Unmanned Autonomous Systems - 31390

Control of Unmanned Aerial Vehicles

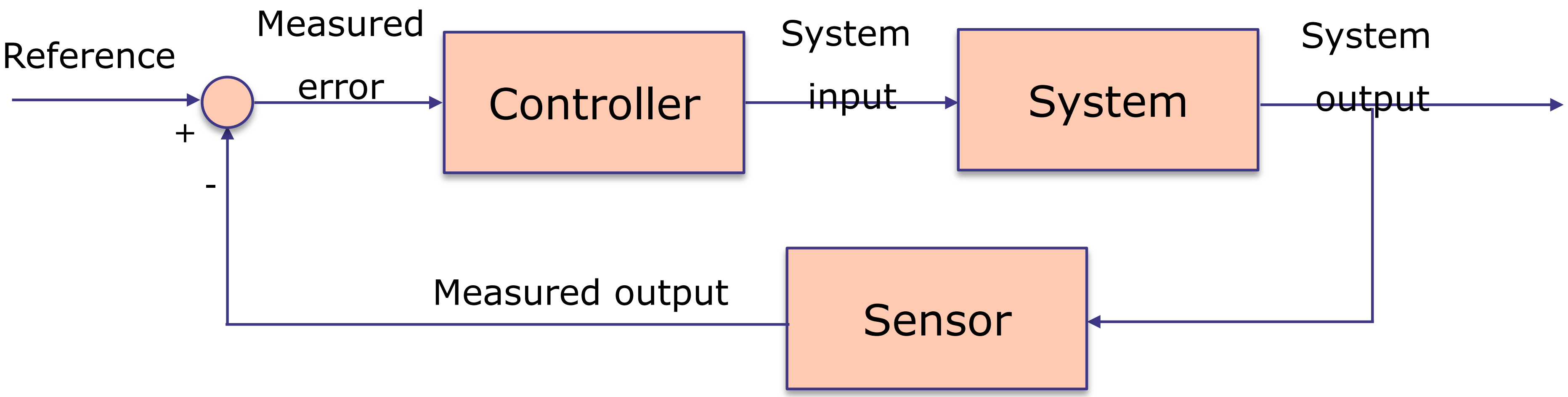
Contributors:
Ananda Nielsen
Tobias Stenbock Andersen
Daniel Saunte Høgh

Matteo Fumagalli
Associate Professor
Automation and Control Group
Department of Electrical Engineering
DTU Lyngby, Building 326

CONTROL

- The process of modifying the behavior of a dynamical system with inputs, through feedback
- Example: Space flight
  - A rocket is a dynamical system with inputs; it has thrusters that can change the heading of the rocket
  - The feedback is the current heading of the rocket
  - The rocket controller:
    - Analyzes the current heading the rocket is having (feedback)
    - Compares to the desired heading
    - Change the thruster settings to achieve the desired heading (input)

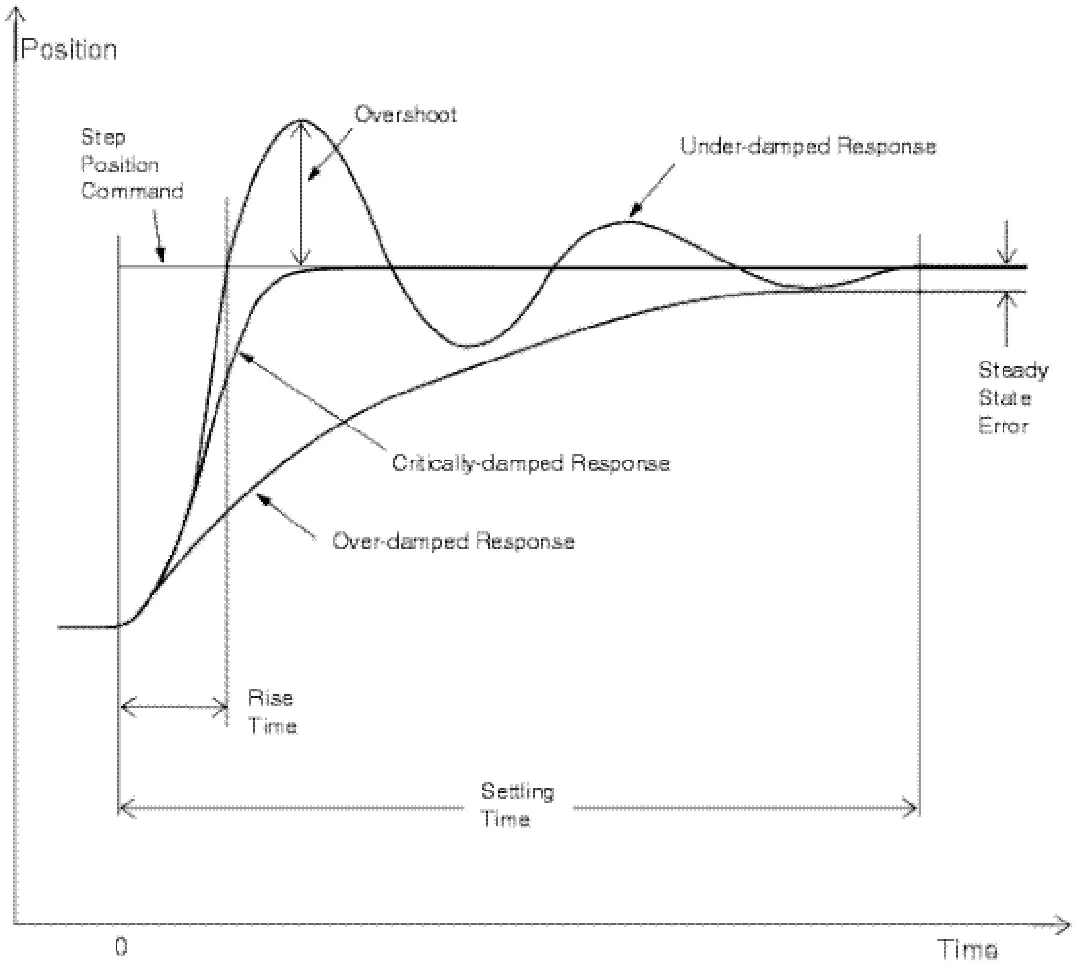
CONTROL



Reference
Measured
error
Controller
System
input
System
System
output
+
-
Measured output
Sensor

CONTROL

Common controller performance metrics



- **Steady state**
  The value of the system at t = ∞
  (or rather the limit of the response as t → ∞)
- **Transient response and steady state response**
  Transient response is the response in the beginning of a step input
  Steady state response is the response when the system has reached SS
- **Steady state error**
  The (constant) error between the target value and steady state output
- **Rise time**
  The time required for the system to reach the target value
- **Settling time**
  The time required for the system to reach steady state
- **Overshoot**
  How much the output “shoots over” the target value

Recap: Basics of Control

CONTROL

Definition

- **Rise time:**
  Not practical to use time when output reaches 100%
  Instead, use time from input is given to 80% is reached
- **Settling time:**
  When has the system settled?
  Usually, when the output stays within some range of the target value.
  In this lecture, we will use 5%.

CONTROL

Feed-Forward Control: We compute the control input without considering the actual state of the system (without measurements)

Once we have defined our trajectory, we could compute the motor torque necessary to perform a certain motion, following the specified trajectory

$$F_{in}^{\star} = m\ddot{x}^{\star} + D\dot{x}^{\star} + F_{ext}$$

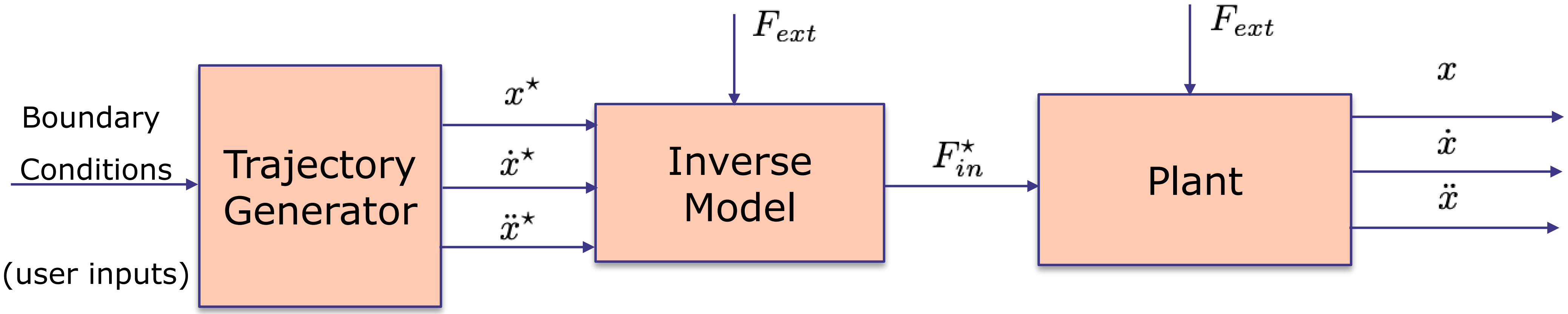
$F_{in}^{\star}$ is the computed input force

$\ddot{x}^{\star}$ is the acceleration profile of our trajectory

$\dot{x}^{\star}$ is the velocity profile of our trajectory

CONTROL

Feed-Forward Control: control scheme



CONTROL

Feedback Control: we compute an error function that our controller tries to minimize

The error can be computed as the difference between the measured value and our reference

$$e_x = x^\star - x$$

The control input $F^\star_{in}$ can be calculated as a function of the error:

$$F^\star_{in} = R(e_x)$$

CONTROL

Proportional Control:

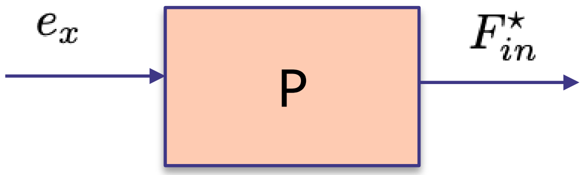
If our aim is to control the position of the moving mass, the error function can be defined as:

$$e_x = x^\star - x$$

And our control input $F_{in}^\star$ can be calculated as:

$$F_{in}^\star = R(e_x) = K_P e_x$$

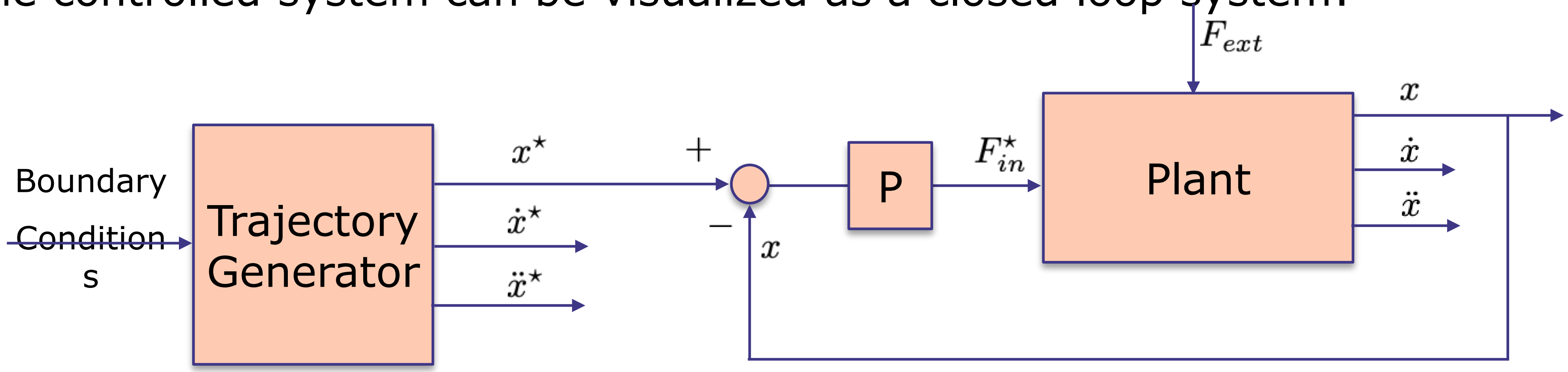
Note that the regulator of the error generates a control that is proportional to the same error. In this case, we call it PROPORTIONAL POSITION CONTROL, with block scheme:



CONTROL

Proportional Control:

The controlled system can be visualized as a closed loop system:



The behavior of the closed loop system is

$$\boxed{m\ddot{x} + D\dot{x} + K_P x} = K_P x^\star - F_{ext}$$

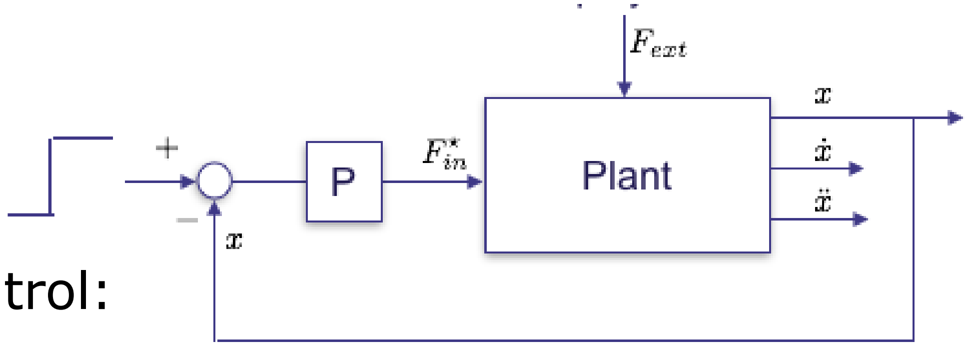
It is equivalent to a mass-spring-damper system, s.t. the external components $K_P x^\star - F_{ext}$

At steady-state ($\dot{x} = 0, \; \ddot{x} = 0$), the error is:

$$e_x = \frac{F_{ext}}{K_P}$$

CONTROL

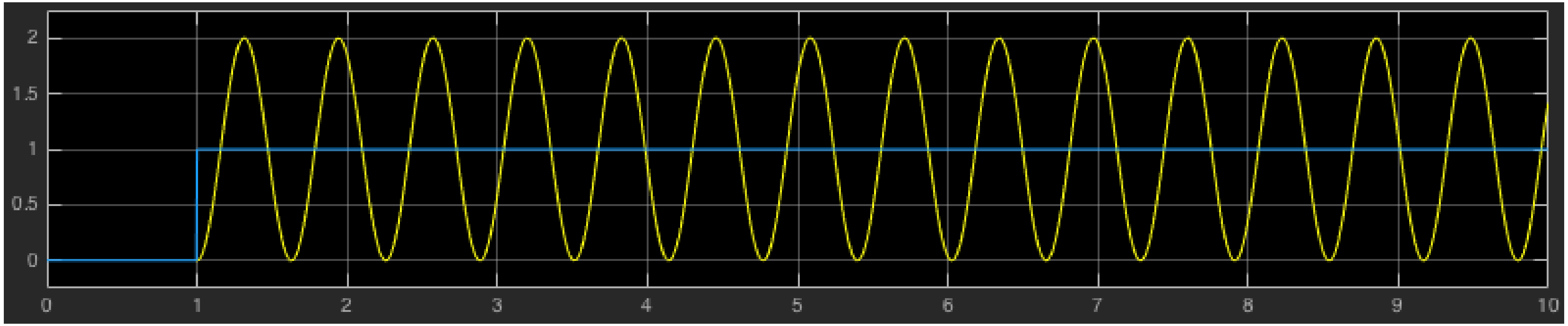
Proportional Control:



