ECE 311: Engineering Electronics — Reference Sheet
Illinois Institute of Technology

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Last Edited: March 8, 2021

Contents

List of Theorems

1 Operational Amplifiers

Defn 1 (Op-Amp)**.** An *op-amp*, (*operational amplifier*), is a active circuit element that is a 2-port network element. An circuit symbol for an op-amp is shown in Figure 1.1.

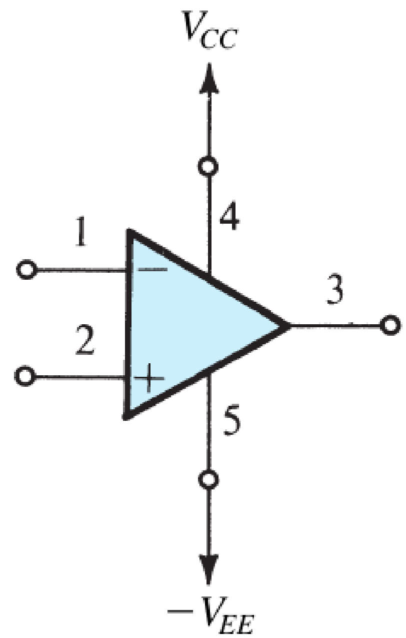

Figure 1.1: Complete Circuit Symbol for an Op-Amp (Sedra et al. 2015, p. 60)

Typically, terminals 4 and 5 are **not** included in circuit schematics, as they are implied to exist, because the Op-Amp requires input power to operate.

Defn 2 (Open-Loop Gain)**.** The *open-loop gain* of an Op-Amp, typically represented with A is the gain in voltage at the output given an input voltage. The equation for the open-loop gain is given in Equation (1.1).

$$A = \frac{V_O}{V_+ - V_-} \tag{1.1}$$

1.1 The Ideal Op-Amp

The ideal Op-Amp is one that has:

1. Infinite input impedance
2. Zero output impedance
3. Zero common-mode gain/infinite common-mode rejection
4. Infinite Open-Loop Gain
5. Infinite bandwidth (There is no distortion of the output signal due to amplification.)

This can be summarized using just two equations, shown in Equations (1.2a) and (1.2b).

$$V_+ = V_- \tag{1.2a}$$

$$I_+ = I_- = 0 \tag{1.2b}$$

When connecting an Op-Amp, even an ideal one, to a circuit, the Closed-Loop Gain becomes a factor.

Defn 3 (Closed-Loop Gain)**.** The *closed-loop gain* occurs when an Op-Amp is connected to a surrounding circuit. It's defining equation is shown in Equation (1.3).

$$G = \frac{V_O}{V_I} \tag{1.3}$$

1.2 Non-Ideal Op-Amps

We have only considered ideal Op-Amps throughout this text. However, in the real world, like everything, op-amps do not behave ideally. In the subsections below, the various non-ideal properties are discussed.

1.2.1 Offset Voltage

The first major non-ideal behavior Op-Amps have is that even when the differential input voltage is zero ($V_+ - V_- = 0$), there is a finite output voltage. This is because of an internal voltage called the Input Offset Voltage.

Defn 4 (Input Offset Voltage)**.** The *input offset voltage* is a non-ideal behavior of Op-Amps. The typical symbol is V_{OS}, and typically has the units µV/°C. For most Op-Amps, V_{OS} is typically in the range $1\,\text{mV} \leq V_{OS} \leq 5\,\text{mV}$.

A way to model a non-ideal Op-Amp that is being affected by an Input Offset Voltage is shown in Figure 1.2.

