Homework 7

Solutions: 28.05.2025

1. In the frequentist paradigm, handling nuisance parameters can a thorny problem. A method that spmetimes works is based on the idea of conditioning. To illustrate this approach, wuppose we measure an event count N that is Poisson distributed with mean $\mu\nu$ where μ is the parameter of interest and ν is a nuisance parameter. Assume that ν is constrained by the auxiliary measurement of a Poisson variate K with mean $\tau\nu$, where τ is a known constant:

$$N \sim Poisson(\mu\nu)$$

$$K \sim Poisson(\tau\nu)$$

In high energy physics one could think of μ as the production cross section for some process of interest and ν as a product of efficiencies, acceptances, and integrated luminosity. One can argue that the sum $MN + K$ provides no information about the ratio μ/τ of the above two Poisson means, or about μ itself. It is therefore interesting to seek inferences that condition on M.

   (a) Find the Conditional Distribution of N given M

   (b) Next, assume that the expectation value of N is the sum of μ and ν instead of their product so we have

$$N \sim Poisson(\mu + \nu)$$

$$K \sim Poisson(\tau\nu)$$

   what is the conditional probability of N given M here?

2. Suppose $\theta \in (a_0, a_p)$ and:

$$\mathbb{P}(a_{j-1} < \theta \leq a_j) = \phi_j \quad (j = 1, \ldots, p)$$

   (a) Define entropy for the discrete case.

   (b) Maximize entropy under the given constraints.

   (c) Express the solution in exponential family form.

3. Consider an exponential decay time t with probability density $f(t|\tau) = e^{-t/\tau}/\tau$.

   (a) Derive Jeffreys' prior for this problem and compute the corresponding posterior.

   (b) Construct equaltailed intervals from this posterior and compute their frequentist coverage. Repeat with a flat prior.

Hint To construct a central interval $[\tau_1, \tau_2]$ with γ credibility (credible interval is the bayesian equivalent to the confidence interval in frequentist statistics) solve the following equations fro τ_1 and τ_2

$$\int_0^{\tau_1} p(\tau|t)d\tau = \frac{1-\gamma}{2}$$

$$\int_{\tau_2}^{\infty} p(\tau|t)d\tau = \frac{1-\gamma}{2}$$

Then to calculate the frequentist coverage of this interval is

$$P\Big(\tau_1(T) \leq \tau \leq \tau_2(T)\Big)$$

4. Let $X_1, \ldots, X_n \sim \mathcal{N}(\mu, 1)$, and suppose the prior on μ is Laplace:

$$\pi(\mu) = \frac{1}{2b} e^{-|\mu|/b}$$

   (a) Write down the log-posterior.

   (b) Show that the MAP estimator corresponds to

$$\arg\min_{\mu} \left[\frac{n}{2}(\mu - \bar{x})^2 + \frac{|\mu|}{b} \right]$$

   (i.e., LASSO).

   (c) What happens as $b \to \infty$?