Limitations of the proportional position control:

Analysis of the STEP response

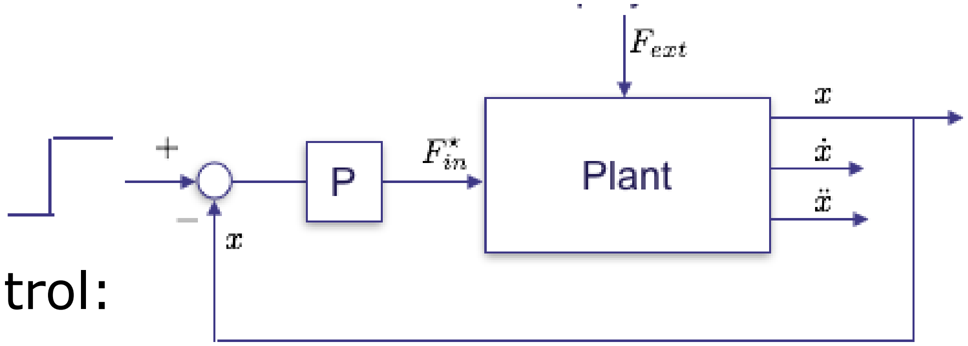
- if the mechanical system is not damped $D = 0$, we may have harmonic oscillations



CONTROL

Proportional Control:

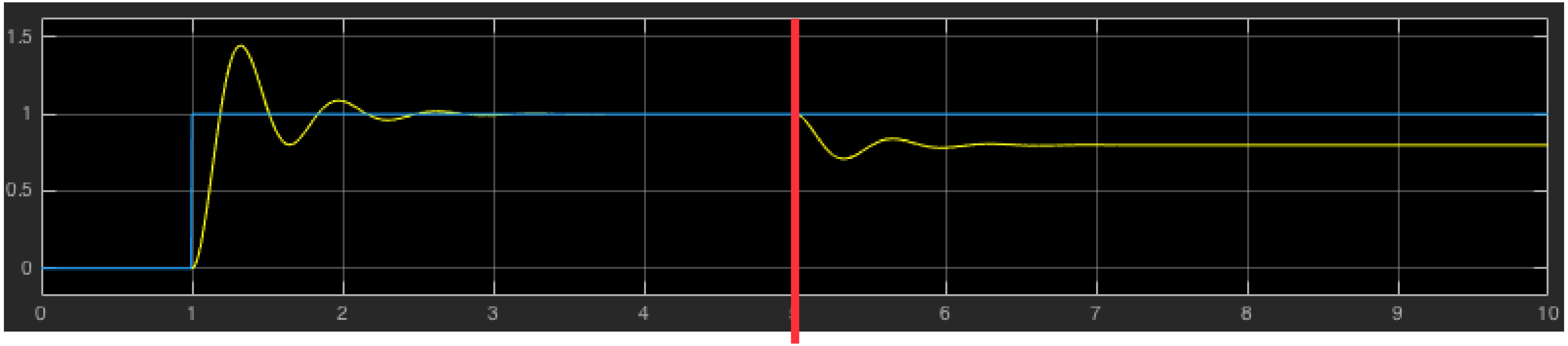
Limitations of the proportional position control:



Analysis of the STEP response

- if an external load is applied, it cannot guarantee steady state null error

$$D \neq 0$$



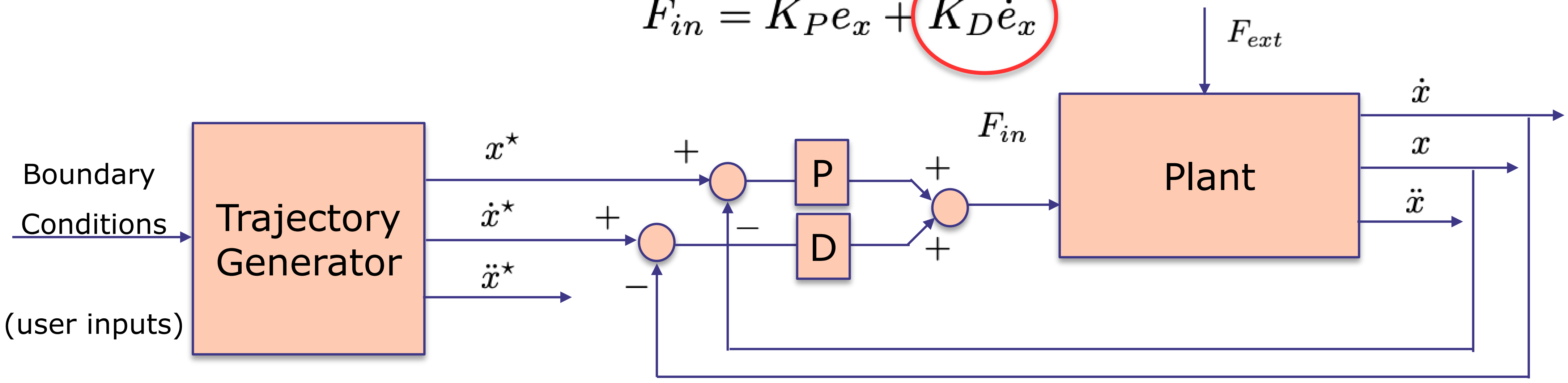
Application of external load

CONTROL

Proportional-Derivative Control:

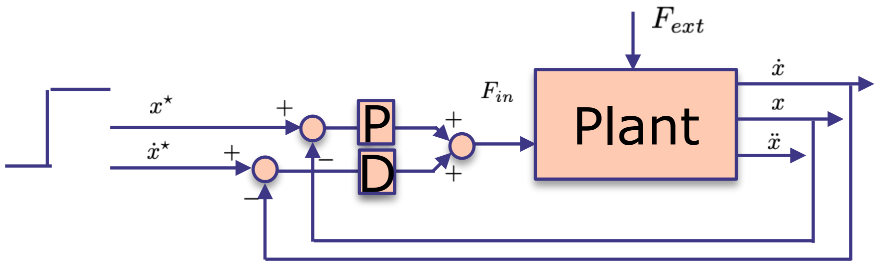
in order to reduce the oscillations of our system, we can add a term that acts like a visc

$$F_{in} = K_P e_x + K_D \dot{e}_x$$



$$m\ddot{x} \boxed{+ (D + K_D)\dot{x} + K_P x} = K_P x^\star + K_D \dot{x}^\star - F_{ext}$$

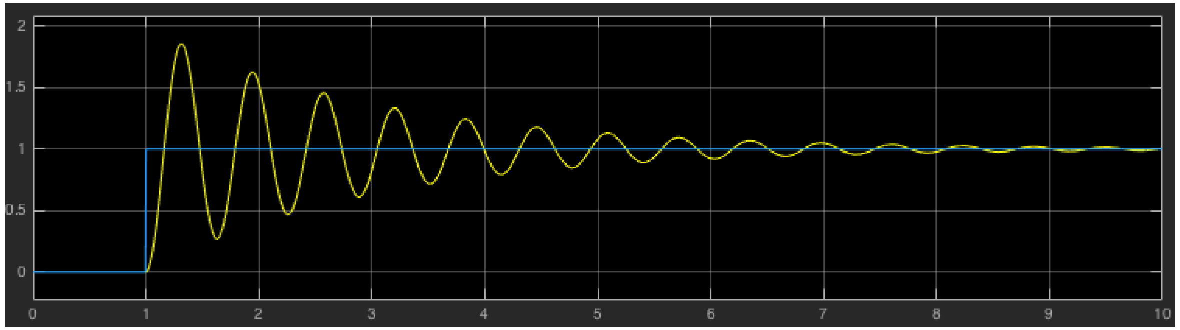
CONTROL



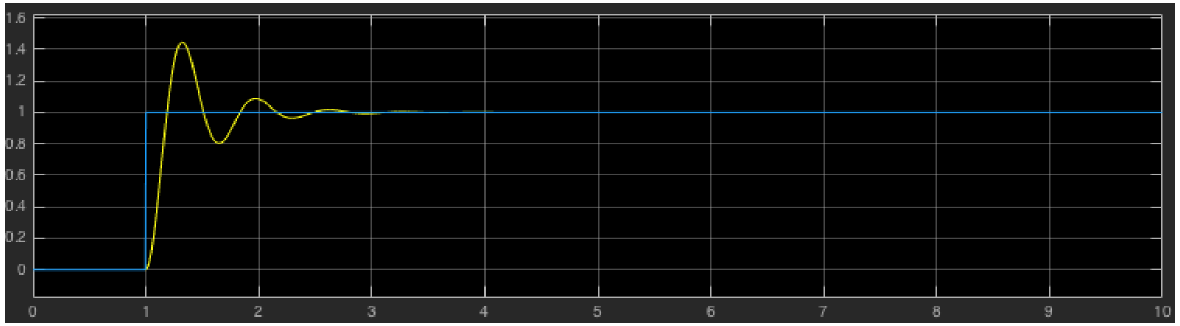
Proportional-Derivative Control:

Analysis of the STEP response

- if the mechanical system is not damped $D = 0$, the derivative gain add damping to the system



Kd=1



Kd=5

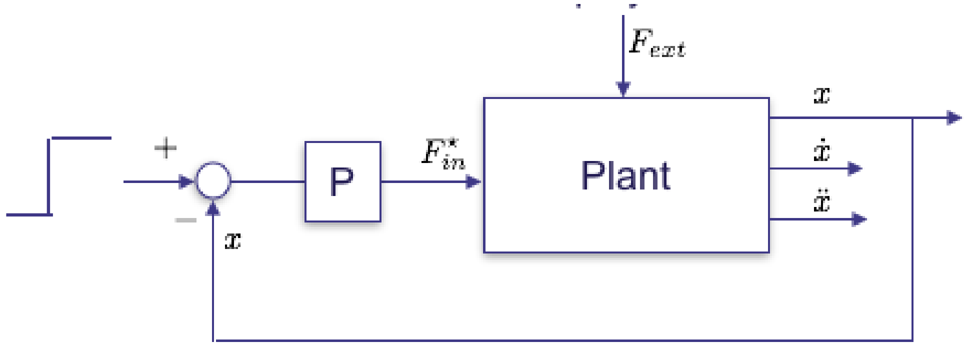
CONTROL

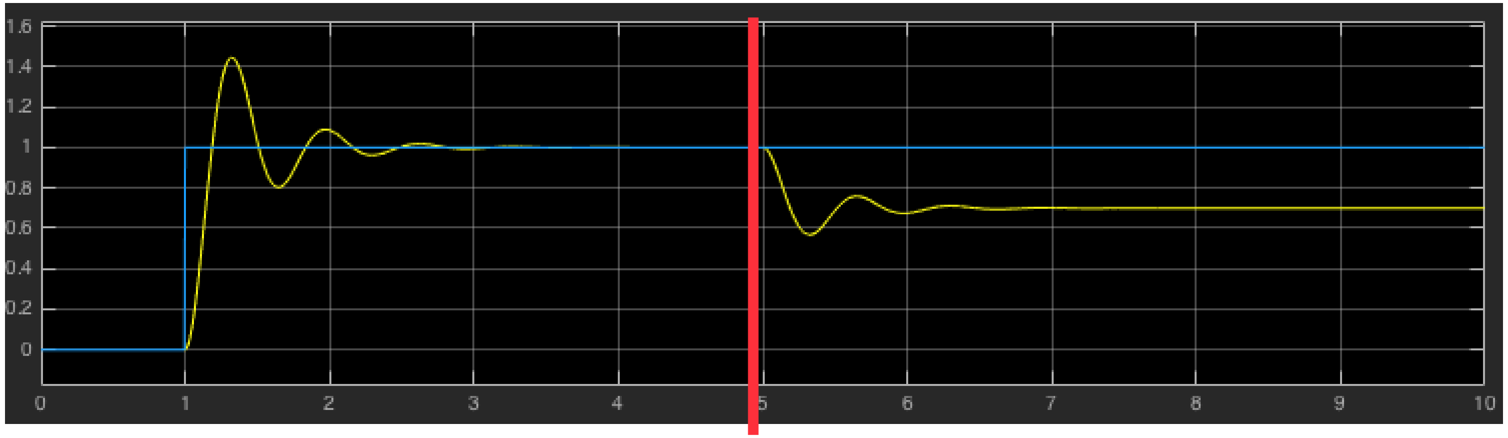
Proportional-Derivative Control:

Limitations of the PD control:

Analysis of the STEP response

- if an external load is applied, it cannot guarantee steady state null error





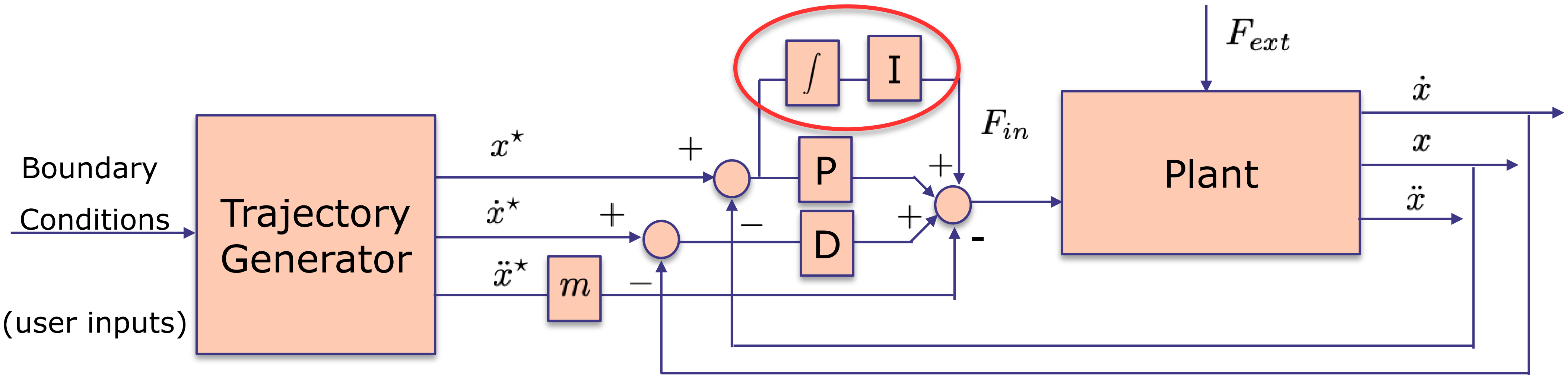
$D = 0$

Kd=5

Application of external load

CONTROL

Proportional-Integral-Derivative Control:

