Unmanned Autonomous Systems - 31390

Modeling of Unmanned Aerial Vehicles

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Recap on Mechanical Modelling

Two major areas of modeling

- **Kinematics**: studies of the motion of a mechanical system, taking into account the constraints of a kinematic structure, without taking into account for forces and torques that govern the motion of the system.

- **Dynamics**: studies the motion of a system as a function of the forces and torques that act on a kinematic structure

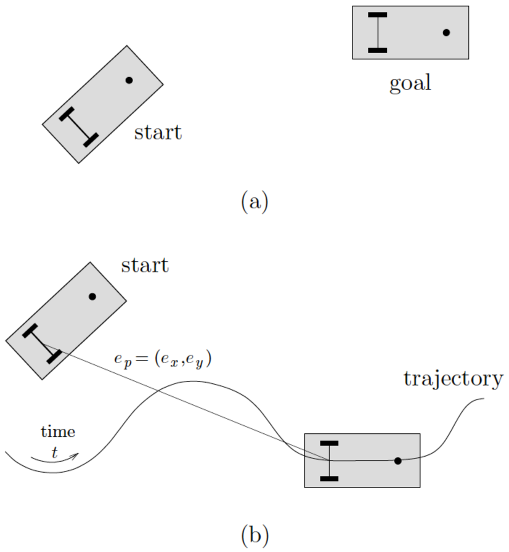
Recap on Mechanical Modelling

In order to study any physical system, we need to create a representation of the system

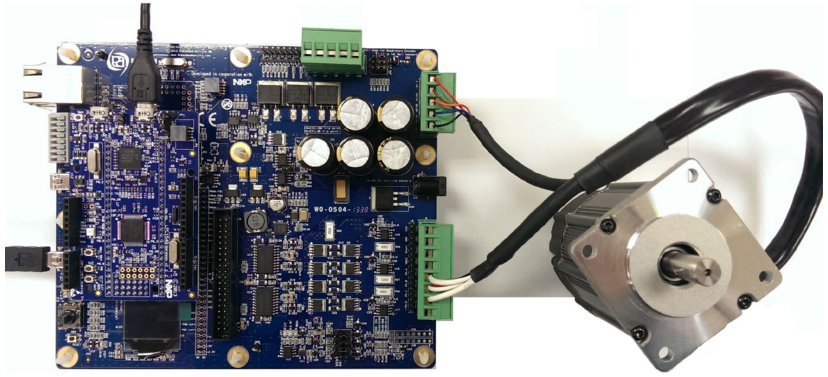
A **model** is a representation, through mathematical equations, of a system.

Example:

wheeled robot motion

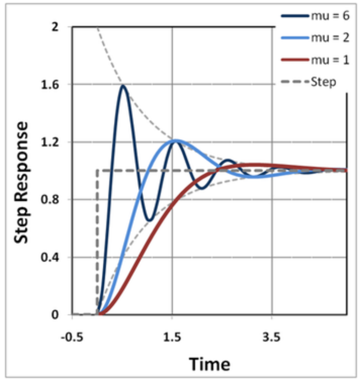


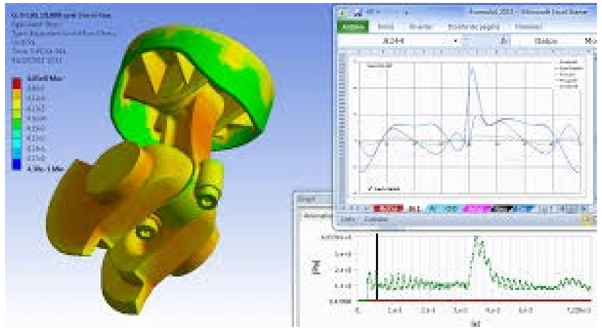
Motor Control

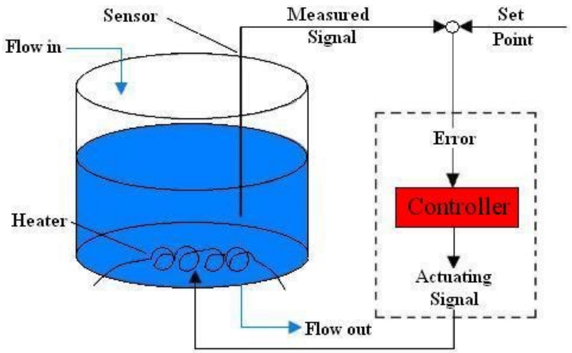


Recap on Mechanical Modelling

- What is, in practice, the purpose of a model?
  - understanding the behavior of a system (e.g. source of vibrations in a precision machine)
  - predicting the behavior of a system, given an external agent, without need to observe it in the real world
  - design of mechanical components based on the actions acting on elements (e.g. design a rod-crank mechanism for a combustion motor of a car
  - synthesis of the controller of a system and off-line regulation of control parameter







Recap on Mechanical Modelling

Kinematics

Recap on Mechanical Modelling, kinematics

In order to study the dynamic behavior of mechanical systems, we need to first model their kinematics

Reminder: the kinematics is the study of the motion of a body, or a set of interconnected bodies without considering the actions acting on the system.

Studying the kinematics of a system means to define the relations among the position, velocity and acceleration of the bodies interconnected in the system

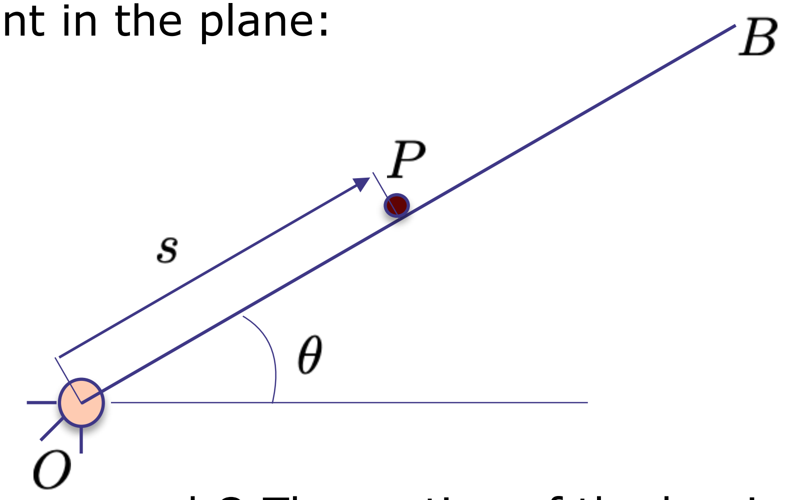
Recap on Mechanical Modelling, kinematics

- interest is on defining quantities such as **position, velocity and acceleration**
- need to specify a reference frame (and a coordinate system in it to actually write the vector expressions)
- Velocity and acceleration depend of the choice of the reference frame
- Only when we go to laws of motion, the reference frame needs to be the inertial frame

Recap on Mechanical Modelling, kinematics

Kinematics of a point in the plane:

a first exercise

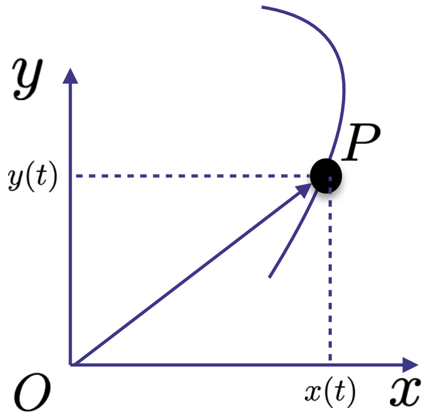


A bar BO can rotate around O The motion of the bar is described by the variable $\theta = \theta(t)$

A point P can slide on the bar BO, and its coordinate on the bar is described by the variable $s = s(t)$

Recap on Mechanical Modelling, kinematics

Kinematics of a point in the plane



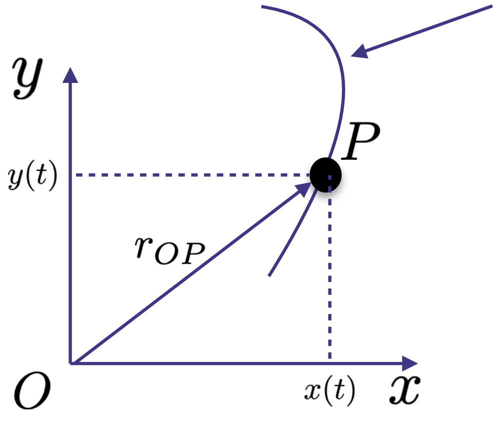
How can we describe the kinematics of a point moving on a trajectory?

