1}{n}\sum_{i=1}^{n}\widehat{X}_{T,i}^2. \tag{4.4.2}$$

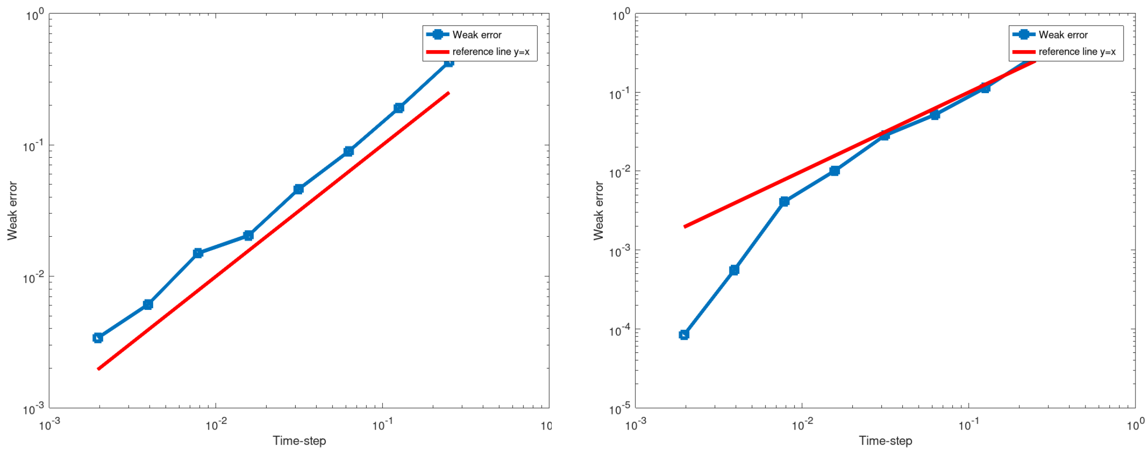
where for each $i = 1, 2, \ldots, n$, we generate $\widehat{X}_{T,i}$ using our numerical scheme.

In this Chapter, we presented some numerical simulation to illustrate our theoretical result. We simulated the trajectories of so-called CIR type process with Poisson jump. We simulated the weak error of symmetric Euler-Maruyama scheme on a moment of order 2.



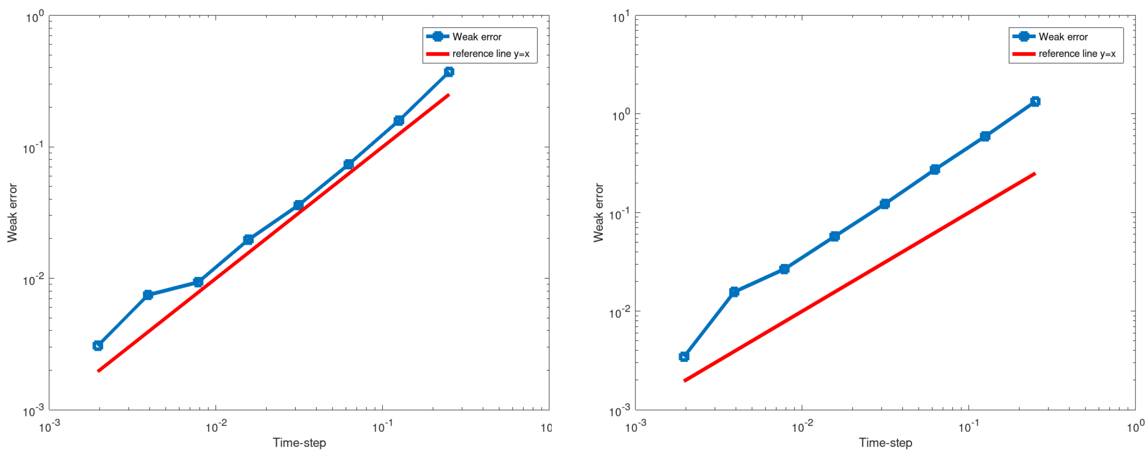
(a) $x_0 = 0.5$,$\alpha = 1.5$, $b = 3$, $\sigma = 0.75$, $\lambda = 2$, $a = 1$ **(b)** $x_0 = 0.5$,$\alpha = 0.56250$, $b = 3$, $\sigma = 0.75$, $\lambda = 2$, $a = 1$

Figure 4.6: Weak error for the above parameters



(a) $x_0 = 0.5$, $\alpha = 2.5$, $b = 3$, $\sigma = 0.75$, $\lambda = 2$, $a = 1$ **(b)** $x_0 = 0.5$,$\alpha = 0.5$, $b = 3$, $\sigma = 0.75$, $\lambda = 2$, $a = 1$

Figure 4.7: Weak error for the above parameters



(a) $x_0 = 0.5$, $\alpha = 1.8$, $b = 3$, $\sigma = 0.75$, $\lambda = 2$, $a = 1$ **(b)** $x_0 = 0.5$,$\alpha = 2.5$, $b = 3$, $\sigma = 2$, $\lambda = 2$, $a = 1$

Figure 4.8: Weak error for the above parameters

Conclusion and future work

The SDE is one of the most important field of stochastic analysis. It is used to model various phenomena such as unstable stock prices, interest rate, or physical subject to thermal fluctuation. In this thesis, we go from so-called CIR model, that is a SDE modelling the evolution of the short time interest rate, and we considered the most general class of so-called CIR type process with Poisson jump, that is a class of SDEs with non-Lipschitz diffusion coefficient and adding a Poisson jump. We studied some properties of such process, establish the link with PDE. We proposed the so-called symmetric Euler-Maruyama scheme to approximate the solution of such SDEs, we establish the weak convergence of such scheme and illustrated our theoretical result by numerical simulation.

For the possible future work, we plan to establish the strong convergence of our scheme and generalized our study on most general class of SDE with non-Lipschitz coefficient and most general jump.

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