Simulation of pi

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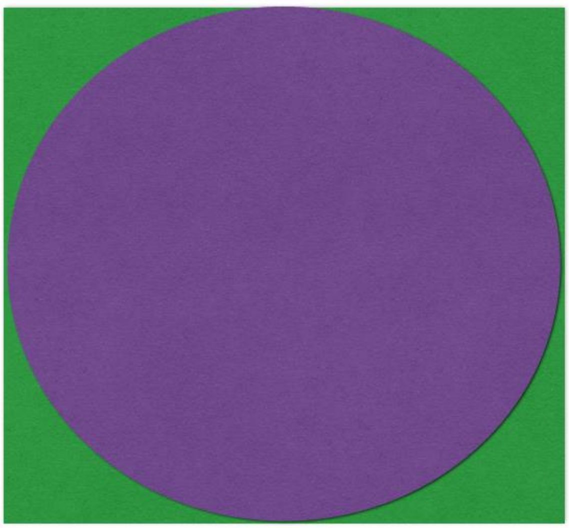
Question

Estimate π using Monte Carlo Methods

Solution 1



1.bb

According to the picture, we can randomly choose points from the square with area of 1and see how many points will be in the circle within the square.

$$p = \frac{\pi}{4}$$

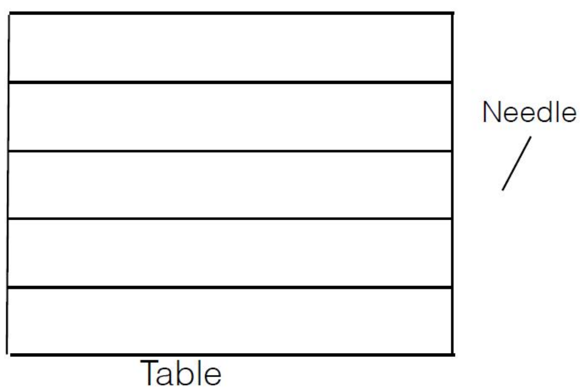
In such case, we will have code like this:

```
set.seed(676)
n<-10000
count<-0
for (i in 1:n){
  point<-runif(2,-1,1)
  if (point[1]**2+point[2]**2 <= 1){
    count<-count+1
  }
```

```
}
spi<-count*4/n
cat("\nOur estimate of the expected pi is ",spi)
```

```
##
## Our estimate of the expected pi is  3.1484
```

Solution 2: Buffon's needles



2.bb

Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?

Suppose the length of needle is l, the distances between two strips is t.

Let x be the distance from the center of the needle to the closest line, and let θ be the acute angle between the needle and the projected line with length x.

The uniform probability density function of x between θ and $\frac{t}{2}$ is:

$$\begin{cases} \dfrac{2}{t} : 0 \leq x \leq \dfrac{t}{2} \\ 0 : elsewhere \end{cases}$$

The uniform probability density function of θ between 0 and $\frac{\pi}{2}$ is:

$$\begin{cases} \dfrac{2}{\pi} : 0 \leq \theta \leq \dfrac{\pi}{2} \\ 0 : elsewhere \end{cases}$$

In such case, the joint probability density function is the product of these two probability density function:

$$\begin{cases} \dfrac{4}{t\pi} : 0 \leq x \leq \dfrac{t}{2}; 0 \leq \theta \leq \dfrac{\pi}{2} \\ 0 : elsewhere \end{cases}$$

The needls crosses a line if:

$$x \leq \frac{l}{2}cos\theta$$

In such case, we have two rules:

(1) We can set an area with width t, and randomly select points in this area.

(2) The probability is not related to horizontal coordinate of the point.

Here we will have two situations:

Situation 1: Short needle

if $l < t$:

$$p = \int_{\theta=0}^{\frac{\pi}{2}} \int_{x=0}^{\frac{l}{2}cos\theta} \frac{4}{t\pi} dx d\theta = \frac{2l}{t\pi}$$

In such case, we will have π like this:

$$\pi = \frac{2l}{tp}$$

In this situation, we will have code like this:

```r
set.seed(676)
l <- 0.5
t <- 1
cross <- numeric() # whether the needle cross the line or not
numberofneedles <- 1000000

for (i in 1:numberofneedles){
  x <- runif(1,0,t) # The vertical coordinate of the point
  theta <- runif(1,0,pi/2)
  if (x <= l/2 * cos(theta) | t-x <= l/2 * cos(theta)){
    cross[i] = 1
  }
  else{
    cross[i] = 0
  }
}

cat("\nThe probability of a needle cross the line is", sum(cross)/numberofneedles)
```

```
##
## The probability of a needle cross the line is 0.318521
```

```r
spi2 <- 2*l/(t * sum(cross) / numberofneedles)

cat("\nOur estimate of the expected pi is", spi2)
```

```
##
## Our estimate of the expected pi is 3.13951
```

Situation 2: Long needle

if $l < t$:

$$When \frac{l}{2}cos(\theta) = \frac{t}{2},$$
$$\theta = arccos(\frac{t}{l})$$

$$\begin{aligned}
p &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{x=0}^{min(\theta)} \frac{4}{t\pi} dx d\theta \; where \; min(\theta) = min(\frac{t}{2}, \frac{l}{2}cos(\theta)) \\
&= \int_{\theta=0}^{arccos(\frac{t}{l})} \int_{x=0}^{\frac{t}{2}} \frac{4}{t\pi} dx d\theta \; + \; \int_{\theta=arccos(\frac{t}{l})}^{\frac{\pi}{2}} \int_{x=0}^{\frac{l}{2}cos(\theta)} \frac{4}{t\pi} dx d\theta \\
&= \frac{2}{\pi} arccos(\frac{t}{l}) \; + \; \frac{2l}{\pi t}(1 \; - \; \sqrt{1 \; - \; \frac{t^2}{l^2}}) \\
\pi &= \frac{2}{p} arccos(\frac{t}{l}) + \frac{2l}{tp}(1 \; - \; \sqrt{1 \; - \; \frac{t^2}{l^2}})
\end{aligned}$$

In such case, we will have code like this:

```r
set.seed(676)
l <- 2
t <- 1
cross <- numeric() # whether the needle cross the line or not
numberofneedles <- 1000000

for (i in 1:numberofneedles){
  x <- runif(1,0,t) # The vertical coordinate of the point
  theta <- runif(1,0,pi/2)
  if (x <= l/2 * cos(theta) | t-x < l/2 * cos(theta)){
    cross[i] = 1
  }
  else{
    cross[i] = 0
  }
}
```

```r
p <- sum(cross)/numberofneedles

cat("\nThe probability of a needle cross the line is", p)
```

```
##
## The probability of a needle cross the line is 0.837051
```

```r
spi3 <- 2 / p * acos(t/l) + 2*l / (t * p) * (1 - sqrt(1 - t^2 / l^2))

cat("\nOur estimate of the expected pi is", spi3)
```

```
##
## Our estimate of the expected pi is 3.142334
```

It seems that using longer needles will get the better estimation. But I don't know if the assumption is true and how to prove it.