Or more generally, how can we describe the motion of a system that consists of interconnected bodies? (e.g. a robot, a car, a production line)

We need a mathematical representation of the position of the bodies and their derivatives (velocity, acceleration, jerk, …)

Recap on Mechanical Modelling, kinematics

Kinematics of a point in the plane



r_{OP} position vector (specifies the position P of the object w.r.t. O)

Note

$$\vec{r}_{OP} = \vec{r}_{OP}(t)$$

$$x = x(t)$$

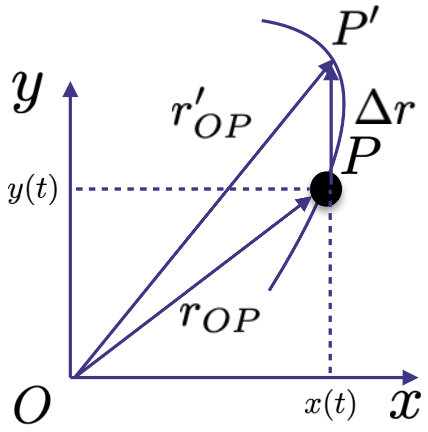
$$y = y(t)$$

$$t = g(x)$$

$$y = y(g(x)) = f(x)$$

Recap on Mechanical Modelling, kinematics

Kinematics of a point in the plane



Velocity vector:

$$\vec{v}_P = \frac{d}{dt}\vec{r}_{OP}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}_{OP}}{\Delta t}$$

Acceleration vector:

$$\vec{a}_P = \frac{d}{dt}\vec{v}_P(t) = \frac{d^2}{dt^2}\vec{r}_{OP}(t)$$

Recap on Mechanical Modelling, kinematics

Speed (scalar) $v_P = \|\overrightarrow{v_P}\|$

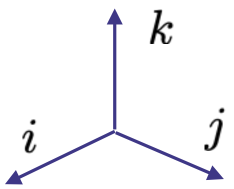
Magnitude of acceleration $a_P = \|\overrightarrow{a_P}\|$

Note: the time derivatives have been considered w.r.t. a reference frame

We can describe them in various coordinate systems:

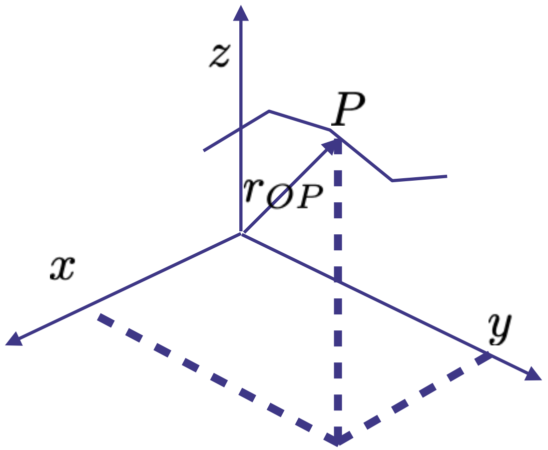
- Cartesian
- Cylindrical

Recap on Mechanical Modelling, kinematics



$\vec{i} \quad \vec{j} \quad \vec{k}$ unit vectors

NOTE: from now on, we call these unit vectors: $i \quad j \quad k$



Cartesian coordinate system

$i \quad j \quad k$ are an orthogonal set

$$i \times j = k$$
$$j \times k = i$$
$$k \times i = j$$

$$\vec{r}_{OP} = x(t)i + y(t)j + z(t)k$$

Recap on Mechanical Modelling, kinematics

The time derivative of the position vector is the velocity

$$\vec{v}_P = \frac{d\vec{r}_{OP}}{dt} = \frac{dx(t)}{dt}i + \frac{dy(t)}{dt}j + \frac{dz(t)}{dt}k + \\ + \frac{di}{dt}x(t) + \frac{dj}{dt}y(t) + \frac{dk}{dt}z(t)$$

the rate of change in a frame in which i, j, k are fixed, gives: $\frac{di}{dt} = \frac{dj}{dt} = \frac{dk}{dt} = 0$

$$\vec{v}_P = \frac{dx(t)}{dt}i + \frac{dy(t)}{dt}j + \frac{dz(t)}{dt}k$$

Similarly $\vec{a}_P = \ddot{x}(t)i + \ddot{y}(t)j + \ddot{z}(t)k$

Recap on Mechanical Modelling, kinematics

Kinematics of **RIGID BODIES**

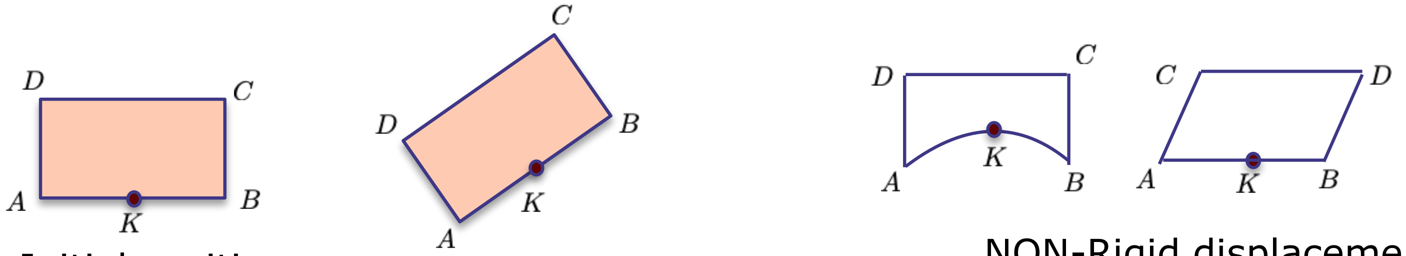
Position: set of vectors defining the position of every point constituting the rigid body

Movement: is the description of the position of the rigid body (and of how all the vectors representing the position of every of its points) over time

DTU

Recap on Mechanical Modelling, kinematics

Definition: the infinitesimal displacement is a displacement in which all the points constituting the body, change their position of an infinitesimal quantity.



Initial position

Rigid displacement

NON-Rigid displacement

Properties of a rigid body:

for any motion of the rigid body

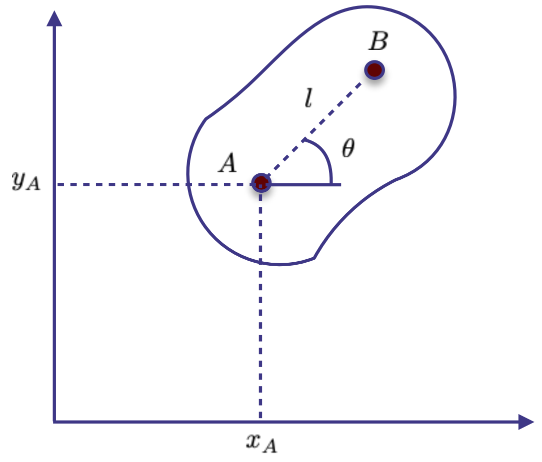
- the distance between any arbitrary couple of points of the rigid body does not change.

- the angles between the segments connecting couples of points of the rigid body do not change

Recap on Mechanical Modelling, kinematics

Kinematics of **RIGID BODIES** in the plane

We can reduce the number of DoF from infinite^2 (the coordinate of every point of the body) to 3: x,y,theta



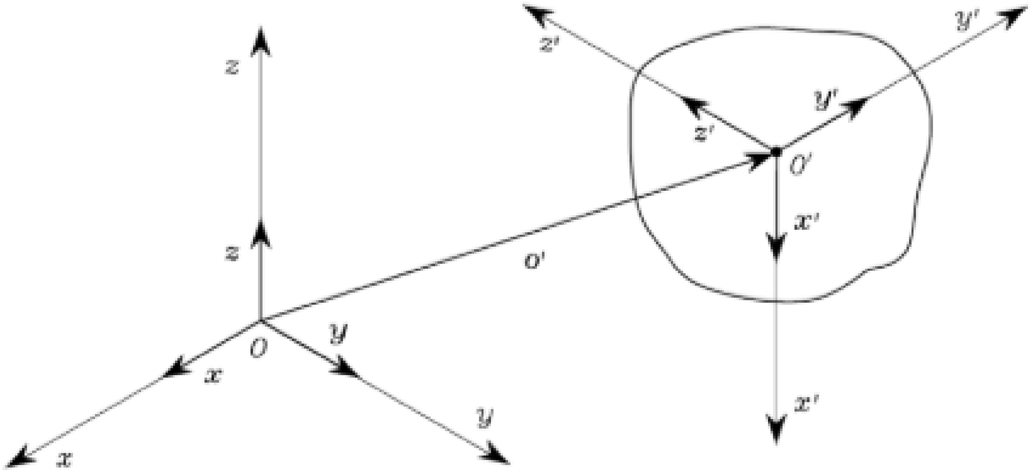
if we know the coordinates (x,y) of point A, we can represent the coordinates of point B by using 1 more coordinate, the rotation θ (provided that we know the distance between A and B)

given θ , every other couple of points will rotate of the same quantity. This means that the rotation is a property of the entire rigid body.

Recap on Mechanical Modelling

Rotations

Recap on Mechanical Modelling, rotations



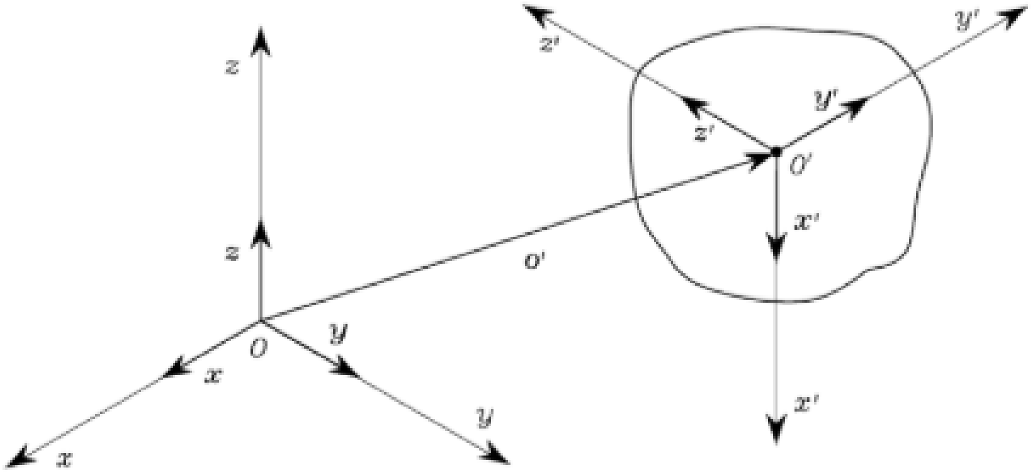
A body is represented in a base frame of reference by a vector that defines the position of the origin of a body fixed frame w.r.t. the base frame of reference

$$\boldsymbol{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$

The orientation of the body w.r.t. the base frame is described by the projection of the unit vectors of the body-fixed frame into the base frame

$$\begin{aligned} \boldsymbol{x}' &= x'_x \boldsymbol{x} + x'_y \boldsymbol{y} + x'_z \boldsymbol{z} \\ \boldsymbol{y}' &= y'_x \boldsymbol{x} + y'_y \boldsymbol{y} + y'_z \boldsymbol{z} \\ \boldsymbol{z}' &= z'_x \boldsymbol{x} + z'_y \boldsymbol{y} + z'_z \boldsymbol{z}. \end{aligned}$$

Recap on Mechanical Modelling, rotations



The rotation of a body can be described, in compact form, by a 3x3 matrix that combines the projection of the three unit vectors

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

Matrix R is a rotation matrix.

