Problem 2e:

Original equation

In[10]:= $x - \frac{\epsilon}{3*x^2} - \frac{3*\epsilon^2}{10*x^4}$

Out[10]= $x - \frac{\epsilon}{3\,x^2} - \frac{3\,\epsilon^2}{10\,x^4}$

Consider unperturbed equation

In[133]:= $\text{FullSimplify}\left[\left(x - \frac{\epsilon}{3*x^2} - \frac{3*\epsilon^2}{10*x^4}\right)*x^4\right]$

Out[133]= $x^5 - \frac{x^2\,\epsilon}{3} - \frac{3\,\epsilon^2}{10}$

In[135]:= $\text{FullSimplify}\left[\left(x - \frac{\epsilon}{3*x^2} - \frac{3*\epsilon^2}{10*x^4}\right)*x^4\right] \,/.\, \epsilon \to 0$

Out[135]= x^5

There is a quintuple root at x =0

Rescale equation

Out[104]= $x^5 - \frac{x^2\,\epsilon}{3} - \frac{3\,\epsilon^2}{10}$

In[136]:= $\text{scaledEq} = \text{Expand}\left[\left(x - \frac{\epsilon}{3*x^2} - \frac{3*\epsilon^2}{10*x^4}\right)*x^4\right] \,/.\, x \to t*\epsilon^{1/3}$

Out[136]= $-\frac{1}{3}\,t^2\,\epsilon^{5/3} + t^5\,\epsilon^{5/3} - \frac{3\,\epsilon^2}{10}$

In[137]:= $\text{scaledEq} = \text{Simplify}\left[\text{scaledEq}/\epsilon^{5/3}\right]$

Out[137]= $-\frac{t^2}{3} + t^5 - \frac{3\,\epsilon^{1/3}}{10}$

In[139]:= $\text{Simplify}[\text{scaledEq}] \,/.\, \epsilon \to 0$

Out[139]= $-\frac{t^2}{3} + t^5$

In[140]:= $\text{Solve}\left[-\frac{t^2}{3} + t^5 == 0,\ t\right]$

Out[140]= $\left\{\{t \to 0\},\ \{t \to 0\},\ \left\{t \to -\left(-\frac{1}{3}\right)^{1/3}\right\},\ \left\{t \to \frac{1}{3^{1/3}}\right\},\ \left\{t \to \frac{(-1)^{2/3}}{3^{1/3}}\right\}\right\}$

Derive gage function

In[141]:=
```
gage = Expand[Collect[Expand[-t^2 + t^5 - ϵ^(1/3) /. t -> {t_o + δ_1 * t_1}], δ_1]] /. t_o → 0
```

Out[141]= $\{-\epsilon^{1/3} - t_1^2\,\delta_1^2 + t_1^5\,\delta_1^5\}$

Balance dominant terms and divide by $\delta_1{}^2$

In[80]:=
```
Expand[(-ϵ^(1/3) - t_1^2 δ_1^2) / δ_1^2]
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Out[80]= $-t_1^2 - \dfrac{\epsilon^{1/3}}{\delta_1^2}$

Let $\frac{\epsilon^{1/3}}{\delta_1{}^2} = 1$ so that $\delta_1 = \epsilon^{1/6}$

From this assume the expansion

$t_o + \epsilon^{1/6} * t_1 + \epsilon^{2/6} * t_2 + \epsilon^{3/6} * t_3 + \epsilon^{4/6} * t_4 + \epsilon^{5/6} * t_5$

Pursue solution using the derived gage function

In[142]:=
```
gagers = Expand[Collect[Expand[
    -t^2/3 + t^5 - 3 ϵ^(1/3)/10 /. t -> {t_o + ϵ^(1/6) * t_1 + ϵ^(2/6) * t_2 + ϵ^(3/6) * t_3 + ϵ^(4/6) * t_4 + ϵ^(5/6) * t_5}], ϵ_1]];
```

In[131]:=
```
Collect[Normal[Series[gagers /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /.
  {t_o → 0, t_1 → 3 ⅈ/Sqrt[10], t_2 -> 0, t_3 -> 0}
```

Out[131]= $\left\{\epsilon^{5/6}\left(\dfrac{243\,i}{100\sqrt{10}} - i\sqrt{\dfrac{2}{5}}\,t_4\right) - i\sqrt{\dfrac{2}{5}}\,\epsilon\,t_5\right\}$

In[132]:=
```
Solve[243 ⅈ/(100 Sqrt[10]) - ⅈ Sqrt[2/5] t_4 == 0, t_4]
```

Out[132]= $\left\{\left\{t_4 \to \dfrac{243}{200}\right\}\right\}$

