Problem 2b:

Naive expansion on original equation

```
govEq = Expand[ε * x^3 + (x - 2)^2]
```

$4 - 4\,x + x^2 + x^3\,\epsilon$

```
naive = Collect[Expand[govEq /. x -> {x0 + ε * x1 + ε^2 * x2}], ε];
```

Extract the O(1) equation

```
Collect[Normal[Series[naive /. Thread[ε → k * ε], {k, 0, 0}]] /. k → 1, ε]
```

$\{4 - 4\,x_o + x_o^2\}$

```
Solve[% == 0, xo]
```

$\{\{x_o \to 2\}, \{x_o \to 2\}\}$

We expect 3 roots but only recovered 2.

Search for last root by rescaling

Use scale x -> t (1/ϵ)

```
Expand[govEq] /. x → t/ε
```

$4 + \frac{t^2}{\epsilon^2} + \frac{t^3}{\epsilon^2} - \frac{4\,t}{\epsilon}$

```
Rs = Expand[(4 + t^2/ε^2 + t^3/ε^2 - 4 t/ε) * ε^2]
```

$t^2 + t^3 - 4\,t\,\epsilon + 4\,\epsilon^2$

```
naiveRS = Collect[Expand[Rs /. t -> {to + ε * t1 + ε^2 * t2}], ε];
```

```
Collect[Normal[Series[naiveRS /. Thread[ε → k * ε], {k, 0, 0}]] /. k → 1, ε]
```

$\{t_o^2 + t_o^3\}$

```
Solve[% == 0, to]
```

$\{\{t_o \to -1\}, \{t_o \to 0\}, \{t_o \to 0\}\}$

We find a double root at 0.

Proceed by using gage function

```
gageRS = Collect[Series[Rs /. t -> {to + ϵ^(1/2) * t1 + ϵ * t2 + ϵ^(3/2) * t3 + ϵ^2 * t4}, {ϵ, 0, 3}], ϵ];
```

Pursue 0 roots

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /. to → 0
```

$\{\epsilon\, t_1^2\}$

$t_1 = 0$

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 2}]] /. k → 1, ϵ] /.
  {to → 0, t1 → 0}
```

$\{\epsilon^2\,(4 - 4\,t_2 + t_2^2)\}$

```
Solve[% == 0, t2]
```

$\{\{t_2 \to 2\}, \{t_2 \to 2\}\}$

Go further

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 3}]] /. k → 1, ϵ] /.
  {to → 0, t1 → 0, t2 → 2}
```

$\{\epsilon^3\,(8 + t_3^2)\}$

```
Solve[% == 0, t3]
```

$\{\{t_3 \to -2\,i\,\sqrt{2}\}, \{t_3 \to 2\,i\,\sqrt{2}\}\}$

Pursue -1 root

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /. to → -1
```

$\{\sqrt{\epsilon}\,t_1 + \epsilon\,(4 - 2\,t_1^2 + t_2)\}$

$t_1 = 0$

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /.
  {to → -1, t1 → 0}
```

$\{\epsilon\,(4 + t_2)\}$

```
Solve[% == 0, t2]
```

$\{\{t_2 \to -4\}\}$

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 2}]] /. k → 1, ϵ] /.
 {t₀ → -1, t₁ → 0, t₂ → -4}
```

$\{\epsilon^{3/2}\, t_3 + \epsilon^2\, (-12 + t_4)\}$

$t_3 = 0$

Compile Solutions

```
Expand[(-1 + ϵ * -4) * 1/ϵ]
```

$-4 - \frac{1}{\epsilon}$

```
Expand[(0 + ϵ * 2 + ϵ^(3/2) * -2 ⅈ √2) * 1/ϵ]
```

$2 - 2\, i \sqrt{2}\, \sqrt{\epsilon}$

```
Expand[(0 + ϵ * 2 + ϵ^(3/2) * 2 ⅈ √2) * 1/ϵ]
```

$2 + 2\, i \sqrt{2}\, \sqrt{\epsilon}$

Note: This equation recovered all the simple roots from the naive expansion and the re-scaled equation.