

1 Convolution

$$r_{y,x}(k) = y(n) * x(-n) \quad (1.1)$$

2 Z-Transform

Signal, $x(n)$	z -Transform, $X(z)$	ROC
$\delta(n)$	1	All z
$\mathcal{U}(n)$	$\frac{1}{1-z^{-1}}$	$ z > 1$
$a^n \mathcal{U}(n)$	$\frac{1}{1-az^{-1}}$	$ z > a $
$na^n \mathcal{U}(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z > a $
$-a^n \mathcal{U}(-n-1)$	$\frac{1}{1-az^{-1}}$	$ z < a $
$-na^n \mathcal{U}(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
$(\cos \omega_0 n) \mathcal{U}(n)$	$\frac{1-z^{-1} \cos \omega_0}{1-2z^{-1} \cos \omega_0 + z^{-2}}$	$ z > 1$
$(\sin \omega_0 n) \mathcal{U}(n)$	$\frac{z^{-1} \sin \omega_0}{1-2z^{-1} \cos \omega_0 + z^{-2}}$	$ z > 1$
$(a^n \cos \omega_0 n) \mathcal{U}(n)$	$\frac{1-az^{-1} \cos \omega_0}{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}}$	$ z > a $
$(a^n \sin \omega_0 n) \mathcal{U}(n)$	$\frac{az^{-1} \sin \omega_0}{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}}$	$ z > a $

Table 2.1: Common Z-Transforms

2.1 Properties of the Z-Transform

Property	Time Domain	z -Domain	ROC
Notation	$x(n)$	$X(z)$	ROC : $r_2 < z < r_1$
	$x_1(n)$	$X_1(z)$	ROC ₁
	$x_2(n)$	$X_2(z)$	ROC ₂
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least the intersection of ROC ₁ and ROC ₂
Time Shifting	$x(n-k)$	$z^{-k} X(z)$	That of $X(z)$, except $z=0$ if $k > 0$ and $z=\infty$ if $k < 0$
Scaling	$a^n x(n)$	$X(a^{-1}z)$	$ a r_2 < z < a r_1$
Time Reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < z < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real Part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary Part	$\text{Im}\{x(n)\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation	$nx(n)$	$-z \frac{dX(z)}{dz}$	$r_2 < z r_1$
Convolution	$x_1 * x_2$	$X_1(z)X_2(z)$	At least, the intersection of ROC ₁ and ROC ₂
2 Sequence Correlation	$r_{x_1 x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1 x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial Value Theorem	If $x(n)$ causal	$x(0) = \lim_{z \rightarrow \infty} X(z)$	
2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j} \oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least, $r_{1l}r_{2l} < a < r_{1u}r_{2u}$
Parsevals Relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$= \frac{1}{2\pi j} \oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

Table 2.2: Z-Transform Properties

2.2 One-Sided \mathcal{Z} -Transform

The *one-sided z -transform* is the same as the \mathcal{Z} -Transform, but is only defined at n values greater than or equal to 0.

$$X(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n} \quad (2.1)$$

2.2.1 Time Delay

If

$$x(n) \xleftrightarrow{z^+} X^+(z)$$

then

$$x(n-k) \xleftrightarrow{z^+} z^{-k} \left[X^+(z) + \sum_{n=1}^k x(-n)z^n \right], \quad k > 0 \quad (2.2)$$

3 DTFT

$$z = e^{j2\pi f} \quad (3.1)$$

$$z = e^{j\omega}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \quad (3.2)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

4 DFT

The N -point DFT is shown as:

$$X_{DFT}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi \frac{k}{N}n} \text{ for } k = 0, 1, 2, \dots, N-1 \quad (4.1)$$

If N is specified, then replace all occurrences of N in Equation (4.1) with that value.

Remark 0.1. If the length, N of the DFT is not specified, it is assumed that $N = \text{length of the signal}$. If the length of the DFT N is greater than the length of the signal, you are sampling the DTFT of the signal.

$$\begin{aligned} x(n) &= A \cos\left(2\pi \frac{k_0}{N}n\right), \quad 0 < k_0 \leq N-1 \\ &= \frac{A}{2} \left(e^{j\frac{2\pi k_0}{N}n} + e^{-j\frac{2\pi k_0}{N}n} \right) \end{aligned} \quad (4.2)$$

$$\begin{aligned} X(k) &= \frac{AN}{2} \left[(\delta(k - k_0) \bmod N) + (\delta(k + k_0) \bmod N) \right] \\ x_{IDFT}(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi \frac{k}{N}n} \text{ for } n = 0, 1, \dots, N-1 \end{aligned} \quad (4.3)$$

$$x_1(n) \otimes x_2(n) = \sum_{k=0}^{N-1} x_1(k)x_2(n - k \bmod N) \quad (4.4)$$

It is important to remember that the modulus (mod) operator yields 0 when the input is a multiple of the divisor.

Defn 1 (Decimation). *Decimation* takes an input signal and compresses it. Decimation uses the symbol $D \in \mathbb{Z}^+$.

$$y(m) = x(mD)$$

If decimation occurs later in the system, then if just the input and output are compared, $y(m)$ appears it was sampled at

$$f = \frac{F_S}{D} \quad (4.5)$$

Thus, when we perform sampling on the input signal, then there is folding at

$$f = \frac{F_S}{2D} \quad (4.6)$$

Property	Time Domain $x(n)$	DFT Domain $X(k)$
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n + N)$	$X(k) = X(k + N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time Reversal	$x(N - n)$	$X(N - k)$
Circular Time Shifting	$x(n - n_0 \bmod N)$	$X(k)e^{-j2\pi \frac{k}{N}n_0}$
Circular Frequency Shift	$x(n)e^{j2\pi ln/N}$	$X(k - l \bmod N)$
Complex Conjugate	$X^*(n)$	$X^*(N - k)$
Circular Convolution	$x(n) \otimes y(n)$	$X(k)Y(k)$
Circular Correlation	$x(n) \otimes y^*(-n)$	$X(k)Y^*(k)$
2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \otimes X_2(k)$
Parsevals Theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k)$

Table 4.1: Properties of the DFT