

# EITF75: Systems and Signals - Reference Sheet

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# 1 Sinusoids

There are several ways to characterize Sinusoids. The first is by dimension:

1. Multidimensional/Multichannel Signals
2. Monodimensional/Monochannel Signals

You can also classify sinusoids by their independent variable (usually time) and the values they take.

1. Continuous-Time Signals or Analog Signals
2. Discrete-Time Signals
3. There is a third way to classify sinusoids and their signals: Digital Signals

**Defn 1** (Continuous-Time Signals). *Continuous-time signals* or *Analog signals* are defined for every value of time and they take on values in the continuous interval  $(a, b)$ , where  $a$  can be  $-\infty$  and  $b$  can be  $\infty$ . Mathematically, these signals can be described by functions of a continuous variable.

For example,

$$x_1(t) = \cos \pi t, x_2(t) = e^{-|t|}, -\infty < t < \infty$$

**Defn 2** (Discrete-Time Signals). *Discrete-time signals* are defined only at certain specified values of time. These time instants **need not** be equidistant, but in practice, they are usually taken at equally spaced intervals for computation convenience and mathematical tractability.

For example,

$$x(t_n) = e^{-|t_n|}, n = 0, \pm 1, \pm 2, \dots$$

A Discrete-Time Signals can be represented mathematically by a sequence of real or complex numbers.

*Remark 2.1.* To emphasize the discrete-time nature of the signal, we shall denote the signal as  $x(n)$ , rather than  $x(t)$ .

*Remark 2.2.* If the time instants  $t_n$  are equally spaced (i.e.,  $t_n = nT$ ), the notation  $x(nT)$  is also used.

## 1.1 Continuous-Time Signals

### 1.1.1 Frequency in Continuous-Time Signals

A simple harmonic oscillation is mathematically described by Equation (1.1).

$$x_a(t) = A \cos(\Omega t + \theta), -\infty < t < \infty \quad (1.1)$$

*Remark.* The subscript  $a$  is used with  $x(t)$  to denote an analog signal.

This signal is completely characterized by three parameters:

1.  $A$ , the *amplitude* of the sinusoid
2.  $\Omega$ , the *frequency* in radians per second (rad/s)
3.  $\theta$ , the *phase* in radians.

Instead of  $\Omega$ , the frequency  $F$  in cycles per second or hertz (Hz) is used.

$$\Omega = 2\pi F \quad (1.2)$$

Plugging (1.2) into (1.1), yields

$$x_a(t) = A \cos(2\pi F t + \theta), -\infty < t < \infty \quad (1.3)$$

### 1.1.2 Properties of Continuous-Time Sinusoidal Signals

The analog sinusoidal signal in equation (1.3) is characterized by the following properties:

- (i) For every fixed value of the frequency  $F$ ,  $x_a(t)$  is periodic.

$$x_a(t + T_p) = x_a(t)$$

where  $T_p = \frac{1}{F}$  is the fundamental period.

- (ii) Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.
- (iii) Increasing the frequency  $F$  results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in the given time interval.

## 1.2 Discrete-Time Signals

### 1.2.1 Frequency in Discrete-Time Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A \cos(\omega n + \theta), n \in \mathbb{Z}, -\infty < n < \infty \quad (1.4)$$

The signal is characterized by these parameters:

1.  $n$ , the sample number. MUST be an integer.
2.  $A$ , the *amplitude* of the sinusoid
3.  $\omega$ , the *angular frequency* in radians per sample
4.  $\theta$ , is the *phase*, in radians.

Instead of  $\omega$ , we use the frequency variable  $f$  defined by

$$\omega \equiv 2\pi f \quad (1.5)$$

Using (1.4) and (1.5) yields

$$x(n) = A \cos(2\pi f n + \theta), n \in \mathbb{Z}, -\infty < n < \infty \quad (1.6)$$

### 1.2.2 Properties of Discrete-Time Sinusoidal Signals

- (i) A discrete-time sinusoid is periodic **ONLY** if its frequency is a rational number.
- (ii) Discrete-time sinusoids whose frequencies are separated by an integer multiple of  $2\pi$  are identical. This leads us to the idea of a Frequency Alias.
- (iii) The highest rate of oscillation in a discrete-time sinusoid is attained when  $\omega = \pm\pi$  or, equivalently,  $f = \pm\frac{1}{2}$ .

