Chapter 2 - Introduction

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1 Abstract

2 Entropy

Entropy $H(S)$, can be thought of as a representation of the average information content of an observation; sometimes referred to as a measure of unpredictability or disorder.

"$H(S)$ is the quantity of surprise you should feel upon reading the result of a measurement" (Fraser and Swinney, 1986) [**?**]. Thus the "entropy of S can be seen as the uncertainty of S" [**?**].

2.1 Shannon Entropy

The Shannon entropy of a random vector X with density function f is given by;

$$\begin{aligned} H &= -\mathbb{E}\{log(f(x))\} \\ &= -\int_{x:f(x)>0} f(x)log(f(x))dx \\ &= -\sum_{x\in\mathbb{R}^d} f(x)log(f(x)) \end{aligned} \tag{1}$$

2.2 Rényi and Tsallis Entropy

These entropies are for the order $q \neq 1$ and the construction of them relies upon the generalisation of the Shannon entropy 1. For a random vector $X \in \mathbb{R}^d$ with density function f, we define;

Rényi entropy

$$\begin{aligned} H_q^* &= \frac{1}{1-q} log\left(\int_{\mathbb{R}^d} f^q(x)dx\right) \qquad (q \neq 1) \\ &= \frac{1}{1-q} log\left(\sum_{x\in\mathbb{R}^d} f^q(x)\right) \end{aligned} \tag{2}$$

Tsallis entropy

$$
\begin{aligned}
H_q &= \frac{1}{q-1}\left(1 - \int_{\mathbb{R}^d} f^q(x)dx\right) \qquad (q \neq 1) \\
&= \frac{1}{q-1}\left(1 - \sum_{x\in\mathbb{R}^d} f^q(x)\right)
\end{aligned}
\tag{3}
$$

When the order of the entropy $q \to 1$, both the Rényi, (2), and Tsallis, (3), entropies tend to the Shannon entropy, (1), this is a special case for when $q = 1$. There are also other special cases, sometimes the Rényi entropy is considered for the special case, $q = 2$, and known as the quadratic Rényi entropy;

$$
\begin{aligned}
H_2^* &= -log\left(\int_{\mathbb{R}^d} f^2(x)dx\right) \\
&= -log\left(\sum_{x\in\mathbb{R}^d} f^2(x)\right)
\end{aligned}
\tag{4}
$$

As $q \to \infty$, the limit of the Rényi entropy exists, and is defined as the minimum entropy, since it's the smallest possible value of H_q^*;

$$H_\infty^* = -\log \sup_{x\in\mathbb{R}^d} f(x)$$

Thus, it follows that; $H_\infty^* \leq H_2^* \leq 2H_\infty^*$.

There is also an approximate relationship between the Shannon entropy and the quadratic Rényi entropy;

$$H_2^* \leq H \leq \log(d) + \frac{1}{d} - e^{-H_2^*}$$

where H_2^* is the quadratic Rényi entropy (4), H is the Shannon entropy (1) and d is the dimension of the distribution.