Homework 2

Solutions: 09.04.2025

1. Let $X_1, \dots, X_n$ be independent and identically distributed random variables whose density is given by

$$p_\theta(x) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \\ 0, & \text{else}, \end{cases}$$

where $\theta > 0$ is an unknown parameter.

   (a) Show that this is an exponential family in canonical form.

   (b) Let $T = T(X_1, \dots, X_n) := -\frac{1}{n}\sum_{i=1}^n \log X_i$. Show that T is a sufficient and complete statistic.

   (c) Prove that

$$\mathbb{E}_\theta[T] = \frac{1}{\theta}, \quad \mathrm{var}_\theta(T) = \frac{1}{n\theta^2}.$$

2. If T is a survival time in $(0, \infty)$ with density f, then one object worth studying is the so-called *hazard function*:

$$\lambda(t) = \lim_{h\to 0} \frac{\mathbb{P}(t \le T \le t+h \mid T > t)}{h} = \frac{f(t)}{1-F(t)} = -\frac{d}{dt}\log(1-F(t)).$$

Consider the one-parameter family of distributions for survival times which is defined via the hazard function

$$\lambda_\theta(t) = \theta\lambda_0(t), \quad \theta \in \mathbb{R}^+,$$

where

$$\lambda_\theta(t) = -\frac{d}{dt}\log(1-F_\theta(t))$$

as above and with $\lambda_0(t)$ the (known) hazard function under the standard treatment.

   (a) Show that this is an exponential family.

   (b) Compute a sufficient statistic for the parameter θ.

3. (Optional) Let $X_1, \dots, X_n$ be i.i.d. Poisson(θ)-distributed. We want to estimate $g(\theta) := \theta\exp(-\theta)$.

   (a) Show that $T(X) := \mathbb{1}(\{X_1 = 1\})$ is an unbiased estimator of $g(\theta)$, where $\mathbb{1}(A)$ denotes the indicator RV on the event A.

   (b) Construct an UMVU estimator of $g(\theta)$.

4. Consider the Beta distribution, a family of continuous probability distributions defined on the interval [0, 1] parameterized by two positive shape parameters, α and β, which appear as exponents of the random variable and control the shape of the distribution. The probability density function is given by:

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)},$$

where $B(\alpha, \beta)$ is the Beta function which serves as a normalization constant to ensure that the total probability is 1.

   (a) Show that the Beta distribution is an exponential family.

   (b) Now assume that $\Theta = \{(\alpha, \beta) \in \mathbb{R}^2_{>0} \ : \ \beta = \alpha + 2\}$. Is the statistic given above still complete? If not, can you find a sufficient and complete statistic for this new model?

5. Let $X_1, X_2, \ldots, X_n, n > 2$, be a random sample from the binomial distribution $b(1, \theta)$.
   (a) Show that $T_n(X) = \sum_{j=1}^{n} X_j$ is a complete sufficient statistic for θ.
   (b) Find the function $\psi(T_n)$ that is the UMVU estimator of θ.
   (c) Let $T_2 = (X_1 + X_2)/2$ and compute $E(T_2)$.
   (d) Determine $E(T_2|T_n = s)$.