Graph Distances: Theory & Computation

Lecture VII for course in Data Structures & Algorithms for AI

Lectures by Hannah Santa Cruz Baur.

Based on *Introduction to Algorithms*, Cormen et al. and material by Rafaella Mulas.

Homework (due with Assignment 3)

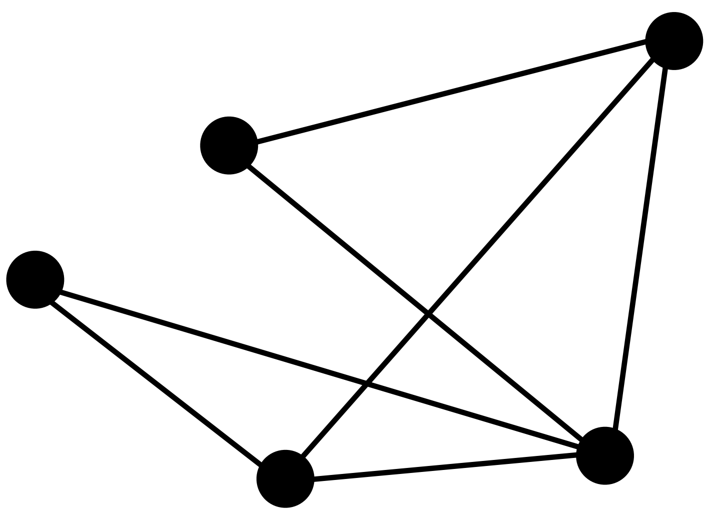
- The square of a directed graph $G = (V, E)$ is the graph $G^2 = (V, E^2)$ such that $(u, v) \in E^2$ if and only if G contains a walk with two edges between u and v.

  Describe an efficient algorithm for computing G^2 from the adjacency matrix of G, and analyze the running time.

- Most graph algorithms that take an adjacency-matrix representation as input require $\Omega(V^2)$ time, but there are some exceptions. Show how to determine whether a directed graph contains a universal sink - a vertex with in-degree $|V| - 1$ and out-degree 0 in $O(V)$ time, given the adjacency matrix.

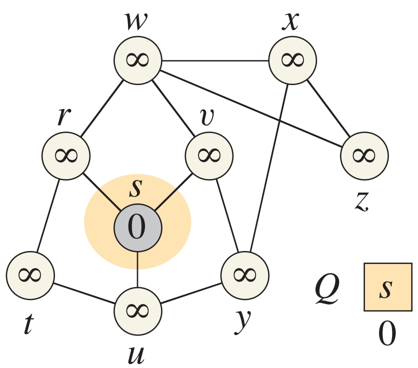
Homework (due with Assignment 3)

- Using matrix multiplication, compute the number of walks on 3 edges between al pairs of vertices, for the graph bellow. Your answer should be a square matrix on the number of vertices.



Breadth First Search

- Given a node, *s*, in a graph, we would like to know the distance to all other nodes of the graph.



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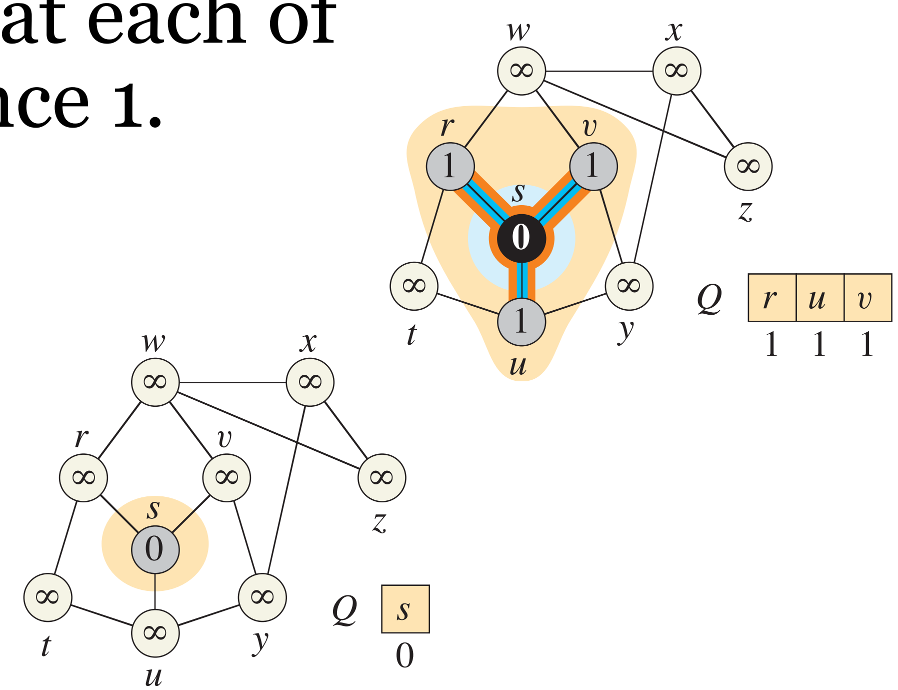
- Given a node, *s*, in a graph, we would like to know the distance to all other nodes of the graph.
- We begin by looking at each of it's neighbors- distance 1.



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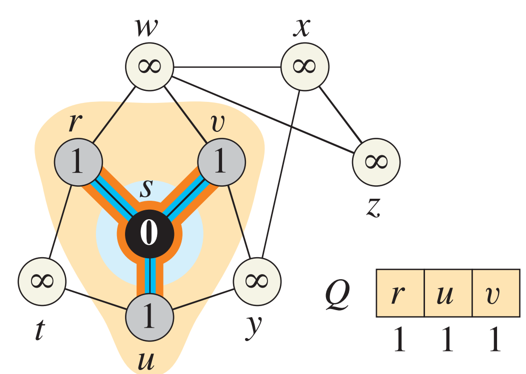
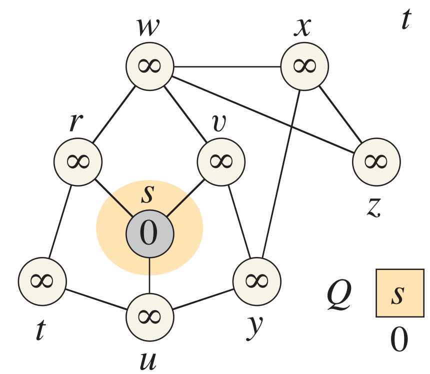
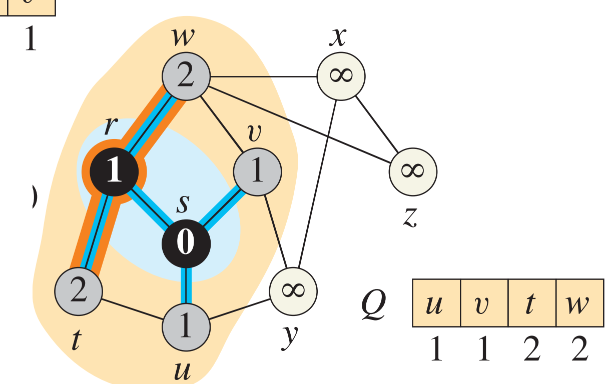


- Next, we expand the frontier - one neighbor at at time.





Now discovering the neighbors of neighbors.