Note that:

The column vectors of R are mutually orthogonal

The column vectors of R have unit norm

$$\boldsymbol{x}'^T\boldsymbol{y}' = 0 \qquad \boldsymbol{y}'^T\boldsymbol{z}' = 0 \qquad \boldsymbol{z}'^T\boldsymbol{x}' = 0$$

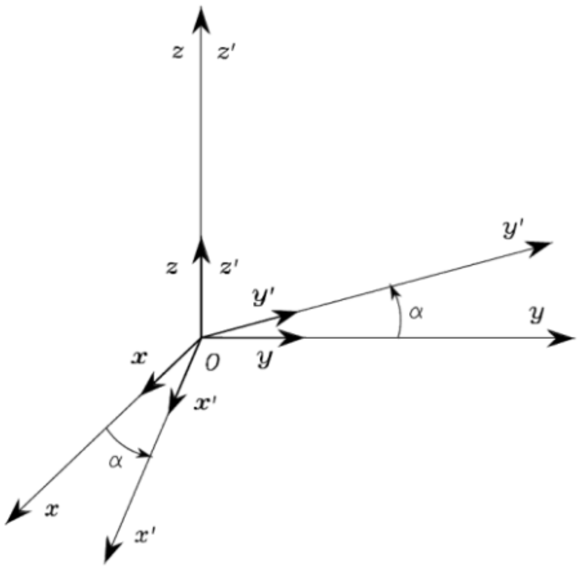
$$\boldsymbol{x}'^T\boldsymbol{x}' = 1 \qquad \boldsymbol{y}'^T\boldsymbol{y}' = 1 \qquad \boldsymbol{z}'^T\boldsymbol{z}' = 1$$

$\Rightarrow$

R is an orthogonal matrix

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3$$

Recap on Mechanical Modelling, rotations



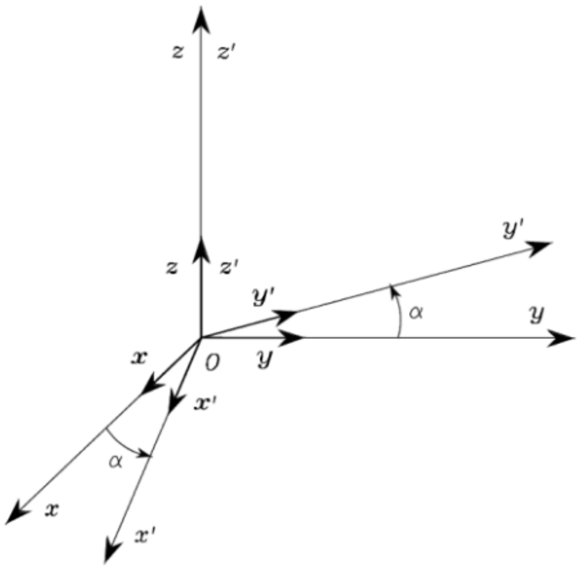
Rotation around axis z, by an angle α

$$\boldsymbol{x}' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{z}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, a rotation by β around axis y, and by γ around axis x are described by matrices

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \qquad \boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Recap on Mechanical Modelling, rotations



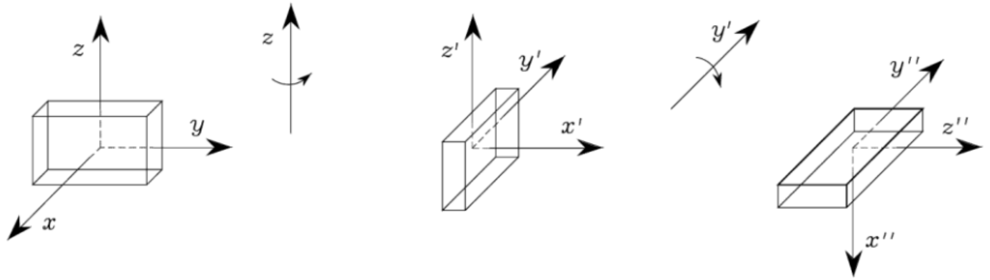
<u>Composition of rotations:</u>

An overall rotation can be expressed by a sequence of partial rotations, with each rotation defined w.r.t. the preceding one.

Composition of successive rotations is achieved by postmultiplication of the rotation matrices following the given order of rotations

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1 \boldsymbol{p}^2$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2$$

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_2^1$$



Recap on Mechanical Modelling, rotations

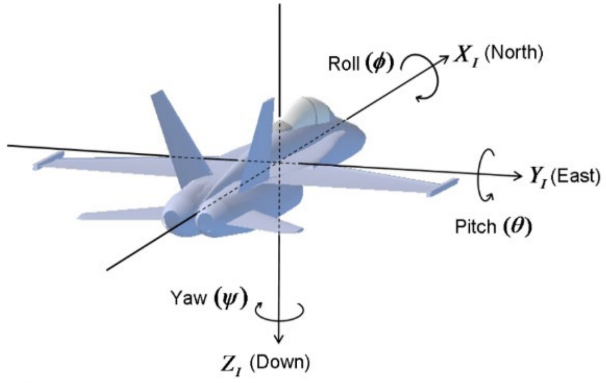
Derivative of a Rotation matrix

$$R(t)R^T(t) = I \xrightarrow{\frac{d}{dt}} \dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0 \xrightarrow{S(t) = \dot{R}(t)R^T(t)} \dot{R}(t) = S(t)R(t)$$

$S(t)$ Is a Skew symmetric matrix, as $S(t) + S^T(t) = 0$

$$S(t) = \begin{bmatrix} 0 & -\omega_z & -\omega_y \\ -\omega_z & 0 & -\omega_x \\ -\omega_y & -\omega_x & 0 \end{bmatrix}$$

Recap on Mechanical Modelling, rotations



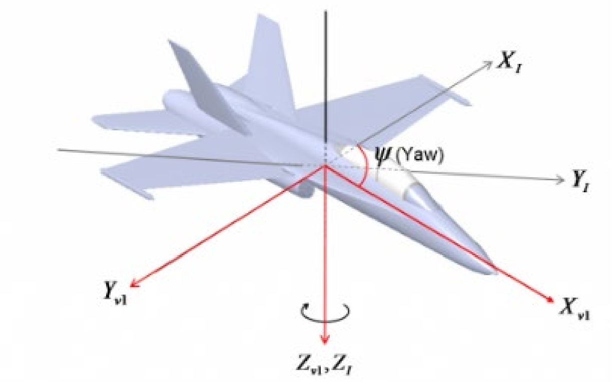


Figure 2 - Yaw rotation into the Vehicle-1 Frame

Minimum representation of orientation

A rotation can be represented with a minimum of 3 parameters

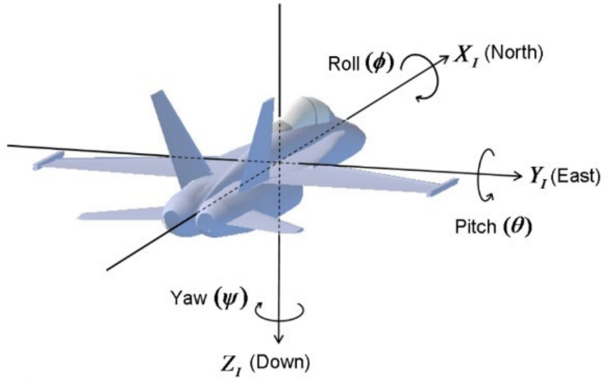
9 elements → 6 orthogonality constraints → 3 parameters, Euler Angles

RPY angles

A set of Euler angles following the order ZYX (Roll-Pitch-Yaw angles)

$$R_I^{v1}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Singularity problems in inverse relation



Recap on Mechanical Modelling, rotations

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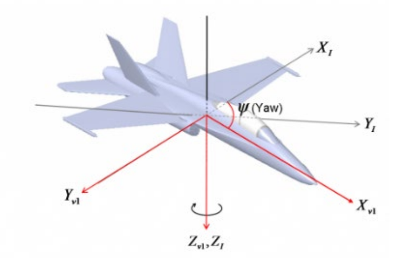


Figure 2 - Yaw rotation into the Vehicle-1 Frame

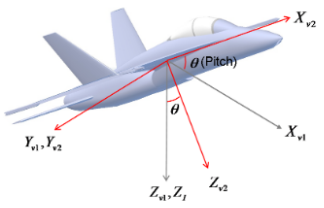


Figure 3 - The Vehicle-3 Frame (Yaw and Pitch Rotation Applied)

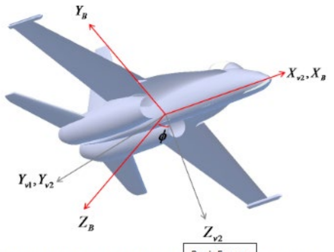


Figure 4 - The Body Frame (Yaw, Pi... applied)

B represents the body-frame

I represents the Inertial frame

$$R_I^B(\phi,\theta,\psi) = R_{v2}^B(\phi)R_{v1}^{v2}(\theta)R_I^{v1}(\psi)$$

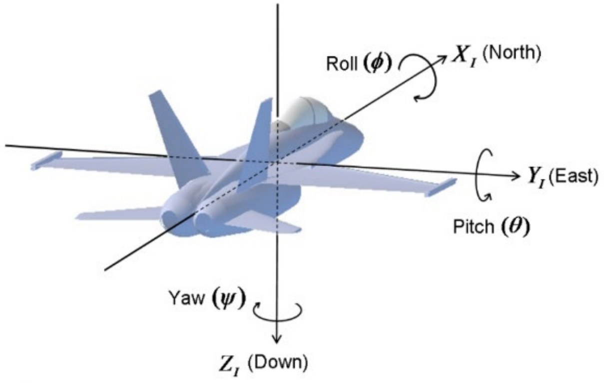
$$R_B^I(\phi,\theta,\psi) = R_I^{v1}(-\psi)R_{v1}^{v2}(-\theta)R_{v2}^B(-\phi).$$

$$R_I^{v1}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{v1}^{v2}(\theta) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$R_{v2}^B(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

Recap on Mechanical Modelling, rotations



How to obtain inertial frame accelerometer data from an accelerometer placed on the body?