### 1.2.3 Frequency Aliases

The concept of a Frequency Alias is drawn from the idea that discrete-time sinusoids whose frequencies are separated by an integer multiple of  $2\pi$  are identical and that frequencies  $|f| > \frac{1}{2}$  are identical. (Properties (ii) and (iii))

**Defn 3** (Frequency Alias). A *frequency alias* is a sinusoid having a frequency  $|\omega| > \pi$  or  $|f| > \frac{1}{2}$ . This is because this sinusoid is *indistinguishable* (*identical*) to one with frequency  $|\omega| < \pi$  or  $|f| < \frac{1}{2}$ .

A *frequency alias* is a sequence resulting from the following assertion based on the sinusoid  $\cos(\omega_0 n + \theta)$ .

It follows that

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta)$$

As a result, all sinusoidal sequences

$$x_k(n) = A \cos(\omega_k n + \theta), k = 0, 1, 2, \dots$$

where

$$\omega_k = \omega_0 + 2k\pi, -\pi \leq \omega_0 \leq \pi$$

are *indistinguishable* (i.e., *identical*).

Because of this, we regard frequencies in the range of  $-\pi \leq \omega \leq \pi$  or  $-\frac{1}{2} \leq f \leq \frac{1}{2}$  as unique, and all frequencies that fall outside of these ranges as aliases.

*Remark 3.1.* It should be noted that there is a difference between discrete-time sinusoids and continuous-time sinusoids here. Continuous-time sinusoids have distinct signals for  $\Omega$  or  $F$  in the entire range  $-\infty < \Omega < \infty$  or  $-\infty < F < \infty$ .

## 1.3 Sampling Rates and Sampling Frequency

Most signals of interest are analog. To process these signals, they must be collected and converted to a digital form, that is, to convert them to a sequence of numbers having finite precision. This is called *analog-to-digital* (*A/D*) *conversion*. Conceptually, we view this conversion as a 3-step process.

1. Sampling
2. Quantization
3. Coding

### 1.3.1 Nyquist Rate

### 1.3.2 Nyquist Frequency

## 1.4 Digital Signals

**Defn 4** (Digital Signals). *Digital signals* are a subset of Discrete-Time Signals. In this case, not only are the values being measured occurring at fixed points in time, the values themselves can only take certain, fixed values.

### 1.4.1 Quantization

**Defn 5** (Quantization). This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-value (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample  $x(n)$  and the quantized output  $x_q(n)$  is called the Quantization Error.

#### 1.4.1.1 Quantization Levels

#### 1.4.1.2 Quantization Error

**Defn 6** (Quantization Error). The *quantization error* of something.

#### 1.4.1.3 Bit Requirements

#### 1.4.1.4 Bit Rate

## 2 Discrete-Time Systems

As discussed in Section 1.2,  $x(n)$  is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is *not defined* at instants between the samples. Also, if  $n$  is not an integer,  $x(n)$  is not defined.

Besides graphical representation of a discrete-time system, there are 3 ways to represent a discrete-time signal.

1. Functional Representation
2. Tabular Representation
3. Sequence Representation

### 2.1 Representing Discrete-Time Systems

#### 2.1.1 Functional Representation

This representation of a discrete-time system is done as a mathematical function.

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \quad (2.1)$$

#### 2.1.2 Tabular Representation

This representation of a discrete-time system is done as a table of corresponding values.

|        |  |     |    |    |   |   |   |   |   |   |     |
|--------|--|-----|----|----|---|---|---|---|---|---|-----|
| $n$    |  | ... | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | ... |
| $x(n)$ |  | ... | 0  | 0  | 0 | 1 | 4 | 1 | 0 | 0 | ... |

#### 2.1.3 Sequence Representation

There are 2 methods of representation for this. The first includes all values for  $-\infty < n < \infty$ . In all cases,  $n = 0$  is marked in the sequence, somehow. I will do this with an underline.

$$x(n) = \{\dots, 0, \underline{0}, 1, 4, 1, 0, 0, \dots\} \quad (2.2)$$

The second only works if all  $x(n)$  values for  $n < 0$  are 0.

$$x(n) = \{\underline{0}, 1, 4, 1, 0, 0, \dots\} \quad (2.3)$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underline{-2}, 5, 0, 4, -1\} \quad (2.4)$$

This is identified as a seven-point sequence.

