Some frequently used models for non-Newtonian fluids

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Viscosity of some fluids

Fluid	Viscosity [cP]
Air (at 18 °C)	0.02638
Benzene	0.5
Water (at 18 °C)	1
Olive oil (at 20 °C)	84
Motor oil SAE 50	540
Honey	2000–3000
Ketchup	50000–70000
Peanut butter	150000–250000
Tar	3×10^{10}
Earth lower mantle	3×10^{25}

Table: Viscosity of some fluids

Models with variable viscosity

General form:

$$\mathbb{T} = -p\mathbb{I} + \underbrace{2\mu(\mathbb{D}, \mathbb{T})\mathbb{D}}_{\mathbb{S}} \tag{2.1}$$

Particular models mainly developed by chemical engineers.

Ostwald–de Waele power law

Wolfgang Ostwald. Über die Geschwindigkeitsfunktion der Viskosität disperser Systeme. I. *Colloid Polym. Sci.*, 36:99–117, 1925

A. de Waele. Viscometry and plastometry. *J. Oil Colour Chem. Assoc.*, 6:33–69, 1923

$$\mu(\mathbb{D}) = \mu_0 \left|\mathbb{D}\right|^{n-1} \tag{2.2}$$

Fits experimental data for: ball point pen ink, molten chocolate, aqueous dispersion of polymer latex spheres

Carreau Carreau–Yasuda

Pierre J. Carreau. Rheological equations from molecular network theories. *J. Rheol.*, 16(1):99–127, 1972
Kenji Yasuda. *Investigation of the analogies between viscometric and linear viscoelastic properties of polystyrene fluids*. PhD thesis, Massachusetts Institute of Technology. Dept. of Chemical Engineering., 1979

$$\mu(\mathbb{D}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1 + \alpha\,|\mathbb{D}|^2)^{\frac{n}{2}}} \tag{2.3}$$

$$\mu(\mathbb{D}) = \mu_\infty + (\mu_0 - \mu_\infty)\,(1 + \alpha\,|\mathbb{D}|^a)^{\frac{n-1}{a}} \tag{2.4}$$

Fits experimental data for: molten polystyrene

Eyring

Henry Eyring. Viscosity, plasticity, and diffusion as examples of absolute reaction rates. *J. Chem. Phys.*, 4(4):283–291, 1936
Francis Ree, Taikyue Ree, and Henry Eyring. Relaxation theory of transport problems in condensed systems. *Ind. Eng. Chem.*, 50(7):1036–1040, 1958

$$\mu(\mathbb{D}) = \mu_\infty + (\mu_0 - \mu_\infty) \frac{\operatorname{arcsinh}(\alpha |\mathbb{D}|)}{\alpha |\mathbb{D}|} \tag{2.5}$$

$$\mu(\mathbb{D}) = \mu_0 + \mu_1 \frac{\operatorname{arcsinh}(\alpha_1 |\mathbb{D}|)}{\alpha_1 |\mathbb{D}|} + \mu_2 \frac{\operatorname{arcsinh}(\alpha_2 |\mathbb{D}|)}{\alpha_2 |\mathbb{D}|} \tag{2.6}$$

Fits experimental data for: napalm (coprecipitated aluminum salts of naphthenic and palmitic acids; jellied gasoline), 1% nitrocelulose in 99% butyl acetate

Cross

Malcolm M. Cross. Rheology of non-newtonian fluids: A new flow equation for pseudoplastic systems. *J. Colloid Sci.*, 20(5):417–437, 1965

$$\mu(\mathbb{D}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + \alpha \left|\mathbb{D}\right|^n} \tag{2.7}$$

Fits experimental data for: aqueous polyvinyl acetate dispersion, aqueous limestone suspension

Sisko

A. W. Sisko. The flow of lubricating greases. *Ind. Eng. Chem.*, 50(12):1789–1792, 1958

$$\mu(\mathbb{D}) = \mu_\infty + \alpha \left|\mathbb{D}\right|^{n-1} \tag{2.8}$$