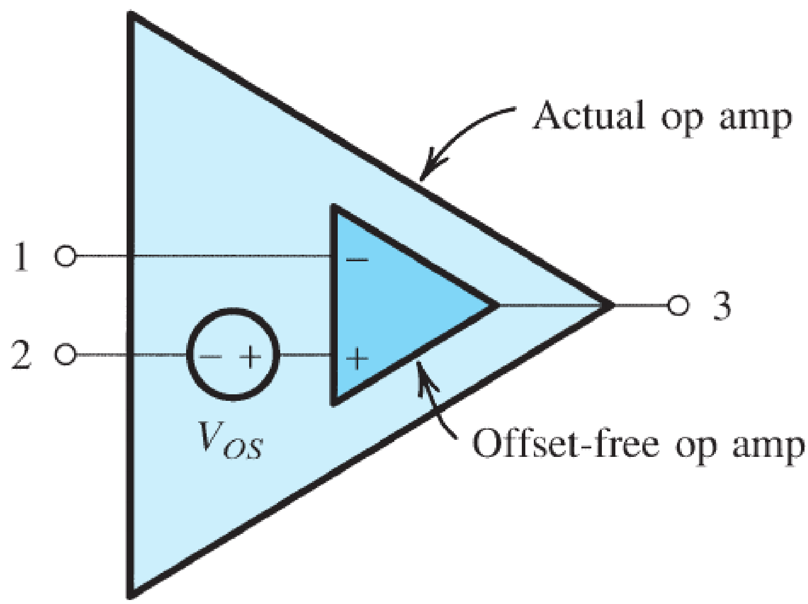

Figure 1.2: Circuit Model for Op-Amp with Input Offset Voltage V_{OS} (Sedra et al. 2015, p. 97)

1.2.2 Input Bias and Offset Currents

The other major non-ideal behavior Op-Amps have is that they have Input Bias Currents and Input Offset Current.

Defn 5 (Input Bias Current)**.** The *input bias currents* are due to the fact that op-amps require DC input currents. These are independent of the fact that a real Op-Amp has finite input resistance. The two input bias currents are: $I_{B,1}$ and $I_{B,2}$. The average value of the input bias current is usually provided on component datasheets. The equation for the average value is given in Equation (1.4).

$$I_B = \frac{I_{B,1} + I_{B,2}}{2} \tag{1.4}$$

Defn 6 (Input Offset Current)**.** The *input offset current* is the difference betweeen the Input Bias Currents. The mathematical symbol for this current is I_{OS}

$$I_{OS} = |I_{B,1} - I_{B,2}| \tag{1.5}$$

A figure depicting the setup of an Op-Amp that has bias currents is shown in Figure 1.3.

2 Semiconductors

2.1 Intrinsic Semiconductors

Defn 7 (Semiconductor)**.** *Semiconductors* are materials whose conductivity is somewhere between that of true conductors, like copper, and insulators, such as glass. Because semiconductors are somewhere between conductors and insulators, they have electrical properties that are easily manipulated through Doping.

Defn 8 (Electron)**.** An *electron* in this scenario is a **free electron**. This means the electron is not bound to any particular atomic nucleus. Such an electron is free to conduct electric current if an electric field is applied.

If an atom is missing electrons due to an electron being free, a Hole can be thought of in its place.

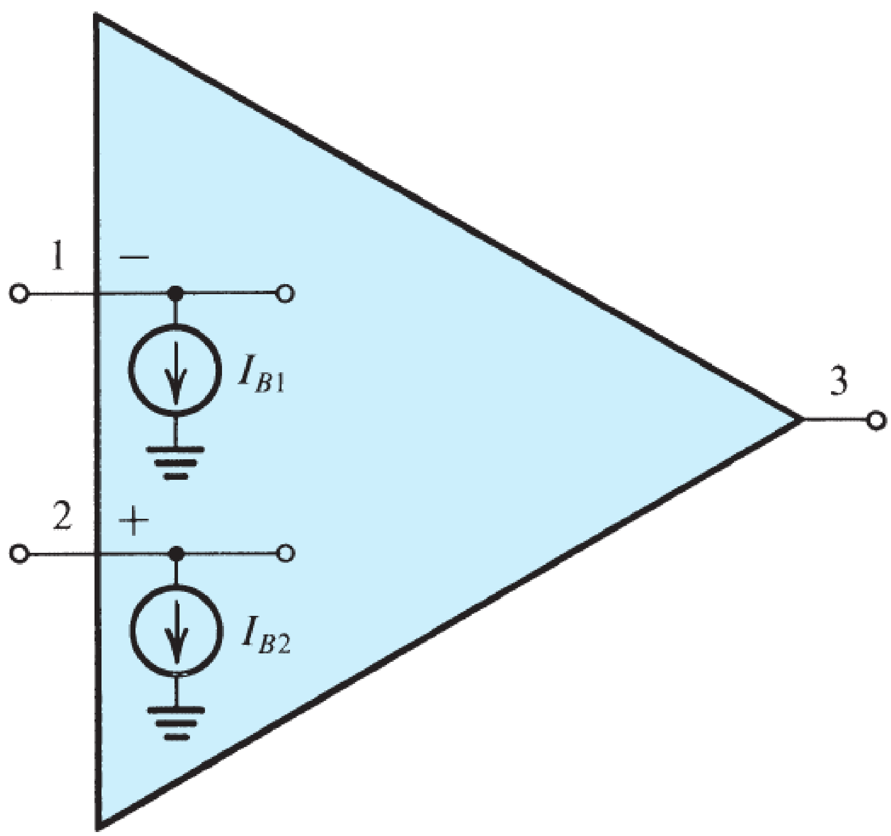

Figure 1.3: Circuit Model for Op-Amp with Input Bias Current and Input Offset Current (Sedra et al. 2015, p. 101)