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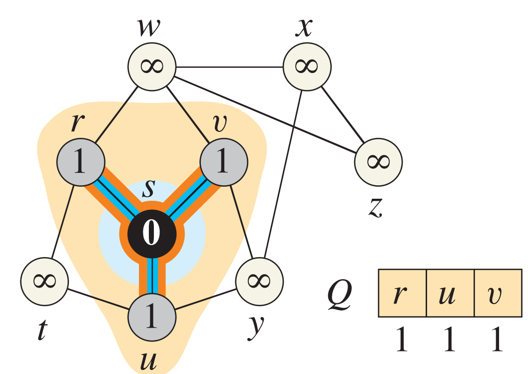

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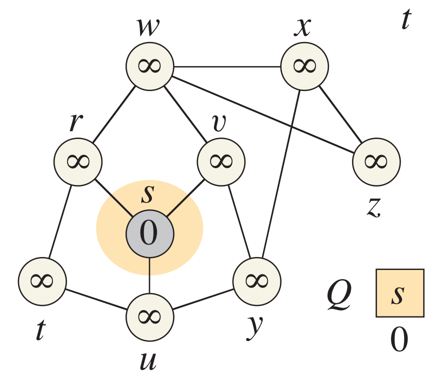

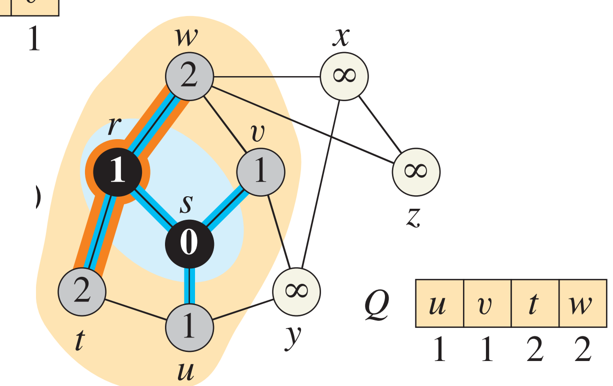

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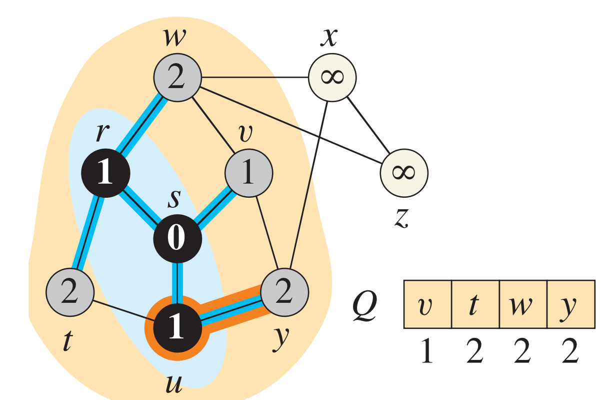

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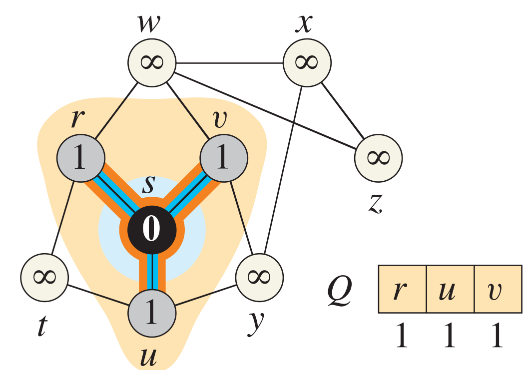

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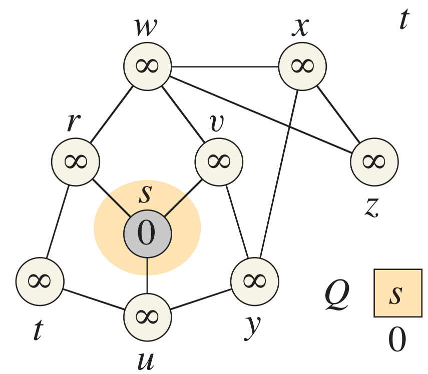

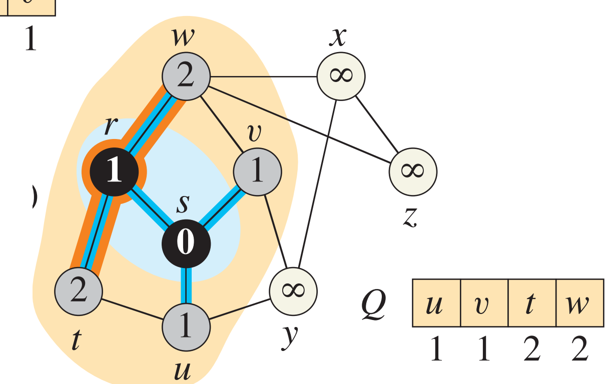

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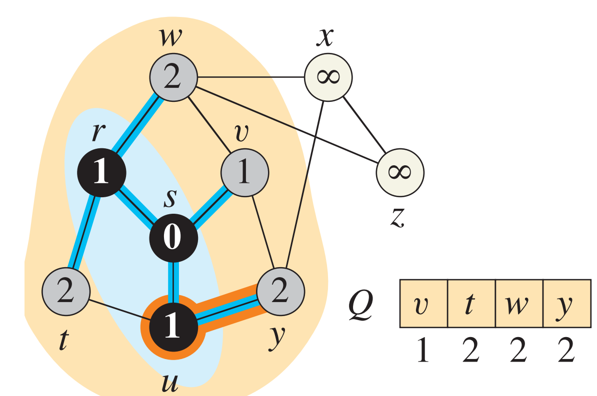
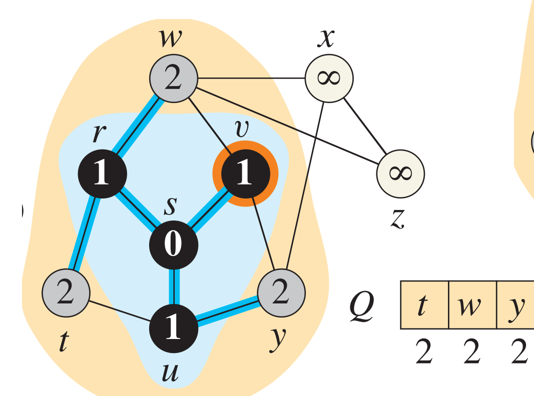
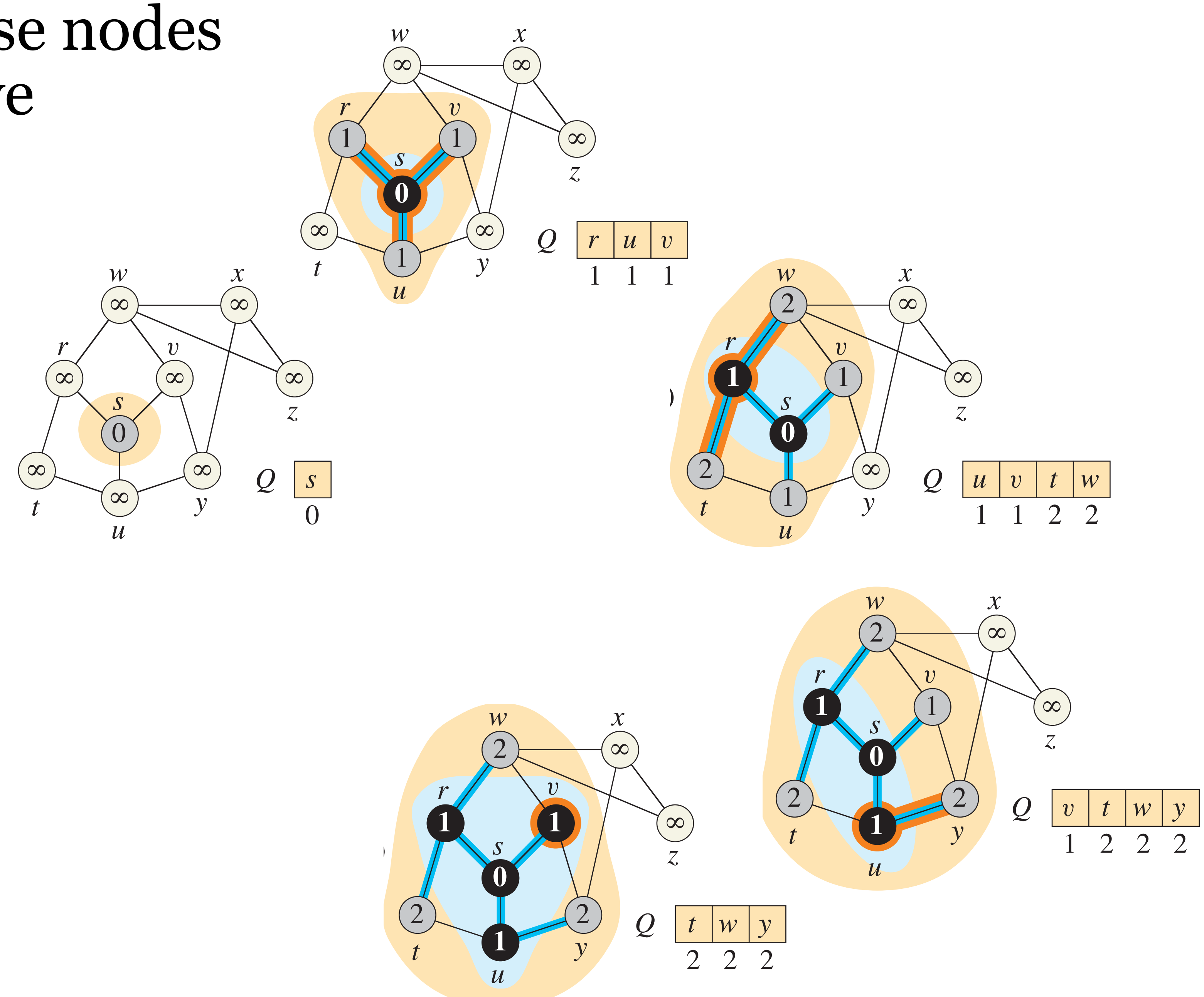




Breadth First Search

- Given a node, *s*, in a graph, we would like to know the distance to all other nodes of the graph.
- In blue, we track those nodes whose neighbors have been discovered.

