Homework 3

Solutions: 17.04.2025

1. Let $X_1, \dots, X_n$ be i.i.d. $\sim p(x|\theta)$, where $p(x|\theta) = \frac{1}{2}e^{-|x-\theta|}$.

   (a) Compute the Cramér-Rao lower bound for the parameter $g(\theta) = \theta$. Then, compute the variance of the estimator $T_n = \frac{1}{n}\sum_{i=1}^n X_i^2$, and compare it with the Cramér-Rao lower bound. (There is no need to verify the technical conditions for the Cramér-Rao lower bound here.)

   (b) The maximum likelihood estimator $\hat{\theta}_n$ of θ is the median of the data $(X_1, \dots, X_n)$. Compute its variance for $n = 5$ using the following lemma: Let $X_1, \dots, X_5$ be i.i.d. with density f and cumulative distribution function F. Then, the density of $X_{(3)}$ is

   $$\frac{5!}{2!2!} f(x)F(x)^2(1 - F(x))^2.$$

   Which estimator is preferable: $\hat{U}_5$ or T_5? You may use the following result: $\int_0^{2\pi} e^{-kx} \cos x \, dx = \frac{\pi}{2}$ for $k \in \mathbb{N}$.

   (c) Justify the above lemma by a heuristic argument. Do this by expressing the event $x \le X_{(3)} \le x + dx$ in terms of $X_1, \dots, X_5$.

   (d) Recall first the definition of weak convergence. Let (X_n) be a sequence of real-valued random variables, and let X be some other real-valued random variable. We say that X_n converges weakly to X if for every continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$, we have that $E[f(X_n)] \to E[f(X)]$. It is known that when f is the density of i.i.d. real-valued random variables $X_1, \dots, X_n$ and when m is such that

   $$\int_{-\infty}^m f(x)\, dx = \frac{1}{2} \int_m^\infty f(x)\, dx = \frac{1}{2},$$

   then the distribution of

   $$\sqrt{n}(\text{median}(X_1, \dots, X_n) - m)$$

   converges weakly to the distribution $\mathcal{N}(0, \frac{1}{4(f(m))^2})$. Using this result, show that the median $\hat{\theta}_n$ reaches asymptotically (i.e., as $n \to \infty$) the Cramér-Rao lower bound.

2. Consider an *iid* sample $X = (X_1, \dots, X_n)$ from a normal distribution $\mathcal{N}(m, \sigma^2)$ with known mean $m \in \mathbb{R}$ and unknown variance $\sigma^2 \in \mathbb{R}_{>0}$. Consider the estimator $T_\alpha(X) = \alpha^{-1}\sum_{j=1}^n (X_j - m)^2$

   (a) Compute bias and variance of the estimator $T_\alpha(X)$.

   (b) Show that $T_n(X)$ is UMVU by proving that it is unbiased and that it saturates the CRLB.

   (c) Compute the MSE for $T_\alpha(X)$ and show that there exists $\alpha > 0$ such that $MSE(T_\alpha) < MSE(T_n)$.

3. Let $(\mathbb{X}, \mathcal{F}, (P_\theta)_{\theta\in\Theta})$ be a statistical model with $\Theta \subseteq \mathbb{R}^k, k \ge 2$, that satisfies the conditions for the Cramér-Rao inequality with a strictly positive-definite Fisher Information matrix $I(\theta)$.

   (a) Show that it always holds that $(I(\theta))_{11}(I(\theta)^{-1})_{11} > 1$. Hint: Write the first unit vector $e_1 = \sum_i \langle e_1, v_i\rangle v_i$ in an orthonormal basis $(v_1, \dots, v_k)$ of eigenvectors of $I(\theta)$.

   (b) Conclude that the Cramér-Rao bound for $g(\theta) = g(\theta_1, \dots, \theta_n) = \theta_1$ is always at least $1/(I(\theta))_{11}$. In which cases does equality hold?

   (c) Interpret this result statistically by comparing it with the model where only θ_1 is unknown, while $\theta_2, \dots, \theta_k$ are known.

(d) Apply the above results to compare the variance of the estimator for the mean of a $\mathcal{N}(m, \sigma^2)$ distribution given an *iid* sample of size n.

4. Let $T(X)$ be an unbiased estimator of $g(\theta) \in \mathbb{R}$ for all $\theta \in \Theta \subseteq \mathbb{R}$, and assume that the statistical model $(\mathbb{X}, \mathcal{F}, \{\mathbb{P}_\theta\}_{\theta \in \Theta})$ satisfies A1-2-3 stated in class.

   (a) Recalling the proof of the CRLB done in class, prove that if $T(X)$ saturates the CRLB, then it holds that
   $$T(X) = g(\theta) + \partial_\theta g(\theta) I(\theta)^{-1} s_\theta(X)$$
   Hint: use that Cauchy-Schwarz inequality for $\mathbb{E}_\theta(fg)$ holds with equality if and only if $f = \alpha(\theta) g$ p_θ-a.s. for an $\alpha(\theta) \in \mathbb{R}$.

   (b) (Optional) Rearranging the expression obtained at the previous point, show that if $T(X)$ satisfies the CRLB, then $p_\theta(x)$ is a 1-dimensional exponential family, i.e., that there exist $c(\theta), d(\theta), h(x)$ such that
   $$p_\theta(x) = \exp(c(\theta) T(x) - d(\theta)) h(x)\,,$$
   and prove that it holds $g(\theta) = \partial_\theta c(\theta) / \partial_\theta d(\theta)$.

   (c) (Optional) Assume now that $c(\theta) = \theta$, i.e., that the exponential family is in canonical form. Prove that $g(\theta) = \partial_\theta d(\theta)$ and $I(x) = \partial_\theta^2 d(\theta)$.

5. Let $X_1, \ldots, X_n$ be i.i.d. random variables drawn from the exponential distribution with unknown rate parameter $\theta > 0$. The probability density function is given by:
   $$p_\theta(x) = \theta e^{-\theta x}, \quad x > 0.$$
   Consider the estimator:
   $$T(X) = \frac{1}{X_1}.$$

   (a) Show that $T(X)$ is an unbiased estimator for θ.

   (b) Compute the variance of $T(X)$. Compare it to the Cramér-Rao lower bound (CRLB) for any unbiased estimator of θ.

   (c) Explain why $T(X)$ does *not* attain the CRLB, even though the model is regular and $T(X)$ appears to estimate θ.

   Hint: Consider whether the expectation $\mathbb{E}_\theta[1/X_1]$ is finite, and whether the estimator is sufficient or makes use of all the data.

   Optional: Find an estimator that is unbiased and efficient for θ in this model, and verify that it achieves the Cramér-Rao lower bound.