The concentration of free Electrons in a material is given the symbol shown in Equation (2.1).

$$n \tag{2.1}$$

Defn 9 (Hole)**.** A *hole* is the lack of an Electron being attached to an atom. These have the same, but opposite charge of an electron.

These are NOT particles in any physical sense. Holes are useful abstractions to use when thinking about current flow in a semiconductor's crystal lattice.

The concentration of free Holes in a material is given the symbol shown in Equation (2.2).

$$p \tag{2.2}$$

Defn 10 (Intrinsic)**.** An *intrinsic* material is one that is pure. It has a regular lattice structure, where atoms are held in place by covalent bonds.

In an Intrinsic material, the concentration of free Electrons and Holes are equal. This is represented by the relation in Equation (2.3).

$$n = p = n_i \tag{2.3}$$

Typically, we express the product of Hole and free Electron concentrations as a product, shown in Equation (2.4).

$$pn = n_i^2 \tag{2.4}$$

Semiconductor physics tells us that n_i is defined by Equation (2.5).

$$n_i = BT^{\frac{3}{2}} e^{\frac{-E_g}{2kT}} \tag{2.5}$$

where

B is a material-dependent parameter.
T is the temperature
k is Boltzmann's constant.

2.2 Doped Semiconductors

Defn 11 (Doping)**.** *Doping* is the process of deliberately adding atomic impurities to alter the electrical/conductivity characteristics of Semiconductors. This is done by substantially increasing the concentration of Electrons or Holes, but without changing the crystal properties of the original semiconductor.

There are two kinds of doping:

1. n Type. This is done by doping with an element with 5 valence electrons, typically phosphorus.
2. p Type. This is done by doping with an element with 3 valence electrons, typically boron.

If the concentration of donor atoms in an n-type doped Semiconductor is N_D, where N_D is much greater than n_i ($N_D \gg p$), then the concentration of **free Electrons** is:

$$n_n \simeq N_D \tag{2.6}$$

By substituting Equation (2.6) into Equation (2.5), we can find the Hole concentration for an n-type Semiconductor.

$$p_n \simeq \frac{n_i^2}{N_D} \tag{2.7}$$

In an n-type doped Semiconductor, the free Electrons have a significantly larger concentration, and are said to be the **majority charge carriers**. The Holes are the **minority charge carriers**.

The complete opposite of this holds true for p-type doped Semiconductors. Meaning,

$$p_p \simeq N_A$$

$$n_p \simeq \frac{n_i^2}{N_A}$$

Similarly oppositely, in an p-type doped Semiconductor, the free Holes have a significantly larger concentration, and are said to be the **majority charge carriers**. The free Electrons are the **minority charge carriers**.

2.3 Current Flow in Semiconductors

2.3.1 Drift Current

Defn 12 (Drift Current)**.** *Drift current* arises when an electric field $\vec{E}$ is applied to a Semiconductor. The Holes are accelerated **in** the direction of $\vec{E}$, and the free Electrons are accelerated in the **opposite** direction of $\vec{E}$.

In the presence of an electric field, because the drift current is made of of Electrons and Holes moving due to a force field, they have a velocity. This velocity is the Drift Velocity.

Defn 13 (Drift Velocity)**.** *Drift velocity* is the velocity that Holes or Electrons gain when an electric field (voltage) is applied to the Semiconductor. There are two separate equations for drift velocity, one for Holes and one for Electrons, shown in Equations (2.8) and (2.9), respectively.

$$v_{p-\text{drift}} = \mu_p \vec{E} \tag{2.8}$$

$$v_{n-\text{drift}} = -\mu_n \vec{E} \tag{2.9}$$

Equation (2.9) is negative because Electrons move in the opposite direction of the electric field $\vec{E}$.
In Equations (2.8) and (2.9), the constants μ_p and μ_n are used. μ_p is the Hole Mobility. μ_n is the Electron Mobility.