  In tan we track discovered nodes with yet unexplored neighbors.



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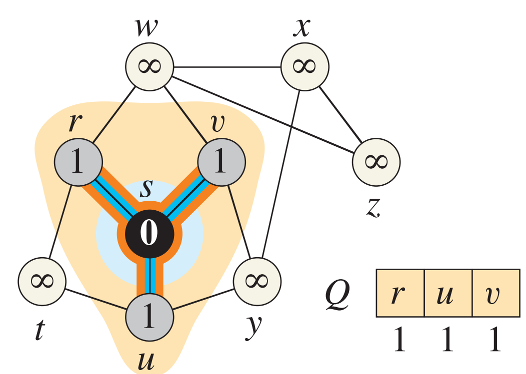

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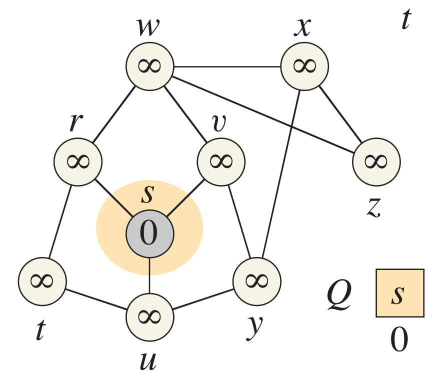
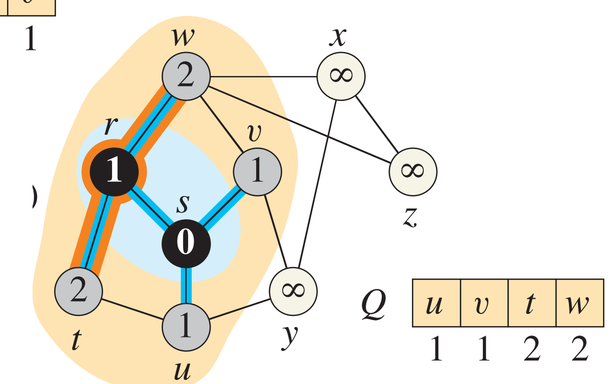
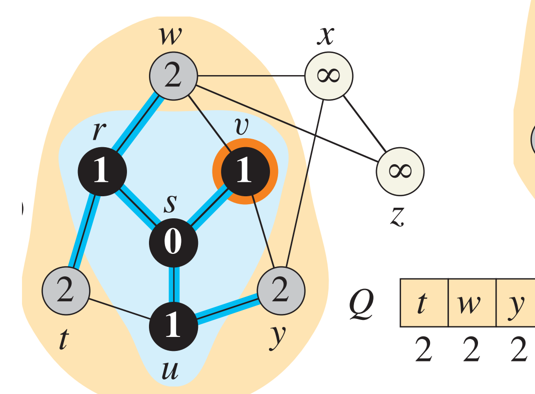
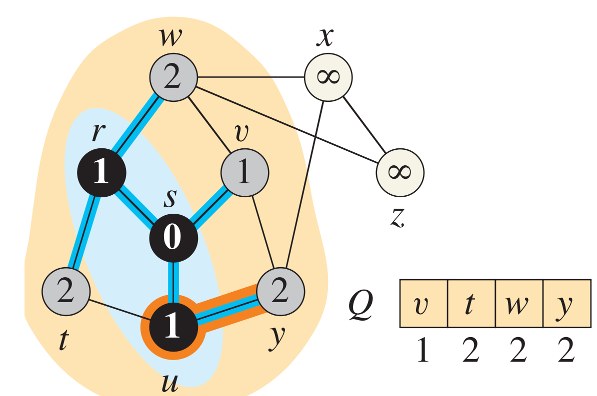




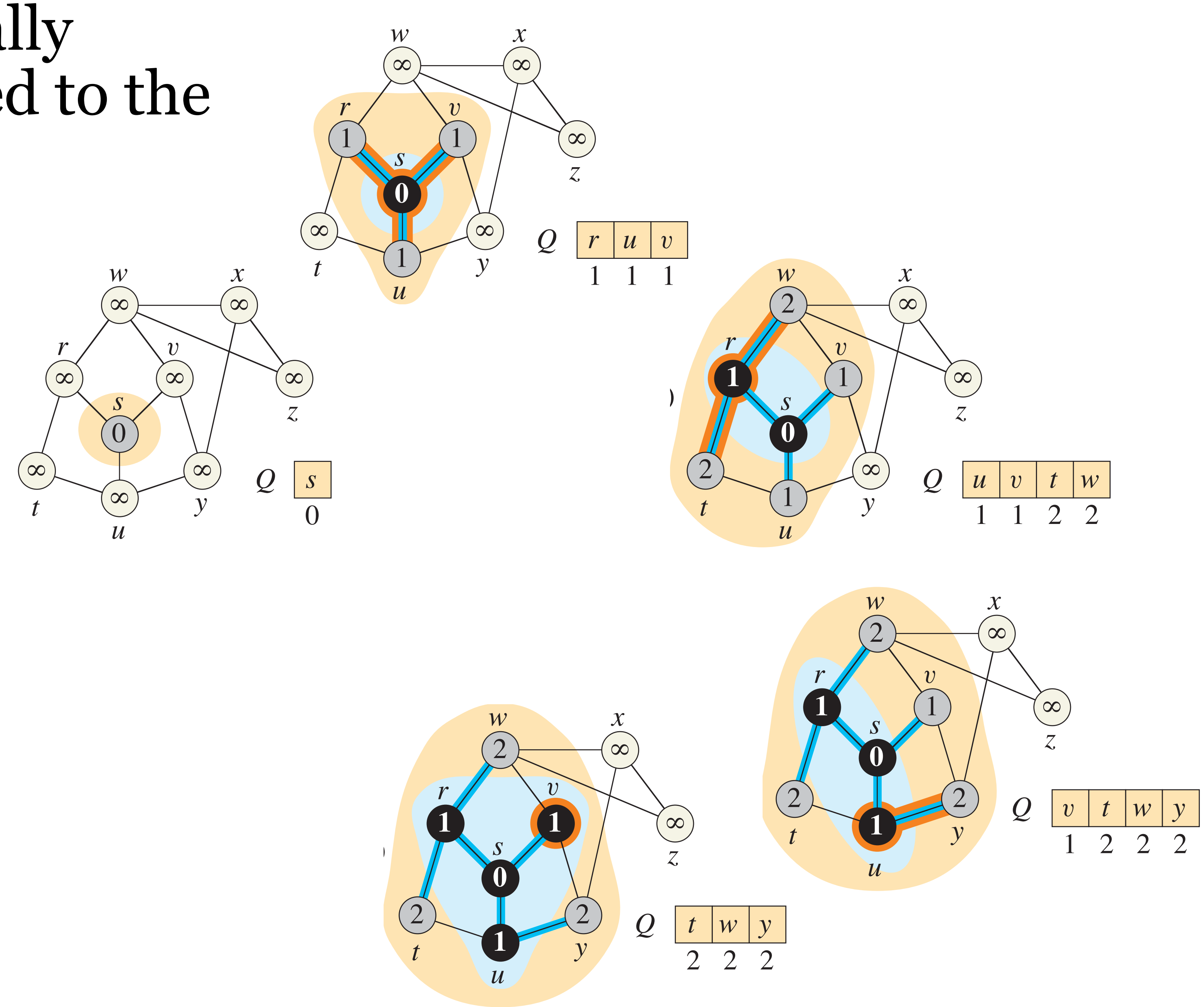
Nodes with yet unexplored neighbors, are kept track of with a queue, Q.