Fits experimental data for: lubricating greases

Barus

C. Barus. Isotherms, isopiestics and isometrics relative to viscosity. *Amer. J. Sci.*, 45:87–96, 1893

$$\mu(\mathbb{T}) = \mu_{\mathrm{ref}} \mathrm{e}^{\beta(p - p_{\mathrm{ref}})} \tag{2.9}$$

Fits experimental data for: mineral oils[1], organic liquids[2]

[1] Michael M. Khonsari and E. Richard Booser. *Applied Tribology: Bearing Design and Lubrication*. John Wiley & Sons Ltd, Chichester, second edition, 2008

[2] P. W. Bridgman. The effect of pressure on the viscosity of forty-four pure liquids. *Proc. Am. Acad. Art. Sci.*, 61(3/12):57–99, FEB-NOV 1926

Ellis

Seikichi Matsuhisa and R. Byron Bird. Analytical and numerical solutions for laminar flow of the non-Newtonian Ellis fluid. *AIChE J.*, 11(4):588–595, 1965

$$\mu(\mathbb{T}) = \frac{\mu_0}{1 + \alpha \left|\mathbb{T}_\delta\right|^{n-1}} \tag{2.10}$$

Fits experimental data for: 0.6% w/w carboxymethyl cellulose (CMC) solution in water, poly(vynil chloride)[3]

[3] T. A. Savvas, N. C. Markatos, and C. D. Papaspyrides. On the flow of non-newtonian polymer solutions. *Appl. Math. Modelling*, 18(1):14–22, 1994

Glen

J. W. Glen. The creep of polycrystalline ice. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 228(1175):519–538, 1955

$$\mu(\mathbb{T}) = \alpha \left|\mathbb{T}_\delta\right|^{n-1} \tag{2.11}$$

Fits experimental data for: ice

Seely

Gilbert R. Seely. Non-newtonian viscosity of polybutadiene solutions. *AIChE J.*, 10(1):56–60, 1964

$$\mu(\mathbb{T}) = \mu_\infty + (\mu_0 - \mu_\infty)\,\mathrm{e}^{-\frac{|\mathbb{T}_\delta|}{\tau_0}} \tag{2.12}$$

Fits experimental data for: polybutadiene solutions

Blatter

Erin C. Pettit and Edwin D. Waddington. Ice flow at low deviatoric stress. *J. Glaciol.*, 49(166):359–369, 2003
H Blatter. Velocity and stress-fields in grounded glaciers – a simple algorithm for including deviatoric stress gradients. *J. Glaciol.*, 41(138):333–344, 1995

$$\mu(\mathbb{T}) = \frac{A}{\left(|\mathbb{T}_\delta|^2 + \tau_0^2\right)^{\frac{n-1}{2}}} \tag{2.13}$$

Fits experimental data for: ice

Bingham Herschel–Bulkley

C. E. Bingham. *Fluidity and plasticity*. McGraw–Hill, New York, 1922

Winslow H. Herschel and Ronald Bulkley. Konsistenzmessungen von Gummi-Benzollösungen. *Colloid Polym. Sci.*, 39(4):291–300, August 1926

$$\begin{aligned} |\mathbb{T}_\delta| > \tau^* \quad &\text{if and only if} \quad \mathbb{T}_\delta = \tau^* \frac{\mathbb{D}}{|\mathbb{D}|} + 2\mu(|\mathbb{D}|)\mathbb{D} \\ |\mathbb{T}_\delta| \leq \tau^* \quad &\text{if and only if} \quad \mathbb{D} = 0 \end{aligned} \tag{2.14}$$

Fits experimental data for: paints, toothpaste, mango jam[4]

[4]Santanu Basu and U.S. Shivhare. Rheological, textural, micro-structural and sensory properties of mango jam. *J. Food Eng.*, 100(2):357–365, 2010