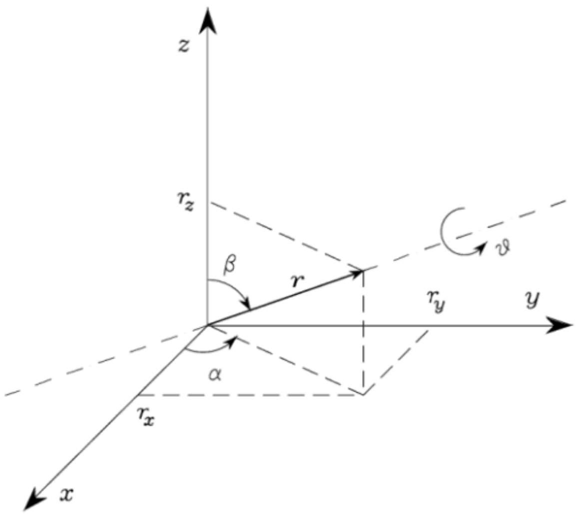
$$\mathbf{v}_I = R_B^I(\phi, \theta, \psi)\mathbf{v}_B.$$

How to convert rate gyro data (rotational velocity of the body, or body angular rates) to derivative of Euler angles

$$D(\phi, \theta, \psi) = \begin{pmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{pmatrix} \longrightarrow \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} p + q\sin(\phi)\tan(\theta) + r\cos(\phi)\tan(\theta) \\ q\cos(\phi) - r\sin(\phi) \\ q\sin(\phi)/\cos(\theta) + r\cos(\phi)/\cos(\theta) \end{pmatrix}$$

http://www.chrobotics.com/library/understanding-euler-angles

Recap on Mechanical Modelling, rotations



Non-minimal representation of orientation: axis-angle

A rotation can be represented with of 4 parameters, defining an axis of rotation (3 parameters) and the rotation around such axis (1 parameter)

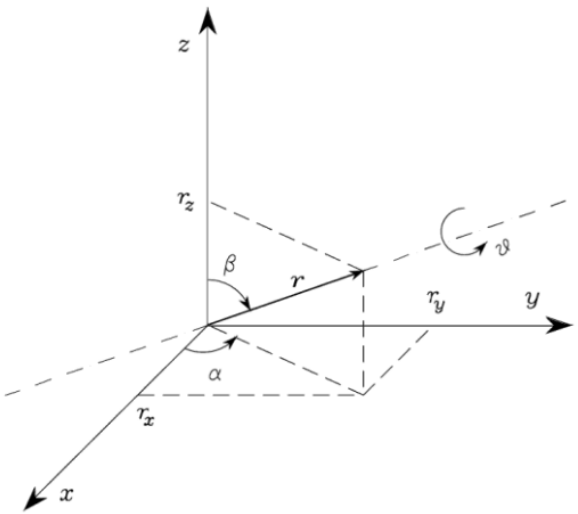
Procedure to compute R, given the axis and the angle

- Align $\boldsymbol{r}$ with z, which is obtained as the sequence of a rotation by $-\alpha$ about z and a rotation by $-\beta$ about y.
- Rotate by ϑ about z.
- Realign with the initial direction of $\boldsymbol{r}$, which is obtained as the sequence of a rotation by β about y and a rotation by α about z.

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha)$$

being $\sin\alpha = \dfrac{r_y}{\sqrt{r_x^2 + r_y^2}}$ $\quad \cos\alpha = \dfrac{r_x}{\sqrt{r_x^2 + r_y^2}}$ $\quad \sin\beta = \sqrt{r_x^2 + r_y^2}$ $\quad \cos\beta = r_z$

Recap on Mechanical Modelling, rotations



Non-minimal representation of orientation: axis-angle

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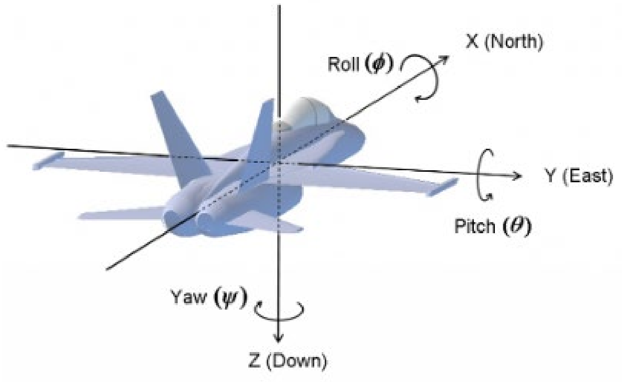
Inverse problem

$$\vartheta=\cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \qquad r=\frac{1}{2sin\vartheta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

Singularity when $\vartheta=0$

Recap on Mechanical Modelling, rotations



<u>Non-minimal representation of orientation: unit quaternion</u>

A rotation can be represented with of 4 parameters, consisting of 1 real and 3 complex numbers.

How can we use quaternions to encode rotations from inertial frame to body-fixed frame?

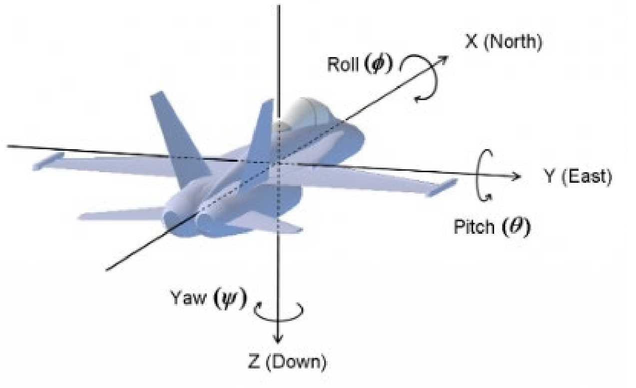
Let the vector q_i^b be defined as the unit-vector quaternion encoding rotation from the inertial frame to the body frame of the sensor

$$\mathbf{q}_i^b = \begin{pmatrix} a & b & c & d \end{pmatrix}^T$$

The elements **b, c,** and **d** are the "vector part" of the quaternion, and can be thought of as a vector about which rotation should be performed.

The element **a** is the "scalar part" that specifies the amount of rotation that should be performed about the vector part.

Recap on Mechanical Modelling, rotations



Non-minimal representation of orientation: unit quaternion

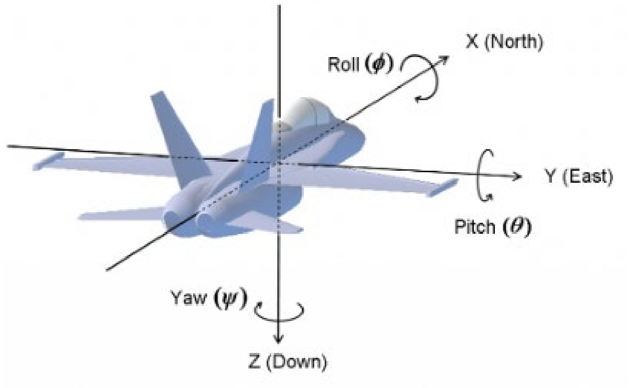
A rotation can be represented with of 4 parameters, consisting of 1 real and 3 complex numbers.

How can we use quaternions to encode rotations from inertial frame to body-fixed frame?

if θ is the angle of rotation and the vector $(v_x \quad v_y \quad v_z)^T$ is a unit vector representing the axis of rotation, then the quaternion elements are defined as

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \cos(0.5\theta) \\ v_x \sin(0.5\theta) \\ v_y \sin(0.5\theta) \\ v_z \sin(0.5\theta) \end{pmatrix}$$

Recap on Mechanical Modelling, rotations



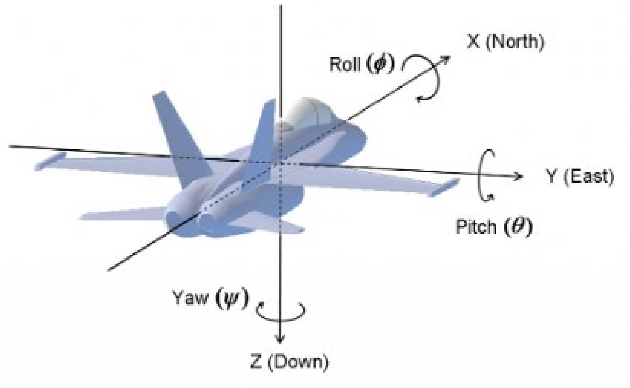
Quaternions

The attitude quaternion q_i^b can be used to rotate an arbitrary 3-element vector from the inertial frame to the body frame using the operation

$$\mathbf{v}_B = \mathbf{q}_i^b \begin{pmatrix} 0 \\ \mathbf{v}_I \end{pmatrix} (\mathbf{q}_i^b)^{-1}$$

- a vector can rotated by treating it like a quaternion with zero real-part and multiplying it by the attitude quaternion and its inverse.
- The inverse of a quaternion is equivalent to its conjugate, which means that all the vector elements (the last three elements in the vector) are negated

Recap on Mechanical Modelling, rotations



Quaternions

The attitude quaternion q_i^b can be used to rotate an arbitrary 3-element vector from the inertial frame to the body frame using the operation

$$\mathbf{v}_B = \mathbf{q}_i^b \begin{pmatrix} 0 \\ \mathbf{v}_I \end{pmatrix} (\mathbf{q}_i^b)^{-1}$$