Breadth First Search

- Given a node, *s*, in a graph, we would like to know the distance to all other nodes of the graph.
- When a node is initially discovered, it is added to the queue.

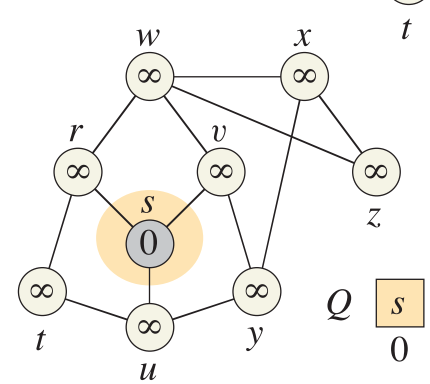


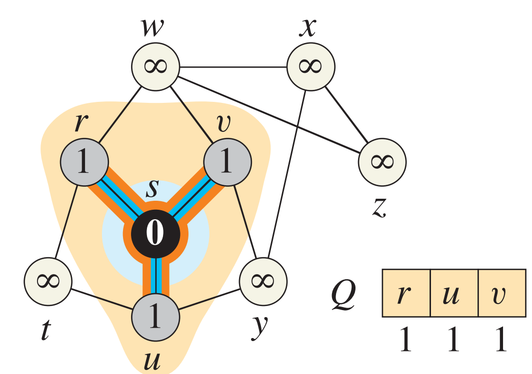
Breadth First Search

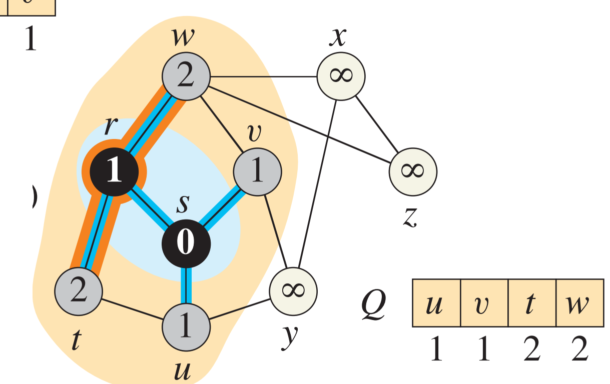
- Given a node, *s*, in a graph, we would like to know the distance to all other nodes of the graph.
- When a node is initially discovered, it is added to the queue.

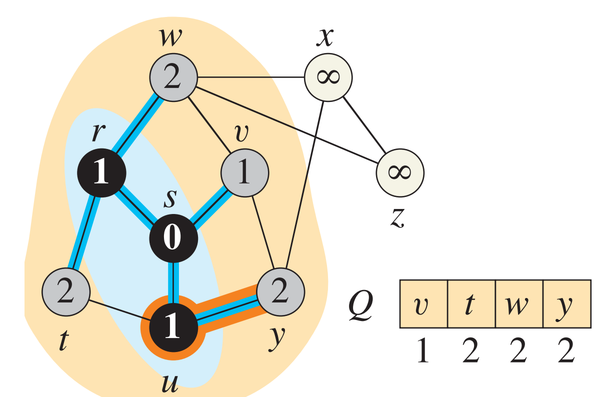
  It's distance to the source node is recorded.

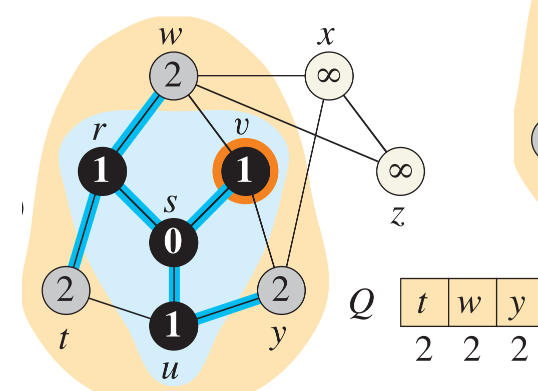
  And the node is colored, grey.







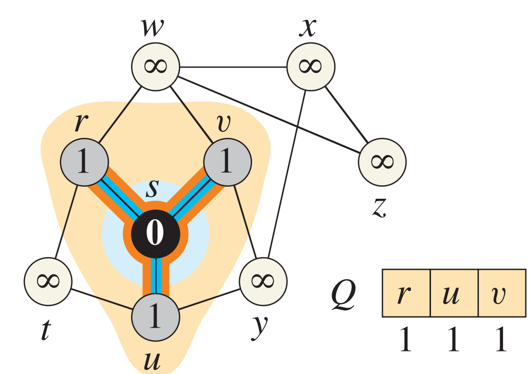




Breadth First Search

- Given a node, *s*, in a graph, we would like to know the distance to all other nodes of the graph.

- When a node is initially discovered, it is added to the queue.



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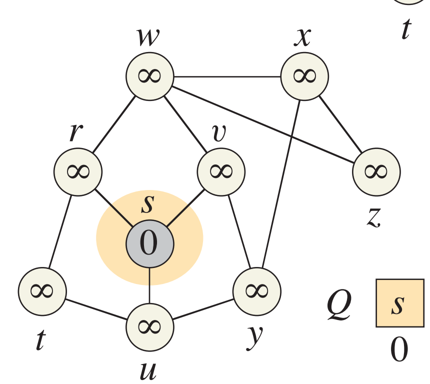
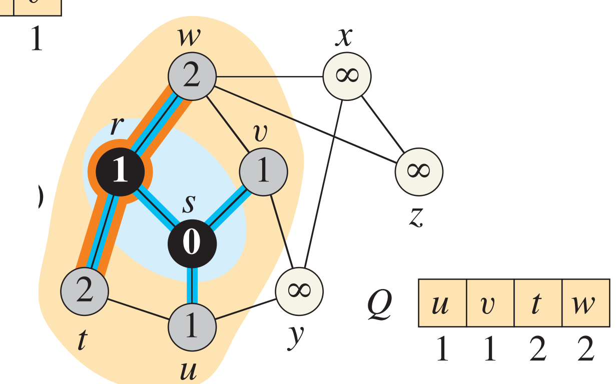
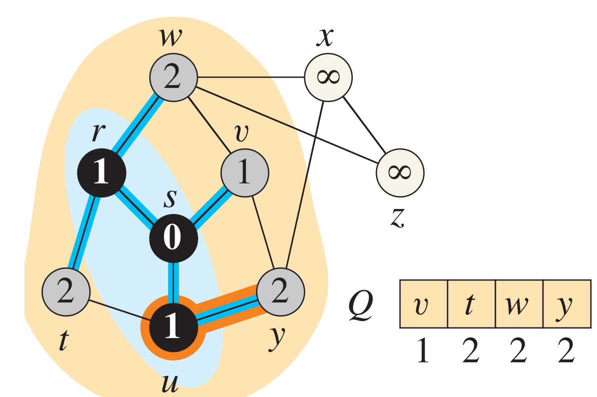
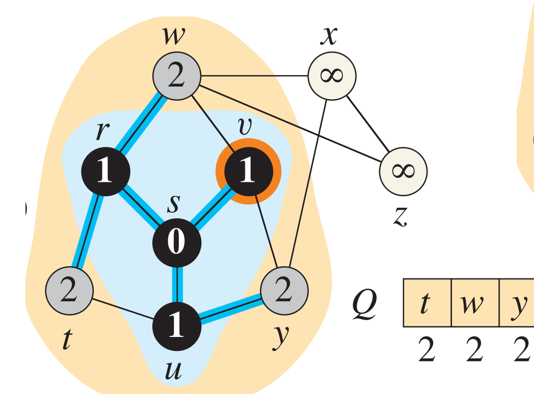




And the node is colored, grey.