Rivlin–Ericksen

R. S. Rivlin and J. L. Ericksen. Stress-deformation relations for isotropic materials. *J. Ration. Mech. Anal.*, 4:323–425, 1955
R. S. Rivlin and K. N. Sawyers. Nonlinear continuum mechanics of viscoelastic fluids. *Annu. Rev. Fluid Mech.*, 3:117–146, 1971
General form:

$$\mathbb{T} = -p\mathbb{I} + \mathfrak{f}(\mathbb{A}_1\mathbb{A}_2\mathbb{A}_3\dots) \tag{3.1}$$

where

$$\mathbb{A}_1 = 2\mathbb{D} \tag{3.2a}$$

$$\mathbb{A}_n = \frac{\mathrm{d}\mathbb{A}_{n-1}}{\mathrm{d}t} + \mathbb{A}_{n-1}\mathbb{L} + \mathbb{L}^\top\mathbb{A}_{n-1} \tag{3.2b}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the usual Lagrangean time derivative and $\mathbb{L}$ is the velocity gradient.

Criminale–Ericksen–Filbey

William O. Criminale, J. L. Ericksen, and G. L. Filbey. Steady shear flow of non-Newtonian fluids. *Arch. Rat. Mech. Anal.*, 1:410–417, 1957

$$\mathbb{T} = -p\mathbb{I} + \alpha_1 \mathbb{A}_1 + \alpha_2 \mathbb{A}_2 + \alpha_3 \mathbb{A}_1^2 \tag{3.3}$$

Fits experimental data for: polymer melts (explains mormal stress differences)

Reiner–Rivlin

M. Reiner. A mathematical theory of dilatancy. *Am. J. Math.*, 67(3):350–362, 1945

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D} + \mu_1\mathbb{D}^2 \tag{3.4}$$

Fits experimental data for: N/A

Maxwell

J. Clerk Maxwell. On the dynamical theory of gases. *Philos. Trans. R. Soc.*, 157:49–88, 1867

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S} \tag{4.1a}$$

$$\mathbb{S} + \lambda_1 \overset{\triangledown}{\mathbb{S}} = 2\mu\mathbb{D} \tag{4.1b}$$

$$\overset{\triangledown}{\mathbb{M}} =_{\mathrm{def}} \frac{\mathrm{d}\mathbb{M}}{\mathrm{d}t} - \mathbb{L}\mathbb{M} - \mathbb{M}\mathbb{L}^{\top} \tag{4.2}$$

Fits experimental data for: N/A

Oldroyd-B

J. G. Oldroyd. On the formulation of rheological equations of state. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 200(1063):523–541, 1950

$$\mathbb{T} = -\pi\mathbb{I} + \mathbb{S} \tag{4.3a}$$

$$\mathbb{S} + \lambda\overset{\triangledown}{\mathbb{S}} = \eta_1\mathbb{A}_1 + \eta_2\overset{\triangledown}{\mathbb{A}}_1 \tag{4.3b}$$

Fits experimental data for: N/A

Oldroyd 8-constants

J. G. Oldroyd. On the formulation of rheological equations of state. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 200(1063):523–541, 1950

$$\mathbb{T} = -\pi \mathbb{I} + \mathbb{S} \tag{4.4a}$$

$$\begin{aligned} \mathbb{S} + \lambda_1 \overset{\triangledown}{\mathbb{S}} + \frac{\lambda_3}{2}\left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D}\right) + \frac{\lambda_5}{2}\left(\operatorname{Tr}\mathbb{S}\right)\mathbb{D} + \frac{\lambda_6}{2}\left(\mathbb{S} : \mathbb{D}\right)\mathbb{I} \\ = -\mu\left(\mathbb{D} + \lambda_2 \overset{\triangledown}{\mathbb{D}} + \lambda_4 \mathbb{D}^2 + \frac{\lambda_7}{2}\left(\mathbb{D} : \mathbb{D}\right)\mathbb{I}\right) \end{aligned} \tag{4.4b}$$