Defn 14 (Hole Mobility)**.** *Hole mobility*, μ_p is a value representing how "easy" it is for Holes to move through the Semiconductor's crystal structure in response to an applied electric field, $\vec{E}$.
The hole mobility for Intrinsic silicon is a known constant, and is shown in Equation (2.10).

$$\mu_p = 480\,\mathrm{cm}^2/(\mathrm{V\,s}) \tag{2.10}$$

Defn 15 (Electron Mobility)**.** *Electron mobility*, μ_n is a value representing how "easy" it is for Electrons to move through the Semiconductor's crystal structure in response to an applied electric field, $\vec{E}$.
The electron mobility for Intrinsic silicon is a known constant, and is shown in Equation (2.11).

$$\mu_n = 1350\,\mathrm{cm}^2/(\mathrm{V\,s}) \tag{2.11}$$

Remark 15.1. Electrons move through a semiconductor's crystal lattice much more easily than Holes do. This can be seen by $\mu_n \approx 2.5\mu_p$.

We are interested in the current flowing through an object due to an applied electric field, so we use:

$$I_p = Aqpv_{p-\text{drift}} \tag{2.12}$$

If we substitute $v_{p-\text{drift}}$ with our knowledge from Equation (2.8), then we have the equation below.

$$I_p = Aqp\mu_p \vec{E}$$

If we divide this equation by the cross-sectional area, A, then we have Equation (2.13).

$$J_p = \frac{I_p}{A} = qp\mu_p \vec{E} \tag{2.13}$$

Similarly, we can find the drift current equation for I_n and the current density equation.

$$I_n = -Aqnv_{p-\text{drift}} \tag{2.14}$$

$$J_n = qn\mu_n \vec{E} \tag{2.15}$$

Then the total drift current is just the addition of the two separate drift currents.

$$J = J_p + J_n \tag{2.16}$$

By factoring out the constant terms, we end up with:

$$\begin{aligned} J &= J_p + J_n \\ &= qp\mu_p \vec{E} + qn\mu_n \vec{E} \\ &= q(p\mu_p + n\mu_n)\vec{E} \\ J &= \sigma \vec{E} \end{aligned}$$

This leads to two important equations, Equations (2.17a) and (2.17b).

$$J = \sigma \vec{E} \tag{2.17a}$$

$$J = \frac{\vec{E}}{\rho} \tag{2.17b}$$

Defn 16 (Conductivity)**.** *Conductivity*, typically represented with σ, is how good a conductor an object is.

$$\sigma = q(p\mu_p + n\mu_n) \tag{2.18}$$

Defn 17 (Resistivity)**.** *Resistivity*, typically represented with ρ, is how good an object is at preventing the flow of current.

$$\rho \equiv \frac{1}{\sigma} = \frac{1}{q(p\mu_p + n\mu_n)}\,\Omega\,\text{cm} \tag{2.19}$$

2.3.2 Diffusion Current

Defn 18 (Diffusion Current)**.** *Diffusion current* arises due to concentration differences in Electrons and Holes in a semiconductor. It travels from the p side to the n side.

The Diffusion Current density is proportional to the slope of the concentration gradient at any given point in the Semiconductor. For example, if a Semiconductor has **holes** added to it such that there is a larger amount of holes at one side of a block than the other, we end up with Equation (2.20).

$$J_p = -qD_p \frac{dp(x)}{dx} \tag{2.20}$$

Similarly, for a free Electron gradient, we have Equation (2.21).

$$J_n = qD_n \frac{dn(x)}{dx} \tag{2.21}$$

Defn 19 (Hole Diffusivity)**.** *Hole diffusivity* or the *hole diffusion constant* is a constant representing how easy it is for a Hole to diffuse through the substrate's crystal lattice. It is given the symbol D_p.

Defn 20 (Electron Diffusivity)**.** *Electron diffusivity* or the *electron diffusion constant* is a constant representing how easy it is for a free Electron to diffuse through the substrate's crystal lattice. It is given the symbol D_n.

There exists a relationship between the diffusivity constants (Definitions 19 and 20) and the mobility constants (Definitions 14 and 15), as seen in Equation (2.22).

$$\frac{D_n}{\mu_n} = \frac{D_p}{\mu_p} = V_T \tag{2.22}$$

2.4 The *pn*-Junction Junction

2.5 The *pn*-Junction Junction with Applied Voltage

2.6 Depletion Layer

Defn 21 (Depletion Layer)**.** The *depletion layer* is the location in the *pn*-Junction where the two differently-doped sides meet. Here, there is a barrier of the opposing carrier on each side. This is visualized in Figure 2.1.