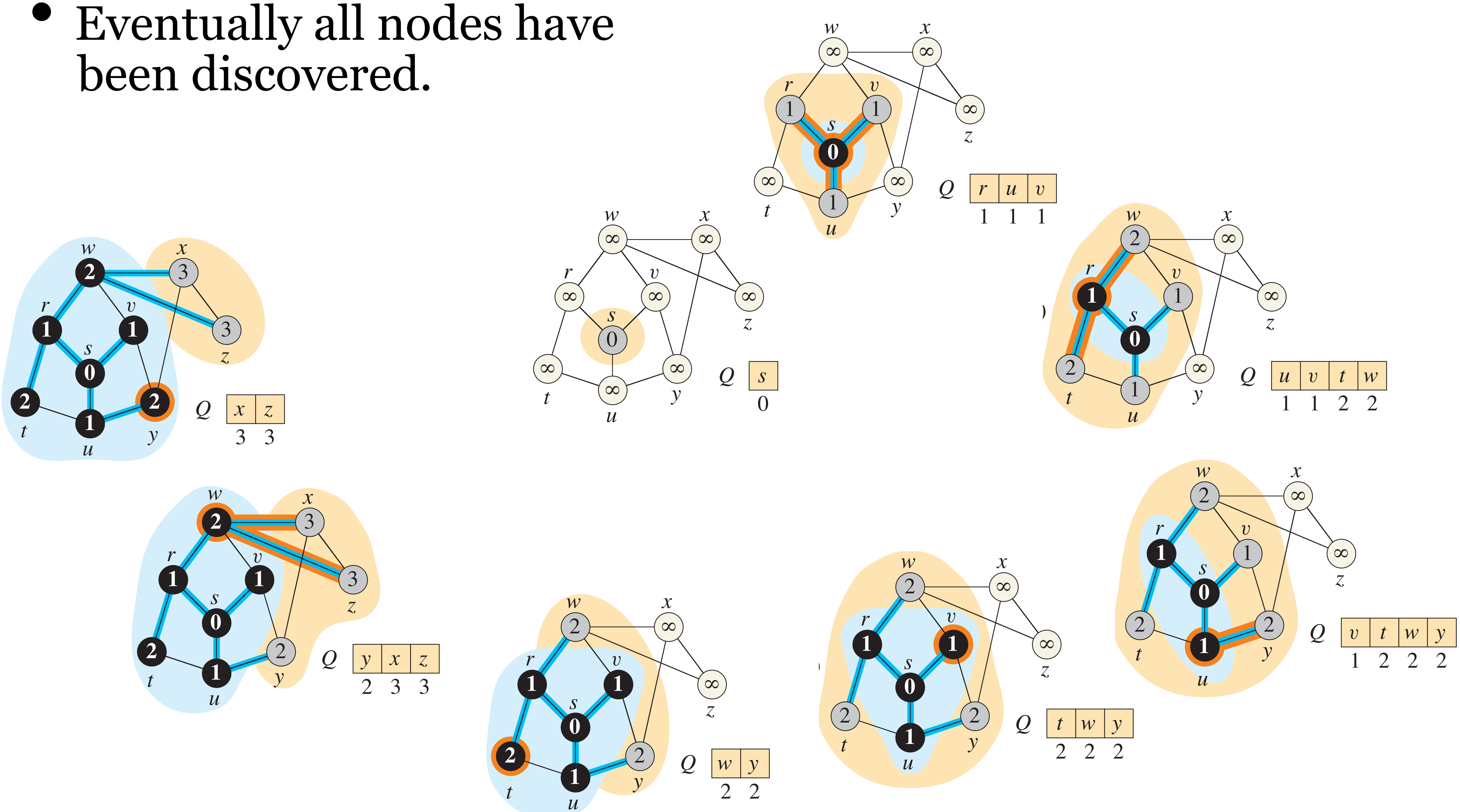


Once the node is dequeued, and it's neighbors discovered, the node is colored black.



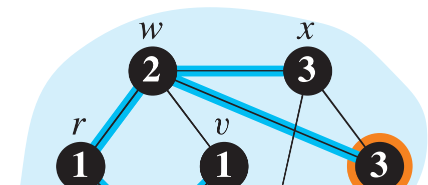
Breadth First Search

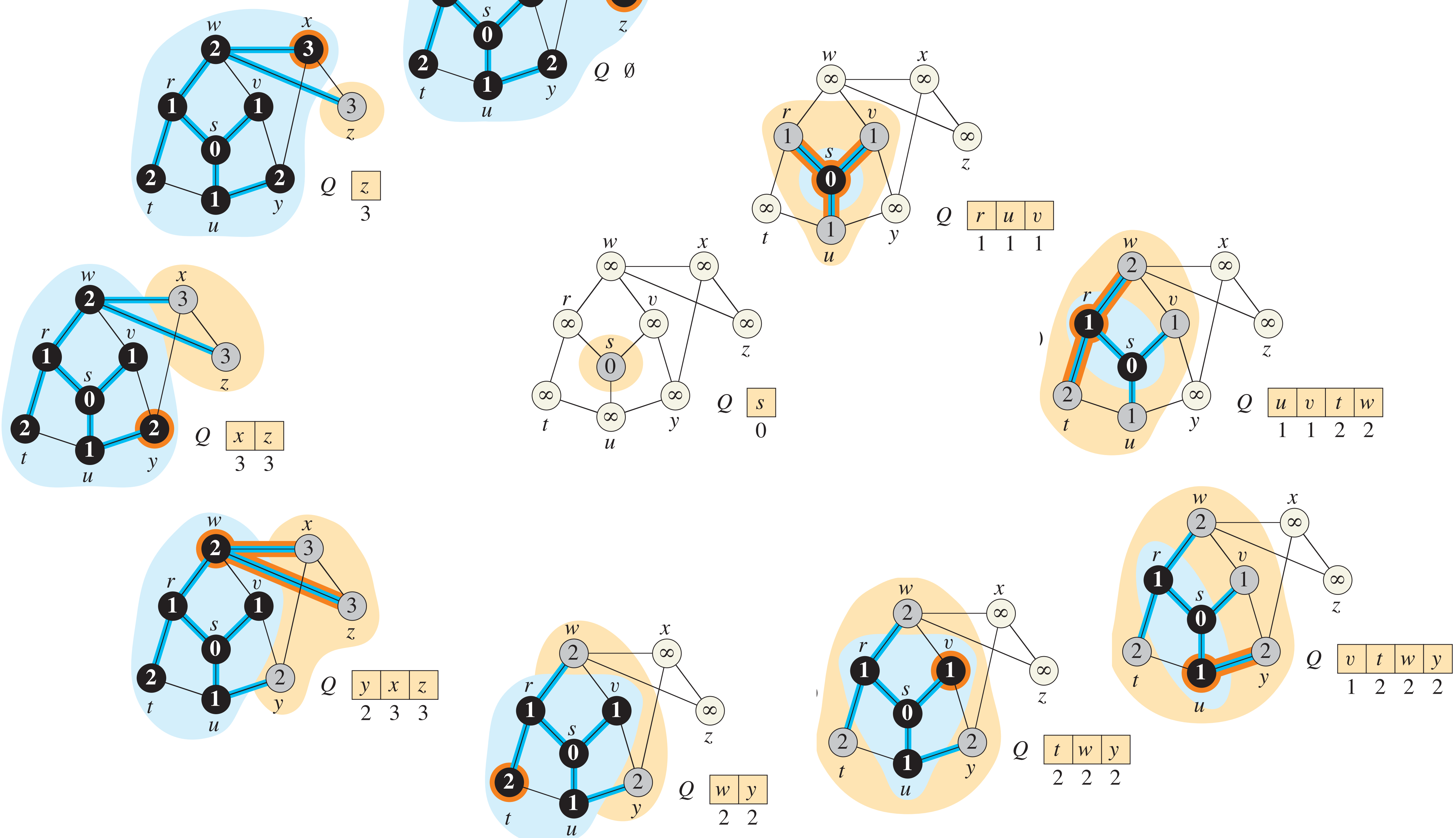
- Given a node, *s*, in a graph, we would like to know the distance to all other nodes of the graph.
- Eventually all nodes have been discovered.



Breadth First Search

- The algorithm finishes when the queue is fully empty.





