Linear Models and SVMs

Optimization for Machine Learning — Homework #1

Monday 8th May, 2023

Part I: Theory

12+2 points

I.1. Convexity

Exercise I.1 (2+1 points)**.** Let $f\colon \mathbb{R}^d \to \mathbb{R}$ and $g\colon \mathbb{R}^d \to \mathbb{R}$ be two convex functions. Show that $f + g$ is convex.

Bonus: Show that $x \mapsto \max(f(x), g(x))$ is convex.

Exercise I.2 (2+1 points)**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear function (i.e. $f(x) = Ax$, for $A \in \mathbb{R}^{m\times n}$) and $g\colon \mathbb{R}^m \to \mathbb{R}$ be convex functions. Show that $g \circ f\colon \mathbb{R}^n \to \mathbb{R}$ is convex.

Bonus: What about if f is affine, i.e. $f(x) = Ax + b$, with $b \in \mathbb{R}^m$?

I.2. (Sub-) Gradients

Exercise I.3 (2 points)**.** Let $\phi\colon \mathbb{R}^d \to \mathbb{R}^p$ be a differentiable basis function. Define the model $f\colon \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}$ as, for $b \in \mathbb{R}$,

$$\begin{array}{rccl} f & : & \mathbb{R}^d \times \mathbb{R}^p & \longrightarrow & \mathbb{R} \\ & & (x, w) & \longmapsto & \langle w, \phi(x)\rangle + b \end{array}$$

1. What is $\nabla_w f(x, w)$?
2. What is $\nabla_x f(x, w)$?

Exercise I.4 (2 points)**.** For $\lambda \geqslant 0$, let $E(w) = E_{\mathrm{s}}(w) + \lambda\|w\|_1$, where E_{s} is assumed to be convex and everywhere differentiable, and

$$\|w\|_1 := \sum_{i=1}^{p} |w_i|$$

is the 1-norm of w.

1. Is E convex?

2. Where is $w \mapsto \|w\|_1$ (not) differentiable?

3. What is $\partial E(w)$ (as a function of $\nabla E_\mathrm{s}(w)$)? (*Hint:* see what happens for $p = 1, 2$ first.)

I.3. Support Vector Machines

Exercise I.5 (4 points)**.** The Support Vector Machines (SVM) algorithm solves a binary classification task. Given N couples samples / targets $\{x_i, t_i\}_{i \in [N]}$, with $x_i \in \mathbb{R}^d$ and $t_i \in \{-1, 1\}$ for each $i \in [N] := \{1, \ldots, N\}$, the goal is to classify the samples, i.e. find the regions where the positive (resp. negative) samples lie. We will do that by finding a **hyperplane that separates** (or **splits**) the dataset, with positive samples on one side of the hyperplane and the negative on the other. We assume that **such an hyperplane exists** (the samples are said to be *linearly separable*). One can picture the case for $d = 2$ or $d = 3$, where points are clustered in two groups and can be separated by a straight line ($d = 2$) or a plane ($d = 3$), see Figure 1a.

A hyperplane in $\mathbb{R}^d$ is represented with a vector $w \in \mathbb{R}^d$ and a *bias* $b \in \mathbb{R}$ with the equation

$$y(x, w) = \langle w, x \rangle + b. \tag{1}$$

For $i \in [N]$, denote $y_i := y(x_i, w) = \langle w, x_i \rangle + b$. Note that y_i **still depends on** (x, w) even if the notation is dropped.

The hyperplane equation (1) splits $\mathbb{R}^d$ into three regions:

- points x such that $y(x) > 0$,
- points x such that $y(x) = 0$ (the hyperplane itself),
- points x such that $y(x) < 0$.

Therefore, we would like to find an hyperplane such that the samples x_i that have a positive target $t_i = 1$ all lie on the side of the hyperplane where $y(x) > 0$, i.e. $t_i = 1 \implies y_i > 0$, and reciprocally all samples x_i such that $t_i = -1$ should be on the side where $y(x) < 0$, i.e. $t_i = -1 \implies y_i < 0$. Then, the target t_i could simply by read from y_i by looking at its sign.