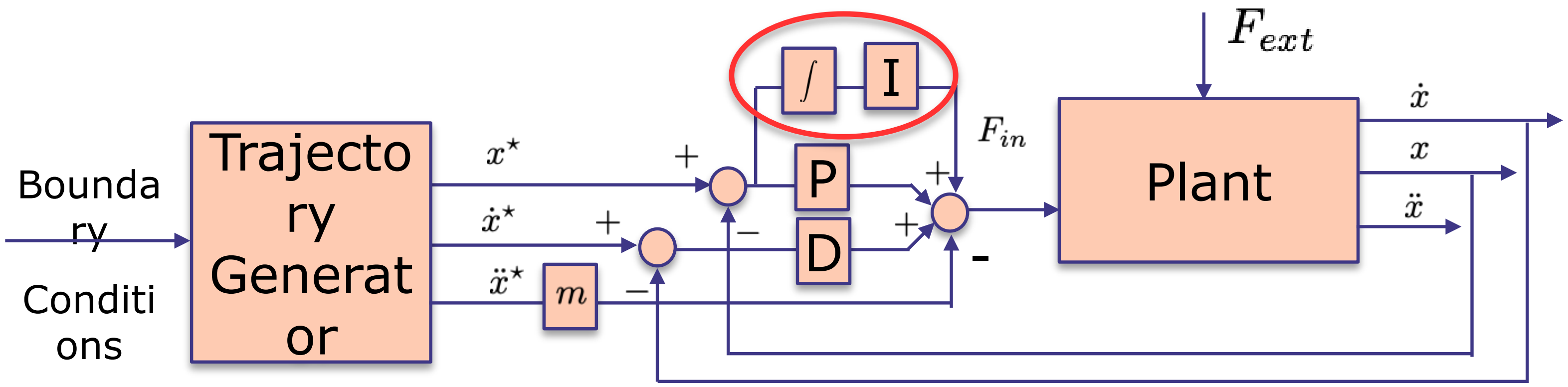


The Integral action allow controlling the steady state error to zero!!!

$$F_{in} = m\ddot{x}^\star + K_D\dot{e}_x + K_P e_x + K_I \int e_x$$

CONTROL

Proportional-Integral-Derivative Control:



Let's assume $D = 0$

Due to the integral action, the controlled closed loop system takes the form:

$$m\ddot{e}_x + K_D\dot{e}_x + K_P e_x + K_I \int e_x = F_{ext}$$

CONTROL

Proportional-Integral-Derivative Control:

$$m\ddot{e}_x + K_D\dot{e}_x + K_P e_x + K_I \int e_x = F_{ext}$$

What does it mean?

Let’s call: $y = \int e_x dt$

The equation of motion of our controlled system becomes:

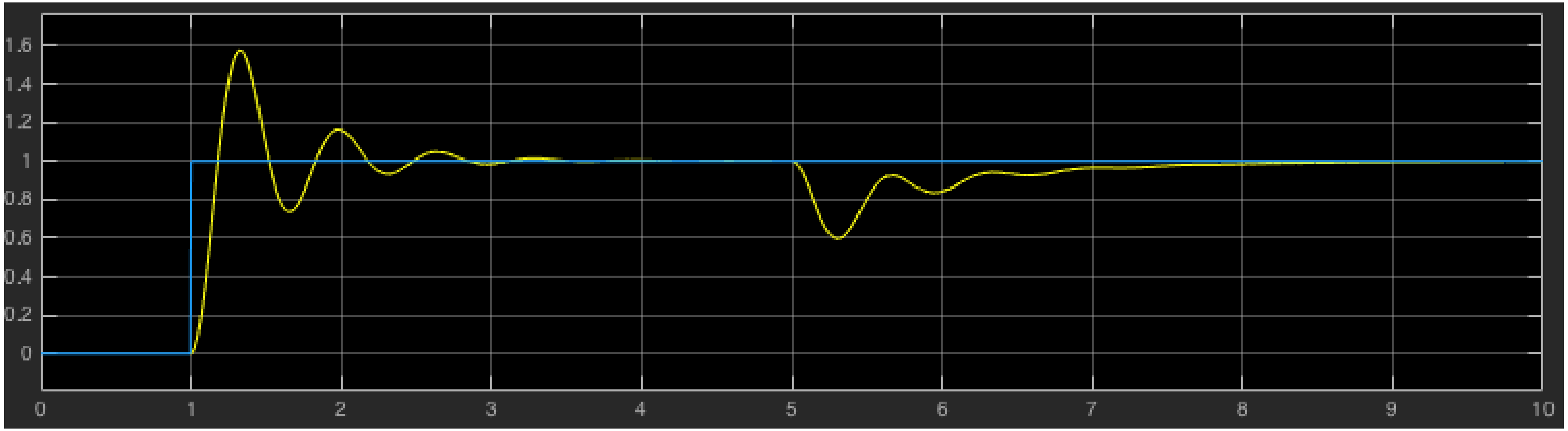
$$m\dddot{y} + K_D\ddot{y} + K_P\dot{y} + K_I y = F_{ext}$$

At steady state: $\dddot{y}, \ddot{y}, \dot{y} = 0$

meaning $e_x = 0$

CONTROL

Proportional-Integral-Derivative Control:



KP=100

KD=5

KI=100

CONTROL

Tuning gains

NOTE that so far we chose gains (KP, Kd and KI) such that the response was…just ok!

In general there exist techniques that allow a proper tuning of the gains of a PID regulator. Some of these techniques are experimental:

- “guess and check” method
- Ziegler Nichols

Recap: Laplace Transform

LAPLACE TRANSFORM

Definition: The Laplace Transform of a signal $f(t),\ t \geq 0$ is the function of the complex variable $s \in \mathbb{C}$

$$F(s) := \mathcal{L}[f(t)](s) = \int_0^{+\infty} f(t)e^{-st}dt$$

LAPLACE TRANSFORM

Typical Leplace Transform of canonical functions

$f(t)$	$\mathcal{L}(s)$
$imp(t)$	1
$step(t)$	$\frac{1}{s}$
$cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$

LAPLACE TRANSFORM

Properties of the Laplace Transform:

LINEARITY

$$\mathcal{L}[\alpha_1 f_1(t) + \alpha_2 f_2(t)](s) = \alpha_1 \mathcal{L}[f_1(t)](s) + \alpha_2 \mathcal{L}[f_2(t)](s)$$

DERIVATION OVER TIME

$$\mathcal{L}[\dot{f}_1(t)](s) = s\mathcal{L}[f_1(t)](s) - f(0)$$

The properties allow us to compare differential and linear relations in time domain, to the equivalent in Laplace domain

LAPLACE TRANSFORM

Inverse Laplace transform

Consider F=N/D, with d>n

$$F(s) = \frac{N(s)}{D(s)} = \frac{n_0 + n_1 s + n_2 s^2 + \cdot + n_n s^n}{d_0 + d_1 s + d_2 s^2 + \cdot + d_d s^d}$$

it is possible to define the inverse transform function

$$f(t) = \mathcal{L}^{-1}[F(s)](t) \ , \quad t \geq 0$$

LAPLACE TRANSFORM

Inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}[F(s)](t) \ , \quad t \geq 0$$

it can be determined using the Heaviside method

LAPLACE TRANSFORM

Consider the Laplace transform of a fuction $f(t)$ with $d > n$ (Strictly proper)

It is possible to calculate the value of $f(t)$ in $t = 0$ and the limit of $f(t)$, for $t \to \infty$, by using the laplace transform of $f(t)$, WITHOUT calculating explicitly

$$\mathcal{L}^{-1}[F(s)](t)$$

THEOREM OF THE INITIAL VALUE

$$f(0) = \lim_{s \to \infty} sF(s)$$

And if the roots of D(s) are equal to 0 or they have real part greater then 0:

THEOREM OF THE FINAL VALUE

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

LAPLACE TRANSFORM

TRANSFER FUNCTION

A function $G(s)$ that describes the input-output relation of a linear system subject to an input $U(s)$:

$$Y(s) = G(s)U(s)$$

NOTE: a TF allows to describe differential equations as algebraic ones, and it allows to evaluate the motion of the output $y(t),\ t \geq 0$, due to an input $u(t),\ t \geq 0$ as:

$$u(t),\ t \geq 0 \ \overset{\mathcal{L}}{\rightarrow}\ U(s) \overset{\cdot G(s)}{\rightarrow} Y(s) \ \overset{\mathcal{L}^{-1}}{\rightarrow}\ y(t)$$

LAPLACE TRANSFORM

TRANSFER FUNCTION

Given the system of the n-th order:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$G(s) = C(sI - A)^{-1}B + D = \frac{N(s)}{D(s)}$$

LAPLACE TRANSFORM

TRANSFER FUNCTION

$$G(s) = C(sI - A)^{-1}B + D = \frac{N(s)}{D(s)}$$

the roots of $D(s)$ are called POLES; the roots of $N(s)$ are called ZEROS

Poles and zeros are singularities of $G(s)$

Poles are the eigenvalues of the state matrix A. This means that the number of poles of a transfer function of a system of the n-th order is n

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

LAPLACE TRANSFORM

TRANSFER FUNCTION

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

Where:

$g \in \mathbb{Z}$ is the type of the transfer function. In general, it indicates the number of singularity ($s = 0$) of the transfer function. More precisely, if $g > 0$ it indicates the number of poles that are equal to zero. If $g < 0$, it indicates the number of zeros that are equal to zero.

$\mu \in \mathbb{R}$ is the gain of the transfer function.

LAPLACE TRANSFORM

TRANSFER FUNCTION

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

Where:

$T_i \in \mathbb{R}$ and $\tau_i \in \mathbb{R}$ are the time constants of the zeros and poles that are real and not null ($s \neq 0$)

$\sigma_{n,i} \in \mathbb{R}^+$ and $\omega_{n,i} \in \mathbb{R}^+$ are the natural frequencies of the complex, conjugated poles

$\zeta_i \in (-1, 1)$ and $\xi_i \in (-1, 1)$ are the damping coefficients of the complex conjugated poles and zeroes that are

Recap: Frequency Response

FREQUENCY RESPONSE

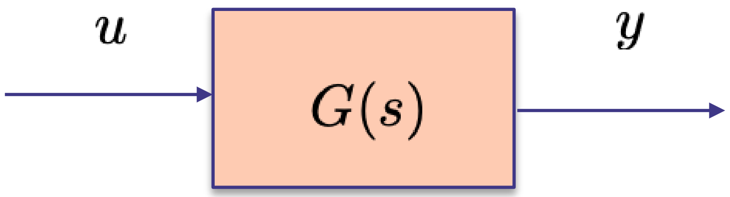
Definition 1 (Frequency Response):

we define "***frequency response***" associated to the tansfer function $G(s)$, the function $G(j\omega)$ of the real variable $\omega \geq 0$

FREQUENCY RESPONSE

Theorem 1 (Theorem of the Frequency Response):

Let us consider the linear dynamic system



$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

With transfer function $G(s)$.

If matrix A of the dynamic system has not eigenvalues on the imaginary axis (it has purely real eigenvalues), then given a sinusoidal input, characterized by pulse ω , amplitude α and phase ϕ

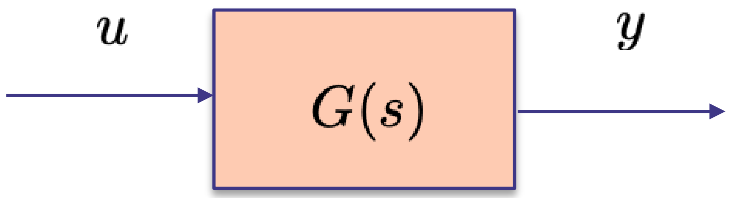
$$u(t) = \alpha sin(\omega t + \phi), \ t \geq 0$$

FREQUENCY RESPONSE

and given an appropriate initial condition $x(0)$, then the output:

$$y(t) = \alpha \cdot \mid G(j\omega) \mid sin(\omega t + \phi + \angle G(j\omega)),\ t \geq 0$$

is also sinusoidal, characterized by the same pulse of the input ω , it is scaled with respect to the input by a factor that depends on the frequency response of the system, evaluated at the pulse ω and shifted, in phase, of a phase shift that depends on the phase shift of the transfer function, evaluated at the pulse ω

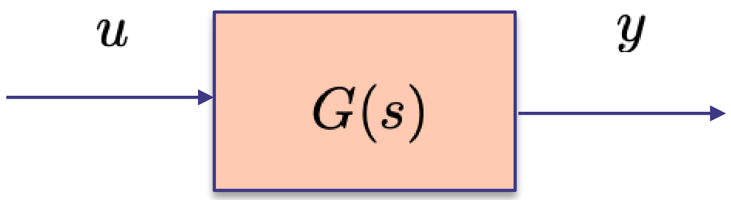


FREQUENCY RESPONSE

Observation: if the dynamic system is asymphtotically stable, then the espression

$$y(t) = \alpha \cdot \mid G(j\omega) \mid sin(\omega t + \phi + \angle G(j\omega)),\ t \geq 0$$

presented in the theorem of the frequency response, represents the steady state response $y_\infty(t)$ that is obtained for every arbitrary initial condition, after the transient is exhausted.



FREQUENCY RESPONSE

NOTE:

The frequency response $G(j\omega)$ is a function with complex values in the real variable $\omega \geq 0$. It can be represented graphycally in 2 ways:

1) drawing the modulus $|G(j\omega)|$ and phase shift $\angle G(j\omega)$, as a function of the real variable $\omega \geq 0$. This type of representation is called: **Bode Diagram** of the modulus and phase of $G(j\omega)$

2) drawing the locus of points in the complex plane, described by $G(j\omega)$, with omega ranging from 0 to $+\infty$, if $G(s)$ has no polse on the imaginary axis. This type of representation is called **polar diagram**.