Breadth First Search

```
BFS(G, s)
    for each vertex u ∈ G.V − {s}
        u.color = WHITE
        u.d = ∞
        u.π = NIL
    s.color = GRAY
    s.d = 0
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    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each vertex v in G.Adj[u]
            if v.color == WHITE
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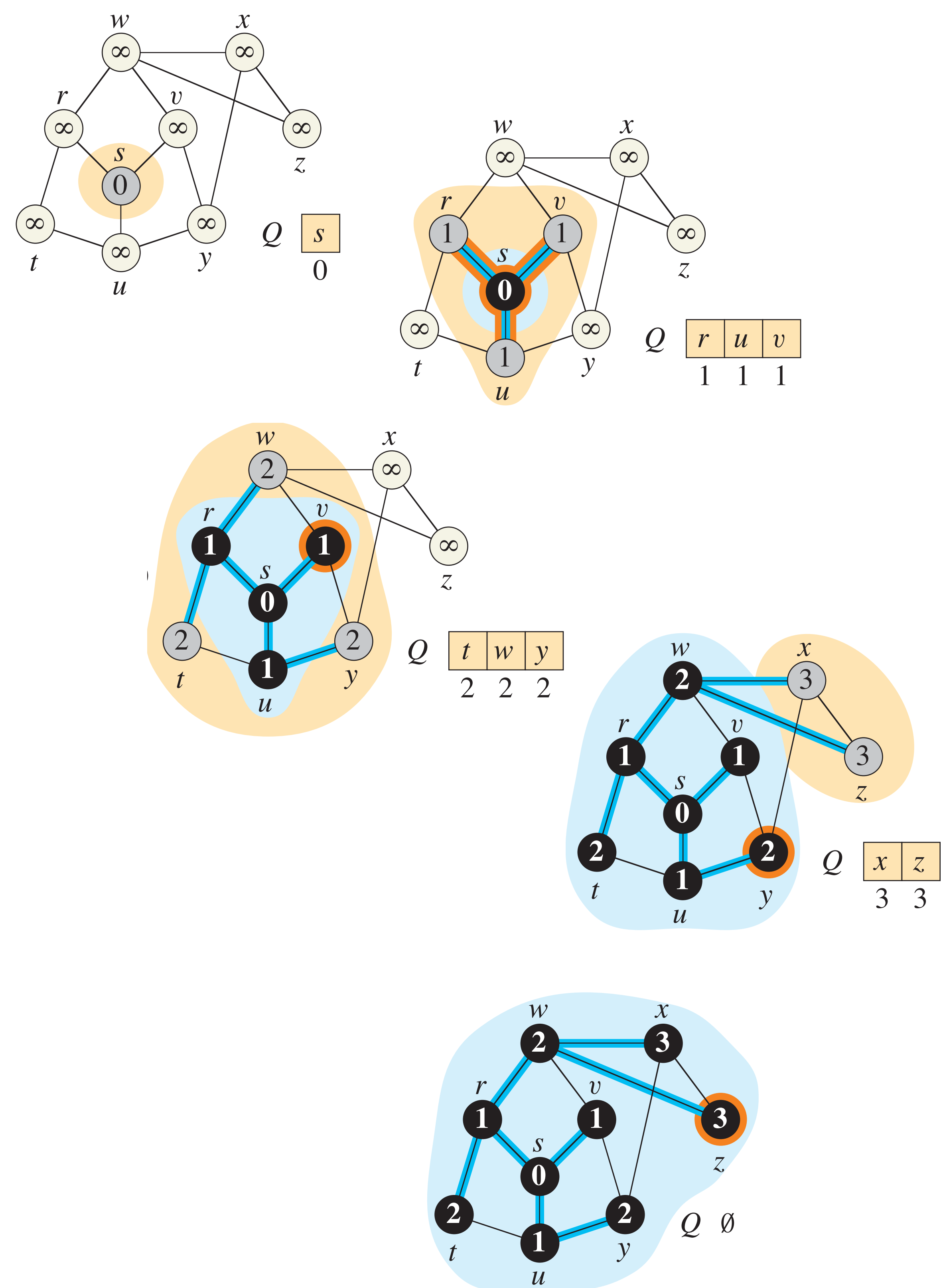

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while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]     // search the neighbors of u
        if v.color == WHITE           // is v being discovered now?
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)             // v is now on the frontier
    u.color = BLACK                   // u is now behind the frontier
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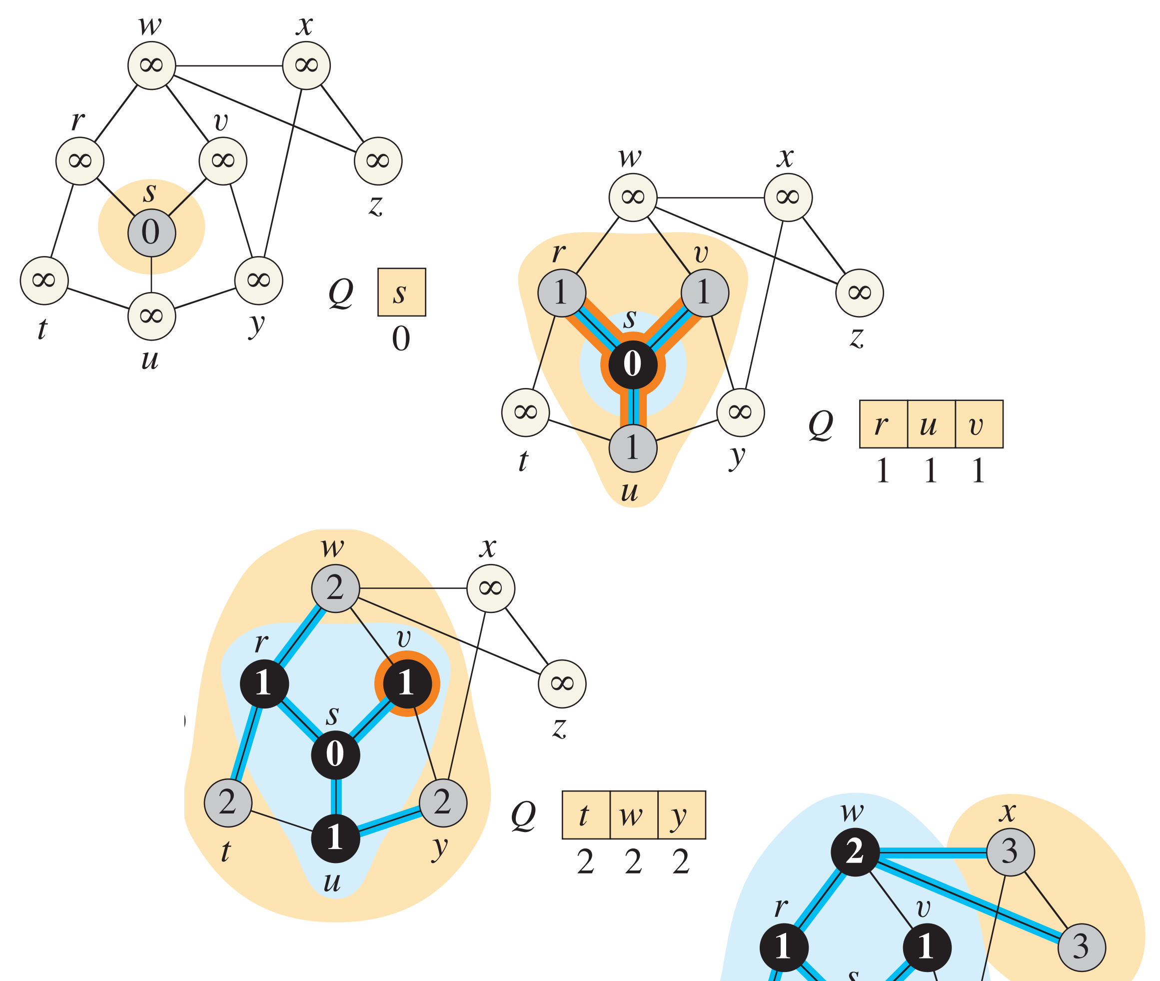

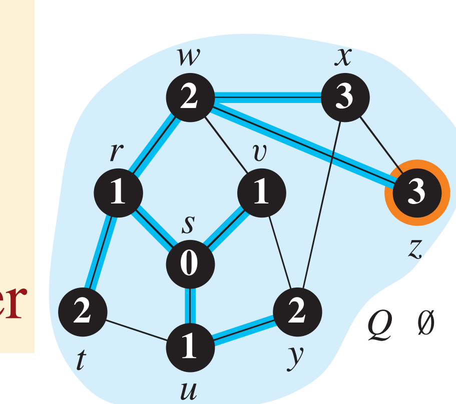

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- What is the running time of BFS?

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- What is the running time of BFS?
    - A vertex is only ENQUEUED if it it is white.

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    - Vertices are only colored white at initialization.

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- What is the running time of BFS?
  - A vertex is only ENQUEUED if it it is white.
  - Vertices are only colored white at initialization.
  - Thus vertices are only ENQUEUED - and DEQUEUED- once.

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- What is the running time of BFS?
    - Vertices are only ENQUEUED - and DEQUEUED- once.
    - Thus queueing operations take *O(V)* time, as does initialization.

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  - The adjacency list of a vertex is scanned only after DEQUEUEING.

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- What is the running time of BFS?
  - Vertices are only ENQUEUED - and DEQUEUED- once.
  - Thus queueing operations take *O(V)* time, as does initialization.
  - The adjacency list of a vertex is scanned only after DEQUEUEING.
  - Thus the adjacency lists are scanned once per vertex.

$$\sum_{V} d(v) = 2|E|$$

Exercise

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- What is the running time of BFS?
  - Vertices are only ENQUEUED - and DEQUEUED- once.
  - Thus queueing operations take *O(V)* time, as does initialization.
  - The adjacency list of a vertex is scanned only after DEQUEUEING. So that all scanning takes *O(E)* time.
  - Total running time for BFS is thus *O(V+E)*.

Homework (due with Assignment 3)

- Give an example of a directed graph, $G=(V, E)$, a source vertex s, and a subset of edges which form a tree, $E_\pi \subseteq E$, with $V(G|_{E_\pi}) = V$. Such that for each vertex, v, the unique path in the graph (V, E_π) from s to v, is a shortest path in G, yet the set of edges E_π cannot be produced by running BFS on G, no matter how the vertices are ordered in the adjacency lists.

- The diameter of a tree is defined as the largest of all distances in the tree. Give an efficient algorithm to compute the diameter of a tree, and analyze the running time of your algorithm.

Correctness of BFS

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- To prove the correctness of BFS, for any graph G and source vertex x, we need to show

  $$v.d = d(s,v)$$

  for any vertex v.

Correctness of BFS

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    u.d = ∞
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s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- To prove the correctness of BFS, for any graph G and source vertex x, we need to show
$$v.d = d(s,v)$$
for any vertex v.

- We begin by showing, a loop invariant: at any point of the algorithm, it holds that:
$$v.d \geq d(s,v)$$
for any vertex v.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

$$v.d \geq d(s,v)$$

for any vertex *v*.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
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            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex v.

- **Initialization:**

  For any vertex $v \neq s$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
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            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

$$v.d \geq d(s,v)$$

for any vertex v.

- **Initialization:**

For any vertex $v \neq s$

$$v.d = \infty > d(s,v)$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
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    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex *v*.

- **Initialization:**

  For any vertex $v \neq s$:

  $$v.d = \infty > d(s, v)$$

  For vertex s :

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
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    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

$$v.d \geq d(s,v)$$

for any vertex v.

- **Initialization:**

For any vertex $v \neq s$:

$$v.d = \infty > d(s,v)$$

For vertex s :

$$s.d = 0 = d(s,s)$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
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    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex v.

- **Initialization:**

  For any vertex $v \neq s$:

  $$v.d = \infty > d(s,v)$$

  For vertex s :

  $$s.d = 0 = d(s,s)$$

  Proving

  $$v.d \geq d(s,v)$$

  for any vertex v.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
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    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

$$v.d \geq d(s,v)$$

for any vertex v.

- **Maintenance:**

Assume the hypothesis still holds for all vertices.

Notice some vertices have now been colored gray, and their $.d$ attribute has been modified.

Notice a **while** iteration will only modify the $.d$ attribute of white vertices.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

$$v.d \geq d(s,v)$$

for any vertex v.

- **Maintenance:**

Consider a gray vertex u and let v be one of it's white neighbors.

By hypothesis,

$$u.d \geq d(s,u)$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex v.

- **Maintenance:**

  Consider a gray vertex u and let v be one of it's white neighbors.

  By hypothesis,

  $$u.d \geq d(s,u)$$

  By definition:

  $$\boxed{v.d = u.d + 1} \geq d(s,u) + 1 \geq d(s,v)$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

$$v.d \geq d(s,v)$$

for any vertex v.

- **Maintenance:**

Consider a gray vertex u and let v be one of it's white neighbors.

By hypothesis,

$$u.d \geq d(s,u)$$

Applying previous corollary:

$$\begin{aligned} v.d &= u.d + 1 \\ &\geq d(s,u) + 1 \\ &\geq d(s,v) \end{aligned}$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
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        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex v.

- **Maintenance:**

  Consider a gray vertex u and let v be one of it's white neighbors.

  By hypothesis,

  $$u.d \geq d(s,u)$$

  Applying previous corollary:

