1

a) $x^2+y^2-2xy+2x+4y+5=0$

$$[x\ y]\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+[2\ 4]\begin{bmatrix}x\\ y\end{bmatrix}+5=0$$

$A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ simétrica, logo ortogonalmente diagonalizável

• valores próprios de A

$$\begin{vmatrix}1-\lambda & -1\\ -1 & 1-\lambda\end{vmatrix}=0 \Leftrightarrow (1-\lambda)(1-\lambda)-1=0$$
$$\Leftrightarrow \cancel{1}-\lambda-\lambda+\lambda^2-\cancel{1}=0$$
$$\Leftrightarrow -2\lambda+\lambda^2=0 \Leftrightarrow \lambda(-2+\lambda)=0$$
$$\Leftrightarrow \lambda=0 \vee \lambda=2$$

Vetores próprios associados aos valores próprios

→ vetor associado a $\lambda=0$

$$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0 \Leftrightarrow \begin{cases}x_1=x_2\\ x_2\in\mathbb{R}\end{cases} \rightarrow \begin{bmatrix}x_2\\ x_2\end{bmatrix}, x_2\in\mathbb{R}\setminus\{0\} \Leftrightarrow \begin{bmatrix}1\\ 1\end{bmatrix}x_2, x_2\in\mathbb{R}\setminus\{0\}$$ Norma $=\sqrt{2}$

→ vetor associado a $\lambda=2$

$$\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0 \Leftrightarrow \begin{cases}x_1=-x_2\\ x_2\in\mathbb{R}\end{cases} \rightarrow \begin{bmatrix}-x_2\\ x_2\end{bmatrix}, x_2\in\mathbb{R}\setminus\{0\} \Leftrightarrow \begin{bmatrix}-1\\ 1\end{bmatrix}x_2, x_2\in\mathbb{R}\setminus\{0\}$$ Norma $=\sqrt{2}$

$$P=\begin{bmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix} \qquad D=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$$

$x=P\hat{x}$

$$\hat{B}=BP=[2\ 4]\begin{bmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}=[-\sqrt{2}+2\sqrt{2}\quad \sqrt{2}+2\sqrt{2}]=[\sqrt{2}\quad 3\sqrt{2}]$$

→ $x^TAx+Bx+\mu=0$

$\Leftrightarrow (P\hat{x})^TA(P\hat{x})+B(P\hat{x})+\mu=0$

$\Leftrightarrow \hat{x}^T\underbrace{P^TAP}_{D}\hat{x}+BP\hat{x}+\mu=0$

$\Leftrightarrow \hat{x}^TD\hat{x}+\hat{B}\hat{x}+\mu=0$

$$[x\ y]\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}+[\sqrt{2}\quad 3\sqrt{2}]\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}+5=0$$

$\Leftrightarrow [2x\quad 0]\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}+\sqrt{2}\hat{x}+3\sqrt{2}\hat{y}+5=0$

$\Leftrightarrow 2x^2+\sqrt{2}x+3\sqrt{2}y+5=0$

$\Leftrightarrow 2\left(x^2+\frac{\sqrt{2}}{2}x\right)+3\sqrt{2}y+5=0$

$\Leftrightarrow 2\left(x^2+\frac{\sqrt{2}}{2}x+\left(\frac{\sqrt{2}}{4}\right)^2\right)-\frac{2}{8}+3\sqrt{2}y+5=0$

$\Leftrightarrow 2x^2+3\sqrt{2}y=-\frac{15}{4}$

$\Leftrightarrow 2\hat{x}^2=-3\sqrt{2}y-\frac{15}{4}$

$\Leftrightarrow -\frac{\sqrt{2}}{3}\hat{x}^2=\underbrace{y+\frac{19\sqrt{2}}{24}}_{y}$

$\Leftrightarrow y=-\frac{\sqrt{2}}{3}\hat{x}^2$ <u>Parábola</u>

b) $4xy - 2x + 6y + 3 = 0$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 3 = 0$$

$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ é simétrica, logo é diagonalizável

<u>valores próprios</u>

$$\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 4 = 0 \Leftrightarrow \lambda = 2 \vee \lambda = -2$$