Quaternion multiplications

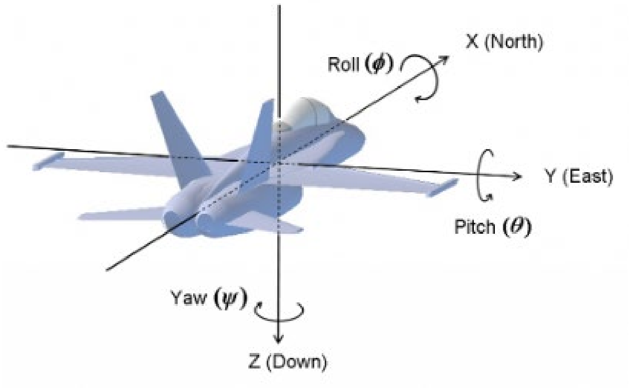
Given two quaternions

$$\mathbf{q}_1 = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \end{pmatrix}^T$$

$$\mathbf{q}_2 = \begin{pmatrix} a_2 & b_2 & c_2 & d_2 \end{pmatrix}^T$$

$$\mathbf{q}_1 \mathbf{q}_2 = \begin{pmatrix} a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2 \\ a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2 \\ a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2 \end{pmatrix}$$

Recap on Mechanical Modelling, rotations



Quaternions

We can reconstruct a rotation matrix from an attitude quaternion.
The rotation matrix from the inertial frame to the body-fixed frame is given by:

$$R_i^b(\mathbf{q}_i^b) = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

We can therefore represent vectors in the inertial frame, as vector in the body-fixed frame as:

$$\mathbf{v}_B = R_i^b(\mathbf{q}_i^b)\mathbf{v}_I$$

http://www.chrobotics.com/library/understanding-quaternions

Recap on Mechanical Modelling, rotations

Derivative of a Rotation matrix

Consider a constant vector p' and the vector $p(t) = R(t)p'$

The derivative of $p(t)$ is $\dot{p}(t) = \dot{R}(t)p'$

Which can be written as

$$\dot{p}(t) = S(t)R(t)p' = \omega(t) \times R(t)p'$$

Where $\omega = [\omega_x \quad \omega_y \quad \omega_z]^T$ is the vector of the angular velocity of frame $R(t)$ with respect to the reference frame at time t

Recap on Mechanical Modelling, rotations

If we represent the the rotation of a frame with Euler angles (e.g. RPY -> $\vec{\theta} = [\varphi \quad \vartheta \quad \psi]^T$), is the derivative of the Euler angles the same as ω ?

NO!

How are they related?

It depends on the angle representation that we chose

$$\vec{\omega} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & c_\theta s_\phi \\ 0 & -s_\phi & c_\theta c_\phi \end{bmatrix} \dot{\vec{\theta}}$$

$$D(\phi, \theta, \psi) = \begin{pmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{pmatrix}$$

Recap on Mechanical Modelling

Dynamics

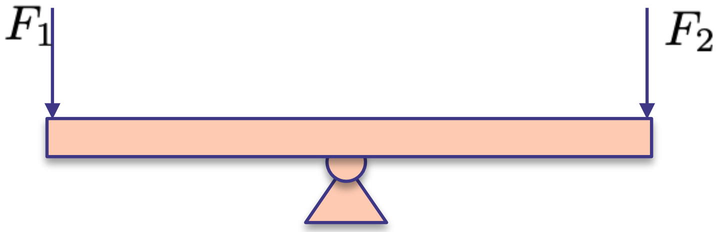
Recap on Mechanical Modelling, dynamics

Discipline of mechanics that studies the equilibrium of a body

In other words, in a static system, all forces and moments are ballanced

… what does it mean?

Example: find the relations and constraint forces that keep the system in equilibrium

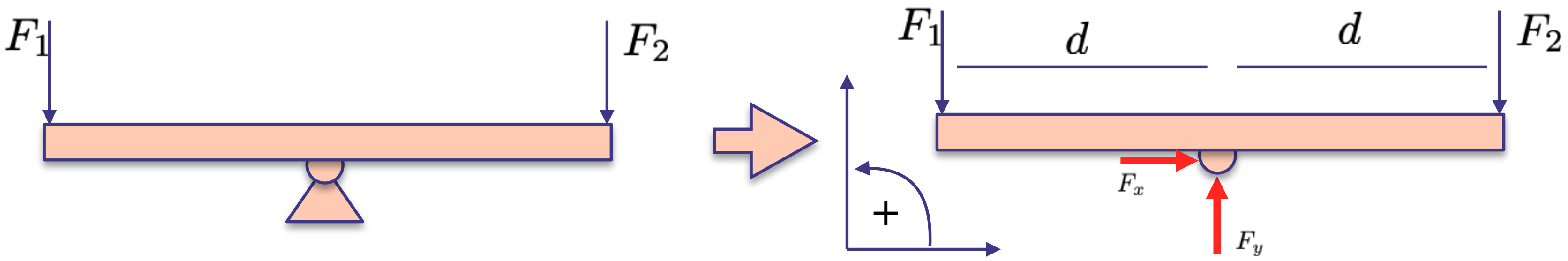


We need to make a ballance of forces in x and y direction, in order to define the constraint forces, and write one or more ballance of moments to make sure we have rotational equilibrium

How does a ballance of forces looks like?

Recap on Mechanical Modelling, dynamics

How does a ballance of forces looks like?



$F_y - F_1 - F_2 = 0$ Ballance of forces in y-direction

$F_x = 0$ Ballance of forces in x-direction

$F_1 d - F_2 d = 0$ Ballance of moments around the pivot

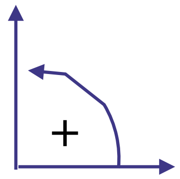
Recap on Mechanical Modelling, dynamics

Basics notions:

Laws of Mechanics

1) An object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an unbalanced force (Law of Inertia)

There exists a frame, with reference to which, an isolated system that is fixed, stays fixed. This frame of reference is called *Inertial frame*.



Recap on Mechanical Modelling, dynamics

Basics notions:

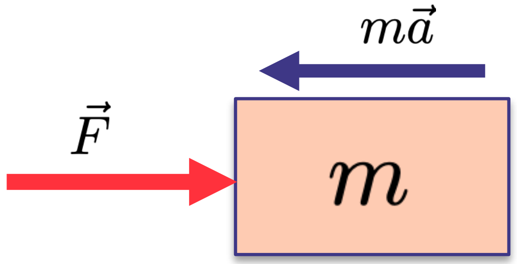
Laws of Mechanics

2) The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force, in the same direction as the net force, and inversely proportional to the mass of the object.

A material point s.t. a force $\vec{F}$ moves with an acceleration $\vec{a}$ that is proportional to the force $\vec{F}$, following:

$$\vec{F} = m\vec{a}$$

m is the inertial mass of the object



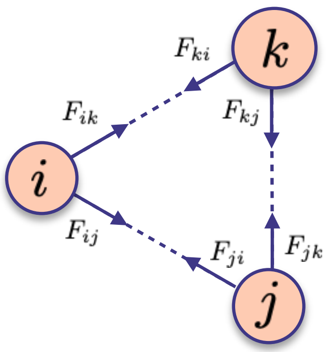
Recap on Mechanical Modelling, dynamics

Basics notions:

Laws of Mechanics

3) For every action, there is an equal and opposite reaction.

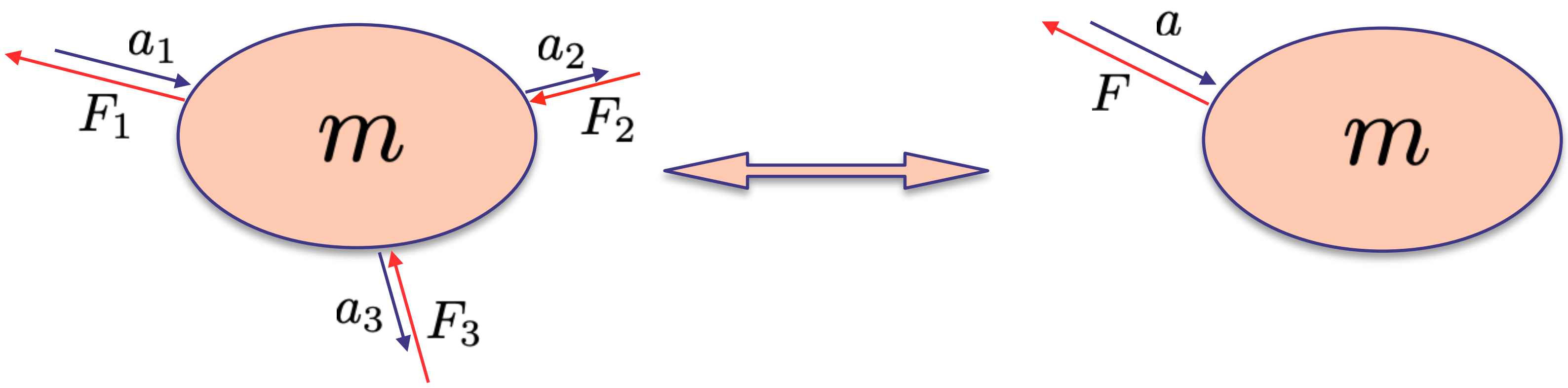
Given a system of material points that interact among each other, the forces exchanged among those points are equal and opposite in couples, and directed on the same line.



