Multivariate Gaussian Distributions:

Gaussian Random Vectors:

Definition: We say that X is a non degenerate d-dimensional Gaussian Random Vector with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\Sigma > 0$ (positive definite) if the law of X has the following density w.r.t. Lebesgue measure:

$$f(x;\mu,\Sigma) = (2\pi)^{-d/2} (\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right) \quad (*)$$

We write $X \sim \mathcal{N}_0(\mu, \Sigma)$

Example: $X_j \overset{i.i.d}{\sim} \mathcal{N}_0(0,1) \Rightarrow X = (X_1, \dots, X_d) \sim \mathcal{N}_0(0, \mathbb{1}_d)$

We substitute in $(*)$ $\mu = 0$ and $\Sigma = \mathbb{1}_d$

$$f_{\underline{X}}(x) = (2\pi)^{-d/2} (\det \mathbb{1}_d)^{-1/2} \cdot \exp\left(-\frac{1}{2} x^T \cdot \mathbb{1}_d^{-1} x\right)$$

Definition: The characteristic function ϕ_X of a random vector $X \in \mathbb{R}^n$:

$$\phi_X(v) = \mathbb{E}\left(\exp(i\langle v, X\rangle)\right) \quad v \in \mathbb{R}^d$$

Recall: (1) Let $Z \sim \mathcal{N}_0(\mu, \sigma^2) \Rightarrow \phi_Z(v) = \exp\left(i \mu v - \frac{\sigma^2 v^2}{2}\right)$

(2) ϕ_X fully describes the density

Example: If $X \sim \mathcal{N}_0(0, \sigma^2 \mathbb{1}_n)$, then

$$\phi_X(v) = \mathbb{E}\left(\exp\left(i \sum_{j=1}^{d} v_j X_j\right)\right) = \mathbb{E}\left(\prod_{j=1}^{d} \exp(i v_j x_j)\right)$$

$$= \prod_{j=1}^{d} \mathbb{E}(\exp(i v_j x_j)) = \prod_{j=1}^{d} \exp\left(-\frac{\sigma^2}{2} v_j^2\right)$$

$$= \exp\left(-\frac{\sigma^2}{2} \| v \|^2\right)$$

Definition: X is a d-dimensional Gaussian random vector if for any $t \in \mathbb{R}^d$ the random vector $\langle t, X \rangle$ has a Gaussian distribution.

↑ including degenerate cases for example δ-functions

Degenerate Gaussian distribution is a Gaussian Distribution where the covariance matrix is not full-rank

- Some directions have zero variance
- Distribution is concentrated in a lower dimensional subspace of $\mathbb{R}^d$

Non degenerate Σ PD / Degenerate Σ PSD

Example: Let $X = \begin{pmatrix} Z \\ Z \end{pmatrix}$, $Z \sim \mathcal{N}_0(0,1)$

Then $X \in \mathbb{R}^2$, but $Cov(X) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

This matrix has rank 1 not 2!

Degenerate 2D Gaussian fully supported on $x=y$ in $\mathbb{R}^2$, has zero variance orthogonal to that line.

$\langle X, e_i \rangle = X_i \sim$ Normal $\quad e_i = (0,0 \ldots \overset{i\text{-th}}{1}, 0\,0)$

$\not\Rightarrow X \sim$ Normal multivariate

Multivariate Gaussian requires that all linear combination of the vector components are Gaussian. not just the marginals.

Notation & Linear Properties:

Let $\mu = (\mathbb{E}(X_1), \ldots, \mathbb{E}(X_d))$ mean vector

$\Sigma_{ij} = Cov(X_i, X_j)$ Covariance matrix.

Then $\mathbb{E}(\langle t, X \rangle) = \langle t, \mu \rangle$

$$Var(\langle t, X \rangle) = Cov\left(\sum_j t_j X_j, \sum_\ell t_\ell X_\ell\right)$$

$$= \sum_{j,\ell} t_j t_\ell Cov(X_j, X_\ell) = \langle \Sigma t, t \rangle$$

↯ Here Σ does not need to be invertible.

