Homework 1

1. Suppose that $X_1, \dots, X_n$ are i.i.d. copies of a random variable X with probability density function for $\theta > 0$
$$p_\theta(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$
   (a) Compute the MLE for θ.
   (b) Compute the method of moments estimator for θ with the same model as in the previous subexercise.

2. Suppose that $X_1, \dots, X_n$ are i.i.d. copies of a random variable X with probability density function with $\theta > 0$ given by
$$p_\theta(x) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Compute the MLE for θ.
   (b) Compute the method of moments estimator for θ.

3. Consider the Gaussian mixture density
$$p_\vartheta(x) = \frac{1}{2\sigma}\phi\left(\frac{x-\mu}{\sigma^2}\right) + \frac{1}{2}\phi\left(x - \mu\right)$$
where $\phi(\,\cdot\,)$ is the density function of the standard normal distribution. Here, the unknown parameters are given by $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_{\geq 0} = \Theta$.
   (a) Let $Z \sim \mathcal{N}(0,1)$, $Y_p \sim Ber(p)$. Write a random variable Z' with density p_θ as a function of independent copies of Z, Y_p for a value of $p > 0$ and $\theta = (\mu, \sigma^2)$.
   (b) For an observation $\mathbf{X} = \{x_1, \dots, x_n\}$ compute the log-likelihood function $L_{\mathbf{X}}(\theta)$.
   (c) Prove that $L_{\mathbf{X}}(\theta)$ is unbounded on the set $\{(x_1, \sigma^2) \;:\; \sigma^2 > 0\}$.

4. Let $X_1, X_2, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ be i.i.d. random variables, where σ^2 is **known** and $\mu \in \mathbb{R}$ is **unknown**. Estimate μ using the **Blackwell-Rao** theorem.

   Hint: Start using the estimator $U = X_1$.

5. Suppose that X, Y and Z are random variables whose joint distribution is continuous with density f_{XYZ}.
   (a) Write down appropriate definitions of
      (i) $f_{XY|Z}$, density of the joint distribution of X and Y given Z, and
      (ii) $f_{X|YZ}$, density of the distribution of X given both Y and Z.
   (b) Assuming the expectations exist, and defining $\mathbb{E}[X|Y,Z] = h(Y,Z)$ where
$$h(y,z) = \int_{-\infty}^{\infty} x f_{X|YZ}(x|y,z)\, dx,$$
   prove that
$$\mathbb{E}[\mathbb{E}[X|Y,Z]|Z] = \mathbb{E}[X|Z],$$