Problem 2d:

Naive expansion on original equation

```
govEq = Expand[ϵ * (x^5 + x^4 - 2 * x^3) + 2 * x^2 - 3 x + 1]
```

$1 - 3\,x + 2\,x^2 - 2\,x^3\,\epsilon + x^4\,\epsilon + x^5\,\epsilon$

```
naive = Collect[Expand[govEq /. x -> {x_o + ϵ * x_1 + ϵ^2 * x_2}], ϵ];
```

Extract the O(1) equation

```
Collect[Normal[Series[naive /. Thread[ϵ → k * ϵ], {k, 0, 0}]] /. k → 1, ϵ]
```

$\{1 - 3\,x_o + 2\,x_o^2\}$

```
Solve[% == 0, x_o]
```

$\left\{\left\{x_o \to \frac{1}{2}\right\}, \{x_o \to 1\}\right\}$

We expect 5 roots but only recovered 2.

Search for last root by rescaling

Use scale x -> t (1/ϵ)

```
Expand[govEq ] /. x → t * 2^(1/3)/ϵ^(1/3)
```

$$1 - 4\,t^3 + \frac{2 \times 2^{2/3}\,t^2}{\epsilon^{2/3}} + \frac{2 \times 2^{2/3}\,t^5}{\epsilon^{2/3}} - \frac{3 \times 2^{1/3}\,t}{\epsilon^{1/3}} + \frac{2 \times 2^{1/3}\,t^4}{\epsilon^{1/3}}$$

```
Rs = Expand[(1 - 4 t^3 + (2 × 2^(2/3) t^2)/ϵ^(2/3) + (2 × 2^(2/3) t^5)/ϵ^(2/3) - (3 × 2^(1/3) t)/ϵ^(1/3) + (2 × 2^(1/3) t^4)/ϵ^(1/3)) * ϵ^(2/3)] /.
  {ϵ^(1/3) → δ, ϵ^(2/3) → δ^2}
```

$$2 \times 2^{2/3}\,t^2 + 2 \times 2^{2/3}\,t^5 - 3 \times 2^{1/3}\,t\,\delta + 2 \times 2^{1/3}\,t^4\,\delta + \delta^2 - 4\,t^3\,\delta^2$$

```
naiveRS = Collect[Expand[Rs /. t -> {t_o + δ * t_1 + δ^2 * t_2}], δ];

Collect[Normal[Series[naiveRS /. Thread[δ → k * δ], {k, 0, 0}]] /. k → 1, δ]
```

$\{2 \times 2^{2/3}\,t_o^2 + 2 \times 2^{2/3}\,t_o^5\}$

```
Solve[% == 0, t_o]
```

$\{\{t_o \to -1\}, \{t_o \to 0\}, \{t_o \to 0\}, \{t_o \to (-1)^{1/3}\}, \{t_o \to -(-1)^{2/3}\}\}$

We find a double root at 0.

Proceed by using gage function

```
gageRS = Collect[Series[Rs /. t -> {tₒ + δ^(1/2) * t1 + δ * t2 + δ^(3/2) * t3 + δ^2 * t4}, {δ, 0, 3}], δ];
```

Pursue 0 roots

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 1}]] /. k → 1, δ] /. tₒ → 0
```

$\{2 \times 2^{2/3}\, \delta\, t_1^2\}$

```
Solve[% == 0, t1]
```

$\{\{t_1 \to 0\}, \{t_1 \to 0\}\}$

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 2}]] /. k → 1, δ] /.
 {tₒ → 0, t1 → 0}
```

$\{\delta^2\, (1 - 3 \times 2^{1/3}\, t_2 + 2 \times 2^{2/3}\, t_2^2)\}$

```
Solve[% == 0, t2]
```

$\left\{\left\{t_2 \to \frac{1}{2 \times 2^{1/3}}\right\}, \left\{t_2 \to \frac{1}{2^{1/3}}\right\}\right\}$

Pursue -1 root

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 1}]] /. k → 1, δ] /. {tₒ → -1}
```

$\{6 \times 2^{2/3}\, \sqrt{\delta}\, t_1 + \delta\, (5 \times 2^{1/3} - 18 \times 2^{2/3}\, t_1^2 + 6 \times 2^{2/3}\, t_2)\}$

```
Solve[6 × 2^(2/3) √δ t1 == 0, t1]
```

$\{\{t_1 \to 0\}\}$

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 1}]] /. k → 1, δ] /.
 {tₒ → -1, t1 → 0}
```

