Question: Can we construct an estimator with smaller Variance?

**Definition**: An unbiased estimator $T^*$ is called Uniform Minimum Variance Unbiased (UMVU) if for any other unbiased estimator $T$ we have

$$Var_\vartheta(T^*) \leq Var_\vartheta(T) \quad \forall \vartheta \in \Theta$$

**Definition**: A sufficient Statistic $T: \mathcal{X} \longrightarrow E$ is complete for a family of probability distributions $\{\mathbb{P}_\vartheta : \vartheta \in \Theta\}$ if the following condition is satisfied

$$\mathbb{E}_\vartheta\, h(S) = 0 \;\forall \vartheta \implies h(S) = 0 \;\; \mathbb{P}_\vartheta\text{-a.s.} \;\forall \vartheta$$

where $h$ is a function of $S$ not depending on $\vartheta$.

Intuition: This property is useful in ensuring uniqueness. Completeness is stronger condition than sufficiency while a sufficient statistic summarize all the information in the data about the parameter, a complete statistic ensures that no non trivial function of the statistic can have a zero expectation for all parameters

## Lemma (Lehmann- Scheffé)

Let $T$ be an unbiased estimator of $g(\theta)$ with $\mathbb{E}_\theta[T^2] < \infty$ $\forall \theta$ Moreover, le $S$ be a sufficient and complete statistic. Then

$$T^* = \mathbb{E}[T \mid S] \text{ is UMVU.}$$

**Proof:** By Blackwell-Rao, $T^*$ is unbiased and

$$\mathrm{Var}_\vartheta(T^*) = MSE(T^*) \leq MSE(T) = \mathrm{Var}_\vartheta(T)$$

Let $T'(S)$ be another unbiased estimator of $g(\vartheta)$. Then

$$\mathbb{E}_\vartheta\big(T^*(S) - T'(S)\big) = \mathbb{E}_\vartheta\big(T^* - g(\vartheta)\big)$$
$$= 0 \quad \forall \vartheta$$

(completeness)
$$\Rightarrow \quad T^* = T' \quad \mathbb{P}_\vartheta\text{- a.s.}$$

Example: Let $X_1, \dots, X_n \sim \mathrm{Bernoulli}(p)$

Goal: Find UMVU estimator for $p$. $(p = g(\vartheta))$

First, we need to identify a statistic which complete and sufficient in order to apply Lehman-Scheffé Thm.

The joint distribution of our data:

$$f(x;p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{1-\sum x_i} (1-p)^{n-\sum x_i}$$

we proved that $S = \sum_{i=1}^{n} x_i$ is a sufficient Statistic by factorization thm. we will prove now that $S$ is also Complete.

$$S = \sum_{i=1}^{n} x_i \sim Bin(n,p)$$

In order to prove that $S$ is Complete i.e, show that if $\mathbb{E}_\vartheta[g(S)] = 0 \ \forall \vartheta$ then $g(S) = 0$.

$$\mathbb{E}_\vartheta[g(\vartheta)] = \sum_{k=0}^{n} g(k)\binom{n}{k}\vartheta^k(1-\vartheta)^{n-k} = 0$$

$$\forall \vartheta \in (0,1)$$

$\Rightarrow$

$$\Rightarrow \sum_{k=0}^{n} g(k)\binom{n}{k}\left(\frac{\vartheta}{1-\vartheta}\right)^{k}\cdot(1-\vartheta)^{n} = 0$$

$\forall\, \vartheta$

$$\Rightarrow (1-\vartheta)^{n}\cdot \sum_{k=0}^{n} g(k)\binom{n}{k}\Big(\underbrace{\frac{\vartheta}{1-\vartheta}}_{=\,\xi}\Big)^{k} = 0$$

$$(1-\vartheta)^{n}\cdot \sum_{k=0}^{n} \underbrace{\overbrace{g(k)\binom{n}{k}}^{a(k)}\cdot \xi^{k}}_{\text{polynomial form}} = 0$$

By Polynomial identity thm

if a polynomial

$\sum_{k=0}^{n} a(k)\cdot \xi^{k} = 0$ for infinitely many $\xi$

then all coefficients $a(k)=0$, which

means that $g(k)=0$ for all $k$.

So $S$ is complete.

We start by a naive estimator

$T = X_1$ for $p$.

$$\mathbb{E}[T] = \mathbb{E}[X_1] = \vartheta \quad \text{(unbiased)}$$

[We use only the first observation
So estimator is not efficient, high
variance]

Since $S$ is sufficient we condit
$T$ with $S$ so we get an est
of lower Variance by B-R
thm.

