Universidade de Aveiro
Departamento de Matemática

Cálculo II – Agrupamento II **2016/2017**

Soluções dos exercícios do capítulo 2 – Equações Diferenciais

1. (a) $df = y\cos(xy)\,dx + x\cos(xy)\,dy|_{P=(0,1)} = dx$;
   (b) $df = (2x - yz)\,dx + (-2y - xz)\,dy + (-2z - xy)\,dz|_{P=(1,1,0)} = 2dx - 2dy - dz$;
   (c) $df = y^3\,dx + 3xy^2\,dy - \pi\sin(\pi z)\,dz|_{P=(1,3,1)} = 27dx + 27dy$.

2. $x = -2$, $y = 3$, $dx \cong -2.02 - (-2) = -0.02$, $dy \cong 3.01 - 3 = 0.01 \Rightarrow df = (2x - 9x^2y^2 + 4)\,dx + (-9x^2y^2 - 6y^2)\,dy \cong 7.38$.

3. (a) $f(x,y,z) = xyz$, $x = 9$, $y = 6$, $z = 4$, $dx \cong 9.02 - 9 = 0.02$, $dy \cong 5.97 - 6 = -0.03$, $dz \cong 4 - 4.01 = -0.01 \Rightarrow df = yz\,dx + xz\,dy + xy\,dz \cong -0.06$.
   (b) $f(9.02, 5.97, 4.01) - f(9,6,4) = -0.063906$.

6. (a) Ordem: 2, variável independente: x, variável dependente: y.
   (b) Ordem: 3, variável independente: t, variável dependente: x.

12. (a) $y(x) = x(-1 + \ln x) + C$, $C \in \mathbb{R}$; (b) $y(x) = \frac{1}{4}x^4 + Ax + B$, $A, B \in \mathbb{R}$.

14. (a) $xy' - y\ln y = 0$; (b) $yy'' - (y')^2 = 0$; (c) $y''' + y' = 0$.

15. Considerando, por exemplo, o integral geral $y = x + c$, $c \in \mathbb{R}$, não é possível obter nenhuma solução do tipo $y = -x + b$, com $b \in \mathbb{R}$, atribuíndo valores reais à constante c do integral geral.

19. $y = -\frac{1}{6}x^3 + 1$.

20. $y = -\frac{1}{6}x^3 + \frac{3}{2}$.

23. (a) $y = Cx$, $C \in \mathbb{R}$; (b) $x^2 + y^2 = C$, $C \in \mathbb{R}^+$; (c) $x\mathrm{e}^{\frac{1}{x}} = Ct\mathrm{e}^{-\frac{1}{t}}$, $C \neq 0$.

24. (a) I. ger. $y = \frac{1}{1 - Cx}$, $C \in \mathbb{R}$ (e s. sing. $y = 0$) $\Rightarrow$ s. PVI: $y = \frac{1}{1+x}$, $x > -1$;
    (b) I. ger. $y = C\mathrm{e}^{-\sqrt{4+x^2}} - 1$, $C \in \mathbb{R} \Rightarrow$ s. PVI: $y = 2\mathrm{e}^{2-\sqrt{4+x^2}} - 1$;
    (c) I. ger. $y = C\sqrt[3]{1 + x^3}$, $C \in \mathbb{R} \Rightarrow$ s. PVI: $y = \sqrt[3]{4(1 + x^3)}$.

26. $(x-1)^2 + (y+5)^2 = C\mathrm{e}^{2\,\mathrm{arctg}\left(\frac{y+5}{x-1}\right)}$, $C \in \mathbb{R}^+$.

27. (a) $y = C\mathrm{e}^{\frac{x^2}{2y^2}}$, $C \in \mathbb{R}$; (b) $y = x\mathrm{e}^{Cy}$, $x > 0$, $C \in \mathbb{R}$.

28. (b) $y = x\mathrm{e}^{Cx}$, $x > 0$, $C \in \mathbb{R}$.

29. (a) $(x-2)^2 + (y-1)^2 = C\mathrm{e}^{2\,\mathrm{arctg}\,\frac{y-1}{x-2}}$, $C > 0$; (b) $(y-x)^2 + 4y = C$, $C \in \mathbb{R}$.