FREQUENCY RESPONSE

BODE DIAGRAMS: Drawing rules

- on the horizontal axis, we represent ω in logaritmic scale
- representation of the modulus $\mid G(j\omega) \mid$: on the vertical axis,we use the linear scale to represent the modulus in dB of the transfer function, as a function of $log(\omega)$, $\omega \geq 0$

$$\mid G(j\omega) \mid_{dB} := 20log \mid G(j\omega) \mid$$

- representation of the phase $\angle G(j\omega)$: it is on a linear scale, in degrees or radiants, as a function of $log(\omega)$, $\omega \geq 0$

NOTE: plotting the Bode diagrams is not easy in general. But some simple rules can be used to plot the asymptotic diagrams, starting from the analytic espression of the transfer function $G(s)$

FREQUENCY RESPONSE

rules for drawing the Asymptotic Bode Diagrams :

1) write the transfer function in the form:

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

2) draw the diagram of the modulus

3) draw the diagram of the phase

FREQUENCY RESPONSE

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1+T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1+\tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

DRAWING THE MODULUS

2.1) on the horizontal axis, point the pulses coincident to the modulus of poles and zeros, both real and complex-conjugated, highlighting whether they are poles or zeros

2.2)start drawing the diagram, starting from the lower frequencies:

- if $g = 0$ draw an horizontal line with ordinate $\mid \mu \mid_{dB}$
- if $g \neq 0$ draw a line inclined of $-20g\ dB/decade$, passing through $\mid \mu \mid_{dB} = 20 log \mid \mu \mid$ when $\omega = 1$

- at any natural pulsation, the inclination of the modulus changes of:
  - $-20\ dB/decade$ for every pole
  - $+20\ dB/decade$ for every zero

FREQUENCY RESPONSE

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1+T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1+\tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

DRAWING THE PHASE

3.1) on the horizontal axis, point the pulses coincident to the modulus of poles and zeros, both real and complex-conjugated, highlighting whether they are poles or zeros

3.2)start drawing the diagram, starting from the lower frequencies:

- if $\mu > 0$ draw a horizontal line with ordinate $-g90^\circ$
- if $\mu < 0$ draw a horizontal line with ordinate $-g90^\circ - 180^\circ$
- at any natural pulsation, the phase shifts of:
  - -90° for every pole with negative real part, or for every zero with positive real part
  - $+90^\circ$ for every pole with real positive part, or for every zero with negative real part

FREQUENCY RESPONSE

STATIC GAIN OF THE TRANSFER FUNCTION

given the transfer function

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s + \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$

the static gain can be calculated as

$$\mu = \left| s^g G(s) \right|_{s=0}$$

FREQUENCY RESPONSE

$$g(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s \frac{s^2}{\omega_{ni}^2}\right)}$$

Difference between asymptotic diagrams and real ones

Let's look at the contribution of two complex conjugated poles of $G(j\omega)$ that we isolate in the TF $G_c(j\omega)$

$$G_c(j\omega) = \frac{1}{1 + 2\frac{\xi}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

In general, the asymptotic representation of the modulus is a good representation of the real bode plot, if

$$|\, \xi \,| > \frac{\sqrt{2}}{2}$$

FREQUENCY RESPONSE

$$g(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1 + \tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s \frac{s^2}{\omega_{ni}^2}\right)}$$

Difference between asymptotic diagrams and real ones

$$G_c(j\omega) = \frac{1}{1 + 2\frac{\xi}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

if $\mid \xi \mid < \frac{\sqrt{2}}{2}$

the modulus of $G_c(j\omega)$ has a peak at the pulsation ω_r , called resonance pulsation, and it is called resonance peak

$$\omega_r = \omega_n \sqrt{1 - 2\xi^2} \qquad \mid G(j\omega) \mid = \frac{1}{2 \mid \xi \mid \sqrt{1 - \xi^2}}$$

$\mid G(j\omega) \mid \to +\infty$ for $\mid \xi \mid \to 0$

FREQUENCY RESPONSE

$$g(s) = \frac{\mu}{s^g} \frac{\prod_i (1+T_i s) \prod_i \left(1 + 2\frac{\zeta_i}{\sigma_{ni}} s \frac{s^2}{\sigma_{ni}^2}\right)}{\prod_i (1+\tau_i s) \prod_i \left(1 + 2\frac{\xi_i}{\omega_{ni}} s \frac{s^2}{\omega_{ni}^2}\right)}$$

Difference between asymptotic diagrams and real ones

$$G_c(j\omega) = \frac{1}{1 + 2\frac{\xi}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

if $|\xi| < \frac{\sqrt{2}}{2}$

the modulus of $G_c(j\omega)$ has a peak at the pulsation ω_r, called resonance pulsation, and it is called resonance peak

$$\omega_r = \omega_n \sqrt{1 - 2\xi^2} \qquad |G(j\omega)| = \frac{1}{2|\xi|\sqrt{1-\xi^2}}$$

$|G(j\omega)| \to +\infty$ for $|\xi| \to 0$

The exact diagram of the phase, if $|\xi| \to 0$, tends to be a step as in the asymptotic diagram

Recap: Control of in the imaginary plane

Analysis and control of mechatronic systems

Step response: the step response of a 1st order system:

$$G(s) = \frac{1}{1+\tau s}$$

The system has 1 pole with negative real part. The system is therefore stable

$$p = -\frac{1}{\tau}$$

Analysis and control of mechatronic systems

Step response: the step response of a 1st order system:

$$G(s) = \frac{1}{1+\tau s}$$

the step response of the system is:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s\,(1+\tau\, s)}\right] = \frac{1}{\tau}\mathcal{L}^{-1}\left[\frac{1}{s\left(s+\frac{1}{\tau}\right)}\right] \\ &= \frac{1}{\tau}\mathcal{L}^{-1}\left[\frac{\tau}{s} - \frac{\tau}{s+\frac{1}{\tau}}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right] = 1 - e^{-\frac{t}{\tau}} \end{aligned}$$

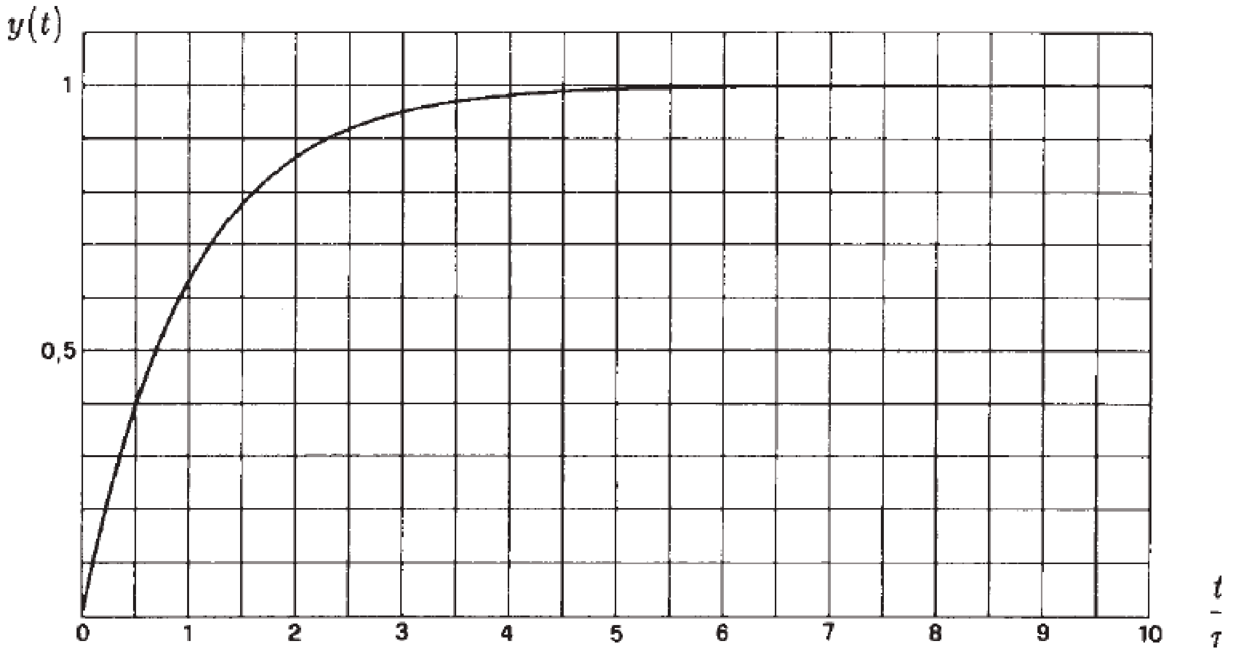
Analysis and control of mechatronic systems

Step response: the step response of a 1st order system:

$$G(s) = \frac{1}{1+\tau s}$$

the step response of the system is:

$$\begin{aligned}
t=\tau &\rightarrow 63,2\% \\
t=2\tau &\rightarrow 86.5\% \\
\underline{t=3\tau} &\rightarrow \underline{95.0\%} \\
t=5\tau &\rightarrow 99.3\% \\
t=7\tau &\rightarrow 99.9\%
\end{aligned}$$



Analysis and control of mechatronic systems

Example

$$a\dot{y}(t) + by(t) = cx(t)$$

The TF G(s) associated to the equation above is:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{c}{as+b}$$

and normalizing:

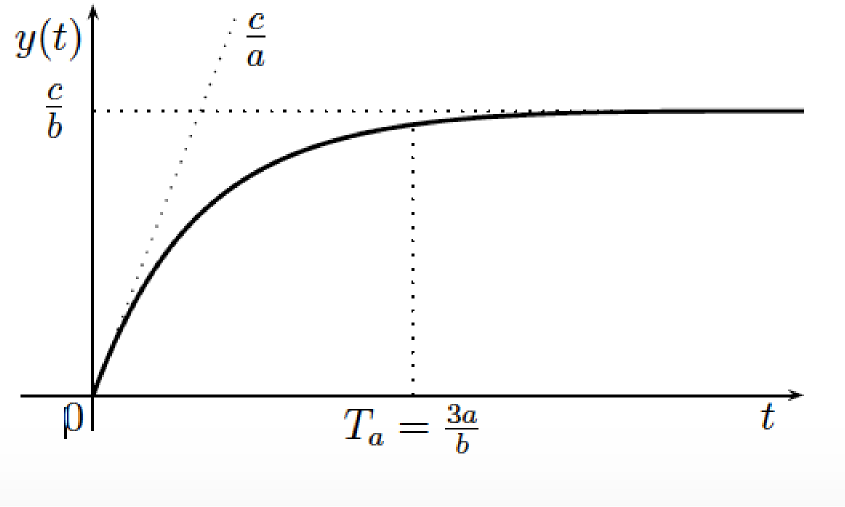
$$G(s) = \frac{c}{b}\frac{1}{\tau s+1} \qquad \tau = \frac{a}{b}$$

Analysis and control of mechatronic systems

Example

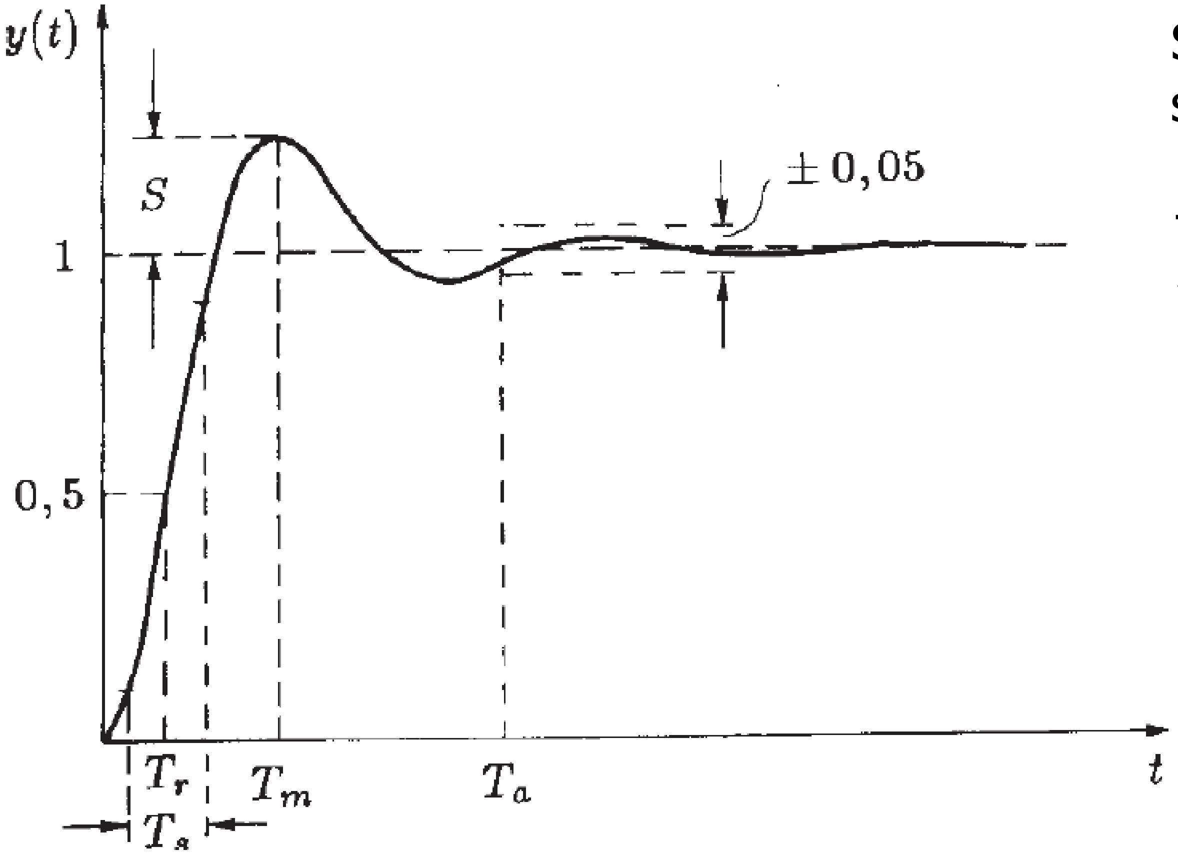
The step response of the system $$G(s) = \frac{c}{b}\frac{1}{\tau s + 1} \qquad \tau = \frac{a}{b}$$

$$y(t) = \frac{c}{b}(1 - e^{-\frac{b}{a}t})$$



Analysis and control of mechatronic systems

Step response: the step response of a 2nd order system:



S = max overshoot (% of the step steady state value)

Tr = delay time (50% of the final value)

Ts = raise time (time from 10 to 90% of the final value)

Ta = settling time (time needed to have a bounded output (+-5% of the final value))

Tm = Time of max overshoot

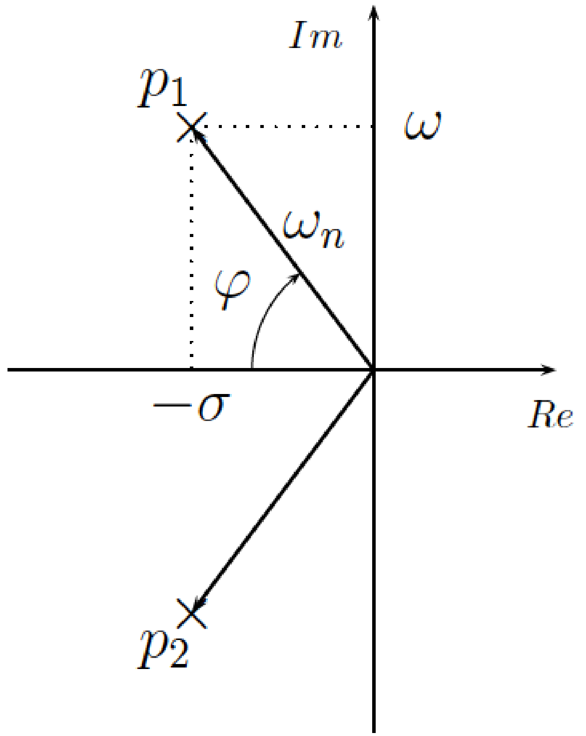
Analysis and control of mechatronic systems

