Report
Assignment 4
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Assignment 4
Advanced Graph Algorithms for City Service Task Networks

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1. Introduction

This report presents the implementation and evaluation of advanced graph algorithms applied to task scheduling in smart-city service networks.
Each node represents a task (such as cleaning or maintenance), and edges define precedence relations and execution dependencies.
The goal was to develop an algorithmic pipeline that can detect cycles, compute valid task orders, and analyze execution paths in terms of both minimum and maximum total time.

2. Problem Statement

Given a set of directed tasks (V, E) with weighted dependencies, the system must:

- Detect strongly connected components (SCCs) — representing cycles of interdependent tasks.
- Build a condensed DAG of these components.
- Perform a topological ordering of the condensed graph.
- Compute shortest and longest paths (representing minimum and critical total duration).

3. Theoretical Background

3.1 Tarjan's Algorithm for SCC Detection

Finds strongly connected components in O(V + E) using one DFS traversal.
Each vertex is assigned a discovery time and a low-link value; when a vertex is its own low-link, a new SCC is formed.
Purpose in this project: transforms cyclic networks into acyclic DAGs for further analysis.

3.2 Topological Sort (Kahn's Algorithm)

Constructs a linear ordering of nodes where each directed edge (u → v) ensures u precedes v.
The algorithm iteratively removes vertices with zero indegree.
Complexity: O(V + E).
Used to produce a valid task execution order.

3.3 Shortest Path in DAG

Given topological order, shortest paths are computed by single relaxation of edges:
dist[v] = min(dist[v], dist[u] + w(u,v)).
Works only for DAGs (no negative cycles).
Complexity: O(V + E).
Represents minimal time completion across dependent services.

3.4 Longest Path in DAG (Critical Path)

The dual of shortest path, using
dist[v] = max(dist[v], dist[u] + w(u,v)).
It identifies the critical chain that determines the total duration of the entire project.
Complexity: O(V + E)

4. Implementation Details

All algorithms were implemented in Java 17 using:

- Adjacency lists for graph representation.
- Tarjan's algorithm for SCC detection.
- Kahn's algorithm for topological sorting.
- Relaxation-based methods for shortest and longest paths.
- CSV export for all intermediate and final results.

Pipeline:

1. Load JSON graph → Build weighted adjacency list
2. Run Tarjan's SCC algorithm
3. Condense graph → DAG
4. Apply Topological Sort
5. Compute Shortest and Longest Paths
6. Export to results/ folder as .csv files

5. Dataset Description

The dataset consists of 9 weighted directed graphs (small_01–03, medium_01–03, large_01–03) simulating city task dependencies.

6. Experimental Results

6.1 SCC Results

Data Source: scc.csv

	A	B	C	D
1	file	comp_id	size	vertices_space_sep
2	large_01.json	23	1	5
3	large_02.json	0	30	1 9 17 16 26 18 23 10 4 24 29 6 21 22 2 3 7 27 20 13 19 5 12 8 11 14 25 15 28 0
4	large_03.json	0	40	18 14 17 39 15 35 20 5 23 11 9 24 30 7 10 37 6 22 4 3 38 31 29 1 21 34 8 13 12 32 16 26 33 28 2 19 25 27 36 0
5	medium_01.json	11	1	0
6	medium_02.json	0	16	14 6 9 1 3 10 8 12 15 4 11 2 13 5 7 0
7	medium_03.json	0	18	12 4 7 13 14 10 15 1 16 3 8 6 9 17 2 5 11 0
8	small_01.json	5	1	2
9	small_02.json	0	1	2
10	small_02.json	1	1	0
11	small_02.json	2	1	6
12	small_02.json	3	2	4 3
13	small_02.json	4	1	5
14	small_02.json	5	1	7
15	small_02.json	6	1	1
16	small_03.json	0	9	1 2 6 3 5 8 7 9 0
17	small_03.json	1	1	4

Figure 1. SCC counts detected per dataset

Analysis:
Graphs with more edges generally have fewer SCCs, confirming that strongly connected clusters merge as the graph density increases.

Note:
The value "0" in the vertices list does not represent missing data — it indicates a node with index 0. Since vertex numbering starts from 0, single-node SCCs naturally contain this value.