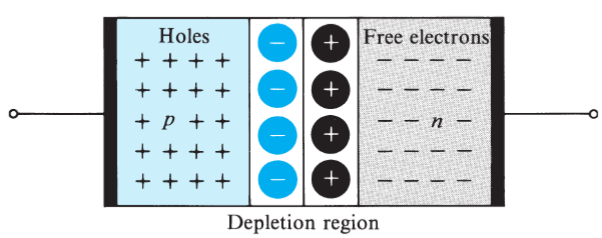

Figure 2.1: Depletion Layer (Sedra et al. 2015, p. 150)

We can find the width of the Depletion Layer using Equation (2.23).

$$W = \sqrt{\frac{2\epsilon_{\text{Si}}}{q}\left(\frac{1}{N_A} + \frac{1}{N_D}\right)V_0} \tag{2.23}$$

The depletion layer "bleeds" into each side of the *pn*-Junction. We can find the distance the depletion layer falls into each side with Equations (2.24a) and (2.24b).

$$x_n = W\left(\frac{N_A}{N_A + N_D}\right) \tag{2.24a}$$

$$x_p = W\left(\frac{N_D}{N_A + N_D}\right) \tag{2.24b}$$

Lastly, the sum of the "bleed" in both directions is equal to the width of the entire Depletion Layer.

$$W = x_n + x_p \tag{2.25}$$

3 Diodes

4 MOS Field-Effect Transistors

A Complex Numbers

Defn A.0.1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted $\mathbb{C}$.

A complex number is written, in Cartesian form, as shown in Equation (A.1) below.

$$z = a \pm ib \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

A.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn A.1.1 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\mathrm{Re}\{z\} = x \tag{A.3}$$

Remark A.1.1.1 (Alternative Notation). The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re\mathfrak{e}$$

Defn A.1.2 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\mathrm{Im}\{z\} = y \tag{A.4}$$

Remark A.1.2.1 (Alternative Notation). The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im\mathfrak{m}$$

A.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$\begin{aligned} z &= x_1 + iy_1 \\ w &= x_2 + iy_2 \end{aligned}$$

A.2.1 Addition

The addition operation, still denoted with the + symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w \coloneqq (x_1 + x_2) + i(y_1 + y_2) \tag{A.5}$$

A.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$\begin{aligned} zw &\coloneqq (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\ &= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned} \tag{A.6}$$

A.3 Complex Conjugates

Defn A.3.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{A.7}$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{A.8}$$

A.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$
2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - xyi + xyi - i^2y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2 \end{aligned}$$

Thus,

$$z\overline{z} = x^2 + y^2 \tag{A.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}$$

Using our solution from Equation (A.9):

$$\begin{aligned} &= \frac{(x + iy)^2}{x^2 + y^2} \\ &= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2} \end{aligned}$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$
\begin{aligned}
&= \frac{(x^2 - y^2) + 2xyi}{x^2 + y^2} \\
&= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2}
\end{aligned}
$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$
\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{A.10}
$$

A.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.

First, start by defining some expressions so that we can prove some of these properties:

$$
\begin{aligned}
z &= x + iy \\
\overline{z} &= x - iy
\end{aligned}
$$

(i) The conjugation operation is commutative.

(ii) The conjugation operation can be distributed over addition and multiplication.

$$
\begin{aligned}
\overline{z + w} &= \overline{z} + \overline{w} \\
\overline{zw} &= \overline{z}\overline{w}
\end{aligned}
$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers ($z, w \in \mathbb{C}$) where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.

We start by simplifying the left-hand side of the equation ($\overline{z + w}$).

$$
\begin{aligned}
\overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\
&= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\
&= (x_1 + x_2) - i(y_1 + y_2)
\end{aligned}
$$

Now, we simplify the other side ($\overline{z} + \overline{w}$).

$$
\begin{aligned}
\overline{z} + \overline{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\
&= (x_1 - iy_1) + (x_2 - iy_2) \\
&= (x_1 + x_2) - i(y_1 + y_2)
\end{aligned}
$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

■

A.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn A.4.1 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in \mathbb{C}$. This is taken from Descarte's definition of a completely real plane.