<u>vetores próprios associados</u>

→ vetor próprio associado a $\lambda = -2$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Leftrightarrow \begin{cases} x_2 = -x_1 \\ x_1 \in \mathbb{R} \end{cases} \rightarrow \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}, x_1 \in \mathbb{R}\setminus\{0\} \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1, x_1 \in \mathbb{R}\setminus\{0\}$$

Norma $\sqrt{2}$

→ vetor próprio associado a $\lambda = 2$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Leftrightarrow \begin{cases} 2x_1 = 2x_2 \\ x_1 \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ x_1 \in \mathbb{R} \end{cases} \rightarrow \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}, x_1 \in \mathbb{R} \Leftrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1, x_1 \in \mathbb{R}\setminus\{0\}$$

Norma $\sqrt{2}$

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$x = P\hat{x}$$

$$\hat{y} = BP = \begin{bmatrix} -2 & 6 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2}+3\sqrt{2} & -\sqrt{2}-3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & -4\sqrt{2} \end{bmatrix}$$

$$\hat{x}^T D\hat{x} + B\hat{x} + \mu = 0$$

$$\Leftrightarrow \begin{bmatrix} \hat{x} & \hat{y} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} 2\sqrt{2} & -4\sqrt{2} \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + 3 = 0$$

$$\Leftrightarrow 2\hat{x}^2 - 2\hat{y}^2 + 2\sqrt{2}\hat{x} - 4\sqrt{2}\hat{y} + 3 = 0$$

$$\Leftrightarrow (2\hat{x}^2 + 2\sqrt{2}\hat{x}) - (2\hat{y}^2 + 4\sqrt{2}\hat{y}) = -3$$

$$\Leftrightarrow 2(\hat{x}^2 + \sqrt{2}\hat{x}) - 2(\hat{y}^2 + 2\sqrt{2}\hat{y}) = -3$$

$$\Leftrightarrow 2\left(\hat{x}^2 + \sqrt{2}x + \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2\right) - 2\left(\hat{y}^2 + 2\sqrt{2}y + (\sqrt{2})^2 - (\sqrt{2})^2\right) = -3$$

$$\Leftrightarrow 2\left[\underbrace{\left(x + \frac{\sqrt{2}}{2}\right)^2}_{x} - \left(\frac{\sqrt{2}}{2}\right)^2\right] - 2\left[\underbrace{(y + \sqrt{2})^2}_{y} - (\sqrt{2})^2\right] = -3$$

$$\Leftrightarrow 2x^2 - 1 - 2y^2 + 4 = -3$$

$$\Leftrightarrow 2x^2 - 2y^2 = -6$$

$$\Leftrightarrow \frac{2x^2}{-6} - \frac{2y^2}{-6} = 1$$

$$\Leftrightarrow -\frac{x^2}{3} + \frac{y^2}{3} = 1 \rightarrow \underline{\text{Hipérbole}}$$

c) $x^2 + 2x + y^2 - 4y = 0$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Leftrightarrow x^2 + y^2 + 2x - 4y = 0$$

$$\Leftrightarrow (x^2 + 2x) + (y^2 - 4y) = 0$$

$$\Leftrightarrow (x^2 + 2x + 1 - 1) + (y^2 - 4y + 4 - 4) = 0$$

$$\Leftrightarrow \underbrace{(x+1)^2}_{x} - 1 + \underbrace{(y-2)^2}_{y} - 4 = 0$$

$$\Leftrightarrow x^2 - 1 + y^2 - 4 = 0$$

$$\Leftrightarrow x^2 + y^2 = 5$$

$$\Leftrightarrow \frac{x^2}{5} + \frac{y^2}{5} = 1 \rightarrow \text{Elipse}$$

(2)

a) $x^2-y^2-z^2+4x-6y-9=0$

$$\begin{bmatrix} x & y & z\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}4&-6&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}-9=0 \Leftrightarrow x^2-y^2-z^2+4x-6y-9=0$$

$\Leftrightarrow (x^2+4x)+(-y^2-6y)-z^2-9=0$

$\Leftrightarrow (x^2+4x+4-4)-(y^2+6y+9-9)-z^2-9=0$

$\Leftrightarrow (x+2)^2-(y-3)^2-z^2-\cancel{9}+\cancel{9}-4=0$

$\Leftrightarrow \tilde{x}^2-\tilde{y}^2-\tilde{z}^2=4$

$\Leftrightarrow \frac{\tilde{x}^2}{4}-\frac{\tilde{y}^2}{4}-\frac{\tilde{z}^2}{4}=1$ Hiperbolóide