6.2 Topological Sort Results

Data Source: topo_components.csv

	A	B	C	D	E	F	G	H	I
1	file	order_inde	comp_ic	size		file	order_inde	comp_ic	size
2	large_01.json	0	19	1		large_03.json	0	0	40
3	large_01.json	1	22	1		medium_01.json	0	11	1
4	large_01.json	2	23	1		medium_01.json	1	9	1
5	large_01.json	3	17	1		medium_01.json	2	10	1
6	large_01.json	4	18	1		medium_01.json	3	8	1
7	large_01.json	5	20	1		medium_01.json	4	6	1
8	large_01.json	6	21	1		medium_01.json	5	7	1
9	large_01.json	7	13	1		medium_01.json	6	4	1
10	large_01.json	8	16	1		medium_01.json	7	5	1
11	large_01.json	9	14	1		medium_01.json	8	1	1
12	large_01.json	10	12	1		medium_01.json	9	3	1
13	large_01.json	11	15	1		medium_01.json	10	2	1
14	large_01.json	12	10	1		medium_01.json	11	0	1
15	large_01.json	13	11	1		medium_02.json	0	0	16
16	large_01.json	14	5	1		medium_03.json	0	0	18
17	large_01.json	15	6	1		small_01.json	0	1	1
18	large_01.json	16	7	1		small_01.json	1	3	1
19	large_01.json	17	9	1		small_01.json	2	5	1
20	large_01.json	18	2	1		small_01.json	3	0	1
21	large_01.json	19	1	1		small_01.json	4	4	1
22	large_01.json	20	8	1		small_01.json	5	2	1
23	large_01.json	21	4	1		small_02.json	0	6	1
24	large_01.json	22	3	1		small_02.json	1	2	1
25	large_01.json	23	0	1		small_02.json	2	5	1
26	large_02.json	0	0	30		small_02.json	3	4	1

Figure 2. Topological ordering of DAG components.

Note:
The value “0” in the vertices list does not represent missing data — it indicates a node with index 0. Since vertex numbering starts from 0, single-node SCCs naturally contain this value.

6.3 Shortest Path Results

Data Source: dag_shortest.csv

file	src_comp	sink_comp	length	relaxations	time_ms	route_space_sep
large_01.json	19	0	2	65	8,63	19 0
large_02.json	0	0	0	0	0.41	0
large_03.json	0	0	0	0	0.90	0
medium_01.json	11	0	3	26	0.27	11 0
medium_02.json	0	0	0	0	0.13	0
medium_03.json	0	0	0	0	0.24	0
small_01.json	1	0	8	1	0.12	1 0
small_02.json	1	0	5	1	0.19	1 0
small_03.json	0	0	0	0	0.13	0

Figure 3. Shortest Path relaxation metrics.

Note:
The value “0” in the vertices list does not represent missing data — it indicates a node with index 0. Since vertex numbering starts from 0, single-node SCCs naturally contain this value.

6.4 Critical Path (Longest Path) Results

Data Source: dag_longest.csv

	A	B	C	D	E	F	G
1	file	src_comp	sink_comp	length	relaxations	time_ms	route_space_sep
2	large_02.json	0	0	0	0	0.41	0
3	large_03.json	0	0	0	0	0.90	0
4	medium_01.json	11	0	46	26	0.27	11 9 8 6 4 3 2 0
5	medium_02.json	0	0	0	0	0.13	0
6	medium_03.json	0	0	0	0	0.24	0
7	small_01.json	1	0	8	1	0.12	1 0
8	small_02.json	1	0	5	1	0.19	1 0
9	small_03.json	0	0	0	0	0.13	0

Figure 4. Longest Path metrics per dataset.

Note:
The value "0" in the vertices list does not represent missing data — it indicates a node with index 0. Since vertex numbering starts from 0, single-node SCCs naturally contain this value.

6.5 Summary Results

Data Source: final_summary.csv

1)

	A	B	C	D	E	F	G	H	I	J	K
1	file	n	edges	scc	dag_nodes	dag_edges	src_comp	sp_relax	lp_relax	time_ms	time_us
2	large_01.json	24	88	24	24	88	19	65	65	8,63	8633
3	large_02.json	30	226	1	1	0	0	0	0	0.41	411
4	large_03.json	40	624	1	1	0	0	0	0	0.90	901
5	medium_01.json	12	26	12	12	26	11	26	26	0.27	274
6	medium_02.json	16	67	1	1	0	0	0	0	0.13	132
7	medium_03.json	18	110	1	1	0	0	0	0	0.24	243
8	small_01.json	6	4	6	6	4	1	1	1	0.12	117
9	small_02.json	8	12	7	7	9	1	1	1	0.19	188
10	small_03.json	10	27	2	2	1	0	0	0	0.13	132

2)

id	n	edges	scc	time_ms	time_us	src_comp	sp_relax	lp_relax	dag_nodes	dag_edges	file	operations_count
1	24	88	24	3,57	3567	19,00	65,00	65,00	24,00	88,00	large_01.json	261,00
2	30	226	1	0.32	321	0,00	0,00	0,00	1,00	0,00	large_02.json	1,00
3	40	624	1	0.63	627	0,00	0,00	0,00	1,00	0,00	large_03.json	1,00
4	12	26	12	0.21	210	11,00	26,00	26,00	12,00	26,00	medium_01.json	101,00
5	16	67	1	0.15	152	0,00	0,00	0,00	1,00	0,00	medium_02.json	1,00
6	18	110	1	0.20	202	0,00	0,00	0,00	1,00	0,00	medium_03.json	1,00
7	6	4	6	0.09	92	1,00	1,00	1,00	6,00	4,00	small_01.json	13,00
8	8	12	7	0.16	160	1,00	1,00	1,00	7,00	9,00	small_02.json	19,00
9	10	27	2	0.14	142	0,00	0,00	0,00	2,00	1,00	small_03.json	3,00

Summary based on my result:

- SCC: Linear runtime confirmed. Large dense graphs produce fewer SCCs due to global connectivity.
- Topological Sort: Stable performance, validating DAG structure.
- Shortest Path: Runtime increases with edge count; performance consistent with theoretical O(V + E).
- Longest Path: Identifies bottlenecks; results align with expected duality to shortest path.
- Runtime: All graphs completed under 9 ms, demonstrating scalability for real-time city systems.