Recap on Mechanical Modelling, dynamics

Basics notions:

If a force F_1 applied to a point produces an acceleration a_1 , and if a force F_2 applied to the same point produces an acceleration a_2 , then the two forces, when applied simultaneously, will produce an acceleration $a = a_1 + a_2$ that correspond to the effect of the overall force $F = F_1 + F_2$



Recap on Mechanical Modelling, dynamics

Types of Forces: some definitions

A force $\vec{F}$ acting on a point, generally depends on:

- $\vec{r}$ the position of the point
- $\vec{v}$ the velocity of point P
- t time

$$\vec{F} = \vec{F}(\vec{r},\ \vec{v},\ t)$$

We define:

ACTIVE: those forces, known a priori, that depend on the position and velocity of a point

REACTIVE: those forces that are a reaction to the motion (e.g. constraint forces)

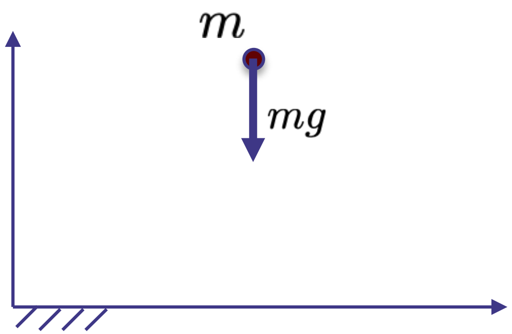
Recap on Mechanical Modelling, dynamics

Types of Forces: some definitions

Constant Force:

A force that does not change over time in both modulus and direction

An example is the weight of an object



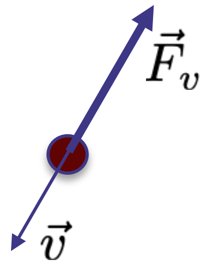
$$\vec{w} = -mg\vec{j}$$

Recap on Mechanical Modelling, dynamics

Types of Forces: some definitions

Viscous Force:

it is not a positional force!



$$\vec{F}_v = -\gamma \vec{v}$$

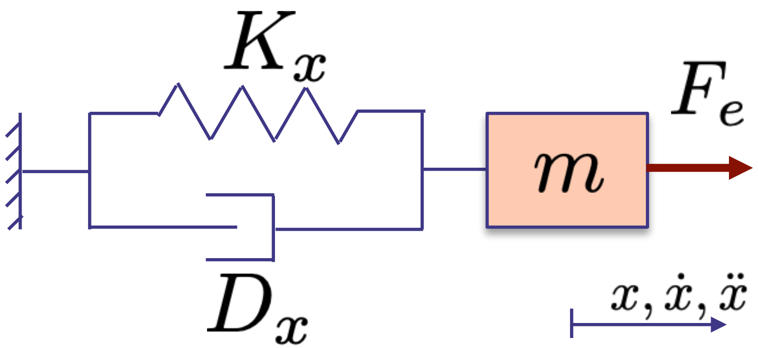
A viscous force is opposed to the motion of the object.

Examples of viscous force are:

- air resistance
- fluid resistance (e.g. water)

Recap on Mechanical Modelling, dynamics

Some exercises (dynamics):



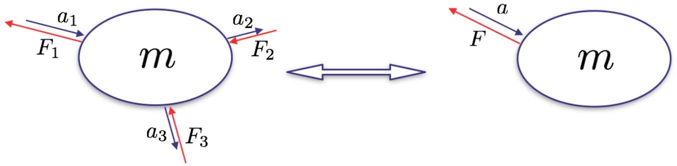
LINEAR

INERTIAL	$\vec{F}_i = m\vec{a}$
VISCOUS	$\vec{F}_v = D_x\vec{v}$
ELASTIC	$\vec{F}_k = K_x\vec{x}$
EXTERNAL	$\vec{F}_e = \vec{F}_e(t)$

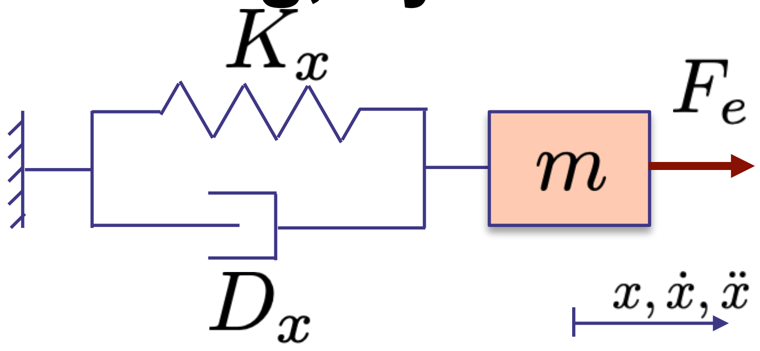
DYNAMICS

Basics notions:

If a force F_1 applied to a point produces an acceleration a_1, and if a force F_2 applied to the same point produces an acceleration a_2, then the two forces, when applied simultaneously, will produce an acceleration $a = a_1 + a_2$ that correspond to the effect of the overall force $F = F_1 + F_2$



Recap on Mechanical Modelling, dynamics



Some exercises (dynamics):

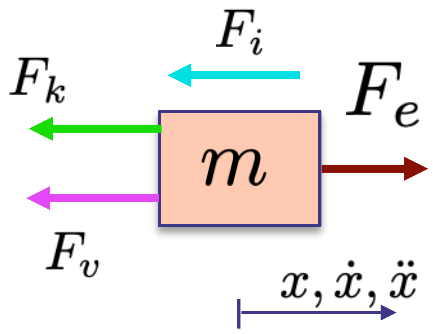
LINEAR

INERTIAL $\vec{F}_i = m\vec{a}$

VISCOUS $\vec{F}_v = D_x\vec{v}$

ELASTIC $\vec{F}_k = K_x\vec{x}$

EXTERNAL $\vec{F}_e = \vec{F}_e(t)$

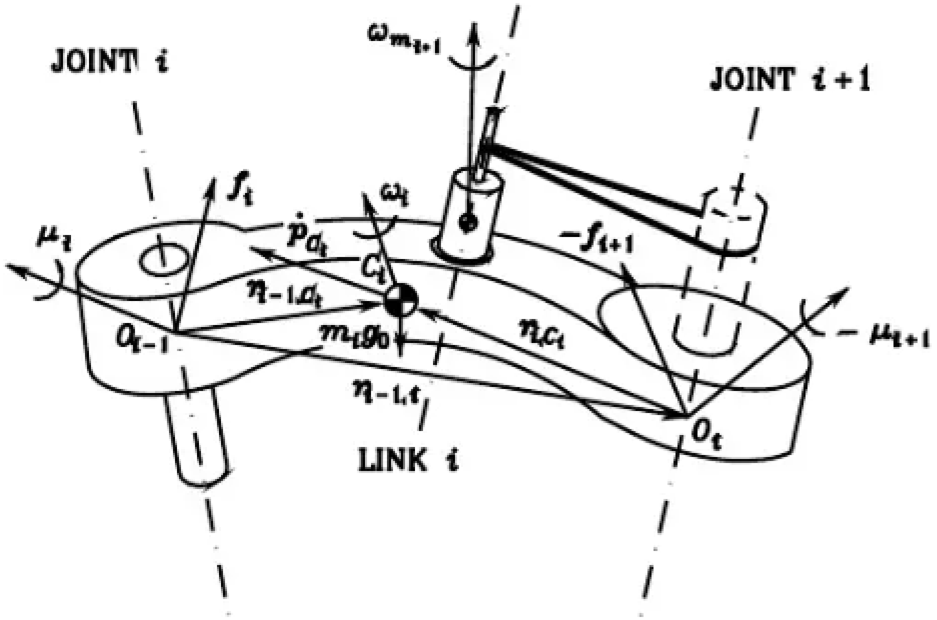


$$\boxed{m\ddot{x} + D_x\dot{x} + K_x x = F_e}$$

EQUATION OF MOTION

Recap on Mechanical Modelling, dynamics

Recursive Newton-Euler: recap from robotic course

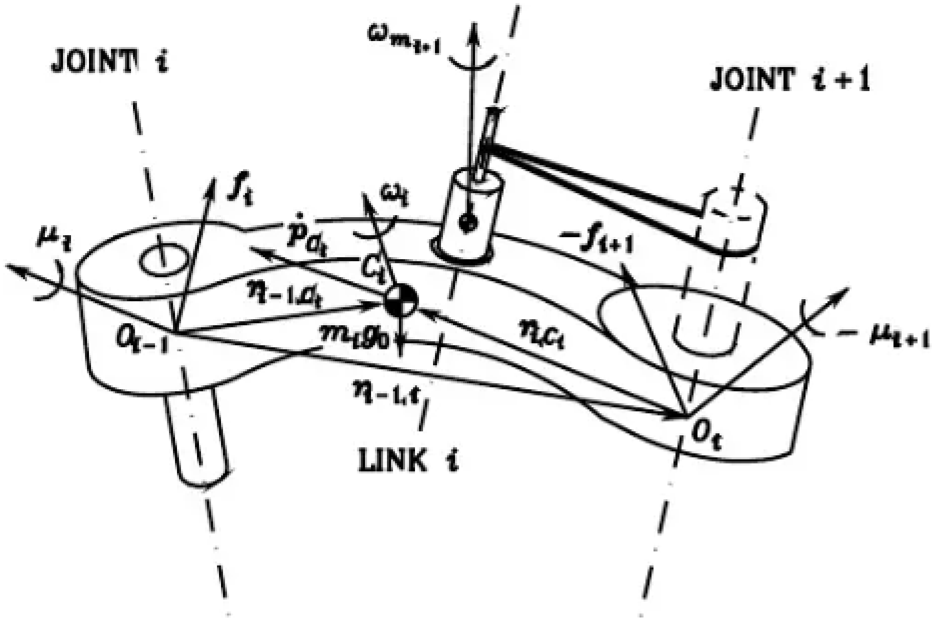


parameters

m_i mass of augmented link,
$\bar{\boldsymbol{I}}_i$ inertia tensor of augmented link,
I_{m_i} moment of inertia of rotor,
$\boldsymbol{r}_{i-1,C_i}$ vector from origin of Frame $(i-1)$ to centre of mass C_i,
$\boldsymbol{r}_{i,C_i}$ vector from origin of Frame i to centre of mass C_i,
$\boldsymbol{r}_{i-1,i}$ vector from origin of Frame $(i-1)$ to origin of Frame i.