Step response: the TF of a 2nd order system is typically:

$$G(s) = \frac{1}{1 + 2\,\delta\,\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}} = \frac{\omega_n^2}{s^2 + 2\,\delta\,\omega_n\,s + \omega_n^2}$$

$\delta = \cos\varphi$ damping coefficient

ω_n natural pulsation



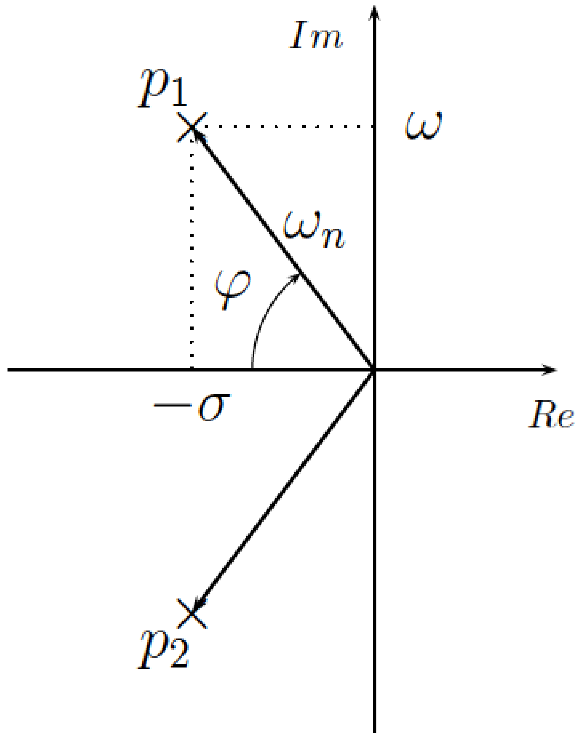
Analysis and control of mechatronic systems

The unit step response of a second order system is:

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{\omega_n^2}{s\,(s^2 + 2\,\delta\,\omega_n\,s + \omega_n^2)}\right] \\
&= 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\,\mathrm{sen}\,(\omega\,t + \varphi)
\end{aligned}
$$

Where

$$
\begin{aligned}
\omega &:= \omega_n\sqrt{1-\delta^2} \\
\sigma &:= \delta\omega_n \\
\varphi &:= \arccos\delta = \arctan\tfrac{\sqrt{1-\delta^2}}{\delta}
\end{aligned}
$$



Analysis and control of mechatronic systems

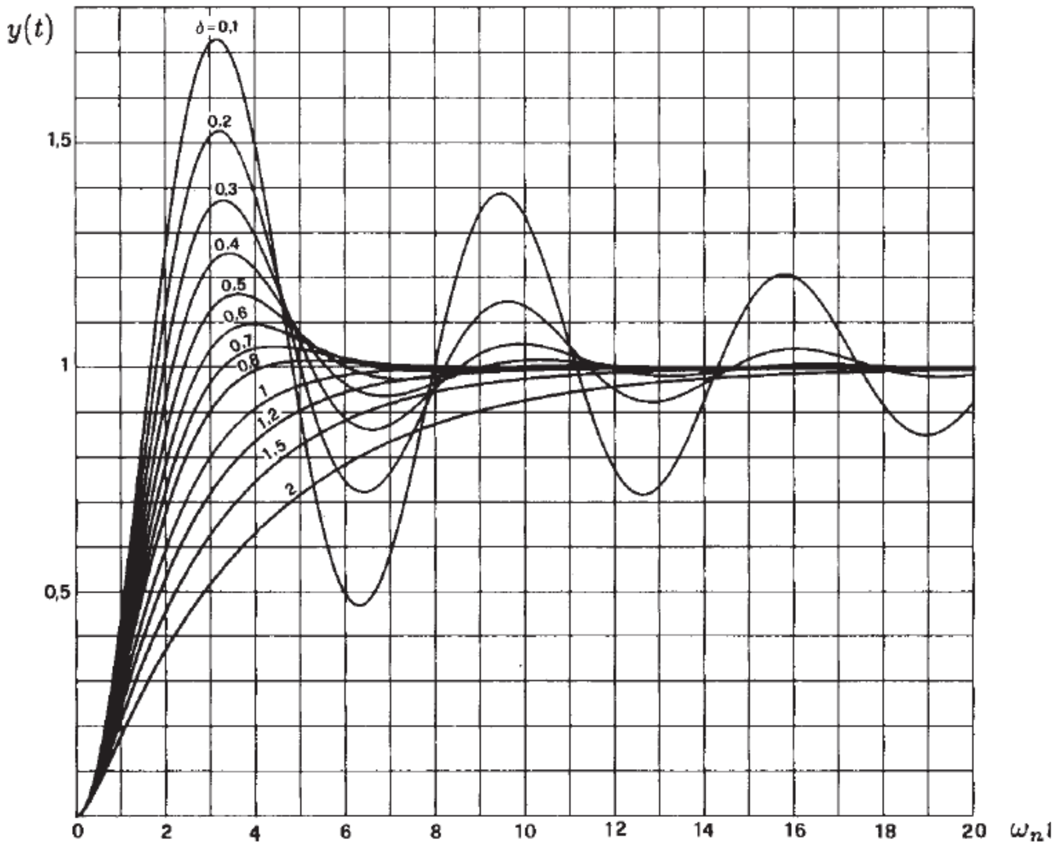
The unit step response of a second order system is:

$$y(t) = \mathcal{L}^{-1}\left[\frac{\omega_n^2}{s\,(s^2 + 2\,\delta\,\omega_n\,s + \omega_n^2)}\right]$$

$$= 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\,\mathrm{sen}\,(\omega\,t + \varphi)$$

$$\omega := \omega_n\sqrt{1-\delta^2}$$
$$\sigma := \delta\omega_n$$
$$\varphi := \arccos\delta = \arctan\frac{\sqrt{1-\delta^2}}{\delta}$$

If $\delta = 1$ there is no overshoot



Analysis and control of mechatronic systems

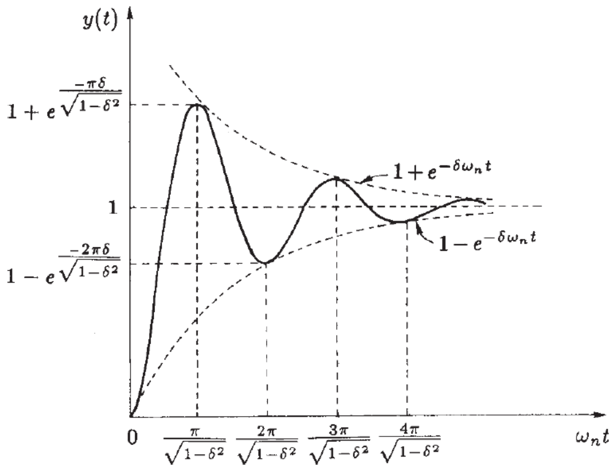
We can determine the maximum overshoot by computing the derivative of y(t)=0:

$$\frac{dy}{dt} = -A\,e^{-\delta\omega_n t}\,\omega\,\cos(\omega\,t+\varphi) + A\,\delta\,\omega_n\,e^{-\delta\omega_n t}\,\mathrm{sen}\,(\omega t+\varphi) \quad \text{with} \quad A = \frac{1}{\sqrt{1-\delta^2}}$$

we obtain: $-\omega_n\sqrt{1-\delta^2}\,\cos(\omega t+\varphi) + \delta\,\omega_n\,\mathrm{sen}\,(\omega t+\varphi) = 0$

which gives $\tan(\omega t+\varphi) = \dfrac{\sqrt{1-\delta^2}}{\delta} \quad \leftrightarrow \quad \omega t = n\,\pi$

$$\boxed{t = \frac{n\,\pi}{\omega_n\sqrt{1-\delta^2}} = \frac{n\,\pi}{\omega}}$$

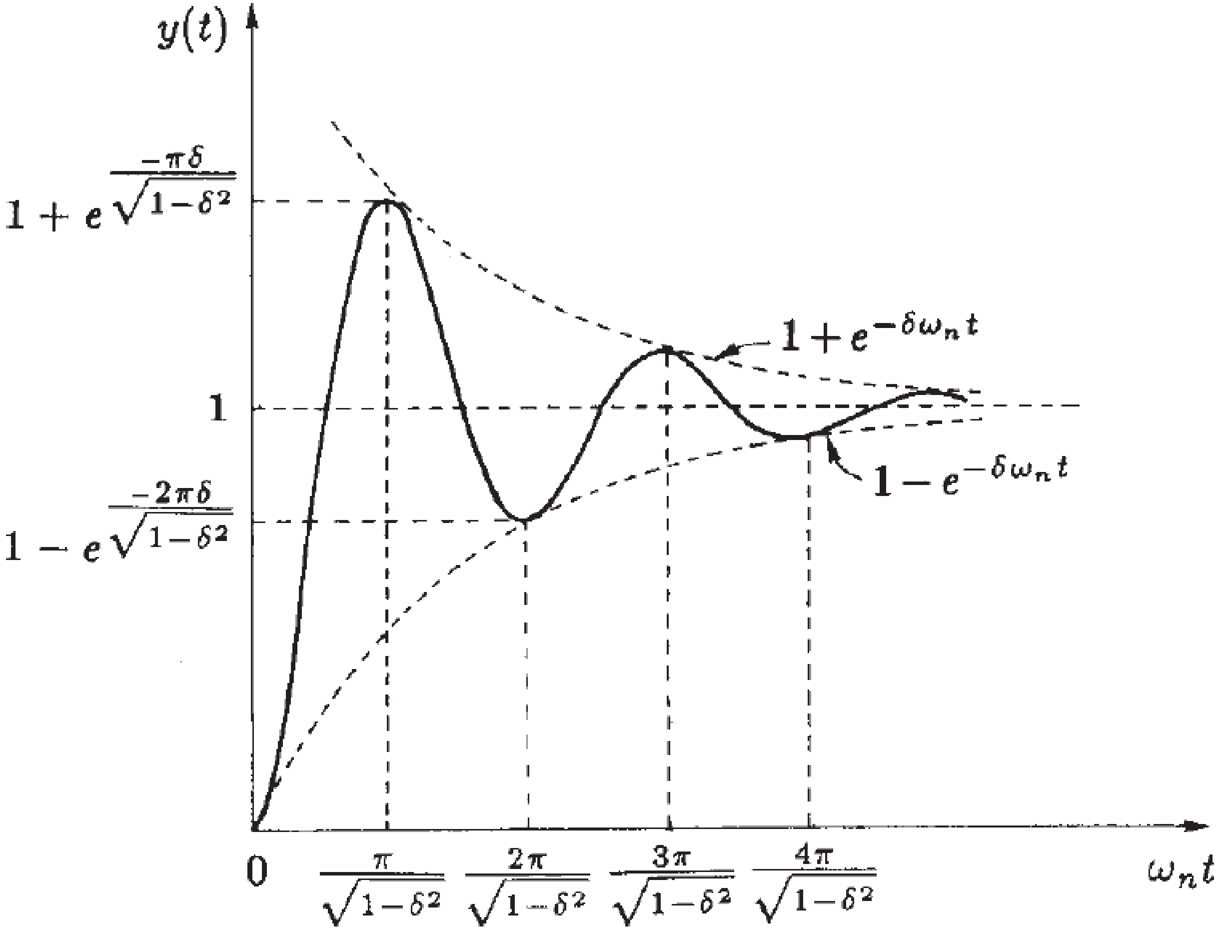


$$\boxed{t = \frac{n\,\pi}{\omega_n\sqrt{1-\delta^2}} = \frac{n\,\pi}{\omega}}$$

$$y(t)\Big|_{\substack{\max\\ \min}} = 1 - \frac{e^{\frac{-n\pi\delta}{\sqrt{1-\delta^2}}}}{\sqrt{1-\delta^2}}\ \mathrm{sen}\,(n\,\pi+\varphi)$$

which means

$$y(t)\Big|_{\substack{\max\\ \min}} = 1 - (-1)^n\ e^{\frac{-n\pi\delta}{\sqrt{1-\delta^2}}}$$



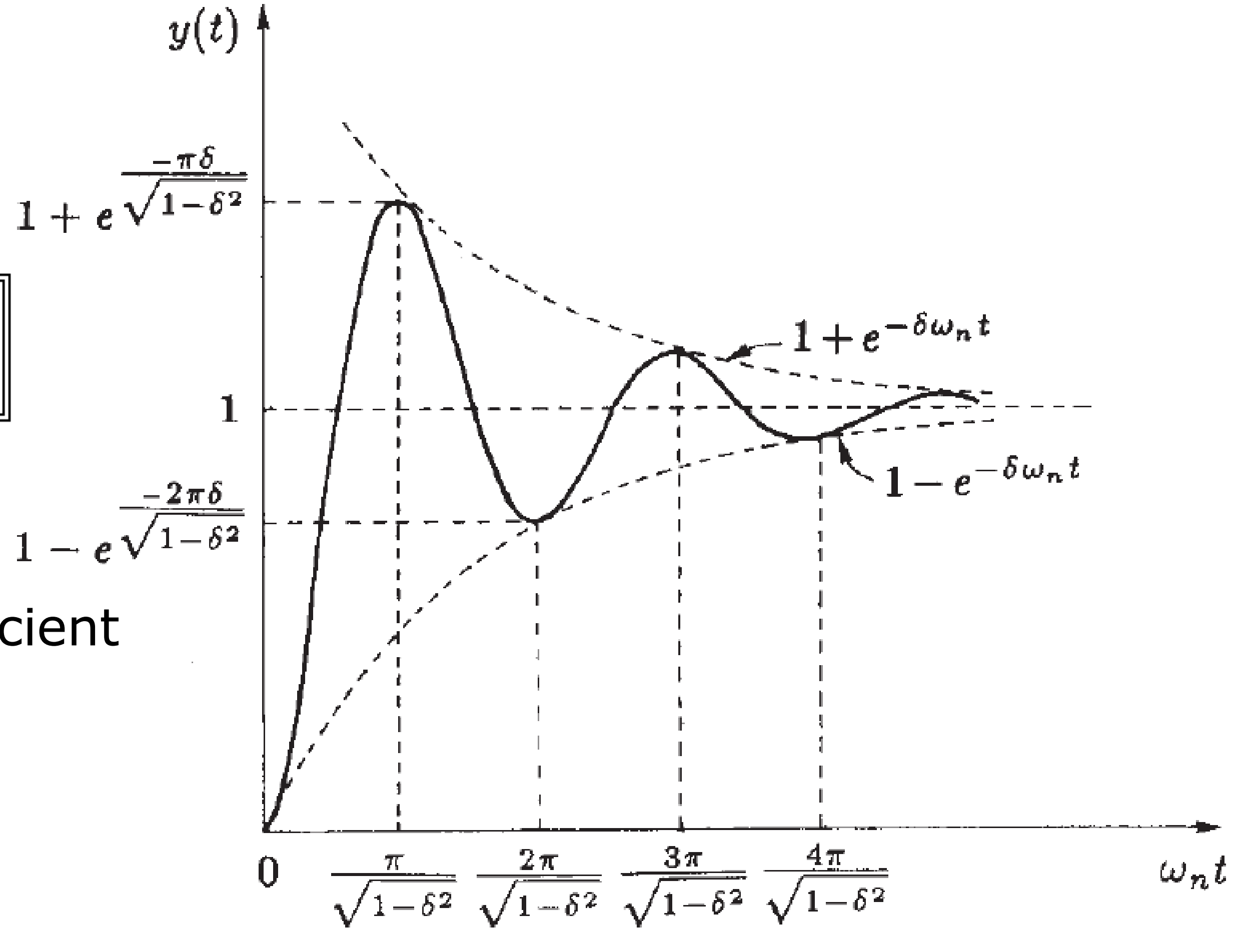
Analysis and control of mechatronic systems

