Problem 2a:

Naive expansion on original equation

```
govEq = Expand[ϵ * (x^3 - x^2) + 4 * x^2 - 3 x - 1]
```

$-1 - 3x + 4x^2 - x^2 \epsilon + x^3 \epsilon$

```
naive = Collect[Expand[govEq /. x -> {x_o + ϵ * x_1 + ϵ^2 * x_2}], ϵ];
```

Extract the O(1) equation

```
Collect[Normal[Series[naive /. Thread[ϵ → k * ϵ], {k, 0, 0}]] /. k → 1, ϵ]
```

$\{-1 - 3x_o + 4x_o^2\}$

```
Solve[% == 0, x_o]
```

$\left\{\left\{x_o \to -\frac{1}{4}\right\}, \{x_o \to 1\}\right\}$

We expect 3 roots but only recovered 2.

Search for last root by rescaling

Use scale x -> t (4/ϵ)

```
Expand[ϵ * (x^3 - x^2) + 4 * x^2 - 3 x - 1] /. x → t * 4 / ϵ
```

$-1 + \frac{64t^2}{\epsilon^2} + \frac{64t^3}{\epsilon^2} - \frac{12t}{\epsilon} - \frac{16t^2}{\epsilon}$

```
Expand[(-1 + 64 t^2/ϵ^2 + 64 t^3/ϵ^2 - 12 t/ϵ - 16 t^2/ϵ) * ϵ^2]
```

$64t^2 + 64t^3 - 12t\epsilon - 16t^2\epsilon - \epsilon^2$

```
naiveRS =
  Collect[Expand[64 t^2 + 64 t^3 - 12 t ϵ - 16 t^2 ϵ - ϵ^2 /. t -> {t_o + ϵ * t_1 + ϵ^2 * t_2}], ϵ];
```

```
Collect[Normal[Series[naiveRS /. Thread[ϵ → k * ϵ], {k, 0, 0}]] /. k → 1, ϵ]
```

$\{64(t_o^2 + t_o^3)\}$

```
Solve[% == 0, t_o]
```

$\{\{t_o \to -1\}, \{t_o \to 0\}, \{t_o \to 0\}\}$

We find a double root at 0.

Proceed by using gage function

```
gageRS = Collect[Series[64 t^2 + 64 t^3 - 12 t ϵ - 16 t^2 ϵ - ϵ^2 /.
      t -> {t_o + ϵ^(1/2) * t_1 + ϵ * t_2 + ϵ^(3/2) * t_3 + ϵ^2 * t_4}, {ϵ, 0, 3}], ϵ];
```

Pursue 0 roots

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /. t_o → 0
```

$\{64\,\epsilon\, t_1^2\}$

$t_1 = 0$

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 2}]] /. k → 1, ϵ] /.
  {t_o → 0, t_1 → 0}
```

$\{\epsilon^2\,(-1 - 12\, t_2 + 64\, t_2^2)\}$

```
Solve[% == 0, t_2]
```

$\{\{t_2 \to -\frac{1}{16}\}, \{t_2 \to \frac{1}{4}\}\}$

Pursue -1 root

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /. t_o → -1
```

$\{64\sqrt{\epsilon}\; t_1 + \epsilon\,(-4 - 128\, t_1^2 + 64\, t_2)\}$

$t_1 = 0$

```
Collect[Normal[Series[gageRS /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /.
  {t_o → -1, t_1 → 0}
```

$\{\epsilon\,(-4 + 64\, t_2)\}$

```
Solve[% == 0, t_2]
```

$\{\{t_2 \to \frac{1}{16}\}\}$

Compile Solutions

```
Expand[(-1 + ϵ * 1/16) * 4/ϵ]
```

$\frac{1}{4} - \frac{4}{\epsilon}$

$$\text{Expand}\left[\left(0+\epsilon * -\frac{1}{16}\right) * \frac{4}{\epsilon}\right]$$

$$-\frac{1}{4}$$

$$\text{Expand}\left[\left(0+\epsilon * \frac{1}{4}\right) * \frac{4}{\epsilon}\right]$$

$$1$$

Note: This equation recovered all the simple roots from the naive expansion and the re-scaled equation.