$\tilde{x}=x+2$
$\tilde{y}=y-3$
$\tilde{z}=z$

ou//

$x^2-y^2-z^2+4x-6y-9=0$

Escrever na forma: $x^TAX+BX+\mu=0$

$$\begin{bmatrix} x & y & z\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}4&-6&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}-9=0$$

Como A é simétrica, A é ortogonalmente diagonalizável

$$A=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

$|A-\lambda I|=0$

$$\Leftrightarrow \begin{vmatrix}1-\lambda&0&0\\0&-1-\lambda&0\\0&0&-1-\lambda\end{vmatrix}=0 \Leftrightarrow (1-\lambda)(-1-\lambda)(-1-\lambda)=0$$

$\Leftrightarrow \lambda=1 \vee \lambda=-1 \vee \lambda=-1$

$\Leftrightarrow \lambda=1 \vee \underbrace{\lambda=-1}_{\text{multiplicidade 2}}$

<u>obter vetores próprios associados</u>

vetor associado a $\lambda=1$

$$\begin{bmatrix}0&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0 \Leftrightarrow \begin{cases}-x_2=0\\-x_3=0\\x_1\in\mathbb{R}\end{cases} \Leftrightarrow \begin{cases}x_1\in\mathbb{R}\\x_2=0\\x_3=0\end{cases} \rightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}x_1,\ x_1\in\mathbb{R}\setminus\{0\}$$

$\|(1,0,0)\|=\sqrt{1}=1$

$\lambda_1=(1,0,0)$

vetor associado a $\lambda=-1$

$$\begin{bmatrix}2&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0 \Leftrightarrow \begin{cases}x_1=0\\x_2\in\mathbb{R}\\x_3\in\mathbb{R}\end{cases} \rightarrow \begin{bmatrix}0\\x_2\\x_3\end{bmatrix}x_2,x_3\in\mathbb{R} \Leftrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix}x_2+\begin{bmatrix}0\\0\\1\end{bmatrix}x_3,\ x_2 \text{ e } x_3\in\mathbb{R}$$

não simultaneamente nulos

$\|(0,1,0)\|=1$ $\lambda_2=(0,1,0) \wedge (0,0,1)$

$\|(0,0,1)\|=1$

$P^TAP=D$

$$P=[\lambda_1\ \lambda_2]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad D=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

- Considerando uma mudança de variável $X=P\hat{X}$

$X^TAX+BX+\mu=0 \overset{X=P\hat{X}}{\Longrightarrow}$

$\Rightarrow (P\hat{X})^TA(P\hat{X})+B(P\hat{X})+\mu=0$

$\Rightarrow \hat{X}^T\underbrace{P^TAP}_{D}\hat{X}+BP\hat{X}+\mu=0$

$$BP=\begin{bmatrix}4&-6&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}4&-6&0\end{bmatrix}$$

$\hat{X}^TD\hat{X}+BP\hat{X}+\mu=0$

$$\begin{bmatrix}\hat{x}&\hat{y}&\hat{z}\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}\hat{x}\\\hat{y}\\\hat{z}\end{bmatrix}+\begin{bmatrix}4&-6&0\end{bmatrix}\begin{bmatrix}\hat{x}\\\hat{y}\\\hat{z}\end{bmatrix}-9=0$$

$\Rightarrow \hat{x}^2-\hat{y}^2-\hat{z}^2+4\hat{x}-6\hat{y}-9=0$

$\Rightarrow (\hat{x}^2+4\hat{x})+(-\hat{y}^2-6\hat{y})-\hat{z}^2-9=0$

$\Rightarrow (\hat{x}^2+4\hat{x}+4-4)+(\hat{y}^2+6\hat{y}+9-9)-\hat{z}^2-9=0$

$\Leftrightarrow \underbrace{(\hat{x}+2)^2}_{x}-4-\underbrace{(\hat{y}+3)^2}_{y}+\cancel{9}-\underbrace{\hat{z}^2}_{z}-\cancel{9}=0$