The max overshoot

$$S = 100\,\frac{(y_{\max} - y_\infty)}{y_\infty}$$

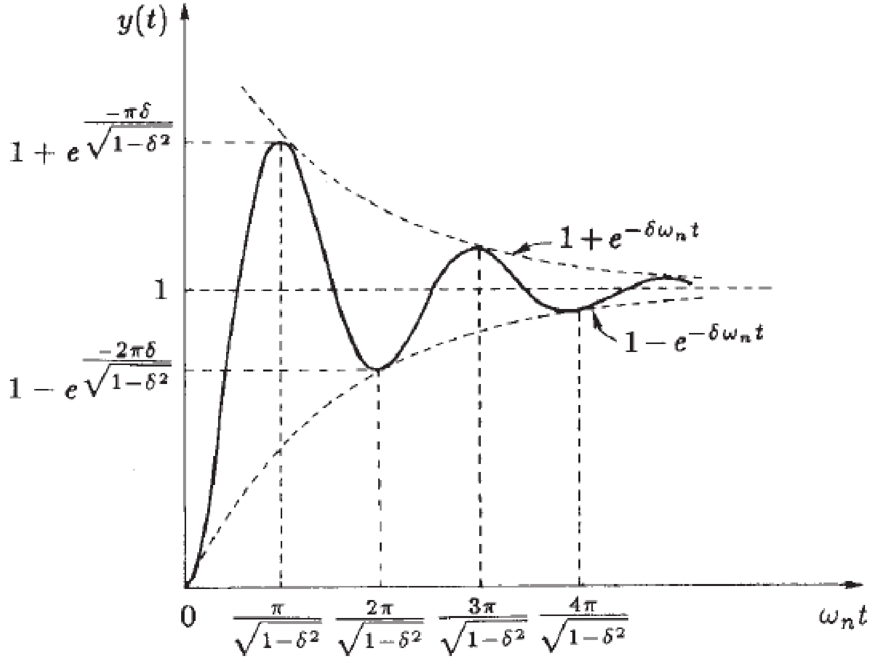
$$\boxed{S = 100\ e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}}$$

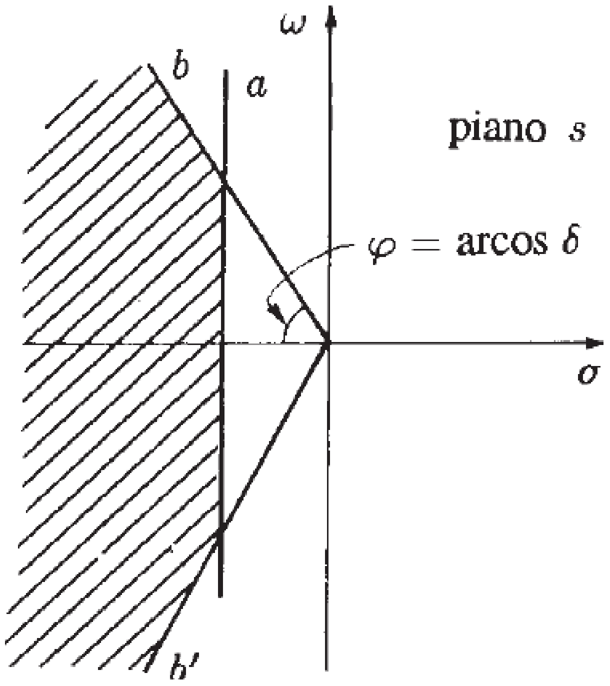
Only depends on the damping coefficient



Analysis and control of mechatronic systems

The max overshoot is constrained to a certain value, if the poles are between the 2 lines with angular coefficient b and b’





Analysis and control of mechatronic systems

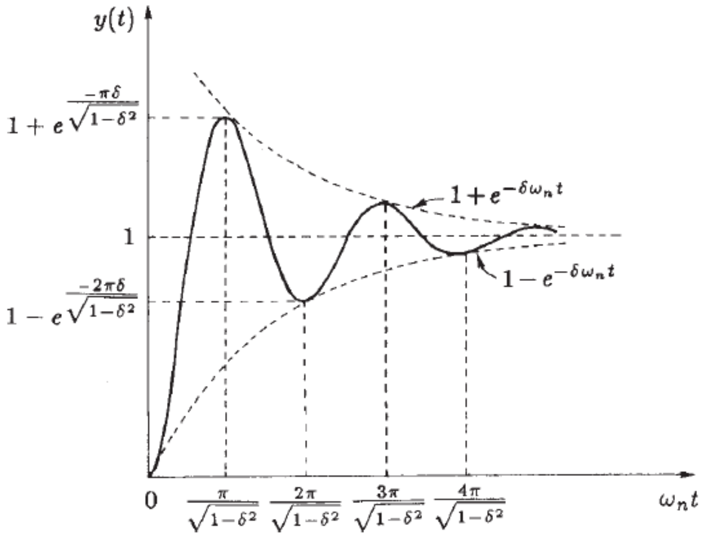
We can constrain the settling time Ta by enforcing that $e^{-\delta\omega_n T_a} = 0.05$

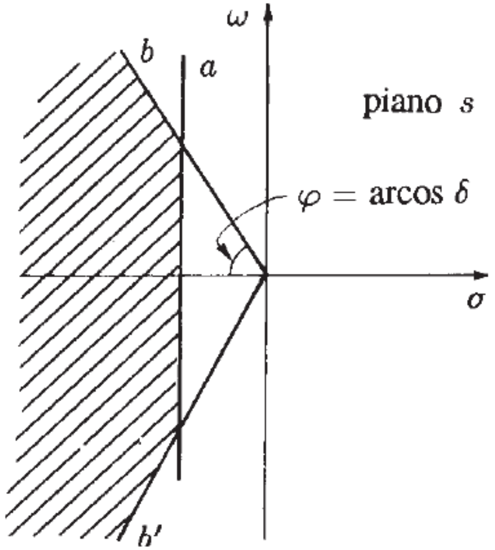
Which means:

$$\delta\,\omega_n\,T_a = 3$$

and therefore:

$$T_a = \frac{3}{\delta\,\omega_n} = \frac{3}{|\sigma|}$$



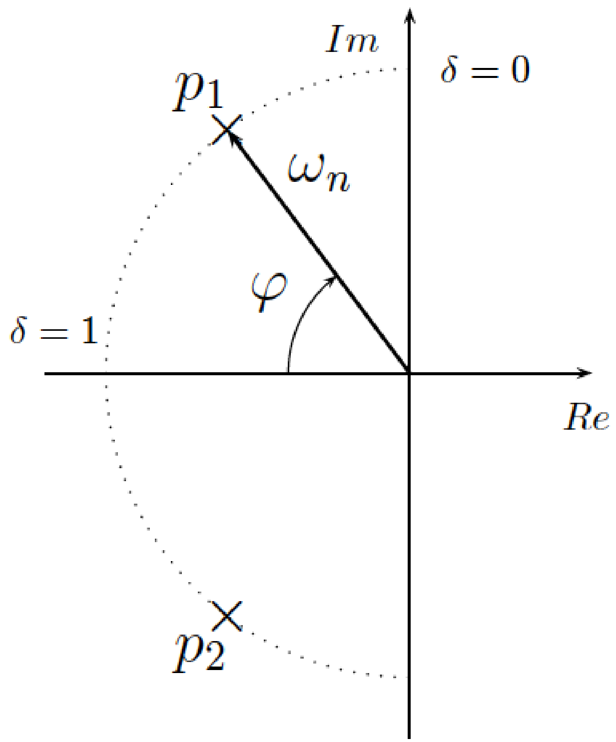


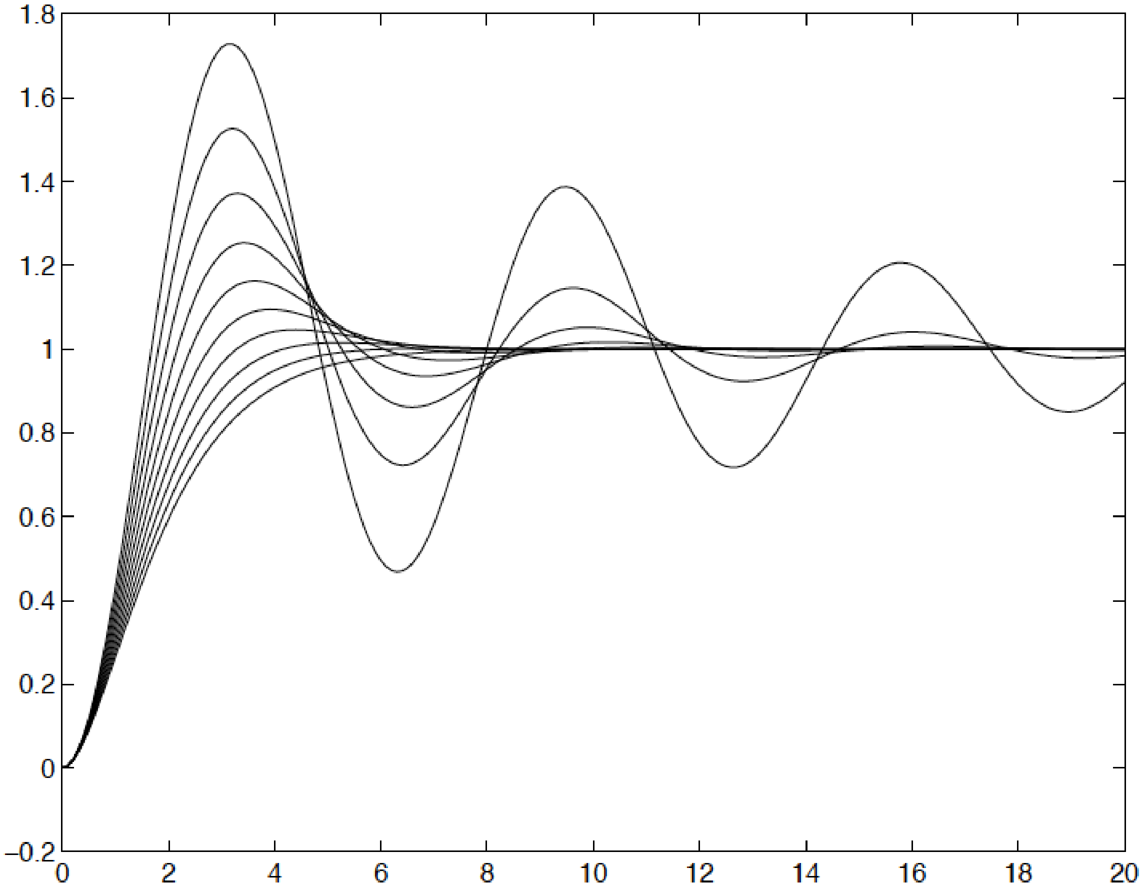
The settling time is smaller then Ta, if