32. $x^2\mathrm{e}^y + xy - y^2 = C$, $C \in \mathbb{R}$.

33. (a) $A = \frac{3}{2}$, $\frac{1}{3}x^3 + \frac{3}{2}x^2y + 2y^2 = C$, $C \in \mathbb{R}$; (b) $A = -2$, $\frac{y}{x^2} - \frac{y}{x} = C$, $C \in \mathbb{R}$.

34. $G(x,y) = x^2\cos y + x^3y - \frac{1}{2}y^2 + \frac{9}{32}$

38. I. ger. $y = \frac{x+C}{x^2}$, $C \in \mathbb{R}$.

43. (a) $y = 1 + x\ln x + Cx$, $C \in \mathbb{R}$; (b) $y = \frac{e^x + C}{x}$, $C \in \mathbb{R}$;
(c) $y = \frac{1}{2}e^{-x} + Ce^x$, $C \in \mathbb{R}$; (d) $y = \frac{2}{5}\cos x + \frac{1}{5}\operatorname{sen} x + Ce^{-2x}$, $C \in \mathbb{R}$.

45. (a) $y = -xe^x$; (b) $y = e^{\frac{4}{3}x} - \frac{1}{4}x - \frac{3}{16}$.

48. (c) I. gen. $y = \frac{e^{-x}}{C-x}$, $C \in \mathbb{R}$ (e s. sing. $y = 0$).

49. (a) I. gen. $y = \frac{1}{x(C-x)}$, $C \in \mathbb{R}$ (e s. sing. $y = 0$);
(b) (também EDO de variáveis separáveis) I. gen. $y = \frac{1}{1+Ce^{-\cos x}}$, $C \in \mathbb{R}$(e s. sing. $y = 0$);
(c) $y = \sqrt[3]{\frac{5x^6}{49-9x^5}}$; (d) $y = \frac{1}{x\sqrt[3]{1-\frac{\pi}{4}+\operatorname{arctg} x}}$.

50. (a) $y = \frac{C}{x^2} - \frac{x^2}{2}$, $C \in \mathbb{R}$; (b) $2y - 3x - 3 = Ce^{x-y}$, $C \in \mathbb{R}$; (c) $4x^3y + 3y^4 = C$, $C \in \mathbb{R}$;
(d) $x = -t^2 + 4 + Ct$, $C \in \mathbb{R}$; (e) $y^2 = \frac{x^2}{C-2x}$, $C \in \mathbb{R}$; (f) $y^4 + Cy^3 = s$, $C \in \mathbb{R}$;
(g) $\frac{1}{2}y^2 + y + 2xy - 2x = C$, $C \in \mathbb{R}$.

54. (i) $a_0(x)(y_1 - y_2)^{(n)} + \cdots + a_{n-1}(x)(y_1 - y_2)' + a_n(x)(y_1 - y_2) = \big(a_0(x)y_1^{(n)} + \cdots + a_{n-1}(x)y_1' + a_n(x)y_1\big) - \big(a_0(x)y_2^{(n)} + \cdots + a_{n-1}(x)y_2' + a_n(x)y_2\big) = b(x) - b(x) = 0$.
(ii) Se y_p é uma solução da equação completa e y_h uma solução da equação homogénea, $a_0(x)(y_p + y_h)^{(n)} + \cdots + a_{n-1}(x)(y_p + y_h)' + a_n(x)(y_p + y_h) = \big(a_0(x)y_p^{(n)} + \cdots + a_{n-1}(x)y_p' + a_n(x)y_p\big) + \big(a_0(x)y_h^{(n)} + \cdots + a_{n-1}(x)y_h' + a_n(x)y_h\big) = b(x) + 0 = b(x)$.
(iii) Sejam y_1 e y_2 soluções particulares da equação homogénea e $c_1, c_2 \in \mathbb{R}$; então, $a_0(x)(c_1y_1 + c_2y_2)^{(n)} + \cdots + a_{n-1}(x)(c_1y_1 + c_2y_2)' + a_n(x)(c_1y_1 + c_2y_2) = c_1\big(a_0(x)y_1^{(n)} + \cdots + a_{n-1}(x)y_1' + a_n(x)y_1\big) + c_2\big(a_0(x)y_2^{(n)} + \cdots + a_{n-1}(x)y_2' + a_n(x)y_2\big) = c_1 \cdot 0 + c_2 \cdot 0 = 0$.