$\Leftrightarrow x^2-y^2-z^2=4$

$\Leftrightarrow \frac{x^2}{4}-\frac{y^2}{4}-\frac{z^2}{4}=1$

Hiperbolóide de 2 folhas

→ 2 valores próprios com o mesmo sinal

→ Os valores próprios com o mesmo sinal e μ têm o mesmo sinal, logo hiperbolóide de 2 folhas

b) $x^2+2y^2+z^2-2x+4y=0$

Escrever na forma: $X^TAX+BX+\mu=0$

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Como A é simétrica, logo A ortogonalmente diagonalizável

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}=0 \Leftrightarrow (1-\lambda)(2-\lambda)(1-\lambda)=0$$
$\Leftrightarrow 1-\lambda=0 \vee 1-\lambda=0 \vee 2-\lambda=0$
$\Leftrightarrow \lambda=1 \vee \lambda=1 \vee \lambda=2$
$\Leftrightarrow \underbrace{\lambda=1}_{\text{multiplicidade } 2} \vee \lambda=2$

obter os vetores próprios associados

→ vetor associado a $\lambda_1=1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \Leftrightarrow \begin{cases} x\in\mathbb{R} \\ y=0 \\ z\in\mathbb{R} \end{cases} \rightarrow \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}, x e z \in\mathbb{R} \Leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}z, x e z \in\mathbb{R}$$

não simultaneamente nulos

Norma = 1

→ vetor associado a $\lambda_2=2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \Leftrightarrow \begin{cases} x=0 \\ y\in\mathbb{R} \\ z=0 \end{cases} \rightarrow \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}, y\in\mathbb{R} \Leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}y, y\in\mathbb{R}\setminus\{0\}$$

Norma = 1

$P^TAP=D$

$$P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Considerando a mudança de variável $X=P\hat{X}$

$X^TAX+BX+\mu=0 \overset{X=P\hat{X}}{\Rightarrow}$

$\Leftrightarrow (P\hat{X})^TAP\hat{X}+B(P\hat{X})+\mu=0$

$\Leftrightarrow \hat{X}^T\underbrace{P^TAP}_{D}\hat{X}+BP\hat{X}+\mu=0$

$$BP=\begin{bmatrix} -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} -2 & 4 & 0 \end{bmatrix}$$

$\hat{X}^TD\hat{X}+BP\hat{X}+\mu=0$

$$\Leftrightarrow \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} + \begin{bmatrix} -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}=0$$

$\Leftrightarrow \hat{x}^2+\hat{y}^2+\hat{z}^2-2\hat{x}+4\hat{y}=0$

$\Leftrightarrow (\hat{x}^2-2\hat{x})+(\hat{y}^2+4\hat{y})+\hat{z}^2=0$

$\Leftrightarrow (\hat{x}^2-2\hat{x}+1-1)+2(\hat{y}^2+2\hat{y}+1-1)+\hat{z}^2=0$

$\Leftrightarrow (\hat{x}-1)^2-1+2(\hat{y}+2)^2-2+\hat{z}^2=0$

$\Leftrightarrow x^2+2y^2+z^2=3$

$\Leftrightarrow \frac{x^2}{3}+\frac{2y^2}{3}+\frac{z^2}{3}=1$

$x=\hat{x}-1$
$y=\hat{y}+2$
$z=\hat{z}$

↓

Elipsóide

- valores próprios têm o mesmo sinal
- λ_1 e μ têm sinais contrários

daí, que seja Elipsóide

c) $x^2+y^2+4x-6y-z=0$

$X^TAX+BX+\mu=0$

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}+\begin{bmatrix} 4 & -6 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

como A é simétrica, logo A é ortogonalmente diagonalizável

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}=0 \Leftrightarrow (1-\lambda)(1-\lambda)(-\lambda)=0$$