$\{\delta\, (5 \times 2^{1/3} + 6 \times 2^{2/3}\, t_2)\}$

```
Solve[5 × 2^(1/3) + 6 × 2^(2/3) t2 == 0, t2]
```

$\left\{\left\{t_2 \to -\frac{5}{6 \times 2^{1/3}}\right\}\right\}$

Pursue $(-1)^{1/3}$ root

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 1}]] /. k → 1, δ] /.
 {tₒ → (-1)^(1/3)}
```

$\{-6\, (-1)^{1/3}\, 2^{2/3}\, \sqrt{\delta}\, t_1 + \delta\, (-5\, (-2)^{1/3} - 18 \times 2^{2/3}\, t_1^2 - 6\, (-1)^{1/3}\, 2^{2/3}\, t_2)\}$

```
Solve[-6 (-1)^(1/3) 2^(2/3) t1 == 0, t1]
```

$\{\{t_1 \to 0\}\}$

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 1}]] /. k → 1, δ] /.
 {t0 → (-1)^(1/3), t1 → 0}
```

$\{\delta\left(-5\,(-2)^{1/3} - 6\,(-1)^{1/3}\,2^{2/3}\,t_2\right)\}$

```
Solve[-5 (-2)^(1/3) - 6 (-1)^(1/3) 2^(2/3) t2 == 0, t2]
```

$\left\{\left\{t_2 \to -\frac{5}{6 \times 2^{1/3}}\right\}\right\}$

Pursue $-(-1)^{2/3}$ root

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 1}]] /. k → 1, δ] /.
 {t0 → -(-1)^(2/3)}
```

$\{6\,(-2)^{2/3}\sqrt{\delta}\,t_1 + \delta\left(5\,(-1)^{2/3}\,2^{1/3} - 18 \times 2^{2/3}\,t_1^2 + 6\,(-2)^{2/3}\,t_2\right)\}$

```
Collect[Normal[Series[gageRS /. Thread[δ → k * δ], {k, 0, 1}]] /. k → 1, δ] /.
 {t0 → -(-1)^(2/3), t1 → 0}
```

$\{\delta\left(5\,(-1)^{2/3}\,2^{1/3} + 6\,(-2)^{2/3}\,t_2\right)\}$

```
Solve[5 (-1)^(2/3) 2^(1/3) + 6 (-2)^(2/3) t2 == 0, t2]
```

$\left\{\left\{t_2 \to -\frac{5}{6 \times 2^{1/3}}\right\}\right\}$

Compile Solutions

$$\text{Expand}\left[\left(-1 + \delta * -\frac{5}{6 \times 2^{1/3}}\right) * \frac{2^{1/3}}{\epsilon^{1/3}}\right] /. \{\delta \to \epsilon^{1/3}\}$$

$-\frac{5}{6} - \frac{2^{1/3}}{\epsilon^{1/3}}$

$$\text{Expand}\left[\left(0 + \delta * \frac{1}{2^{1/3}}\right) * \frac{2^{1/3}}{\epsilon^{1/3}}\right] /. \{\delta \to \epsilon^{1/3}\}$$

1

$$\text{Expand}\left[\left(0 + \delta * \frac{1}{2 \times 2^{1/3}}\right) * \frac{2^{1/3}}{\epsilon^{1/3}}\right] /. \{\delta \to \epsilon^{1/3}\}$$

$\frac{1}{2}$

$$\text{Expand}\left[\left((-1)^{1/3} + \delta * - \frac{5}{6 \times 2^{1/3}}\right) * \frac{2^{1/3}}{\epsilon^{1/3}}\right] \text{ /. } \{\delta \text{ -> } \epsilon^{1/3}\}$$

$$-\frac{5}{6} + \frac{(-2)^{1/3}}{\epsilon^{1/3}}$$

$$\text{Expand}\left[\left(-(-1)^{2/3} + \delta * - \frac{5}{6 \times 2^{1/3}}\right) * \frac{2^{1/3}}{\epsilon^{1/3}}\right] \text{ /. } \{\delta \text{ -> } \epsilon^{1/3}\}$$

$$-\frac{5}{6} - \frac{(-1)^{2/3}\, 2^{1/3}}{\epsilon^{1/3}}$$

Note: This equation recovered all the simple roots from the naive expansion and the re-scaled equation.