$X_1 = 1$ or $0$

$$\mathbb{E}[T \mid S] = \mathbb{E}[X_1 \mid S = s]$$

$$= \mathbb{P}[X_1 = 1 \mid S = s]$$

$$= \frac{\mathbb{P}[X_1 = 1, S = s]}{\mathbb{P}[S = s]} \quad (*)$$

$$\mathbb{P}[S=s] = \binom{n}{s}\theta^s(1-\theta)^{n-s}$$

$$\mathbb{P}[X_1=1,\ S=s] = \mathbb{P}[X_1=1]\cdot\mathbb{P}\left[\sum_{j=2}^{n} X_j = s-1\right]$$

$\mathbb{P}[X_1=1]=\theta$ Bernoulli

$\mathbb{P}\left[\sum_{j=2}^{n} x_j = s-1\right]$ Binomial$(n-1,\theta)$

$$\mathbb{P}\left[\sum_{j=2}^{n} x_j = s-1\right] = \binom{n-1}{s-1}\theta^{s-1}\cdot(1-\theta)^{n-s}$$

$$(*)\ \mathbb{P}[X_1=1 \mid S=s] = \frac{\cancel{\theta}\cdot\binom{n-1}{s-1}\cancel{\theta^{s-1}}\cancel{(1-\theta)^{n-s}}}{\binom{n}{s}\cancel{\theta^{s}}\cancel{(1-\theta)^{n-s}}}$$

$$= \binom{n-1}{s-1}\Big/\binom{n}{s} = \frac{s}{n} \Rightarrow \mathbb{E}[X_1 \mid S] = \frac{S}{n}$$

"$T^*$" estimator.

$Var(T^*) \leq Var(T_1)$ By B-R thm.

By L-S thm $T^*$ is UMVU estimator

## Completeness of exponential family:

Definition: A $k$-dimensional exponential family is a dominated statistical model $(\mathcal{X}, \mathcal{F}, \{\mathbb{P}_\theta\}_{\theta \in \Theta})$ with densities of the form

$$p_\theta(x) = \exp\left(\sum_{j=1}^{k} c_j(\theta) T_j(x) - d(\theta)\right) \cdot h(x)$$

where $(T_1(x), \ldots, T_k(x))$ is a statistic in $\mathbb{R}^k$.

**Lemma:** Let $(\mathcal{X}, \mathcal{F}, \{\mathbb{P}_\theta\}_{\theta \in \Theta})$ be a $k$-dim exponential family with densities

$$p_\theta(x) = \exp\left(\sum c_j(\theta) T_j(x) - d(\theta)\right) \cdot h(x).$$

if the set (natural parameter space)

$$\mathcal{C} = \{c_1(\theta), \ldots, c_k(\theta) : \theta \in \Theta\} \subseteq \mathbb{R}^k$$

contains an open set in $\mathbb{R}^k$, then $S = (T_1, \ldots, T_k)$ is complete.

**Example:** Bernoulli

$X_1, \dots, X_n \sim Ber(\vartheta)$

$$\mathbb{P}[X_1 = x_1, \dots, X_n = x_n] = \prod_{i=1}^{n} \vartheta^{x_i} (1-\vartheta)^{1-x_i}$$

$$= \vartheta^{\sum x_i} (1-\vartheta)^{1-\sum x_i}$$

$S = \sum x_i \qquad S \sim Bin(n, \vartheta)$

$$\mathbb{P}[S=s] = \binom{n}{s} \vartheta^{s} (1-\vartheta)^{n-s}$$

$$= \exp\Big(\underbrace{\log\left(\frac{\vartheta}{1-\vartheta}\right)}_{c(\vartheta)} \cdot \underbrace{s}_{T(x)} + \underbrace{n \ln(1-\vartheta)}_{-d(\vartheta)}\Big) \underbrace{\binom{n}{s}}_{h(x)}$$

we need to identify that the natural parameter space contains an open set

$$c(\vartheta) = \frac{\vartheta}{1-\vartheta} \qquad \vartheta \in (0,1)$$

As $\vartheta$ varies, $c(\vartheta)$ spans $\mathbb{R}$

$$c(\vartheta) \to -\infty, \quad \vartheta \to 0$$

$$c(\vartheta) \to +\infty \quad \vartheta \to 1$$

Thus the natural parameter space $\mathcal{C}$ is $\mathbb{R}$, which is an open set.