A finite-duration sequence where  $x(n) = 0$  for all  $n < 0$  is represented as

$$x(n) = \{0, 1, 4, 1\} \quad (2.5)$$

This is identified as a four-point sequence.

## 2.2 Elementary Discrete-Time Signals

The following signals are basic signals that appear often and play an important role in signal processing.

### 2.2.1 Unit Impulse Signal

**Defn 7** (Unit Impulse Signal). The *unit impulse signal* or *unit sample sequence* is denoted as  $\delta(n)$  and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \quad (2.6)$$

This function is a signal that is zero everywhere, except at  $n = 0$ , where its value is 1.

*Remark 7.1.* This signal is different that the analog signal  $\delta(t)$ , which is also called a unit impulse, and is defined to be 0 everywhere except  $t = 0$ . The discrete unit impulse sequence is much less mathematically complicated.

### 2.2.2 Unit Step Signal

**Defn 8** (Unit Step Signal). The *unit step signal* is denoted as  $u(n)$  or as  $\mathcal{U}(n)$  and is defined as

$$\mathcal{U}(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \quad (2.7)$$

### 2.2.3 Unit Ramp Signal

**Defn 9** (Unit Ramp Signal). The *unit ramp signal* is denoted as  $u_r(n)$  and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \quad (2.8)$$

### 2.2.4 Exponential Signal

**Defn 10** (Exponential Signal). The *exponential signal* is a sequence of the form

$$x(n) = a^n \text{ for all } n \quad (2.9)$$

If  $a$  is real, then  $x(n)$  is a real signal. When  $a$  is complex valued ( $a \equiv b \pm cj$ ), it can be expressed as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n (\cos \theta n + j \sin \theta n) \end{aligned} \quad (2.10)$$

This can be expressed by graphing the real and imaginary parts

$$\begin{aligned} x_R(n) &\equiv r^n \cos \theta n \\ x_I(n) &\equiv r^n j \sin \theta n \end{aligned} \quad (2.11)$$

or by graphing the amplitude function and phase function.

$$\begin{aligned} |x(n)| &= A(n) \equiv r^n \\ \angle x(n) &= \phi(n) \equiv \theta n \end{aligned} \quad (2.12)$$

## 2.3 Classification of Discrete-Time Signals

In order to apply some mathematical methods to discrete-time signals, we must characterize these signals.

### 2.3.1 Energy Signal

**Defn 11** (Energy Signal). The energy  $E$  of a signal  $x(n)$  is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad (2.13)$$

The energy of a signal can be finite or infinite. If  $E$  is finite ( $0 < E < \infty$ ), then  $x(n)$  is called an *energy signal*.

### 2.3.2 Power Signal

**Defn 12** (Power Signal). The average power of a discrete time signal  $x(n)$  is defined as

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2 \quad (2.14)$$

This means that there are 2 potential outcomes:

1. If  $E$  is finite,  $P = 0$
2. If  $E$  is infinite,  $P$  may be either finite or infinite

If  $P$  is finite and nonzero, the signal is called a *power signal*.

### 2.3.3 Periodic and Aperiodic Signals

A signal  $x(n)$  is periodic with period  $N$  ( $N > 0$ ) if and only if

$$x(n+N) = x(n) \text{ for all } n \quad (2.15)$$

The smallest value of  $N$  for which (2.15) holds is called the fundamental period. If there is no value of  $N$  that satisfies (2.15), the signal is called *nonperiodic* or *aperiodic*.

