Homework 5

Solutions: 21.05.2025

1. Let X_i are conditionally independent given a parameter θ. For each of the distributions:

   (a) $X_i|\theta \sim \text{Bern}(\theta)$;

   (b) $X_i|\theta \sim N(\mu, \theta)$ with μ known;

   (c) $X_i|\theta \sim \text{Maxwell}(\theta)$, the Maxwell distribution with parameter θ so that

   $$f(x_i|\theta) = \left(\frac{2}{\pi}\right)^{1/2} \theta^{3/2} x_i^2 \exp\left\{-\frac{\theta x_i^2}{2}\right\}, \quad x_i > 0$$

   and $E(X_i|\theta) = \frac{2\sqrt{2}}{\sqrt{\pi\theta}}$, $\text{Var}(X_i|\theta) = \frac{3\pi-8}{\pi\theta}$; answer the following questions.

   i. Show that $f(x_i|\theta)$ belongs to the 1-parameter exponential family and for $X = (X_1, \ldots, X_n)$ state the sufficient statistic for learning about θ.

   ii. By viewing the likelihood as a function of θ, which generic family of distributions (over θ) is the likelihood a kernel of?

   iii. By first finding the corresponding posterior distribution for θ given $x = (x_1, \ldots, x_n)$, show that this family of distributions is conjugate with respect to the likelihood $f(x|\theta)$.

   Hint: To find the kernel of the posterior distribution, you need to simplify the posterior distribution as possible so as the kernel depends only on the parameter.

   (a) $X|\theta \sim \text{Po}(\theta)$.

   $$\begin{aligned} f(x|\theta) = P(X = x|\theta) &= \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \ldots \\ &\propto \theta^x e^{-x}. \end{aligned}$$

   (b) $Y|\theta, \beta \sim \text{Beta}(\beta\theta, \beta)$.

   $$\begin{aligned} f(y|\theta, \beta) &= \frac{\Gamma(\beta\theta + \beta)}{\Gamma(\beta\theta)\Gamma(\beta)} y^{\beta\theta - 1}(1 - y)^{\beta - 1}, \quad y \in [0, 1] \\ &\propto y^{\beta\theta - 1}(1 - y)^{\beta - 1}. \end{aligned}$$

2. Let $X_1, \ldots, X_n$ be exchangeable so that the X_i are conditionally independent given a parameter θ. Suppose that $X_i|\theta$ is geometrically distributed with probability density function

   $$f(x_i|\theta) = (1 - \theta)^{x_i - 1}\theta, \quad x_i = 1, 2, \ldots$$

   (a) Show that $f(x|\theta)$, where $x = (x_1, \ldots, x_n)$, belongs to the 1-parameter exponential family. Hence, or otherwise, find the conjugate prior distribution and corresponding posterior distribution for θ.

   (b) Show that the posterior mean for θ can be written as a weighted average of the prior mean of θ and the maximum likelihood estimate, $\bar{x}^{-1}$.

   (c) Suppose now that the prior for θ is instead given by the probability density function

   $$f(\theta) = \frac{1}{2B(\alpha + 1, \beta)} \theta^{\alpha}(1 - \theta)^{\beta - 1} + \frac{1}{2B(\alpha, \beta + 1)} \theta^{\alpha - 1}(1 - \theta)^{\beta},$$

   where $B(\alpha, \beta)$ denotes the Beta function evaluated at α and β. Show that the posterior probability density function can be written as

   $$f(\theta|x) = \lambda f_1(\theta) + (1 - \lambda) f_2(\theta)$$

where

$$\lambda = \frac{(\alpha+n)\beta}{(\alpha+n)\beta + (\beta - n + \sum_{i=1}^{n} x_i)\alpha}$$

and $f_1(\theta)$ and $f_2(\theta)$ are probability density functions.

3. Let X_i for $i = 1, \ldots, n$ are conditionally independent given a parameter θ. Suppose that $X_i|\theta$ is distributed as a double-exponential distribution with probability density function

$$f(x_i|\theta) = \frac{1}{2\theta} \exp\left\{-\frac{|x_i|}{\theta}\right\}, \quad -\infty < x_i < \infty$$

for $\theta > 0$.

(a) Find the conjugate prior distribution and corresponding posterior distribution for θ following observation of $x = (x_1, \ldots, x_n)$.

(b) Consider the transformation $\phi = \theta^{-1}$. Find the posterior distribution of $\phi|x$.

Assume that X is a random n-vector from the multivariate normal distribution

$$X \sim N(Z\beta, \sigma^2 I_n),$$

where Z is an $n \times p$ known matrix of rank $r \leq p < n$, β is a p-vector of unknown parameters, I_n is the identity matrix of order n, and $\sigma^2 > 0$ is unknown.

Find the UMVU estimators of

4. (a) $(l^\top \beta)^2$

(b) $\frac{l^\top \beta}{\sigma}$

(c) $\left(\frac{l^\top \beta}{\sigma}\right)^2$

for $l^\top \beta$.