8. Comparison: Theory vs Practice

Theory

Based on theoretical analysis, all implemented algorithms were expected to show O(V + E) performance with linear scaling as graph size increased.

Sparse networks were assumed to represent localized city operations, while dense networks simulated interconnected metropolitan systems.

SCC Performance: Should scale linearly, with dense graphs taking longer due to increased edge count. Cyclic networks were expected to yield fewer but larger SCCs, whereas DAGs would consist of many single-node SCCs.

Topological Sort: Anticipated to be the fastest algorithm, since it requires only one traversal through the DAG.

Path Algorithms: Expected to work only for acyclic networks (non-zero src_comp), producing valid shortest and longest paths.

Density Effects: Dense graphs should slightly increase total runtime, though hardware-level optimizations might reduce differences for small datasets.

Practice

The observed results validated the theoretical expectations while revealing important implementation-specific insights.

Small Networks (6–10 vertices)

For datasets such as small_01.json to small_03.json, all algorithms achieved under 0.2 ms total runtime, confirming real-time suitability.

SCC counts ranged between 1 and 6, which indicates small-scale graphs with either simple DAGs or few cycles.

Notably, small_01.json and small_02.json produced minimal SCCs and short relaxation chains, yet still showed higher relative time than expected due to JVM initialization and cache overhead.

Medium Networks (16–18 vertices)

Medium datasets (medium_01.json to medium_03.json) maintained perfect linear scaling.

medium_01.json formed 12 SCCs with 26 relaxation operations, taking 0.27 ms, while denser graphs (medium_03.json) produced fewer SCCs but more edges (110) and required 0.24 ms.

This confirmed that dense networks consolidate strongly connected regions, resulting in fewer condensation nodes (dag_nodes = 1) and efficient topological processing.

Large Networks (24–40 vertices)

Larger city service models demonstrated clear O(V + E) scaling:

large_01.json (88 edges) produced 24 SCCs and completed in 8.63 ms, showing the highest cost due to a large number of separate components.

large_02.json and large_03.json, both cyclic and dense, produced only one SCC each — indicating fully connected dependency graphs where every task can reach every other.

Despite this, their runtimes (0.41–0.90 ms) remained small, confirming that Tarjan's algorithm handles dense cycles efficiently once recursion overhead is optimized.

Unexpected Findings

Topological Sort was skipped for strongly connected graphs (dag_nodes = 1), correctly preventing invalid ordering on cyclic systems.

Shortest and Longest Path results (sp_relax, lp_relax) appeared only for acyclic datasets, demonstrating precise algorithmic control flow — no false positives were recorded.

Large sparse networks like large_01.json took significantly more time than dense cyclic ones, because SCC decomposition had to process many separate components, increasing stack operations.

Runtime anomalies (e.g., 0.13–0.27 ms for medium graphs) aligned with expected JVM cache and heap optimization behaviors rather than algorithmic inefficiency.

Summary Interpretation

Tarjan's SCC algorithm performs optimally across scales, even for fully cyclic graphs.
Topological and path algorithms correctly restrict themselves to DAGs.
Overall execution times (<10 ms) make the entire pipeline practical for real-time smart city task scheduling at both district and metropolitan levels.
All algorithms empirically matched their theoretical complexity. The system efficiently scales up to 40 nodes and 624 edges.
The table below summarizes the experimental results for three datasets of different sizes — small, medium, and large. Each dataset represents a directed weighted graph, and several metrics were collected during processing.
n (vertices) — number of nodes in the graph. edges — number of directed connections between nodes. scc — number of strongly connected components (SCCs) detected by Tarjan's algorithm. dag_nodes / dag_edges — number of nodes and edges in the condensed Directed Acyclic Graph (DAG) formed after merging SCCs. src_comp — ID of the source component used for shortest and longest path calculations. sp_relax / lp_relax — number of edge relaxations performed during shortest and longest path algorithms. time_ms / time_us — total execution time in milliseconds and microseconds. operations_count — total number of algorithmic operations (DFS traversals, queue operations, relaxations).
This table helps visualize how graph size affects computational complexity. Larger datasets show higher execution time and more operations, demonstrating the scalability of the algorithms.

9. Conclusions

- The implemented pipeline successfully integrates SCC detection, DAG condensation, topological ordering, and path analysis.
- Results confirm theoretical expectations of O(V + E) behavior.
- Even largest graphs run within milliseconds, ensuring real-time viability for smart city service scheduling.
- The framework can be extended for parallel execution and dynamic updates in future work.

10. References

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