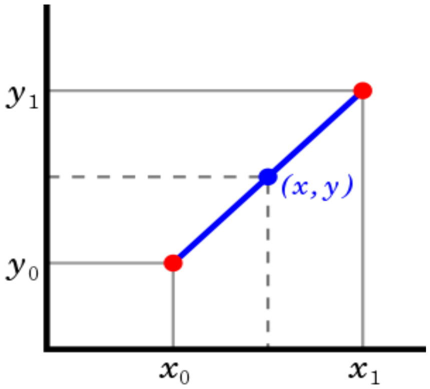
Linear Interpolation

Linear interpolation is the process of computing an arbitrary value within a range, or an arbitrary position along a line in geometric terms.

This document attempts to unify the geometric and parametric definitions of linear interpolation.



The location of an arbitrary point lying on a given line segment is defined by the two-dimensional coordinates (x, y) which satisfy the following equation:

$$y = ax + b$$

Where a denotes the rate of positional change in y with respect to x and b a constant representing the value of y when x is zero, also known as the y intercept.

This means that the concrete points characterised by the coordinate pairs (x_0, y_0) and (x_1, y_1) define the following system of equations:

$$\begin{cases} ax_0 + b = y_0 \ (1) \\ ax_1 + b = y_1 \ (2) \end{cases}$$

In this case, subtracting equation (1) from equation (2) yields:

$$\begin{cases} ax_0 + y_0 = y_0 & (1) \\ ax_1 - ax_0 + y_0 - y_0 = y_1 - y_0 & (2) - (1) \end{cases}$$

$$\begin{cases} ax_0 + y_0 = y_0 & (1) \\ ax_1 - ax_0 = y_1 - y_0 & (2) - (1) \end{cases}$$

$$\begin{cases} ax_0 + y_0 = y_0 & (1) \\ a(x_1 - x_0) = y_1 - y_0 & (2) - (1) \end{cases}$$

$$\begin{cases} ax_0 + y_0 = y_0 & (1) \\ a = \dfrac{y_1 - y_0}{x_1 - x_0} & (2) - (1) \end{cases}$$

Thus we computed the line segment slope a to be:

$$\frac{y_1 - y_0}{x_1 - x_0}$$

By definition, the slope a of a line or rate of change in y with respect to x is constant for each and every point lying on that line.

Consequently, the value of the slope a obtained above is identical for any arbitrary coordinate pair (x, y), in addition to (x_1, y_1) as in the previous example. This leads to the following equality:

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = a$$

Which, through multiplying both leftmost sides by $(x - x_0)$ may equivalently be expressed as:

$$y - y_0 = (x - x_0)\,\frac{y_1 - y_0}{x_1 - x_0}$$

Or:

$$y = y_0 + (x - x_0)\,\frac{y_1 - y_0}{x_1 - x_0}$$

$$y = y_0\frac{x_1 - x_0}{x_1 - x_0} + (x - x_0)\,\frac{y_1 - y_0}{x_1 - x_0}$$

Here, expanding each individual product term composing the final sum yields:

$$y = \frac{y_0 x_1 - y_0 x_0}{x_1 - x_0} + \frac{x y_1 - x y_0 - x_0 y_1 + x_0 y_0}{x_1 - x_0}$$

$$y = \frac{y_0 x_1 - y_0 x_0 + x y_1 - x y_0 - x_0 y_1 + x_0 y_0}{x_1 - x_0}$$

$$y = \frac{y_0 x_1 + x y_1 - x y_0 - x_0 y_1}{x_1 - x_0}$$

$$y = \frac{y_0(x_1 - x) + y_1(x - x_0)}{x_1 - x_0}$$

$$y = y_0\frac{x_1 - x}{x_1 - x_0} + y_1\frac{x - x_0}{x_1 - x_0} \quad (3)$$

Let the variable μ be defined by the quotient:

$$\mu = \frac{x - x_0}{x_1 - x_0}$$

Then:

$$1 - \mu = 1 - \frac{x - x_0}{x_1 - x_0}$$

$$1 - \mu = \frac{x_1 - x_0}{x_1 - x_0} - \frac{x - x_0}{x_1 - x_0}$$

$$1 - \mu = \frac{x_1 - x_0 - (x - x_0)}{x_1 - x_0}$$

$$1 - \mu = \frac{x_1 - x_0 - x + x_0)}{x_1 - x_0}$$

$$1 - \mu = \frac{x_1 - x}{x_1 - x_0}$$

As a result, the fractional terms in equation (3) may be replaced as follows:

$$y = y_0(1 - \mu) + y_1\mu \, (3)$$

Thus expressing the value of y as a blend, or proportional mix between the two extrema y_0 and $y_1$by a factor, or interpolation coefficient of μ.

The equivalence exposed between the geometric and parametric definitions of linear interpolation allows one to understand both temporally evolving arbitrary positions along a line defined by two extremities and blends of two values progressing from minimum to maximum as one and the same.

In fact, the interpolation coefficient μ, or “***mu***” according to Paul Bourke in his 1999 article: *Linear Interpolation Methods* “defines where to estimate the value on the interpolated line, it is 0 at the first point and 1 and the second point” and may be used as a basic parameter in computing linear, but also cosine or cubic interpolated output as shown in the following snippet of C code:

```
double LinearInterpolate(
   double y1, double y2,
   double mu
)
{
   return( y1 * (1 - mu)+ y2 * mu );
}
```