```
Collect[Normal[Series[gagers /. Thread[ϵ → k * ϵ], {k, 0, 1}]] /. k → 1, ϵ] /.
  {t_o → -(-1/3)^(1/3), t_1 → 0, t_2 → 3/10 (-1)^(2/3) 3^(1/3)}
```

Out[116]= $\left\{\epsilon^{1/3}\left(-\dfrac{3}{10} - \left(-\dfrac{1}{3}\right)^{1/3} t_2\right) - \left(-\dfrac{1}{3}\right)^{1/3}\sqrt{\epsilon}\,t_3 + \epsilon^{2/3}\left(3\,t_2^2 - \left(-\dfrac{1}{3}\right)^{1/3} t_4\right) + \epsilon\left(10\left(-\dfrac{1}{3}\right)^{2/3} t_2^3 + 3\,t_3^2 + 6\,t_2\,t_4\right) + \epsilon^{5/6}\left(6\,t_2\,t_3 - \left(-\dfrac{1}{3}\right)^{1/3} t_5\right)\right\}$

In[117]:= $\text{Solve}\left[\left(-\frac{3}{10}-\left(-\frac{1}{3}\right)^{1/3} t_2\right) == 0,\ t_2\right]$

Out[117]= $\left\{\left\{t_2 \to \frac{3}{10} (-1)^{2/3}\, 3^{1/3}\right\}\right\}$

$\text{Collect}[\text{Normal}[\text{Series}[\text{gagers} \mathbin{/.} \text{Thread}[\epsilon \to k * \epsilon], \{k, 0, 1\}]] \mathbin{/.} k \to 1, \epsilon] \mathbin{/.}$
$\left\{t_o \to \frac{1}{3^{1/3}},\ t_1 \to 0,\ t_2 \to \frac{3 \times 3^{1/3}}{10}\right\}$

Out[119]= $\left\{\epsilon^{1/3}\left(-\frac{3}{10}+\frac{t_2}{3^{1/3}}\right)+\frac{\sqrt{\epsilon}\ t_3}{3^{1/3}}+\epsilon^{2/3}\left(3\,t_2^2+\frac{t_4}{3^{1/3}}\right)+\epsilon\left(\frac{10\,t_2^3}{3^{2/3}}+3\,t_3^2+6\,t_2\,t_4\right)+\epsilon^{5/6}\left(6\,t_2\,t_3+\frac{t_5}{3^{1/3}}\right)\right\}$

In[120]:= $\text{Solve}\left[-\frac{3}{10}+\frac{t_2}{3^{1/3}} == 0,\ t_2\right]$

Out[120]= $\left\{\left\{t_2 \to \frac{3 \times 3^{1/3}}{10}\right\}\right\}$

$\text{Collect}[\text{Normal}[\text{Series}[\text{gagers} \mathbin{/.} \text{Thread}[\epsilon \to k * \epsilon], \{k, 0, 1\}]] \mathbin{/.} k \to 1, \epsilon] \mathbin{/.}$
$\left\{t_o \to \frac{(-1)^{2/3}}{3^{1/3}},\ t_1 \to 0,\ t_2 \to -\frac{3}{10} (-3)^{1/3}\right\}$

Out[121]= $\left\{\epsilon^{1/3}\left(-\frac{3}{10}+\frac{(-1)^{2/3}\,t_2}{3^{1/3}}\right)+\frac{(-1)^{2/3}\sqrt{\epsilon}\ t_3}{3^{1/3}}+\epsilon^{2/3}\left(3\,t_2^2+\frac{(-1)^{2/3}\,t_4}{3^{1/3}}\right)+\epsilon\left(-\frac{10\,(-1)^{1/3}\,t_2^3}{3^{2/3}}+3\,t_3^2+6\,t_2\,t_4\right)+\epsilon^{5/6}\left(6\,t_2\,t_3+\frac{(-1)^{2/3}\,t_5}{3^{1/3}}\right)\right\}$

In[122]:= $\text{Solve}\left[-\frac{3}{10}+\frac{(-1)^{2/3}\,t_2}{3^{1/3}} == 0,\ t_2\right]$

Out[122]= $\left\{\left\{t_2 \to -\frac{3}{10} (-3)^{1/3}\right\}\right\}$

Compile Solutions

$t \to \left\{t_o + \epsilon^{1/6} * t_1 + \epsilon^{2/6} * t_2 + \epsilon^{3/6} * t_3 + \epsilon^{4/6} * t_4 + \epsilon^{5/6} * t_5\right\}$

$\left\{t_o \to 0,\ t_1 \to \frac{3\,i}{\sqrt{10}},\ t_2 \to 0,\ t_3 \to 0,\ t_4 \to \frac{243}{200}\right\}$

$\left\{t_o \to 0,\ t_1 \to -\frac{3\,i}{\sqrt{10}},\ t_2 \to 0,\ t_3 \to 0,\ t_4 \to \frac{243}{200}\right\}$

$\left\{t_o \to -\left(-\frac{1}{3}\right)^{1/3},\ t_1 \to 0,\ t_2 \to \frac{3}{10} (-1)^{2/3}\, 3^{1/3}\right\}$

$\left\{t_o \to \frac{1}{3^{1/3}},\ t_1 \to 0,\ t_2 \to \frac{3 \times 3^{1/3}}{10}\right\}$

$\left\{t_o \to \frac{(-1)^{2/3}}{3^{1/3}},\ t_1 \to 0,\ t_2 \to -\frac{3}{10} (-3)^{1/3}\right\}$