$\Leftrightarrow \lambda=1 \vee \lambda=1 \vee \lambda=0$

$\Leftrightarrow \underbrace{\lambda=1}_{\text{multiplicidade } 2} \vee \lambda=0$

obter os vetores próprios associados

→ vetor associado a $\lambda=1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \Leftrightarrow \begin{cases} x\in\mathbb{R} \\ y\in\mathbb{R} \\ z=0 \end{cases} \rightarrow \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}, x \text{ e } y \in\mathbb{R} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}x+\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}y, x \text{ e } y\in\mathbb{R}$$

não simultaneamente nulos

Norma = 1

→ vetor associado a $\lambda=0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \Leftrightarrow \begin{cases} x=0 \\ y=0 \\ z\in\mathbb{R} \end{cases} \rightarrow \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z\in\mathbb{R} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}z, z\in\mathbb{R}\setminus\{0\}$$

Norma = 1

$P^TAP=D$

$$P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

considerando a mudança de variável $X=P\hat{X}$

$X^TAX+BX+\mu=0$

$\Leftrightarrow \hat{X}^T\underbrace{P^TAP}_{D}\hat{X}+BP\hat{X}+\mu=0$

$$BP=\begin{bmatrix} 4 & -6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 4 & -6 & -1 \end{bmatrix}$$

$\hat{X}^TD\hat{X}+BP\hat{X}+\mu=0$

$$\Leftrightarrow \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}+\begin{bmatrix} 4 & -6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}=0$$

$\Leftrightarrow \hat{x}^2+\hat{y}^2+4\hat{x}-6\hat{y}-\hat{z}=0$

$\Leftrightarrow (\hat{x}^2+4\hat{x})+(\hat{y}^2-6\hat{y})-\hat{z}=0$

$\Leftrightarrow (\hat{x}^2+4\hat{x}+4-4)+(\hat{y}^2-6y+9-9)-\hat{z}=0$

$\Leftrightarrow (\hat{x}+2)^2-4+(\hat{y}-3)^2-9-\hat{z}=0$

$\Leftrightarrow x^2+y^2=z+13$

$\Leftrightarrow x^2+y^2=z^2$

$x=\hat{x}+2$
$y=y-3$
$z=z+13$

Paraboloide Elíptico

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

d) $x^2+4y^2+4xy-2x-4y+2z+1=0$

$$\begin{bmatrix} x & y & z\end{bmatrix}\underbrace{\begin{bmatrix}1&2&0\\2&4&0\\0&0&0\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}-2&-4&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+1=0$$

como A é simétrica, logo é ortogonalmente diagonalizável

$|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda&2&0\\2&4-\lambda&0\\0&0&-\lambda\end{vmatrix}=0 \Leftrightarrow (1-\lambda)(4-\lambda)(-\lambda)-(-\lambda)\times2\times2=0$$

$\Leftrightarrow -\lambda\left((1-\lambda)(4-\lambda)-4\right)=0$

$\Leftrightarrow \lambda=0 \vee (\cancel{4}-\lambda-4\lambda+\lambda^2-\cancel{4})=0$

$\Leftrightarrow \lambda=0 \vee \lambda(-5+\lambda)=0$

$\Leftrightarrow \lambda=0 \vee \lambda=0 \vee \lambda=5$

Obter os vetores próprios associados

→ vetor associado $\lambda=0$

$$\begin{bmatrix}1&2&0\\2&4&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=0 \Leftrightarrow \begin{cases}x=-2y\\y\in\mathbb{R}\\z\in\mathbb{R}\end{cases} \rightarrow \begin{bmatrix}-2y\\y\\z\end{bmatrix}, y e z \in\mathbb{R} \Leftrightarrow \begin{bmatrix}-2\\1\\0\end{bmatrix}y+\begin{bmatrix}0\\0\\1\end{bmatrix}z, \ y e z \in \mathbb{R}$$

não simultaneamente nulos

$\|\lambda_1\|=\sqrt{4+1}=\sqrt{5}$ $\quad \lambda_1\,(-2\sqrt{5},\sqrt{5},0)$

$\|\lambda_2\|=1$ $\quad \lambda_2\,(0,0,1)$

→ Vetor associado $\lambda=5$

$$\begin{bmatrix}-4&2&0\\2&-1&0\\0&0&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=0 \Leftrightarrow \begin{cases}x\in\mathbb{R}\\y=2x\\z=0\end{cases} \rightarrow \begin{bmatrix}x\\2x\\0\end{bmatrix}, x\in\mathbb{R} \rightarrow \begin{bmatrix}1\\2\\0\end{bmatrix}x, x\in\mathbb{R}\setminus\{0\}$$

