Midterm-like Exercises

The text of the assignment may continue on the back. Each exercise's solution must be cleanly and legibly written on a separate sheet of paper (a different sheet for each exercise, to allow for the separation of sheets for grading), explaining the reasoning in detail with mathematically valid arguments, and checking the assumptions of the applied theorems. Points may be deducted for non-compliance with these criteria. Sentences that provide a heuristic explanation of the intended strategy are welcome, but they are not sufficient; we expect demonstrations. The use of non-programmable calculators, tables and 4 A4 pages of formulas and notes is allowed, but not electronic devices. Numerical results must be given with 3 significant figures.

1. You are a statistician working RustyTM, a company manufacturing ultra low-cost washers for combustion engines. It is important to ensure to the customers that these parts as durable as advertised. The production process is rather crude and therefore many parts are rejected before getting to the customers. The lifetime (in work hours) of a randomly chosen washer taken from the production line (before the quality control) is well modeled by a sample from the following probability density function.

$$f_\lambda(x) = \begin{cases} \frac{1}{2\sqrt{\lambda x}} e^{-\sqrt{\frac{x}{\lambda}}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

   Basic integral calculus shows that a random variable X with the above density has the following properties:

$$\mathbb{E}[X^{1/2}] = \sqrt{\lambda}; \quad \mathbb{E}[X] = 2\lambda; \quad \mathbb{E}[X^{3/2}] = 6\sqrt{\lambda}; \quad \mathbb{E}[X^2] = 24\lambda^2.$$

   As part of the normal factory protocol, a random selection of $n = 200$ washers was taken out of the production line and destructively tested to measure their lifetime.

   (a) Derive the expression of the Maximum Likelihood (ML) estimator of λ, denoted by $\hat{\lambda}_{ML}$. What is the value of the maximum likelihood estimate of λ for the collected data?

   (b) Show that the Method-of-Moments (MM) estimator of λ based on the first moment is given by

$$\hat{\lambda}_{MM} = \frac{1}{2n} \sum_{i=1}^{n} X_i.$$

   (c) Compute the bias and variance of the method of moments estimator of question above.

   (d) Compute the bias of the maximum likelihood estimator. Is the estimator unbiased?

   (e) One can also compute the exact expression for variance of the maximum likelihood estimator, but this is rather tedious and results in a complicated expression. However, the conditions ensuring the maximum likelihood estimator is asymptotically normal hold (you do not need to check this). Therefore we know the ML estimator is asymptotically unbiased and its variance is asymptotically characterized by the $I^{-1}(\lambda)$, the Fisher information number. Compute the Fisher information number and give an asymptotic (large n) expression for the variance of the maximum likelihood estimator.

(f) Given your answers above which estimator will you prefer for a large sample size n? Why?

2. Let X be a random sample with $P(X=-1) = 2p(1-p)$ and $P(X=k) = p^k(1-p)^{3-k}, k = 0, 1, 2, 3$, where $p \in (0,1)$.

   (a) Consider an estimator $\hat{p} = f(X)$ of p. Write the conditions for this estimator to be unbiased.

   (b) Solving the above conditions, identify the family of unbiased estimators $f(X)$ of p.

   (c) Is there an estimator in this familty that is UMVU for p?

   (d) Repeat the process above to discuss if there exists an UMVU estim ator for $p(1-p)$

3. Let $(X_1, \ldots, X_n)$ be a random sample drawn from a distribution with density:

$$f(x;\theta) = \theta(1+x)^{-(1+\theta)}\mathbb{I}_{(0,+\infty)}(x), \quad x \in \mathbb{R}, \quad \theta > 0.$$

   (a) In the case where $\theta > 1$, estimate θ using the method of moments.

   (b) If they exist, find the maximum likelihood estimators for θ and $1/\theta$.

   (c) If it exists, find a sufficient and complete statistic and determine its distribution.

   (d) If they exist, find the UMVUE for θ and $1/\theta$.

   (e) Determine the Cramér-Rao lower bound for unbiased estimators of $1/\theta$.

   (f) Compare this quantity with the mean square error of the UMVUE for $1/\theta$.

4. Let $(X_1, \ldots, X_n)$ be a random sample from a Bernoulli distribution $Be(p)$, with $0 < p < 1$.

   (a) Show that $\hat{p}_n = \bar{X}_n$ is a consistent estimator for p and that its variance is $\sigma^2 = p(1-p)$.

   (b) Determine the asymptotic distribution of $\sqrt{n}(\hat{p}_n - p)$, considering the estimator $V_n = \hat{p}_n(1-\hat{p}_n)/(n-1)$ for the variance σ^2 and show that it is a consistent estimator for σ^2.