$$\delta\,\omega_n \geq \frac{3}{T_a}$$

Analysis and control of mechatronic systems

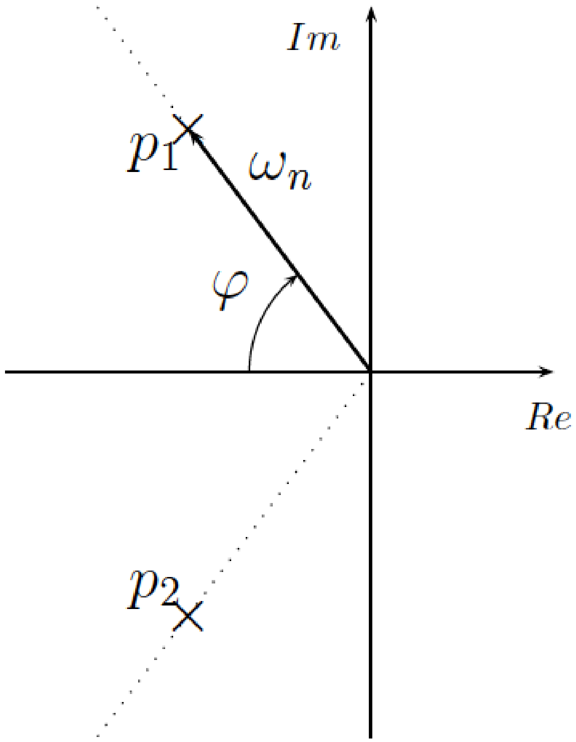
constant ω_n

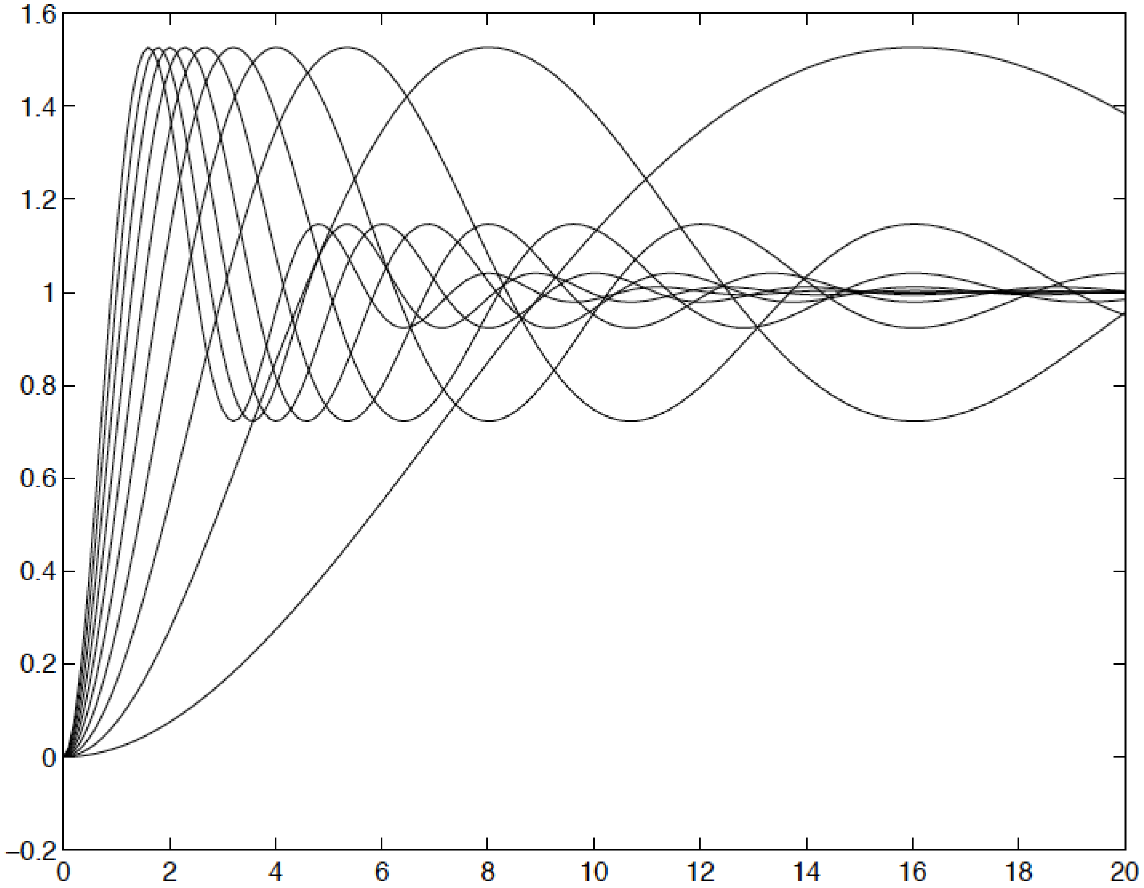




Analysis and control of mechatronic systems

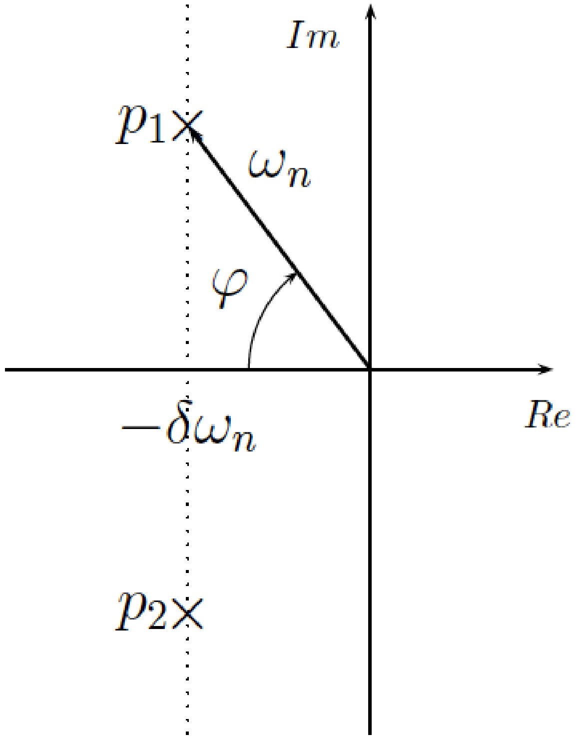
constant δ

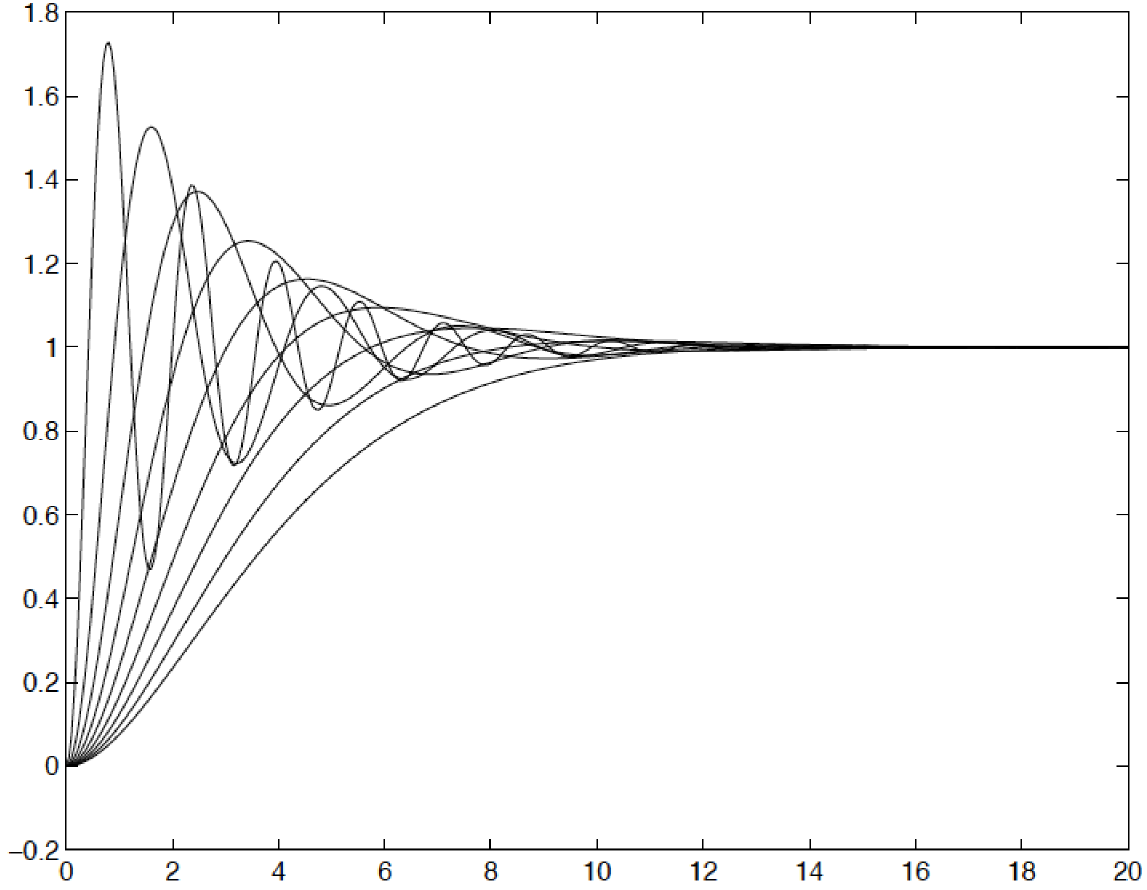




Analysis and control of mechatronic systems

constant T_a

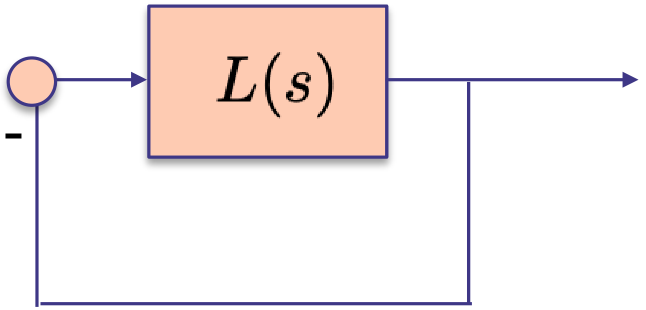




Analysis and control of mechatronic systems

ROOT LOCUS

It is the locus of the roots of closed loop function, as a function of the forward loop gain



$$L(s) = \rho \frac{N(s)}{D(s)} = \rho \frac{\prod_i (s + z_i)}{\prod_k (s + pk)}$$

The root locus is the locus described, in the complex plane, by the roots of the polinomial, as a function of the gain $-\infty < \rho < +\infty$

$$\Theta(s) = D(s) + \rho N(s) = 0$$

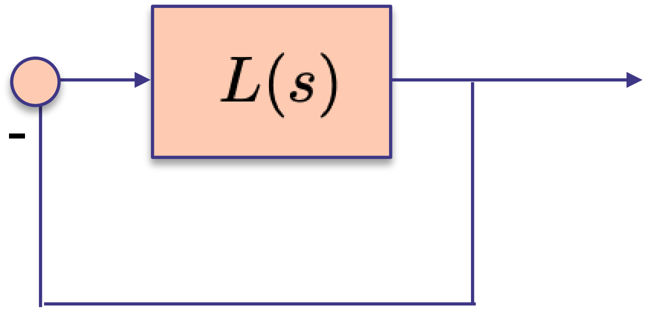
direct root locus $0 < \rho < +\infty$

inverse root locus $-\infty < \rho < 0$

Analysis and control of mechatronic systems

ROOT LOCUS

Given the system



$$L(s) = \frac{\rho}{(s+1)(s+2)}$$

$$\Theta(s) = (s+1)(s+2) + \rho = s^2 + 3s + 2 + \rho = 0 \longrightarrow s_{1,2} = \frac{-3 \pm \sqrt{1-4\rho}}{2}$$

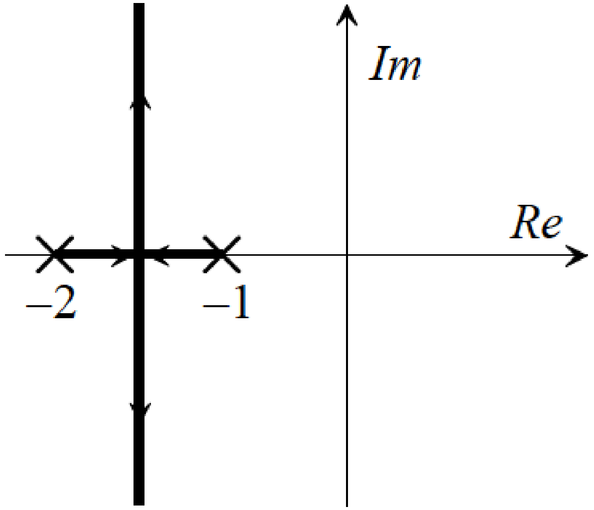
Analysis and control of mechatronic systems

ROOT LOCUS

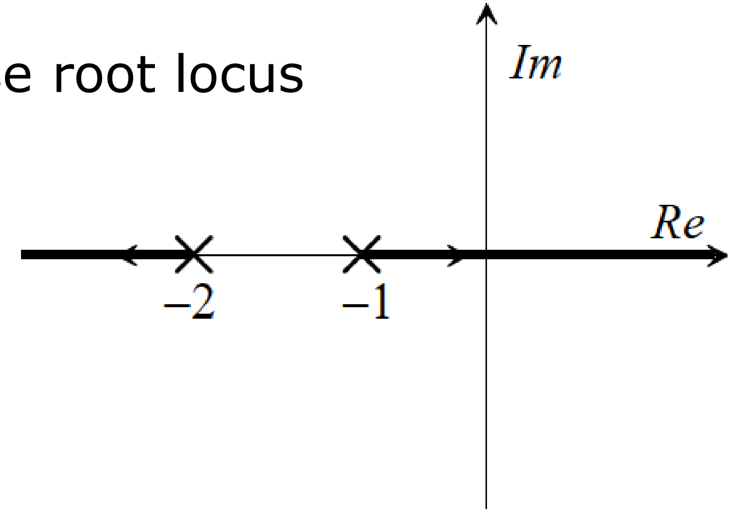
$$s_{1,2} = \frac{-3 \pm \sqrt{1-4\rho}}{2}$$

$$\begin{cases} \rho = 0 \\ 0 < \rho < 1/4 \\ \rho = 1/4 \\ \rho > 1/4 \\ \rho < 0 \end{cases}$$

Direct root locus



Inverse root locus



Analysis and control of mechatronic systems

ROOT LOCUS

Why using the root locus?

It is a visual representation of the system dynamics, which allow to analyze controlled systems

It is easy to plot, thanks to some simple **rules (next slides)**

It can be used to design the controller!!!

Analysis and control of mechatronic systems

ROOT LOCUS Drawing rules

1) given m=number of zeroes of L(s) and n=number of poles of L(s), the direct and inve

2) the RL is symmetric about the real axis

3) each branch starts from one of the poles of L(s), given $\rho = 0$

4) for $|\rho| \rightarrow \infty$, the branches of the RL end, either on the zeroes of L(s), or they go t

5) all the asymptotes intersect on one point, on the real axis xa

$$x_a = \frac{\sum_i z_1 - \sum_k p_k}{n - m}$$

Analysis and control of mechatronic systems

ROOT LOCUS Drawing rules

6) the n-m asymptotes have the following slopes:

$$\theta_{ah} = \begin{cases} \frac{180^\circ + h360^\circ}{n-m} & \text{direct locus} \\ \frac{h360^\circ}{n-m} & \text{inverse locus} \end{cases}$$

7) if n-m>=2 the sum of the real parts of the poles of the closed loop transfer functionis conserved (rule of the center of mass)

Analysis and control of mechatronic systems

ROOT LOCUS Drawing rules

8) the value of the gain ρ , for a particular root $\bar{s}$ can be determined as:

$$| \rho | = \frac{\prod_k | \bar{s} + p_k |}{\prod_i | \bar{s} + z_i |}$$

9) the points, on the real axis, where the real poles become complex and conjugated are x_d , such that:

$$\sum_k \frac{1}{x_d + p_k} - \sum_i \frac{1}{x_d + z_i} = 0$$

Analysis and control of mechatronic systems

ROOT LOCUS   Tuning a PID

$$G(s) = \frac{1}{1+\tau s}$$

$$G_P(s) = K_P$$

$$G_{PI}(s) = K_P \frac{1+T_i s}{T_i s}$$

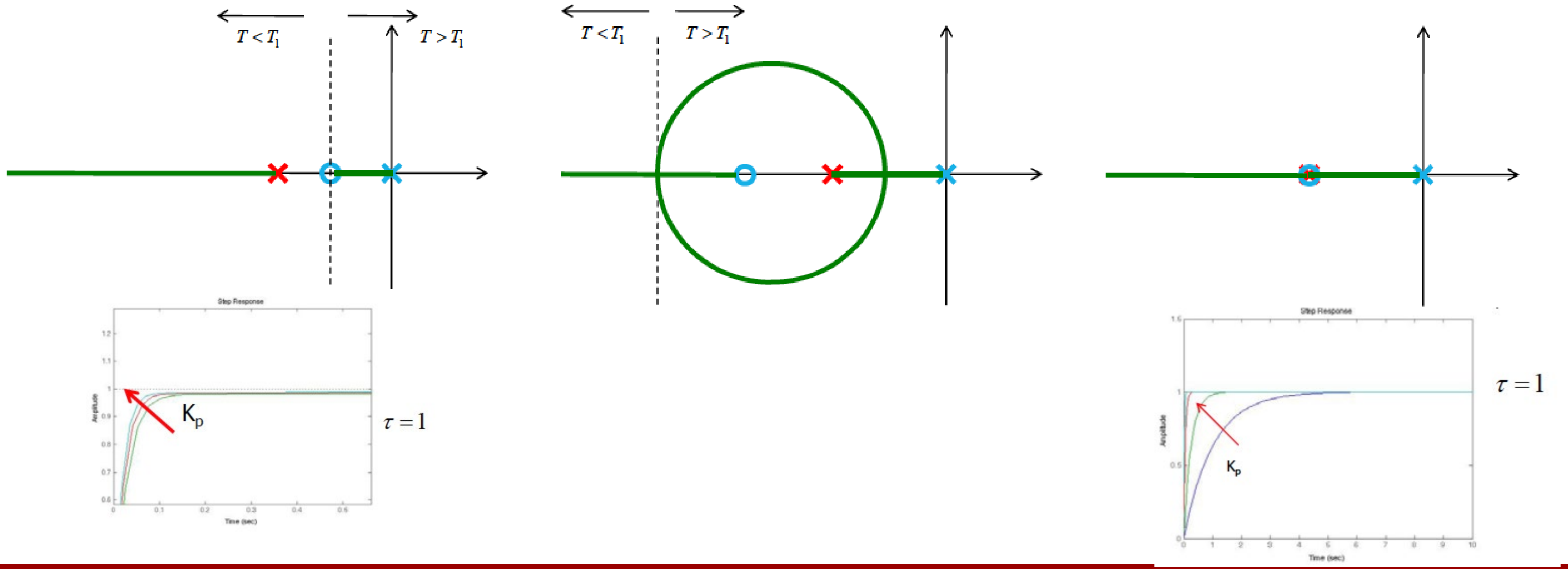
How to choose Ti?