$\|\lambda_3\|=\sqrt{5}$ $\quad \lambda_3\,(\sqrt{5},2\sqrt{5},0)$

$P^TAP=D$

$$P=\begin{bmatrix}\frac{\sqrt5}{5}&0&-\frac{2\sqrt5}{5}\\\frac{2\sqrt5}{5}&0&\frac{\sqrt5}{5}\\0&1&0\end{bmatrix} \qquad D=\begin{bmatrix}5&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\uparrow \lambda_3 \quad \uparrow \lambda_2 \quad \uparrow \lambda_1$

Consideremos a mudança de variável $X=P\hat{X}$

$X^TAX+BX+\mu=0$

$\Leftrightarrow \hat{X}^T\underbrace{P^TAP}_{D}\hat{X}+BP\hat{X}+\mu=0$

$$BP=\begin{bmatrix}-2&-4&2\end{bmatrix}\begin{bmatrix}\frac{\sqrt5}{5}&0&-\frac{2\sqrt5}{5}\\\frac{2\sqrt5}{5}&0&\frac{\sqrt5}{5}\\0&1&0\end{bmatrix}=\begin{bmatrix}-\frac{10\sqrt5}{5}&2&0\end{bmatrix}=\begin{bmatrix}-2\sqrt5&2&0\end{bmatrix}$$

$\hat{X}^TDX+BP\hat{X}+\mu=0$

$$\Leftrightarrow\begin{bmatrix}\hat{x}&\hat{y}&\hat{z}\end{bmatrix}\begin{bmatrix}5&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\hat{x}\\\hat{y}\\\hat{z}\end{bmatrix}+\begin{bmatrix}-2\sqrt5&2&0\end{bmatrix}\begin{bmatrix}\hat{x}\\\hat{y}\\\hat{z}\end{bmatrix}+1$$

$\left(\frac{2\sqrt5}{5}\right)$ → Dividir por 2 e levar ao quadrado

sendo

$x=\hat{x}-\frac{\sqrt5}{5}$

$y=\hat{y}$

$\Leftrightarrow 5\hat{x}^2-2\sqrt5\hat{x}+2\hat{y}+1=0$

$\Leftrightarrow (5\hat{x}^2-2\sqrt5\hat{x})+2\hat{y}+1=0$

$\Leftrightarrow 5(\hat{x}^2-\frac{2\sqrt5}{5}\hat{x}+1/5+1/5)+2\hat{y}+1=0$

$\Leftrightarrow 5(x-\frac{\sqrt5}{5})^2-\cancel{1}+2\hat{y}+\cancel{1}=0$

$\Leftrightarrow 5x^2+2y=0$

$\Leftrightarrow y=-\frac{5}{2}x^2$

Cilindro Parabólico

$\cdot\ y=ax^2$

③ $5x^2+5y^2+2xy+2x-2y+\alpha=0$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\alpha=0$$

$|A-\lambda I|=0$

$$\begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix}=0 \Leftrightarrow (5-\lambda)(5-\lambda)-1=0$$
$$\Leftrightarrow 25-10\lambda+\lambda^2-1=0$$
$$\Leftrightarrow \lambda^2-10\lambda+24=0$$
$$\Leftrightarrow \lambda=6 \vee \lambda=4$$

$$\lambda=\frac{10\pm\sqrt{100-96}}{2} \Leftrightarrow \lambda=\frac{10\pm 2}{2} \Leftrightarrow \lambda=6 \vee \lambda=4$$

<u>vetores associados</u>

→ vetor associado a $\lambda=6$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0 \Leftrightarrow \begin{cases} x=y \\ y\in\mathbb{R} \end{cases} \rightarrow \begin{bmatrix} y \\ y \end{bmatrix}, y\in\mathbb{R} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}y, y\in\mathbb{R}\setminus\{0\}$$

$\|(1,1)\|=\sqrt{2}$ $\qquad \lambda_1=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

→ vetor associado a $\lambda=4$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0 \Leftrightarrow \begin{cases} x=-y \\ y\in\mathbb{R} \end{cases} \rightarrow \begin{bmatrix} -y \\ y \end{bmatrix}, y\in\mathbb{R} \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}y, y\in\mathbb{R}\setminus\{0\}$$