Fits experimental data for: N/A

Burgers

J. M. Burgers. Mechanical considerations – model systems – phenomenological theories of relaxation and viscosity. In *First report on viscosity and plasticity*, chapter 1, pages 5–67. Nordemann Publishing, New York, 1939

$$\mathbb{T} = -\pi\mathbb{I} + \mathbb{S} \tag{4.5a}$$

$$\mathbb{S} + \lambda_1 \overset{\triangledown}{\mathbb{S}} + \lambda_2 \overset{\triangledown\triangledown}{\mathbb{S}} = \eta_1 \mathbb{A}_1 + \eta_2 \overset{\triangledown}{\mathbb{A}}_1 \tag{4.5b}$$

Fits experimental data for: N/A

Giesekus

H. Giesekus. A simple constitutive equation for polymer fluids based on the concept of deformation-dependent tensorial mobility. *J. Non-Newton. Fluid Mech.*, 11(1-2):69–109, 1982

$$\mathbb{T} = -\pi \mathbb{I} + \mathbb{S} \tag{4.6a}$$

$$\mathbb{S} + \lambda \overset{\triangledown}{\mathbb{S}} - \frac{\alpha \lambda_2}{\mu} \mathbb{S}^2 = -\mu \mathbb{D} \tag{4.6b}$$

Fits experimental data for: N/A

Phan-Thien–Tanner

N. Phan Thien. Non-linear network viscoelastic model. *J. Rheol.*, 22(3):259–283, 1978

N. Phan Thien and Roger I. Tanner. A new constitutive equation derived from network theory. *J. Non-Newton. Fluid Mech.*, 2(4):353–365, 1977

$$\mathbb{T} = -\pi\mathbb{I} + \mathbb{S} \tag{4.7a}$$

$$Y\mathbb{S} + \lambda\overset{\triangledown}{\mathbb{S}} + \frac{\lambda\xi}{2}\left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D}\right) = -\mu\mathbb{D} \tag{4.7b}$$

$$Y = \mathrm{e}^{-\varepsilon\frac{\lambda}{\mu}\operatorname{Tr}\mathbb{S}} \tag{4.7c}$$

Fits experimental data for: N/A

Johnson–Segalman

M. W. Johnson and D. Segalman. A model for viscoelastic fluid behavior which allows non-affine deformation. *J. Non-Newton. Fluid Mech.*, 2(3):255–270, 1977

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S} \tag{4.8a}$$

$$\mathbb{S} = 2\mu\mathbb{D} + \mathbb{S}' \tag{4.8b}$$

$$\mathbb{S}' + \lambda\left(\frac{\mathrm{d}\mathbb{S}'}{\mathrm{d}t} + \mathbb{S}'\left(\mathbb{W} - a\mathbb{D}\right) + \left(\mathbb{W} - a\mathbb{D}\right)^{\top}\mathbb{S}'\right) = 2\eta\mathbb{D} \tag{4.8c}$$

Fits experimental data for: spurt

Johnson–Tevaarwerk

K. L. Johnson and J. L. Tevaarwerk. Shear behaviour of elastohydrodynamic oil films. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 356(1685):215–236, 1977

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S} \tag{4.9a}$$

$$\overset{\triangledown}{\mathbb{S}} + \alpha \sinh \frac{\mathbb{S}}{s_0} = 2\mu\mathbb{D} \tag{4.9b}$$

Fits experimental data for: lubricants

Kaye–Bernstein–Kearsley–Zapas

B. Bernstein, E. A. Kearsley, and L. J. Zapas. A study of stress relaxation with finite strain. *Trans. Soc. Rheol.*, 7(1):391–410, 1963

I-Jen Chen and D. C. Bogue. Time-dependent stress in polymer melts and review of viscoelastic theory. *Trans. Soc. Rheol.*, 16(1):59–78, 1972

$$\mathbb{T} = \int_{\xi=-\infty}^{t} \frac{\partial W}{\partial \mathrm{I}}\mathbb{C} + \frac{\partial W}{\partial \mathrm{II}}\mathbb{C}^{-1} \, \mathrm{d}\xi \tag{5.1}$$

Fits experimental data for: polyisobutylene, vulcanised rubber

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