Recap on Mechanical Modelling, dynamics

Recursive Newton-Euler: recap from robotic course

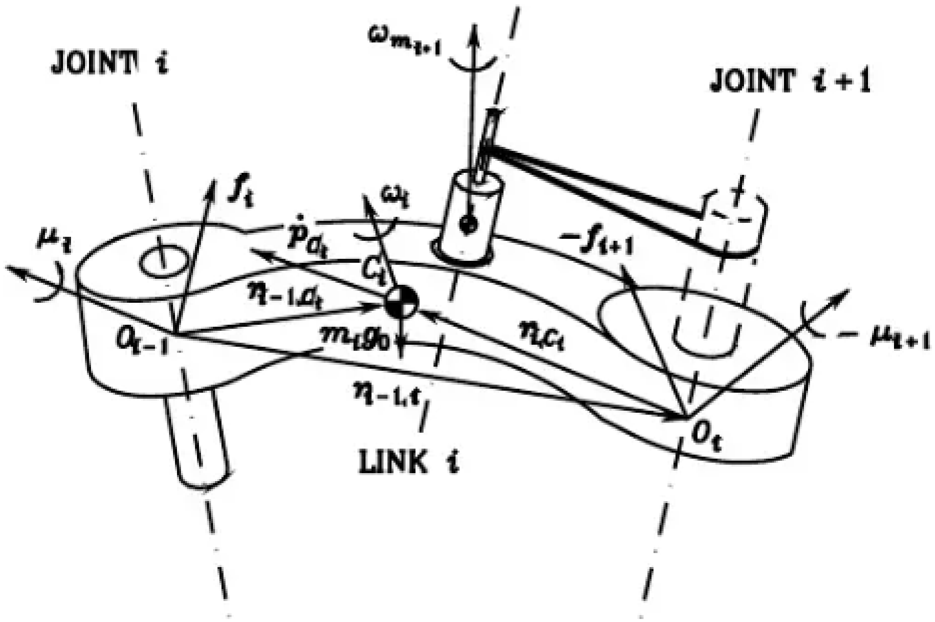


Kinematic variables (velocities and accelerations to be considered)

$\dot{\boldsymbol{p}}_{C_i}$ linear velocity of centre of mass C_i,
$\dot{\boldsymbol{p}}_i$ linear velocity of origin of Frame i,
$\boldsymbol{\omega}_i$ angular velocity of link,
$\boldsymbol{\omega}_{m_i}$ angular velocity of rotor,
$\ddot{\boldsymbol{p}}_{C_i}$ linear acceleration of centre of mass C_i,
$\ddot{\boldsymbol{p}}_i$ linear acceleration of origin of Frame i,
$\dot{\boldsymbol{\omega}}_i$ angular acceleration of link,
$\dot{\boldsymbol{\omega}}_{m_i}$ angular acceleration of rotor,
$\boldsymbol{g}_0$ gravity acceleration.

Recap on Mechanical Modelling, dynamics

Recursive Newton-Euler: recap from robotic course

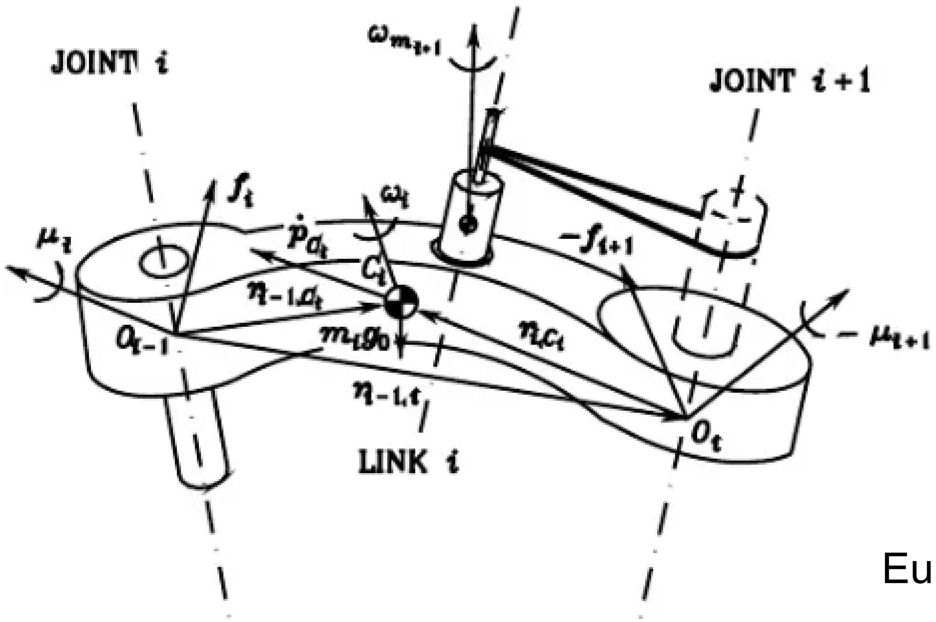


Dynamic variables (forces and moments to be considered)

$\boldsymbol{f}_i$ force exerted by Link $i-1$ on Link i,
$-\boldsymbol{f}_{i+1}$ force exerted by Link $i+1$ on Link i,
$\boldsymbol{\mu}_i$ moment exerted by Link $i-1$ on Link i with respect to origin of Frame $i-1$,
$-\boldsymbol{\mu}_{i+1}$ moment exerted by Link $i+1$ on Link i with respect to origin of Frame i.

Recap on Mechanical Modelling, dynamics

Recursive Newton-Euler: recap from robotic course



Quantities expressed w.r.t. base frame

Newton Equations (translational motion of CoM)

$$\boldsymbol{f}_i - \boldsymbol{f}_{i+1} + m_i \boldsymbol{g}_0 = m_i \ddot{\boldsymbol{p}}_{C_i}$$

Euler Equations (rotational motion of CoM)

$$\boldsymbol{\mu}_i + \boldsymbol{f}_i \times \boldsymbol{r}_{i-1,C_i} - \boldsymbol{\mu}_{i+1} - \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,C_i} = \frac{d}{dt}(\bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i + k_{r,i+1} \dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}})$$

Recap on Mechanical Modelling, dynamics

Euler Equations (rotational motion of CoM)

$$\boldsymbol{\mu}_i + \boldsymbol{f}_i \times \boldsymbol{r}_{i-1,C_i} - \boldsymbol{\mu}_{i+1} - \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,C_i} = \frac{d}{dt}(\bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i + k_{r,i+1}\dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}})$$

It is convenient to express the inertia tensor in the current frame, to have a constant tensor.

$$\begin{aligned}
\frac{d}{dt}(\bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i) &= \dot{\boldsymbol{R}}_i \bar{\boldsymbol{I}}_i^i \boldsymbol{R}_i^T \boldsymbol{\omega}_i + \boldsymbol{R}_i \bar{\boldsymbol{I}}_i^i \dot{\boldsymbol{R}}_i^T \boldsymbol{\omega}_i + \boldsymbol{R}_i \bar{\boldsymbol{I}}_i^i \boldsymbol{R}_i^T \dot{\boldsymbol{\omega}}_i \\
&= \boldsymbol{S}(\boldsymbol{\omega}_i) \boldsymbol{R}_i \bar{\boldsymbol{I}}_i^i \boldsymbol{R}_i^T \boldsymbol{\omega}_i + \boldsymbol{R}_i \bar{\boldsymbol{I}}_i^i \boldsymbol{R}_i^T \boldsymbol{S}^T(\boldsymbol{\omega}_i) \boldsymbol{\omega}_i + \boldsymbol{R}_i \bar{\boldsymbol{I}}_i^i \boldsymbol{R}_i^T \dot{\boldsymbol{\omega}}_i \\
&= \bar{\boldsymbol{I}}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \boldsymbol{\mu}_i + \boldsymbol{f}_i \times \boldsymbol{r}_{i-1,C_i} & - \boldsymbol{\mu}_{i+1} - \boldsymbol{f}_{i+1} \times \boldsymbol{r}_{i,C_i} = \bar{\boldsymbol{I}}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\bar{\boldsymbol{I}}_i \boldsymbol{\omega}_i) \\
& + k_{r,i+1}\ddot{q}_{i+1} I_{m_{i+1}} \boldsymbol{z}_{m_{i+1}} + k_{r,i+1}\dot{q}_{i+1} I_{m_{i+1}} \boldsymbol{\omega}_i \times \boldsymbol{z}_{m_{i+1}}
\end{aligned}$$

Modelling of a Quadrotor UAV

For studying the mathematical model of the Quad-rotor, we need to make some assumptions for system dynamics. Some of these assumptions are as below.

- Quad-rotor’s structure is symmetric around.
- Quad-rotor UAV consists of a rigid frame equipped with four rotors.
- No mass changes during motion (time invariant).
- COG (Center of Gravity) is fixed at the origin of the Quad-rotor.
- Thrust and drag constants are proportional to the square value of the motor’s speed.

Modelling of a Quadrotor UAV

Kinematics

Linear position and velocity

$$\mathbf{x} = (x, y, z)^\top$$

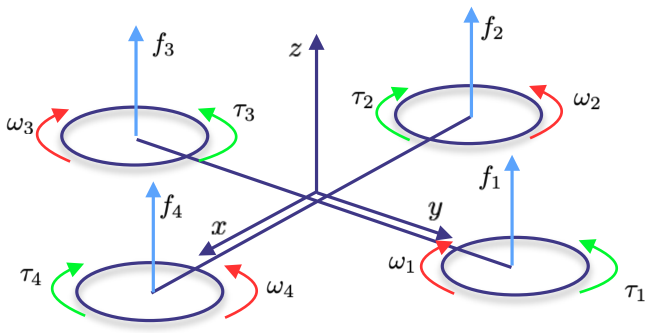
$$\dot{\mathbf{x}} = (\dot{x}, \dot{y}, \dot{z})^\top$$

Angular position and velocity

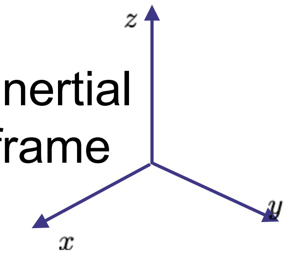
$$\theta = (\phi, \theta, \psi)^\top$$

$$\dot{\theta} = (\dot{\phi}, \dot{\theta}, \dot{\psi})^\top$$

Body-fixed frame



Inertial frame



Modelling of a Quadrotor UAV

Kinematics

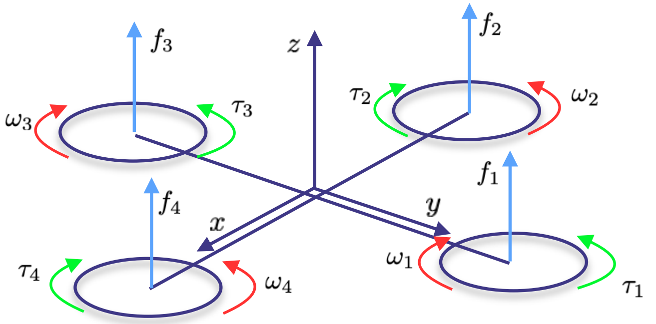
$$\theta = (\phi, \theta, \psi)^\top$$
$$\dot{\theta} = (\dot{\phi}, \dot{\theta}, \dot{\psi})^\top$$