$\|(-1,1)\|=\sqrt{2}$ $\qquad \lambda_2=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

$P^TAP=D$

$$P=\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad D=\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

Fazendo a mudança de variável $x=P\hat{x}$

$$X^TAX+BX+\alpha=0$$
$$\Leftrightarrow \hat{X}^T\underbrace{P^TAP}_{D}X+BP\hat{X}+\alpha=0$$

$$BP=\begin{bmatrix} 2 & -2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}=\begin{bmatrix} 0 & -2\sqrt{2} \end{bmatrix}$$

<u>E fica:</u>

$$\hat{X}^TD\hat{X}+BP\hat{X}+\alpha=0$$
$$\Leftrightarrow \begin{bmatrix} \hat{x} & \hat{y} \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}+\begin{bmatrix} 0 & -2\sqrt{2} \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}+\alpha=0$$

$$\Leftrightarrow 6\hat{x}^2+4\hat{y}^2-2\sqrt{2}\hat{y}+\alpha=0$$
$$\Leftrightarrow 6\hat{x}^2+(4\hat{y}^2-2\sqrt{2}\hat{y})+\alpha=0$$
$$\Leftrightarrow 6\hat{x}^2+4\left(\hat{y}^2-\frac{2\sqrt{2}}{4}y+\frac{2}{16}-\frac{2}{16}\right)+\alpha=0$$
$$\Leftrightarrow 6\hat{x}^2+4\left(\hat{y}^2-\frac{2\sqrt{2}}{4}+\frac{2}{16}\right)-\frac{1}{2}+\alpha=0$$
$$\Leftrightarrow 6\hat{x}^2+4\left(\hat{y}-\frac{\sqrt{2}}{4}\right)^2-\frac{1}{2}+\alpha=0$$
$$\Leftrightarrow 6x^2+4y^2\underbrace{-\frac{1}{2}+\alpha}_{\mu}=0$$

<u>sendo</u>

$\hat{x}=\hat{x}$

$y=y-\frac{\sqrt{2}}{4}$

<u>Elipse</u> $\lambda_1x^2+\lambda_2y^2+\mu=0$

$$-\frac{1}{2}+\alpha<0 \Leftrightarrow \alpha<\frac{1}{2}$$

④ a)

$A=\begin{bmatrix}0&2\\2&0\end{bmatrix}$ $P=\begin{bmatrix}\frac{\sqrt2}{2}&-\frac{\sqrt2}{2}\\\frac{\sqrt2}{2}&\frac{\sqrt2}{2}\end{bmatrix}$

$|A-\lambda I|=0$

$\begin{vmatrix}-\lambda&2\\2&-\lambda\end{vmatrix}=0 \Leftrightarrow \lambda^2-4=0$

$\Leftrightarrow \lambda=2 \vee \lambda=-2$

· Para que P seja ortogonal o produto interno é zero

$(\frac{\sqrt2}{2},\frac{\sqrt2}{2})\cdot(-\frac{\sqrt2}{2},\frac{\sqrt2}{2})=0 \Leftrightarrow 0=0$

$\cdot D=\begin{bmatrix}2&0\\0&-2\end{bmatrix}$

b) $4xy+x+y=0$

$\begin{bmatrix}x&y\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$

$A=\begin{bmatrix}0&2\\2&0\end{bmatrix}$ $B=\begin{bmatrix}1&1\end{bmatrix}$ $P=\begin{bmatrix}\frac{\sqrt2}{2}&-\frac{\sqrt2}{2}\\\frac{\sqrt2}{2}&\frac{\sqrt2}{2}\end{bmatrix}$

$BP=\begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt2}{2}&-\frac{\sqrt2}{2}\\\frac{\sqrt2}{2}&\frac{\sqrt2}{2}\end{bmatrix}$

$=\begin{bmatrix}\sqrt2&0\end{bmatrix}$

$X^TDX+BPX+\mu=0$

$\Leftrightarrow \begin{bmatrix}x&y\end{bmatrix}\begin{bmatrix}2&0\\0&-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}\sqrt2&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$