The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

A.4.1 Modulus of a Complex Number

Defn A.4.2 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\overline{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{A.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in \mathbb{C}$). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (A.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\overline{z}\overline{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \end{aligned}$$

Note how we never needed to define z or w, so this is as general a result as possible. ■

A.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right) \\ &= |z|^2|w|^2 \end{aligned}$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

A.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

A.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (A.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{A.12}$$

Example A.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$\begin{aligned}
|5iz + 10| &= 7 \\
|5i(z + \frac{10}{5i})| &= 7 \\
|5i(z + \frac{2}{i})| &= 7 \\
|5i(z + \frac{2}{i}\frac{-i}{-i})| &= 7 \\
|5i(z - 2i)| &= 7
\end{aligned}$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$\begin{aligned}
|5i||z - 2i| &= 7 \\
5|z - 2i| &= 7 \\
|z - 2i| &= \frac{7}{5}
\end{aligned}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

A.5.1 Annulus

Defn A.5.1 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (A.13).

$$r_1 \leq |z - a| \leq r_2 \tag{A.13}$$

In Equation (A.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

A.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (A.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{A.14}$$

Remark. Note that in the definition of polar form (Equation (A.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$\begin{aligned}
r &= |z| \\
\theta &= \arg(z)
\end{aligned}$$

Example A.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned} \cos(\theta) &= \frac{-\sqrt{3}}{2} \\ \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned} r &= 2 \\ \theta &= \frac{5\pi}{6} \\ \arg(z) &= \frac{5\pi}{6} + 2\pi k \end{aligned}$$

A.6.1 Converting Between Cartesian and Polar Forms

Using Equation (A.14) and Equation (A.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{A.15}$$

A.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

A.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \tag{A.16}$$

A.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big) \tag{A.17}$$

A.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph A.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn A.6.1 (de Moiver's Law)**.** Given a complex number z, $z \in \mathbb{C}$ and a rational number n, $n \in \mathbb{Q}$, the exponentiation of z^n is defined as Equation (A.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{A.18}$$

A.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{A.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (A.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{A.20}$$

A w that satisfies Equation (A.20) is an nth root of z.

Example A.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example A.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (A.19), we can find the general form of the roots.

$$\begin{aligned}
z &= 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)\\
z^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right)\\
&= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right)
\end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \ldots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

k = 2

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(2)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(2)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)$$

k = 3

$$\begin{aligned}\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(3)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(3)}{18}\right)\right) &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + \frac{36\pi}{18}\right) + i\sin\left(\frac{\pi}{18} + \frac{36\pi}{18}\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + 2\pi\right) + i\sin\left(\frac{\pi}{18} + 2\pi\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)\end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned} z_1^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right) \\ z_2^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right) \\ z_3^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)\end{aligned}$$

A.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn A.8.1 (Argument)**.** The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example A.4: Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) &= \tan(\theta) = \frac{-i}{-1} \\ &= \frac{\pi}{4}\end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z) = \frac{5\pi}{4} + 2\pi k$$

Thus, the argument of $z = -1 - i$ is $\arg(z) = \frac{5\pi}{4} + 2\pi k$.

Defn A.8.2 (Principal Argument)**.** The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi < \operatorname{Arg}(z) \leq \pi$.

Example A.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$
\begin{aligned}
z &= -1 - i \\
\arg(z) = \tan(\theta) &= \frac{-i}{-1} \\
&= \frac{\pi}{4}
\end{aligned}
$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\operatorname{Arg}(z) = \frac{5\pi}{4}$.

A.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (A.21).

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big) \tag{A.21}$$

If instead of e as the base, we have some value a, then we have Equation (A.22).

$$
\begin{aligned}
a^z &= e^{z\ln(a)} \\
&= e^{\operatorname{Re}\{z\ln(a)\}}\Big(\cos\big(\operatorname{Im}\{z\ln(a)\}\big) + i\sin\big(\operatorname{Im}\{z\ln(a)\}\big)\Big)
\end{aligned} \tag{A.22}
$$

In the case of Equation (A.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example A.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$
\begin{aligned}
x &= -1 \\
y &= \sqrt{3}
\end{aligned}
$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$
\begin{aligned}
\left|e^{-1+i\sqrt{3}}\right| &= \left|e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right| \\
&= e^{-1} \\
\arg\left(e^{-1+i\sqrt{3}}\right) &= \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\
&= \sqrt{3} + 2\pi k \\
\operatorname{Arg}\left(e^{-1+i\sqrt{3}}\right) &= \operatorname{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\
&= \sqrt{3}
\end{aligned}
$$