We need to know the angular velocity ω of the body-fixed frame

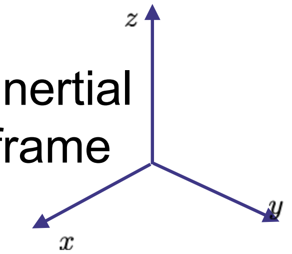
$$\mathbf{R}(\gamma) = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi) = \begin{pmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{pmatrix}$$

$$\vec{\omega} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & c_\theta s_\phi \\ 0 & -s_\phi & c_\theta c_\phi \end{bmatrix} \vec{\dot{\theta}}$$

Body-fixed frame



Inertial frame



Modelling of a Quadrotor UAV

Forces

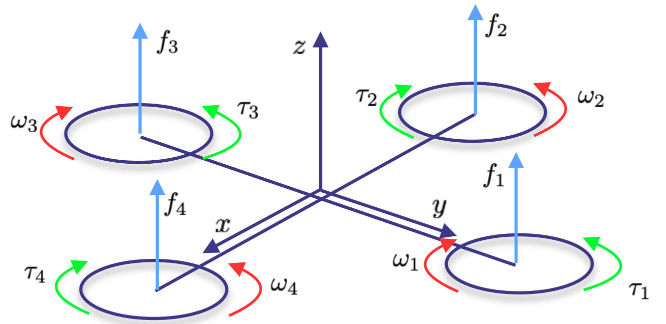
Propeller total trust

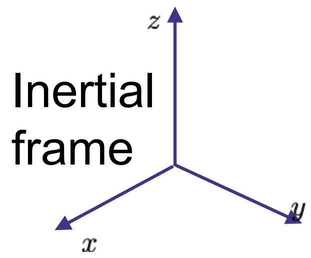
$$F_B = \sum_{i=1}^{4} f_i = k \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} \omega_i^2 \end{bmatrix}$$

Viscous damping

$$F_D = \begin{bmatrix} -k_d \dot{x} \\ -k_d \dot{y} \\ -k_d \dot{z} \end{bmatrix}$$

Body-fixed frame





Modelling of a Quadrotor UAV

Torques

Propellers' drag torques

$$\tau_D = \frac{1}{2} R \rho C_D A v^2 = \frac{1}{2} R \rho C_D A (\omega R)^2 = b\omega^2$$

total torque for one propeller around its z-axis:

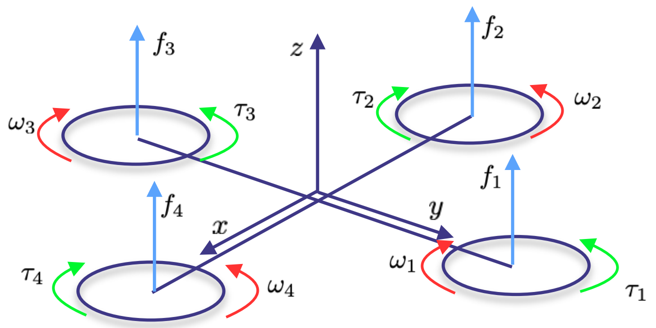
$$\tau_z = b\omega^2 + I_M \dot{\omega}$$

Assumption: $\dot{\omega} \approx 0$

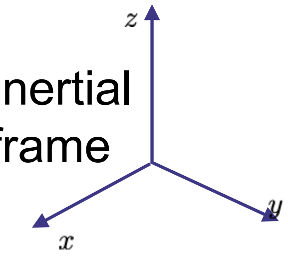
total torque for one propeller around z-axis of the body:

$$\tau_z = (-1)^{i+1} b\omega_i^2 \qquad \tau_\psi = b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2)$$

Body-fixed frame

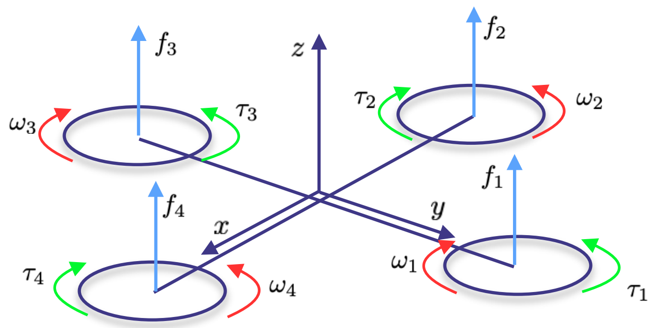


Inertial frame



Modelling of a Quadrotor UAV

Body-fixed frame



Torques

Torque on x-axis of the rigid body

$$\tau_\phi = \sum r \times T = L(k\omega_1^2 - k\omega_3^2) = Lk(\omega_1^2 - \omega_3^2)$$

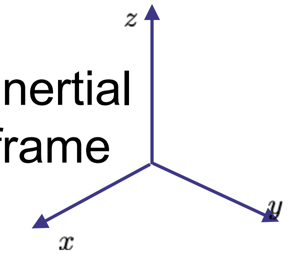
Torque on x-axis of the rigid body

$$\tau_\theta = \sum r \times T = L(k\omega_2^2 - k\omega_4^2) = Lk(\omega_2^2 - \omega_4^2)$$

Total body torque

$$\tau_B = \begin{bmatrix} Lk(\omega_1^2 - \omega_3^2) \\ Lk(\omega_2^2 - \omega_4^2) \\ b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \end{bmatrix}$$

Inertial frame



Modelling of a Quadrotor UAV

Body-fixed frame

Equations of motion

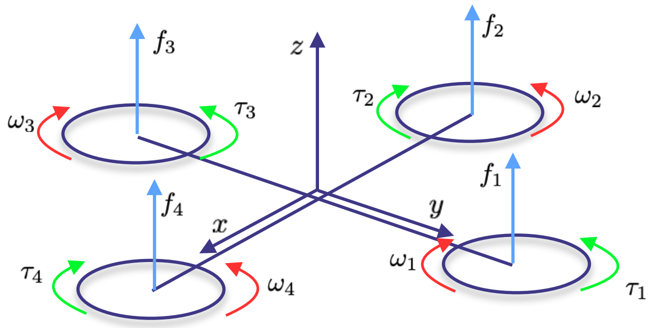
Linear dynamics (in the inertial frame):

$$\mathbf{m}\ddot{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + RF_B + F_D$$

Rotational Dynamics (in body-fixed frame)

$$I\dot{\omega} + \omega \times (I\omega) = \tau \qquad (\tau = \tau_B)$$

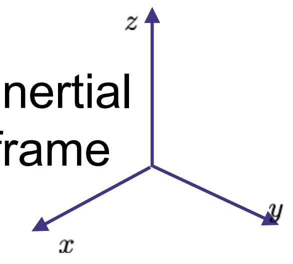
$$\dot{\omega} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = I^{-1}(\tau_B - \omega \times (I\omega))$$



$$F_B = \sum_{i=1}^{4} f_i = k \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} \omega_i^2 \end{bmatrix}$$

$$F_D = \begin{bmatrix} -k_d\dot{x} \\ -k_d\dot{y} \\ -k_d\dot{z} \end{bmatrix}$$

Inertial frame



$$\mathbf{m} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

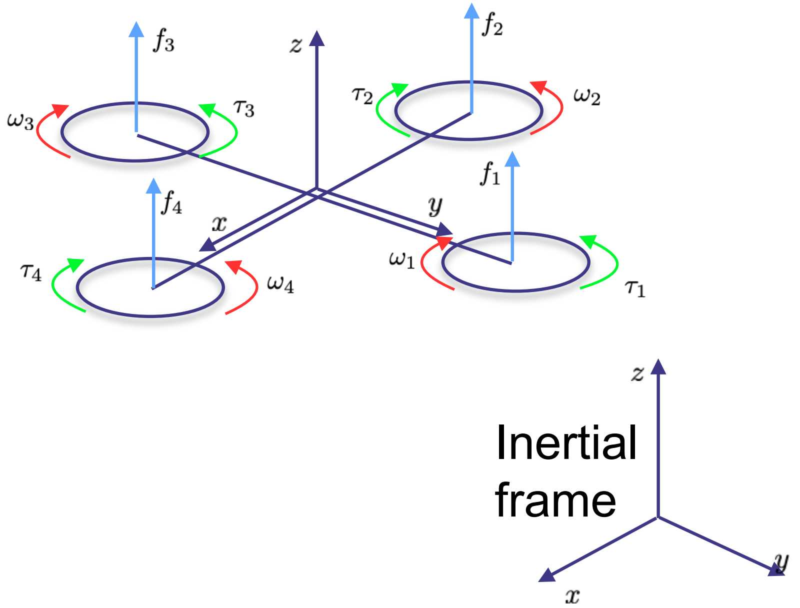
Modelling of a Quadrotor UAV

Equations of motion

Rotational Dynamics (in body-fixed frame)

$$\dot{\omega} = \begin{bmatrix} \tau_\phi I_{xx}^{-1} \\ \tau_\theta I_{yy}^{-1} \\ \tau_\psi I_{zz}^{-1} \end{bmatrix} - \begin{bmatrix} \frac{I_{yy}-I_{zz}}{I_{xx}} \omega_y \omega_z \\ \frac{I_{zz}-I_{xx}}{I_{yy}} \omega_x \omega_z \\ \frac{I_{xx}-I_{yy}}{I_{zz}} \omega_x \omega_y \end{bmatrix}$$

Body-fixed frame



Inertial frame