With this requirement, the product $t_i y_i$ should **always be positive**, and the loss we define is

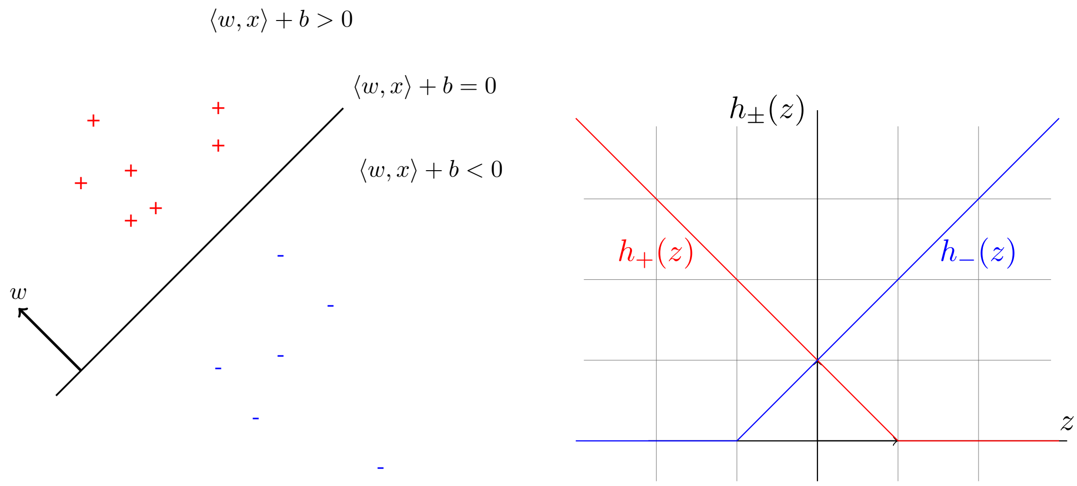
$$E(w) = \sum_{i=1}^{N} \max(0, 1 - t_i y_i) = \sum_{i=1}^{N} \max\big(0, 1 - t_i(\langle w, x_i \rangle + b)\big), \tag{2}$$

which has the effect of pushing the product $t_i y_i$ towards the greatest value possible, for all $i \in [N]$.

Let $\mathcal{I}_+ = \{i \in [N] \mid t_i = +1\}$ and $\mathcal{I}_- = \{i \in [N] \mid t_i = -1\}$ be the sets of the indices of the positive and negative samples. The loss can be further written as

$$E(w) = \sum_{i \in \mathcal{I}_+} \max(0, 1 - y_i) + \sum_{i \in \mathcal{I}_-} \max(0, 1 + y_i) =: \sum_{i \in \mathcal{I}_+} h_+(y_i) + \sum_{i \in \mathcal{I}_-} h_-(y_i)$$

The functions h_+ and h_- are plotted in Figure 1b.



(a) Possible solution found by SVM. Points with $t_i = +1$ (in red) are on the side where $y(x) > 0$, points with $t_i = -1$ (in blue) are on the side where $y(x) < 0$.

(b) In red, the loss function for the samples that have $t_i = +1$. In blue, the loss function for the samples with $t_i = -1$.

Figure 1: SVM illustration, $d = 2$.

1. Why is the loss $w \mapsto E(w)$ convex? (*Hint*: if f and g are convex, then $\max(f, g)$ is convex).

Recall that the function $h_+ : \mathbb{R} \ni z \mapsto \max(0, 1 - z)$ has the following subgradient (see Exercise Sheet #2, I.2):

$$\partial h_+(z) = \begin{cases} \{-1\} & \text{if } z < 1, \\ [-1, 0] & \text{if } z = 1, \\ \{0\} & \text{if } z > 1. \end{cases}$$

2. Show that the subgradient of the function $h_- : \mathbb{R} \ni z \mapsto \max(0, 1 + z)$ is

$$\partial h_-(z) = \begin{cases} \{0\} & \text{if } z < -1, \\ [0, 1] & \text{if } z = -1, \\ \{1\} & \text{if } z > -1. \end{cases}$$

3. Compute the subgradient of the loss E at $w \in \mathbb{R}^p$.

4. What should be the (sub-) gradient descent algorithm to minimize E?

Part II: Programming *8 points*

Exercise II.1 (8 points)**.** This exercise implements some material from Exercise I.5. The data is generated with the helper function `gen_binary_data` in `ex02/utils.py`. The dimension is set to $d = 2$ in order to visualize the result at the end. The generation simply draws some random points on the plane, draws an hyperplane, and classify the points depending on the sign of $\langle w, x \rangle + b$. Therefore, the training data is linearly separable.

1. Implement the loss from (2).
2. Implement the (sub-) gradient algorithm derived in I.5.4
3. Visualize the solution found by the algorithm. What happens if the data is not linearly separable?