<u>Theorem</u> 1 (a) The law of a n-dimensional Gaussian vector is uniquely determined by m, Σ

(b) Let $X \sim \mathcal{N}_d(\mu, \Sigma) \Rightarrow Y = AX + b$

$$\sim \mathcal{N}(A\mu + b, A\Sigma A^T)$$

<u>Lemma</u>: Let A be an orthogonal matrix

$X \sim \mathcal{N}_d(0, \mathbb{1}_d) \Rightarrow Y = AX \sim \mathcal{N}_d(0, \mathbb{1}_d)$

$\Rightarrow Y_j \overset{\text{i.i.d}}{\sim} \mathcal{N}_d(0, 1)$

<u>Proof</u> (a) This follows from Lemma and Gaussian 1d

$$\phi_X(u) = \mathbb{E}(\exp(i\langle u, X\rangle)) = \phi_{\langle u, X\rangle}(1)$$

$$= \exp(i\langle u, m\rangle - \frac{1}{2}\langle \Sigma u, u\rangle)$$

(b) $\phi_Y(u) = \mathbb{E}(\exp(i\langle u, Y\rangle)) = \mathbb{E}(\exp(i\langle u, AX + b\rangle))$

$$= \exp(i\langle u, b\rangle)\, \mathbb{E}(\exp(i\langle A^T u, X\rangle))$$

$$= \exp(i\langle u, b\rangle + i\langle A^T u, X\rangle - \frac{1}{2}\langle \Sigma A^T u, A^T u\rangle)$$

$$= \exp(i\langle u, \underbrace{Am + b}_{m'}\rangle - \frac{1}{2}\langle \underbrace{A\Sigma A^T}_{\Sigma'} u, u\rangle)$$

Remark: if $m = 0$, $\Sigma = \mathbb{1}_d$ $Z \sim \mathcal{N}_0(0, \mathbb{1}_d)$

$\Leftrightarrow$ $Z = (Z_1, \dots, Z_n)$ $Cov(x_j, x_l) = 0 \Rightarrow$ $x_j \perp\!\!\!\perp x_l$

$Z_j \overset{i.i.d}{\sim} \mathcal{N}_0(0, 1)$

Theorem 2 (a) Fix $m \in \mathbb{R}^d$ $\Sigma \geq 0$ symmetric

Then d-dim random vector X with law $\mathcal{N}_0(m, \Sigma)$

(b) Let Σ be invertible, then X has density

$f_{\mu, \Sigma}$

Lemma: Let $\Sigma \geq 0$ then $\exists$ $A = \Sigma^{1/2}$ with the same rank of Σ s.t. $A^T A = \Sigma$

Proof: (a) Let $Z = (Z_1, \dots Z_n)$ with $Z_j \overset{i.i.d}{\sim} \mathcal{N}_0(0,1)$

choose $A = \Sigma^{1/2}$. Then by Thm 1 the RV $X = AZ + m \sim \mathcal{N}_0(m, \Sigma)$

(b) (HW)

Note A RV $X \sim \mathcal{N}_0(m, \Sigma)$ can be constructed as follows let $A = \Sigma^{1/2}$

then $X = A \cdot Z + b$ for $Z = (Z_1, \dots, Z_{d'})$

$Z_j \overset{iid}{\sim} \mathcal{N}_0(0,1)$

Distributions related to Gaussians:

Recall: let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}_0(0,1)$

then $Y = \sum_{j=1}^{n} X_j^2 \sim \chi^2(n)$

$$f_Y(y) = \frac{1}{\Gamma(n/2)\, 2^{n/2}} \cdot y^{n/2 - 1} \cdot e^{-y/2}$$

Theorem: (Cochran) let

$E_1 \oplus \dots \oplus E_k$ be an orthogonal decomposition of $\mathbb{R}^n$ with respective dimensions

$r_1, \dots, r_k$ $(\sum r_k = n)$. Further denote

by Π_{E_j} the projection on E_j

Then The RV $Y_j = \Pi_{E_j} X$ are

mutually independent and $\|Y_j\|^2 \sim \chi^2(r_j)$

Proof: Let u_ℓ be an orthonormal basis of $\mathbb{R}^n$ with

$E_j = \mathrm{Span}_{\ell \in \{r_1 + \dots r_j, \dots, r_1 + \dots r_{j-1}\}} u_\ell$

Consider $A^T = \begin{pmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_n \\ \vdots & & \vdots \end{pmatrix}$ A is orthogonal

from lemma 1 $(Ax_j) = \langle x, u_j \rangle = Z_j \overset{iid}{\sim} \mathcal{N}(0,1)$

$\Rightarrow \Pi_{E_1} X = \sum_{j=1}^{r_1} \langle x, u_j \rangle u_j = \sum_{j=1}^{r_1} Z_j u_j$

$\Rightarrow \Pi_{E_j} X \perp\!\!\!\perp \Pi_{E_k} X$ since they are functions of disjoint sets of indep r.v.

Now $\|\Pi_{E_1} X\|^2 = \sum_{j=1}^{r_1} \langle x, u_j \rangle^2 = \sum_{j=1}^{r_1} Z_j^2 \sim \chi^2(r_1)$