### 2.3.4 Symmetric and Antisymmetric Signals

A real-valued signal  $x(n)$  is called *symmetric* or *even* if

$$x(n) = x(-n) \quad (2.16)$$

On the other hand, a signal  $x(n)$  is called *antisymmetric* or *odd* if

$$x(n) = -x(-n) \quad (2.17)$$

## 2.4 Discrete-Time Signal Manipulations

### 2.4.1 Transformation of the Independent Variable (Time)

It is important to note that Shifting in Time and Folding are not commutative. For example,

$$\text{TD}_k\{\text{FD}[x(n)]\} = \text{TD}_k[x(-n)] = x(-n+k) \quad (2.18)$$

whereas

$$\text{FD}\{\text{TD}_k[x(n)]\} = \text{FD}[x(n-k)] = x(-n-k) \quad (2.19)$$

**2.4.1.1 Shifting in Time** A signal  $x(n)$  may be shifted in time by replacing the independent variable  $n$  by  $n-k$ , where  $k$  is an integer. If  $k$  is a positive integer, the time shift results in a delay of the signal by  $k$  units of time (moves left). If  $k$  is a negative integer, the time shift results in an advance of the signal by  $|k|$  units of time (moves right).

This could be denoted by

$$\text{TD}_k[x(n)] = x(n-k) \quad (2.20)$$

You cannot advance a signal that is being generated in real-time. Because that would involve signal samples that haven't been generated yet. So, you can only advance a signal that is stored on something. However, you can always introduce a delay to a signal.



**2.4.1.2 Folding** Another useful modification of the time base is to replace  $n$  with  $-n$ . The result is a *folding* or *reflection* of the original signal around  $n = 0$ .

This could be denoted by

$$\text{FD}[x(n)] = x(-n) \quad (2.21)$$

## 2.4.2 Addition, Multiplication, and Scaling

Amplitude modifications include Addition, Multiplication, and Amplitude Scaling.

**2.4.2.1 Addition** The *sum* of 2 signals  $x_1(n)$  and  $x_2(n)$  is a signal  $y(n)$  whose value at any instant is equal to the sum of the values of these two signals at that instant.

$$y(n) = x_1(n) + x_2(n), \quad -\infty < n < \infty \quad (2.22)$$

**2.4.2.2 Multiplication** The *product* of two signals  $x_1(n)$  and  $x_2(n)$  is a signal  $y(n)$  whose value at any instant is equal to the product of the values of these two signals at that instant.

$$y(n) = x_1(n)x_2(n), \quad -\infty < n < \infty \quad (2.23)$$

**2.4.2.3 Amplitude Scaling** *Amplitude scaling* of a signal by a constant  $A$  is accomplished by multiplying every sample by  $A$ . Consequently, we obtain

$$y(n) = Ax(n), \quad -\infty < n < \infty \quad (2.24)$$

## 3 Convolutions

**Defn 13** (Convolution). The *convolution* operator.

$$y(t) = \sum_{k=-\infty}^{\infty} x(k) * h(n-k) \quad (3.1)$$

## 4 The $\mathcal{Z}$ -Transform

The  $\mathcal{Z}$ -Transform plays the same role in the analysis of Discrete-Time Signals and LTI systems as the Laplace Transform does in the analysis of Continuous-Time Signals and LTI systems.

### 4.1 The $\mathcal{Z}$ -Transform

**Defn 14** ( $\mathcal{Z}$ -Transform). The *z-transform* is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

*Remark 14.1.* For convenience, the  $z$ -transform of a signal  $x(n)$  is denoted by

$$X(z) \equiv \mathcal{Z}\{x(n)\} \quad (4.2)$$

and the relationship between  $x(n)$  and  $X(z)$  is indicated by

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z) \quad (4.3)$$

#### 4.1.1 Region of Convergence

**Defn 15** (ROC). The *ROC* or *region of convergence* is the region for which the infinite power series in the  $z$ -transform has a convergent solution.

*Remark 15.1. Any time we cite a  $z$ -transform, we should also indicate its ROC*

**Example 4.1: Simple  $\mathcal{Z}$ -Transform.**

Determine the  $z$ -transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n \mathcal{U}(n)$$

The  $z$ -transform is the infinite power series

$$\begin{aligned} X(z) &= 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^{-2} + \cdots + \left(\frac{1}{2}\right)^n z^{-n} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n \end{aligned}$$

Because this is an infinite geometric series, we can solve with with our equivalency:

$$1 + A + A^2 + \cdots + A^n + \cdots = \frac{1}{1-A} \text{ if } |A| < 1$$

Thus,  $X(z)$  converges to

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC} : |z| > \frac{1}{2}$$

| Signal                    | ROC   |
|---------------------------|---|
| Finite-Duration Signals   |   |
| Causal                    | Entire $z$ -plane except $z = 0$                  |
| Anticausal                | Entire $z$ -plane except $z = \infty$             |
| Two-Sided                 | Entire $z$ -plane except $z = 0$ and $z = \infty$ |
| Infinite-Duration Signals |   |
| Causal                    | $ z  > r_2$                                       |
| Anticausal                | $ z  < r_1$                                       |
| Two-Sided                 | $r_2 <  z  < r_1$                                 |