Example A.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^{0}\big(\cos(-1) + i\sin(-1)\big) \\ &= 1\big(\cos(-1) + i\sin(-1)\big) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1\big(\cos(-1)+i\sin(-1)\big)} \\ &= e^{-1\big(\cos(1)-i\sin(1)\big)} \\ &= e^{-\cos(1)+i\sin(1)} \end{aligned}$$

If we refer back to Equation (A.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big) \end{aligned}$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} \left|e^{-e^{-i}}\right| &= \left|e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right| \\ &= e^{-\cos(1)} \\ \arg\left(e^{-e^{-i}}\right) &= \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) + 2\pi k \\ \operatorname{Arg}\left(e^{-e^{-i}}\right) &= \operatorname{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) \end{aligned}$$

Example A.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^{i}$?

Use Equation (A.22) to simplify this to a base of e, where we can use the usual Equation (A.21) to solve this.

$$\begin{aligned} a^{z} &= e^{z\ln(a)} \\ 1^{i} &= e^{i\ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^2} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

A.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.23}$$

A.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn A.10.1 (Complex Logarithm)**.** The *complex logarithm* is defined in Equation (A.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e |w| + i \arg(w) \end{aligned} \tag{A.24}$$

Remark A.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example A.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (A.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e |z| + i \arg(z) \\ &= \log_e |-1| + i \arg(-1) \\ &= \log_e(1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \ldots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example A.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (A.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e |w| + i \arg(w) \\ \ln(1) &= \log_e |1| + i \arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e |2\pi k i| + i \arg(2\pi k i) \\ &= \log_e(2\pi |k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases} \right) \end{aligned}$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases}\right)$.

A.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\overline{z}) \tag{A.25}$$

A.11 Complex Trigonometry

For the equations below, $z \in \mathbb{C}$. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{A.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{A.27}$$

Example A.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

We start by using the defintion of complex cosine Equation (A.26).

$$\begin{aligned} \cos(z) &= 5 \\ \frac{e^{iz} + e^{-iz}}{2} &= 5 \\ e^{iz} + e^{-iz} &= 10 \\ e^{iz}\left(e^{iz} + e^{-iz}\right) &= e^{iz}(10) \\ {e^{iz}}^2 + 1 &= 10e^{iz} \\ {e^{iz}}^2 - 10e^{iz} + 1 &= 0 \end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned} e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 4}}{2} \\ &= \frac{10 \pm \sqrt{96}}{2} \\ &= \frac{10 \pm 4\sqrt{6}}{2} \\ &= 5 \pm 2\sqrt{6} \end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned} iz &= \ln(5 \pm 2\sqrt{6}) \\ &= \log_e\left|5 \pm 2\sqrt{6}\right| + i\arg(5 \pm 2\sqrt{6}) \\ &= \log_e\left|5 \pm 2\sqrt{6}\right| + i(0 + 2\pi k) \\ &= \log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi k i \\ z &= \frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi k i\right) \\ &= \frac{-i}{-i}\frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right|\right) + 2\pi k \\ &= 2\pi k - i\log_e\left|5 \pm 2\sqrt{6}\right| \end{aligned}$$

Thus, $z = 2\pi k - i\log_e \left|5 \pm 2\sqrt{6}\right|$.

A.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x \pm iy) = \big(\cos(x)\cosh(y)\big) \mp i\big(\sin(x)\sinh(y)\big) \tag{A.28}$$

$$\sin(x \pm iy) = \big(\sin(x)\cosh(y)\big) \pm i\big(\cos(x)\sinh(y)\big) \tag{A.29}$$

Example A.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (A.28).

$$\begin{aligned}\cos(x+iy) &= \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big)\\ \cos(2+3i) &= \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)\end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (A.21), we can solve this.

$$\begin{aligned}e^z &= e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big)\\ x &= \cos(2)\cosh(3)\\ y &= -\sin(2)\sinh(3)\\ e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} &= e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)\end{aligned}$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned}|z| &= \Big|e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)\Big|\\ &= e^{\cos(2)\cosh(3)}\\ \arg(z) &= \arg(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3) + 2\pi k\\ \operatorname{Arg}(z) &= \operatorname{Arg}(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3)\end{aligned}$$

A.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned}\overline{\cos(x)} &= \cos(x)\\ &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\end{aligned} \tag{A.30}$$

$$\begin{aligned}\overline{\sin(x)} &= \sin(x)\\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\end{aligned} \tag{A.31}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{B.24}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.26}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.27}$$

C Calculus

C.1 L'Hôpital's Rule

L'Hôpital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcilable results.