  $$\begin{aligned} v.d &= u.d + 1 \\ &\geq d(s,u) + 1 \\ &\geq d(s,v) \end{aligned}$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex v.

- **Maintenance:**

  Proving after a **while** iteration, it still holds that

  $$v.d \geq d(s,v)$$

  for any vertex v.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex *v*.

- **Termination:**

  After finishing the **while** loop, it holds that

  $$v.d \geq d(s,v)$$

  for any vertex *v*.

  Proving the loop invariant.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- To prove the correctness of BFS, for any graph G and source vertex x, we need to show

$$v.d = d(s,v)$$

  for any vertex v.

- We have shown, at any point of the algorithm, it holds that:

$$v.d \geq d(s,v)$$

  for any vertex v.

- Now let us prove a more precise property regarding the $.d$ attribute.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- We claim another loop invariant: at any point of the algorithm, if the queue, Q, contains vertices $(v_1, \ldots v_r)$ ordered such that v_1 is the head, it holds that:

$$v_r.d \le v_1.d + 1$$

$$v_i.d \le v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

  $$v_r.d \leq v_1.d + 1$$

  $$v_i.d \leq v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Initialization:**

  Queue contains only s, so statement holds.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

  $$v_r.d \le v_1.d + 1$$

  $$v_i.d \le v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

  During a **while** iteration, one vertex will be DEQUEUED and others will be ENQUEUED.

  Let us show DEQUEUEING maintains the loop invariant.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

  $$v_r.d \leq v_1.d + 1$$

  $$v_i.d \leq v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

  After DEQUEUEING, v_2 becomes the head.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
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        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

$$v_r.d \le v_1.d + 1$$

$$v_i.d \le v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

After DEQUEUEING, v_2 becomes the head.

By hypothesis,

$$v_2.d \quad \ge v_1.d$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1,\ldots v_r)$, with v_1 the head, it holds that:

$$v_r.d \le v_1.d + 1$$

$$v_i.d \le v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

After DEQUEUEING, v_2 becomes the head.

By hypothesis,

$$v_2.d + 1 \ge v_1.d + 1$$
$$\ge v_r.d$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \dots v_r)$, with v_1 the head, it holds that:

$$v_r . d \leq v_1 . d + 1$$

$$v_i . d \leq v_{i+1} . d \ \text{ for } \ i = 1, \dots, r-1$$

- **Maintenance:**

After DEQUEUEING, v_2 becomes the head.

By hypothesis,

$$v_2 . d + 1 \geq v_1 . d + 1$$

$$\geq v_r . d$$

Proving the loop invariant holds.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

  $$v_r . d \leq v_1 . d + 1$$

  $$v_i . d \leq v_{i+1} . d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

  If the queue is empty before ENQUEUING, the loop invariant automatically holds after the ENQUEUE operation, since the queue has a single vertex.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
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while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

$$v_r . d \leq v_1 . d + 1$$

$$v_i . d \leq v_{i+1} . d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

If the queue is empty before ENQUEUING, the loop invariant automatically holds after the ENQUEUE operation, since the queue has a single vertex.

Assume the queue were not initially empty.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1,\ldots v_r)$, with v_1 the head, it holds that:

$$v_r.d \leq v_1.d + 1$$

$$v_i.d \leq v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

After ENQUEUING, the tail is now vertex v_{r+1}. We know it is a neighbor of the most recently dequeued vertex, u, which up to the last iteration satisfied:

$$v_r.d \leq u.d + 1$$

and $u.d \leq v_2.d$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1,\ldots v_r)$, with v_1 the head, it holds that:

$$v_r.d \leq v_1.d + 1$$

$$v_i.d \leq v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

After ENQUEUING, the tail is now vertex v_{r+1}. We know it is a neighbor of the most recently dequeued vertex, u, which this iteration satisfies:

$$v_r.d \leq u.d + 1$$

and $u.d \leq \cancel{v_2.d}\ v_1.d$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1,\ldots v_r)$, with v_1 the head, it holds that:

$$v_r.d \le v_1.d + 1$$

$$v_i.d \le v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

After ENQUEUEING, the tail is now vertex v_{r+1}. We know it is a neighbor of the most recently dequeued vertex, u, which this iteration satisfies:

$$v_r.d \le u.d + 1$$

$$u.d \le v_1.d$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1,\ldots v_r)$, with v_1 the head, it holds that:

$$v_r.d \le v_1.d + 1$$

$$v_i.d \le v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Maintenance:**

After ENQUEUEING, the tail is now vertex v_{r+1}. We know it is a neighbor of the most recently dequeued vertex, u, which this iteration satisfies:

$$v_r.d \le u.d + 1 = v_{r+1}.d$$

$$v_{r+1}.d = u.d + 1 \le v_1.d + 1$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1,\dots v_r)$, with v_1 the head, it holds that:

  $$v_r.d \le v_1.d + 1$$

  $$v_i.d \le v_{i+1}.d \text{ for } i = 1,\dots,r-1$$

- **Maintenance:**

  After ENQUEUING, the tail is now vertex v_{r+1}. We know it is a neighbor of the most recently dequeued vertex, u, which this iteration satisfies:

  $$v_r.d \le v_{r+1}.d$$

  $$v_{r+1}.d \le v_1.d + 1$$

  Proving the loop invariant.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

$$v_r . d \leq v_1 . d + 1$$

$$v_i . d \leq v_{i+1} . d \text{ for } i = 1, \ldots, r-1$$

- **Termination:**

  After finishing the **while** loop, the loop invariant holds as the queue is empty.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- **Loop invariant**: if the queue $Q=(v_1, \ldots v_r)$, with v_1 the head, it holds that:

$$v_r.d \leq v_1.d + 1$$

$$v_i.d \leq v_{i+1}.d \text{ for } i = 1,\ldots,r-1$$

- **Corollary:**

If v_i is enqueued before v_j, then

$$v_i.d \leq v_j.d$$

when v_j is enqueued.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- To prove the correctness of BFS, for any graph G and source vertex x, we need to show

  $$v.d = d(s,v)$$

  for any vertex v.

- We have shown, at any point of the algorithm, it holds that:

  $$v.d \geq d(s,v)$$

  for any vertex v.

- And that if v_i is enqueued before v_j, then

  $$v_i.d \leq v_j.d$$

  when v_j is enqueued.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- We have shown, at any point of the algorithm, it holds that:

$$v.d \geq d(s,v)$$

for any vertex v.

- And that if v_i is enqueued before v_j, then

$$v_i\,.\,d \leq v_j\,.\,d$$

when v_j is enqueued.

- Let us now prove the correctness of the algorithm, by contradiction. Assume there is a vertex v such that:

$$v\,.\,d > d(s, v)$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v.d > d(s, v)$$

That has smallest $d(s,v)$ value.

Correctness of BFS