$\Leftrightarrow 2\hat{x}^2-2\hat{y}^2+\sqrt2 x=0$

$\Leftrightarrow (2\hat{x}^2+\sqrt2 x)-2\hat{y}^2=0$

$\Leftrightarrow 2(\hat{x}^2+\frac{\sqrt2}{2}x+1/8-1/8)-2\hat{y}^2=0$

$\Leftrightarrow 2(\hat{x}+\sqrt2/4)^2-1/4-2\hat{y}^2=0$

$\Leftrightarrow 2x^2-2y^2=\frac14$

$\Leftrightarrow x^2-y^2=\frac18$

Sendo

$x=\hat{x}+\frac{\sqrt2}{4}$

$y=\hat{y}$

↳ Hipérbole

⑤ $A_1(0,1,1)$ $d=D+1$

$P_1(x,y,z)$

$\vec{AP}=P-A=(x,y,z)-(0,1,1)=(x,y-1,z-1)$ $d=\|\vec{AP}\|=\sqrt{x^2+(y-1)^2+(z-1)^2}$

$\vec{OP}=P-O=(x,y,z)-(0,0,0)=(x,y,z)$ $D=\|\vec{OP}\|=\sqrt{x^2+y^2+z^2}$

$d=D+1$

$\Leftrightarrow \|\vec{AP}\|=\|\vec{OP}\|+1$

$\Leftrightarrow \left(\sqrt{x^2+(y-1)^2+(z-1)^2}\right)^2=\left(\sqrt{x^2+y^2+z^2}+1\right)^2$

$\Leftrightarrow x^2+(y-1)^2+(z-1)^2=1+2\sqrt{x^2+y^2+z^2}+x^2+y^2+z^2$

$\Leftrightarrow \cancel{x^2}+\cancel{y^2}-2y+\cancel{1}+\cancel{z^2}-2z+1=\cancel{1}+2\sqrt{x^2+y^2+z^2}+\cancel{x^2}+\cancel{y^2}+\cancel{z^2}$

$\Leftrightarrow -2y-2z+1=2\sqrt{x^2+y^2+z^2}$

$\Leftrightarrow ((-2y+1)-2z)^2=(2\sqrt{x^2+y^2+z^2})^2$

$\Leftrightarrow (-2y+1)^2-4z(-2y+1)+4z^2=4(x^2+y^2+z^2)$

$\Leftrightarrow \cancel{4y^2}-4y+1+8zy-4z+\cancel{4z^2}=4x^2+\cancel{4y^2}+\cancel{4z^2}$

$\Leftrightarrow -4x^2+8zy-4y-4z+1=0$

$\Leftrightarrow 4x^2-8zy+4y+4z-1=0$

Acabar

⑥

Seja $X=(x,y,z)$ um ponto de $\mathbb{R}^3$

Seja $A=(0,0,-2)$

A distância entre (x,y,z) e $z+18=0$ é dada pela forma da distância entre plano e o ponto

$d(X,P)=\frac{|z+18|}{\sqrt{1^2}}=|z+18|$

Os pontos de $\mathbb{R}^3$ que queremos são os pontos que satisfazem:

$d(A,X)=\frac{1}{3}d(X,P)$

$\Leftrightarrow \sqrt{x^2+y^2+(z+2)^2}=\frac{1}{3}|z+18|$

$\Leftrightarrow x^2+y^2+(z+2)^2=\left(\frac{1}{3}|z+18|\right)^2$

$\Leftrightarrow x^2+y^2+(z+2)^2=\frac{1}{9}|z+18|^2$

$\Leftrightarrow x^2+y^2+(z+2)^2-\frac{1}{9}(z+18)^2=0$

$\Leftrightarrow x^2+y^2+z^2+4z+4-\frac{1}{9}(z^2+36z+324)=0$

$\Leftrightarrow 9x^2+9y^2+9z^2+36z+36-z^2-36z-324=0$

$\Leftrightarrow 9x^2+9y^2+8z^2-288=0$

$\Leftrightarrow 9x^2+9y^2+8z^2=288$

$\Leftrightarrow \frac{x^2}{32}+\frac{y^2}{32}+\frac{z^2}{36}=1$

<u>Elipsóide</u>