61. (a) $y = c_1e^{-x} + c_2e^{-3x}$, $c_1, c_2 \in \mathbb{R}$; (b) $y = c_1 + c_2x + c_3\cos x + c_4\operatorname{sen} x$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$; (c) $y = c_1 + c_2e^x + c_3e^{-x} + c_4e^{3x}$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$; (d) $y = c_1e^{-x}\cos 2x + c_2e^{-x}\operatorname{sen} 2x$, $c_1, c_2 \in \mathbb{R}$; (e) $y = c_1 + c_2\cos x + c_3\operatorname{sen} x$, $c_1, c_2, c_3 \in \mathbb{R}$; (f) $y = c_1\cos x + c_2\operatorname{sen} x + c_3x\cos x + c_4x\operatorname{sen} x$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

62. (a) $y = c_1e^x + c_2xe^x + xe^x\ln|x|$, $c_1, c_2 \in \mathbb{R}$; (b) $y = c_1e^x + c_2e^x\cos x + c_3e^x\operatorname{sen} x + e^x\big(\ln|\frac{1+\operatorname{sen} x}{\cos x}| - x\cos x + \operatorname{sen} x\ln|\cos x|\big)$, $c_1, c_2, c_3 \in \mathbb{R}$; (c) $y = c_1e^x + c_2e^{2x} + e^x\big(xe^x - (1 + e^x)\ln|1 + e^x|\big)$, $c_1, c_2 \in \mathbb{R}$; (d) $y = ce^{-\operatorname{sen} x} + \operatorname{sen} x - 1$, $c \in \mathbb{R}$.

65. (a) $y = c_1e^{4x} + c_2e^{-x} - x^2 + \frac{3}{2}x - \frac{13}{8}$, $c_1, c_2 \in \mathbb{R}$; (b) $y = c_1e^{4x} + c_2e^{-x} - \frac{1}{17}(3\operatorname{sen} x - 5\cos x)$, $c_1, c_2 \in \mathbb{R}$; (c) $y = ce^{-2x} + \frac{1}{8}(4x^3 - 6x^2 + 18x - 5)$, $c \in \mathbb{R}$; (d) $y = ce^x + \frac{1}{4}e^{3x}(2x^2 - 2x + 3)$, $c \in \mathbb{R}$; (e) $y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\operatorname{sen} x + \cos x)$, $c_1, c_2 \in \mathbb{R}$; (f) $y = c_1 + c_2e^{-x} + \frac{1}{3}x^3 - x^2 + 6x$, $c_1, c_2 \in \mathbb{R}$; (g) $y = c_1 + c_2\cos x + c_3\operatorname{sen} x - \frac{1}{2}x\operatorname{sen} x$, $c_1, c_2, c_3 \in \mathbb{R}$; (h) $y = c_1 + c_2x + c_3e^x + c_4e^{-x} - \frac{1}{12}x^4 - x^2 + \frac{1}{2}xe^x$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

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Soluções dos exercícios dos capítulo 3 e 4 – Transformada de Laplace (inversa)

3. Transformada de Laplace

6. (1) $\frac{2a^2}{s(s^2+4a^2)}$, $s > 0$; (2) $\frac{s^2+2a^2}{s(s^2+4a^2)}$, $s > 0$; (3) $\frac{3}{4}\left(\frac{1}{s^2+a^2} + \frac{1}{s^2+9a^2}\right)$, $s > 0$; (4) $\frac{1}{4}\left(\frac{3s}{s^2+a^2} + \frac{s}{s^2+9a^2}\right)$, $s > 0$; (5) $\frac{6a}{s^4} + \frac{2b}{s^3} + \frac{c}{s^2} + \frac{d}{s}$, $s > 0$.