Lemma C.0.1 (L'Hôpital's Rule)**.** *If the equation*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) *holds.*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.3}$$

Defn C.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Quotient Rule

If

$$f(x) = \frac{g(x)}{h(x)}$$

then,

$$f'(x) = \frac{\frac{d\,g(x)}{dx} h(x) - g(x) \frac{d\,h(x)}{dx}}{\big(h(x)\big)^2} \tag{C.4}$$

C.3.2 Chain Rule

Defn C.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.5}$$

C.4 Useful Integrals

$$\int \cos(x)\,dx = \sin(x) \tag{C.6}$$

$$\int \sin(x)\,dx = -\cos(x) \tag{C.7}$$

$$\int x\cos(x)\,dx = \cos(x) + x\sin(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x\sin(x)\,dx = \sin(x) - x\cos(x) \tag{C.9}$$

Equation (C.9) simplified with Integration by Parts.

$$\int x^2\cos(x)\,dx = 2x\cos(x) + (x^2-2)\sin(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int x^2\sin(x)\,dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{C.11}$$

Equation (C.11) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\,dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x}\sin(\beta x)\,dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.13}$$

$$\int e^{\alpha x}\,dx = \frac{e^{\alpha x}}{\alpha} \tag{C.14}$$

$$\int xe^{\alpha x}\,dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{C.15}$$

Equation (C.15) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha+\beta x} = \int \frac{1}{\alpha+\beta x}\,dx = \frac{1}{\beta}\ln(\alpha+\beta x) \tag{C.16}$$

$$\int \frac{dx}{\alpha^2+\beta^2x^2} = \int \frac{1}{\alpha^2+\beta^2x^2}\,dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{C.17}$$

$$\int \alpha^x\,dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.18}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{C.19}$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule). *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\big[b(t),t\big]\frac{db(t)}{dt} - f\big[a(t),t\big]\frac{da(t)}{dt} \tag{C.20}$$

C.6 Laplace's Equation

Laplace's Equation is used to define a harmonic equation. These functions are twice continuously differentiable $f : U \to R$, where U is an open subset of $\mathbb{R}^n$, that satisfies Equation (C.21).

$$\frac{\partial^2 f}{\partial {x_1}^2} + \frac{\partial^2 f}{\partial {x_2}^2} + \cdots + \frac{\partial^2 f}{\partial {x_n}^2} = 0 \tag{C.21}$$

This is usually simplified down to

$$\nabla^2 f = 0 \tag{C.22}$$

D Laplace Transform

D.1 Laplace Transform

Defn D.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$

D.2 Inverse Laplace Transform

Defn D.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\, ds \tag{D.2}$$

D.3 Properties of the Laplace Transform

D.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (D.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{D.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{D.3b}$$

D.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (D.4).

$$\mathcal{L}\big\{x(\alpha t)\big\} = \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \tag{D.4}$$

D.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (D.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\big\{x(t-a)\big\} = X(s)e^{-as} \tag{D.5}$$

D.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\big\{X(s-a)\big\} = x(t)e^{at} \tag{D.6}$$

D.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\, d\lambda\right\} = \frac{1}{s}X(s) \tag{D.7}$$

D.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\big\{x(t) * v(t)\big\} = X(s)V(s) \tag{D.8}$$

D.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (D.9) holds.

$$X(s) = X(\omega)\big|_{\omega=\frac{s}{j}} \tag{D.9}$$

D.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem D.1 (Intial Value Theorem)**.** *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Theorem D.2 (Final Value Theorem)**.** *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

D.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t-T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n\,\mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t-T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at}\,\mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at}\,\mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)\,\mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)\,\mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)\,\mathcal{U}(t)$	$\begin{cases} a: \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb}+\frac{re^{-j\theta}}{s+a+jb}\right) \\ c: \frac{As+B}{s^2+2as+c} \begin{cases} r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$
$e^{-at}\left(A\cos(\sqrt{c-a^2}t)+\frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

D.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0)-X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0)+X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n \frac{d^n}{ds^n}X(s)\ n\in\mathbb{N}$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s)-\sum_{0}^{n-1} s^{n-1-i}\frac{d^i}{dt^i}x(t)\big\rvert_{t=0^-}\ n\in\mathbb{N}$

References

[Sed+15] Adel S. Sedra et al. *Microelectronic Circuits*. English. Seventh. Oxford University Press, 2015. ISBN: 9780190853464.