Analysis and control of mechatronic systems

ROOT LOCUS Tuning a PI

$$G_{PI}(s) = K_P \frac{1 + T_i s}{T_i s}$$

How to choose Ti?



Analysis and control of mechatronic systems

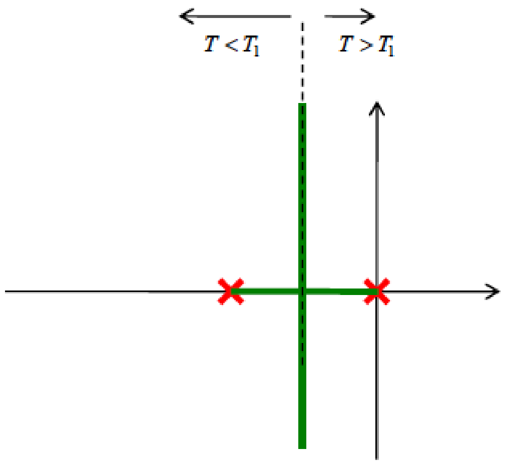
ROOT LOCUS Tuning a PD

$$G_{PD}(s) = K_P \frac{1 + T_d s}{1 + T_d/N s}$$

How to choose Td?

Plant

$$G(s) = \frac{1}{s(1 + s\tau)}$$



With this plant, we can't arbitrarily choose a Kp that gives the desired settling time

Analysis and control of mechatronic systems

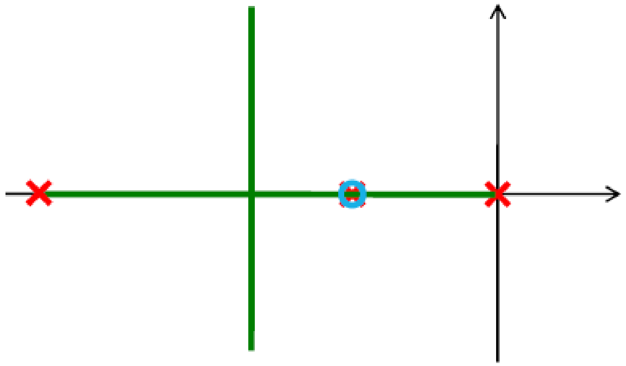
ROOT LOCUS Tuning a PD

$$G_{PD}(s) = K_P \frac{1 + T_d s}{1 + T_d/N s}$$

Plant

$$G(s) = \frac{1}{s(1 + s\tau)}$$

How to choose Td?



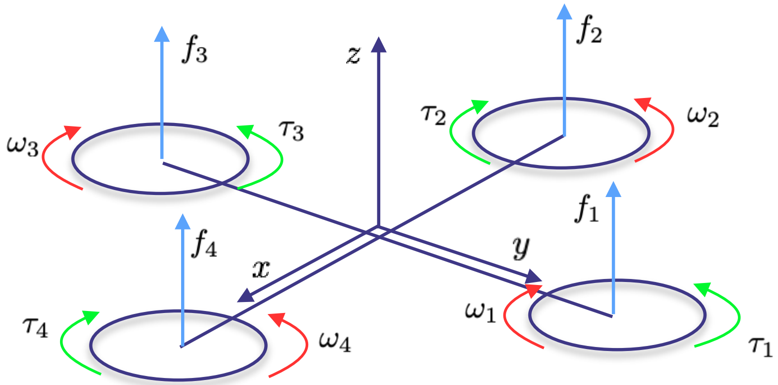
We choose Td and N such that:

- the zero cancels the pole.
- N big enough to move one of the real pole far enough

Attitude control of a quadrotor UAV

ATTITUDE CONTROL OF A QUADROTOR UAV

Body-fixed frame



Control of the attitude

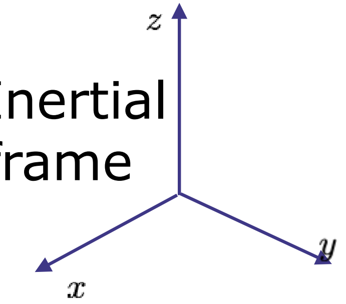
Let's consider the rotational dynamics:

$$\dot{\vec{\omega}} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = I^{-1}(\vec{\tau}_B - \vec{\omega} \times (I\vec{\omega}))$$

with $\vec{\tau}_B = \begin{bmatrix} Lk(\omega_1^2 - \omega_3^2) \\ Lk(\omega_2^2 - \omega_4^2) \\ b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \end{bmatrix}$

Since we are always considering the square of the propeller angular velocity, ω_i^2 , we can refer to it as a new control input τ_i

Inertial frame



Let's also remember that:

$$\vec{\omega} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & c_\theta s_\phi \\ 0 & -s_\phi & c_\theta c_\phi \end{bmatrix} \dot{\vec{\theta}}$$

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

We want to use a PD control scheme on the Euler angles R-P-Y

However, our quadcopter uses a gyroscope, meaning that the only measurement we have about rotations are the rotational velocities $\dot{\phi} \quad \dot{\theta} \quad \dot{\psi}$

Let's assume we want to control our quadrotor in a way that it can hover, meaning that we want to stabilise the quadrotor in a horizontal position

$$\begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} -I_{xx}(k_d\dot{\phi} + k_p \int_0^T \dot{\phi} dt) \\ -I_{yy}(k_d\dot{\theta} + k_p \int_0^T \dot{\theta} dt) \\ -I_{zz}(k_d\dot{\psi} + k_p \int_0^T \dot{\psi} dt) \end{bmatrix}$$

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

More in general, we want to control the quadrotor's rotation over a certain orientation reference position

$$\begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} -I_{xx}(k_d \dot{e}_\phi + k_p e_\phi) \\ -I_{yy}(k_d \dot{e}_\theta + k_p e_\theta) \\ -I_{zz}(k_d \dot{e}_\psi + k_p e_\psi) \end{bmatrix}$$

being

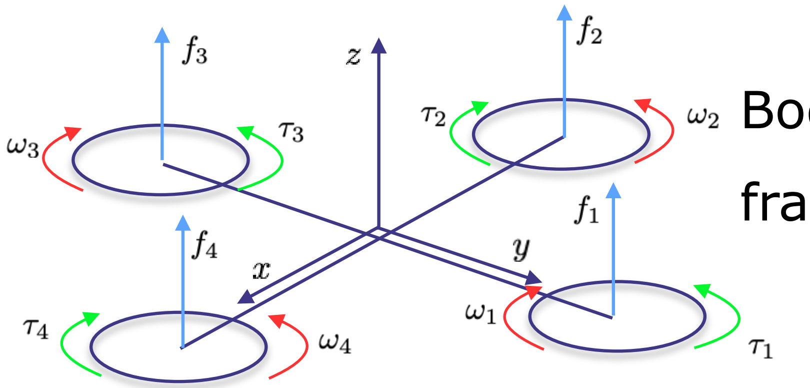
$\dot{e}_x = \dot{x} - \dot{x}^\star$ and $e_x = \int_0^T \dot{x} dt - x^\star$

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

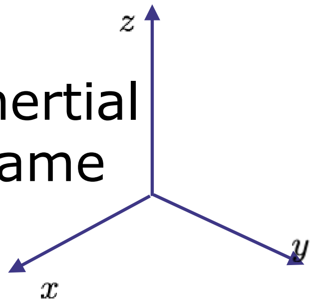
Our model

$$\dot{\vec{\omega}} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = I^{-1}(\vec{\tau}_B - \underbrace{\vec{\omega} \times (I\vec{\omega})}_{\text{Second order term}})$$



Body-fixed frame

Inertial frame



Control input

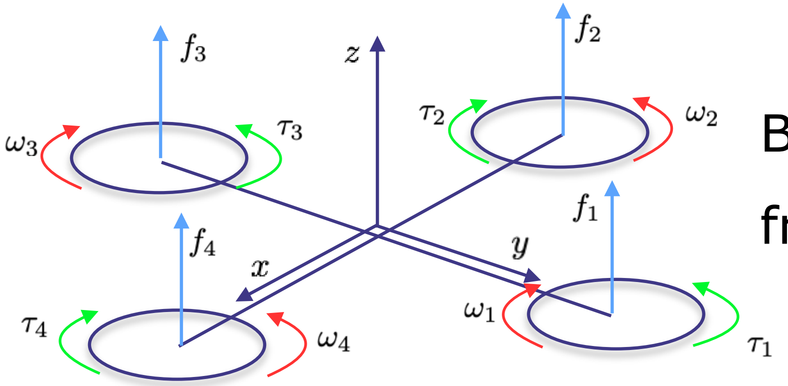
$$\vec{\tau}_B = \begin{bmatrix} Lk(\gamma_1 - \gamma_3) \\ Lk(\gamma_2 - \gamma_4) \\ b(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4) \end{bmatrix} = \begin{bmatrix} -I_{xx}(k_d\dot{\phi} + k_p \int_0^T \dot{\phi}dt) \\ -I_{yy}(k_d\dot{\theta} + k_p \int_0^T \dot{\theta}dt) \\ -I_{zz}(k_d\dot{\psi} + k_p \int_0^T \dot{\psi}dt) \end{bmatrix}$$

3 equations, 4 unknowns $\gamma_1, \gamma_2, \gamma_3, \gamma_4$

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

$$\vec{\tau}_B = \begin{bmatrix} Lk(\gamma_1 - \gamma_3) \\ Lk(\gamma_2 - \gamma_4) \\ b(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4) \end{bmatrix} = \begin{bmatrix} -I_{xx}(k_d\dot{\phi} + k_p \int_0^T \dot{\phi} dt) \\ -I_{yy}(k_d\dot{\theta} + k_p \int_0^T \dot{\theta} dt) \\ -I_{zz}(k_d\dot{\psi} + k_p \int_0^T \dot{\psi} dt) \end{bmatrix}$$

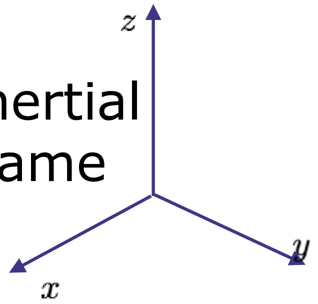


Body-fixed frame

We can constrain this imposing that out inputs should keep the quadcopter floating (not falling)

$$T = mg$$

Inertial frame



Note that in case the orientation controller is perturbed, there is no guarantee that the trust points up.

We need do consider that the projection of the trust on the vertical axis z (of the inertial frame) is equal to mg

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

Note that in case the orientation controller is perturbed, there is no guarantee that the trust points up.

We need do consider that the projection of the trust on the vertical axis z (of the inertial frame) is equal to mg

Therefore we need to enforce a constraint that keeps into account the orientation of the UAV.

$$T = \frac{mg}{cos\theta cos\phi} = k\sum \gamma_i \longrightarrow \sum \gamma_i = \frac{mg}{kcos\theta cos\phi}$$

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

Let's call

$$\begin{bmatrix} u_\phi \\ u_\theta \\ u_\psi \end{bmatrix} = \begin{bmatrix} k_d\dot{\phi} + k_p \int_0^T \dot{\phi}dt \\ k_d\dot{\theta} + k_p \int_0^T \dot{\theta}dt \\ k_d\dot{\psi} + k_p \int_0^T \dot{\psi}dt \end{bmatrix}$$

the control inputs deriving from our PD controller for hovering

Our 4 equations with the 4 unknown can then be written as

$$\begin{bmatrix} Lk(\gamma_1 - \gamma_3) \\ Lk(\gamma_2 - \gamma_4) \\ b(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4) \\ \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \end{bmatrix} = \begin{bmatrix} -I_{xx}u_\phi \\ -I_{yy}u_\theta \\ -I_{zz}u_\psi \\ \frac{mg}{kcos\theta cos\phi} \end{bmatrix}$$

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

$$\begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} = \begin{bmatrix} -I_{xx}u_\phi \\ -I_{yy}u_\theta \\ -I_{zz}u_\psi \\ \frac{mg}{kcos\theta cos\phi} \end{bmatrix}$$

That we can rewrite as:

$$A\vec{\gamma} = \vec{U} \longrightarrow \vec{\gamma} = A^{-1}\vec{U}$$

ATTITUDE CONTROL OF A QUADROTOR

Control of the attitude

$$\gamma_1 = \frac{mg}{k4cos\theta cos\phi} - \frac{2bu_\phi I_{xx} + u_\psi I_{zz} kL}{4bkL}$$

$$\gamma_2 = \frac{mg}{k4cos\theta cos\phi} - \frac{2bu_\theta I_{yy} - u_\psi I_{zz} kL}{4bkL}$$

$$\gamma_3 = \frac{mg}{k4cos\theta cos\phi} - \frac{-2bu_\phi I_{xx} + u_\psi I_{zz} kL}{4bkL}$$

$$\gamma_4 = \frac{mg}{k4cos\theta cos\phi} - \frac{-2bu_\theta I_{yy} - u_\psi I_{zz} kL}{4bkL}$$

ALTITUDE CONTROL OF A QUADROTOR

Control of the altitude

$$\sum \gamma_i = \frac{\mathrm{mg} + f_{c,z}}{\mathrm{k}\cos\vartheta\ \cos\varphi}$$

$$f_{c,z} = PID(e_z)$$

ALTITUDE CONTROL OF A QUADROTOR

Control of the altitude

$$\sum\gamma_i = \frac{\mathrm{mg}+f_{c,z}}{\mathrm{k}\cos\vartheta\ \cos\varphi}$$

$$f_{c,z} = m\ PID(e_z)$$

POSITION CONTROL OF A QUADROTOR

Control of the position

Goal: to control x and y in the inertial frame, by using roll and pitch references

NOTE: to control x we need to pitch; to control y, we need to roll

$$\begin{bmatrix}\phi^\star \\ \theta^\star\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 \\ -1 & 0 & 0\end{bmatrix} R \begin{bmatrix} PID(e_x) \\ PID(e_y) \\ 0 \end{bmatrix}$$