8. Dado qualquer $s \in \mathbb{R}$, $y = t(t-s)$ é uma parábola com mínimo $\frac{s^2}{4}$; defina-se $m = \mathrm{e}^{\frac{s^2}{4}}$. A transformada de Laplace de $f(t) = \mathrm{e}^{t^2}$ é o limite do integral $\int_0^T f(t)\mathrm{e}^{-st}\,dt = \int_0^T \mathrm{e}^{t(t-s)}\,dt \geq \int_0^T \mathrm{e}^{\frac{s^2}{4}}\,dt = mT \to +\infty$, que diverge quando $T \to +\infty$.

12. $\lim\limits_{t\to+\infty} \mathrm{e}^{-st}f(t) = 0 \Leftrightarrow \lim\limits_{t\to+\infty} |\mathrm{e}^{-st}f(t)| = \lim\limits_{t\to+\infty} \mathrm{e}^{-st}|f(t)| = 0$; em particular, isso acontece se $\lim\limits_{t\to+\infty} \mathrm{e}^{-st}|f(t)| \leq \lim\limits_{t\to+\infty} \mathrm{e}^{-st}M\mathrm{e}^{at} = \lim\limits_{t\to+\infty} M\mathrm{e}^{(a-s)t} = 0 \Leftrightarrow a - s < 0 \Leftrightarrow s > a$.

14. Para $t \geq 0$, tem-se que (a) $|\frac{1}{1+t}| \leq 1 = 1 \cdot \mathrm{e}^{0t}$ e (b) $|\frac{\mathrm{e}^{at}}{1+t}| \leq \mathrm{e}^{at} = 1 \cdot \mathrm{e}^{at}$.

24. $\int_0^{+\infty} t^{10}\mathrm{e}^{-2t}\,dt = \mathcal{L}\{t^{10}\}(2) = \frac{10!}{2^{11}}$.

26. (1) $-\frac{3}{2}\frac{s}{s^2+\frac{9}{4}}$, $s > \frac{3}{2}$; (2) $\frac{2s^2}{(s-2)^3}$, $s > 2$; (3) $\frac{8s}{s^2+4} - \frac{1}{s+1}$, $s > 0$.

27. (1) $\frac{24s}{s^2+64} + \frac{2}{s^3} - \frac{48}{s+2}$, $s > 2$; (2) $\frac{3}{(s-6)^2+9}$, $s > 6$; (3) $\frac{1}{(s-10)^3} + \frac{1}{(s+2)^3}$, $s > 10$; (4) $\frac{4}{s} + \frac{1}{s^2} + \frac{10}{s^3} - \frac{30!\pi}{(s+3)^{31}}$, $s > 0$; (5) $\frac{10+\mathrm{e}^{-\pi s}}{s^2+1}$, $s > 0$; (6) $\frac{6\mathrm{e}^{-8s}}{(s-4)^4}$, $s > 4$.

4. Transformada de Laplace Inversa

9. (1) $f(t) = \operatorname{sen} 5t$, $t \geq 0$; (2) $f(t) = 3\mathrm{e}^{4t}$, $t \geq 0$; (3) $f(t) = 3\mathrm{e}^{-2t}\cos(6t)$, $t \geq 0$; (4) $f(t) = \frac{1}{\sqrt{2}}\mathrm{e}^{-2t}\operatorname{sen}(\sqrt{2}t)$, $t \geq 0$; (5) $f(t) = \mathrm{e}^{2t}\big(3\cos(3t) + \frac{5}{3}\operatorname{sen}(3t)\big)$, $t \geq 0$; (6) $f(t) = \frac{1}{3}\mathrm{e}^{t}\big(4 + \mathrm{e}^{-1}H_1\big) + \frac{1}{3}\mathrm{e}^{-2t}\big(8 - \mathrm{e}^{2}H_1\big)$, $t \geq 0$; (7) $f(t) = \frac{1}{4}t\operatorname{sen} 2t$, $t \geq 0$.

10. $y(t) = \frac{1}{3}(5\mathrm{e}^{-2t} + \mathrm{e}^{t})$, $t \geq 0$. Contudo, pelo teorema sobre a existência e unicidade do PVI, $t \in \mathbb{R}$.