```
BFS(G, s)
    for each vertex u ∈ G.V − {s}
        u.color = WHITE
        u.d = ∞
        u.π = NIL
    s.color = GRAY
    s.d = 0
    s.π = NIL
    Q = ∅
    ENQUEUE(Q, s)
    while Q ≠ ∅
        u = DEQUEUE(Q)
        for each vertex v in G.Adj[u]
            if v.color == WHITE
                v.color = GRAY
                v.d = u.d + 1
                v.π = u
                ENQUEUE(Q, v)
        u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v.d > d(s, v)$$

That has smallest $d(s,v)$ value.

  - Notice $v \neq s$, or both values would be zero.
  - And there must exist some path connecting v and s, or $d(s, v) = \infty$, larger than any value.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v.d > d(s, v)$$

That has smallest $d(s,v)$ value.

  - Notice $v \neq s$, or both values would be zero.
  - And there must exist some path connecting v and s, or $d(s, v) = \infty$, larger than any value.
  - Thus there must exist some shortest path, of length at least one. Let u be the vertex preceding v on this path.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v\,.\,d > d(s, v)$$

That has smallest *d(s,v)* value.

- Let u be the vertex preceding v on a shortest path from s to v.
- By construction:

$$d(s, v) = d(s, u) + 1$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v.d > d(s, v)$$

That has smallest $d(s,v)$ value.

- Let u be the vertex preceding v on a shortest path from s to v.
- By construction:

$$d(s, v) = d(s, u) + 1$$

- By hypothesis, $d(s, u) = u.d$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v.d > d(s, v)$$

That has smallest $d(s,v)$ value.

- Let u be the vertex preceding v on a shortest path from s to v.
- By construction:

$$d(s, v) = d(s, u) + 1$$

- By hypothesis, $d(s, u) = u.d$
- Yielding:

$$v.d > d(s, v) = u.d + 1$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v.d > d(s,v)$$

That has smallest $d(s,v)$ value.

- Then v has a neighbor u, such that:

$$v.d > u.d + 1 \qquad (*)$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Specifically, consider the vertex v satisfying:

$$v.d > d(s,v)$$

That has smallest $d(s,v)$ value.

- Then v has a neighbor u, such that:

$$v.d > u.d + 1 \qquad (*)$$

- Consider the time when u has just been DEQUEUED. We will show wether v is white, gray or black, $(*)$ implies a contradiction.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is white.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is white.
- Then the algorithm sets:

$$v.d = u.d + 1$$

Contradicting our hypothesis.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is black.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
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while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is black.
- This means v was ENQUEUED before u.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is black.
- This means v was ENQUEUED before u.
- Earlier we proved:

  If v is ENQUEUED before u, then
  $$v.d \le u.d$$
  when u is ENQUEUED.

- Contradicting our hypothesis.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is gray.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
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ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is gray.
- Then there exists some other vertex w, that was ENQUEUED before u, such that:

$$v.d = w.d + 1$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
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    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is gray.
- Then there exists some other vertex w, that was ENQUEUED before u, such that:

$$v.d = w.d + 1$$

- As we just recalled though:

If w is ENQUEUED before u, then

$$w.d \leq u.d$$

when u is ENQUEUED.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- Assume v has a neighbor u, such that:

$$v.d > u.d + 1$$

- Let it be the time when u has just been DEQUEUED, assume v is gray.
- Then there exists some other vertex w, that was ENQUEUED before u, such that:

$$v.d = w.d + 1$$

- As we just recalled though:

  If w is ENQUEUED before u, then
  $$w.d \leq u.d$$
  when u is ENQUEUED.

- Implying: $v.d \leq u.d + 1$ **!!!**

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- We've showed that if we assume:
  - There exists a vertex *v* satisfying:

    $$v.d > d(s, v)$$

  - And choose the specific vertex with smallest *d(s,v)* value.
  - Then this vertex *v* must have a neighbor *u*, such that:

    $$v.d > u.d + 1$$

  - Which yields a contradiction.

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- We've showed that if we assume:
  - There exists a vertex v satisfying:

$$v.d > d(s, v)$$

  - And choose the specific vertex with smallest $d(s,v)$ value.
  - Then this vertex v must have a neighbor u, such that:

$$v.d > u.d + 1$$

  - Which yields a contradiction.
- Thus proving for all vertices,

$$v.d \leq d(s, v)$$

Correctness of BFS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- In conclusion we have proven both:

$$v.d \leq d(s, v)$$

$$v.d \geq d(s, v)$$

  for any vertex v.

- Implying: $v.d = d(s, v)$ for all vertices v in the graph, proving the correctness of the algorithm.

Exercise

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- After running BFS, how can you form a shortest path from the source s to any vertex v?

Exercise

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each vertex v in G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

- After running BFS, how can you form a shortest path from the source s to any vertex v?

- Construct the path recursively, by taking the shortest path from s to $v.\pi$, and adding the edge $(v.\pi, v)$.