Table 4.1: Characteristic Families of Signals with Their Corresponding ROCs

#### 4.1.2 The One-Sided $\mathcal{Z}$ -Transform

**Defn 16** (One-Sided  $\mathcal{Z}$ -Transform). The *one-sided  $z$ -transform* is the same as the  $\mathcal{Z}$ -Transform, but is only defined at  $n$  values greater than or equal to 0.

$$X(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n} \quad (4.4)$$

The One-Sided  $\mathcal{Z}$ -Transform is generally used when there are initial conditions on a causal signal. This captures the normal causal portion of the signal, while also showing the effect of the initial condition.

## 4.2 The Inverse $\mathcal{Z}$ -Transform

This is the formal definition of The Inverse  $\mathcal{Z}$ -Transform.

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz \quad (4.5)$$

where the integrals is a contour integral over a closed path  $C$  that encloses the origin and lies within the region of convergence of  $X(z)$ .

There are 3 methods that are often used for the evaluation of the inverse  $z$ -transform in practice:

1. Direct evaluation of (4.5).
2. Expansion into a series of terms, in the variable  $sz$  and  $z^{-1}$ .
3. Partial-fraction expansion and table lookup.

#### 4.2.1 The Inverse $\mathcal{Z}$ -Transform by Contour Integration

**Defn 17** (Cauchy's Integral Theorem). Let  $f(z)$  be a function of the complex variable  $z$  and  $C$  be a closed path in the  $z$ -plane. If the derivative  $\frac{df(z)}{dz}$  exists on and inside the contour  $C$  and if  $f(z)$  has no poles at  $z = z_0$ , then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \quad (4.6)$$

More generally, if the  $(k+1)$ -order derivative of  $f(z)$  exists and  $f(z)$  has no poles at  $z = z_0$ , then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k} dz = \begin{cases} \left. \frac{1}{(k-1)!} \frac{d^{k-1}f(z)}{dz^{k-1}} \right|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \quad (4.7)$$

#### 4.2.2 The Inverse $\mathcal{Z}$ -Transform by Power Series Expansion

#### 4.2.3 The Inverse $\mathcal{Z}$ -Transform by Partial-Fraction Expansion

### 4.3 Properties of the $\mathcal{Z}$ -Transform

| Property   | Time Domain                                | $z$ -Domain   | ROC  |
|--|--|---|--|
| Notation   | $x(n)$<br>$x_1(n)$<br>$x_2(n)$             | $X(z)$<br>$X_1(z)$<br>$X_2(z)$                                  | ROC : $r_2 <  z  < r_1$<br>ROC <sub>1</sub><br>ROC <sub>2</sub>        |
| $\mathcal{Z}$ -Transform Linearity                 | $a_1x_1(n) + a_2x_2(n)$                    | $a_1X_1(z) + a_2X_2(z)$   | At least the intersection of ROC <sub>1</sub> and ROC <sub>2</sub>     |
| $\mathcal{Z}$ -Transform Time Shifting             | $x(n - k)$                                 | $z^{-k}X(z)$  | That of $X(z)$ , except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$ |
| $\mathcal{Z}$ -Domain Scaling                      | $a^n x(n)$                                 | $X(a^{-1}z)$  | $ a r_2 <  z  <  a r_1$  |
| $\mathcal{Z}$ -Transform Time Reversal             | $x(-n)$                                    | $X(z^{-1})$   | $\frac{1}{r_1} <  z  < \frac{1}{r_2}$                                  |
| Conjugation  | $x^*(n)$                                   | $X^*(z^*)$  | ROC  |
| Real Part  | $\text{Re}\{x(n)\}$                        | $\frac{1}{2}[X(z) + X^*(z^*)]$                                  | Includes ROC   |
| p Imaginary Part                                   | $\text{Im}\{x(n)\}$                        | $\frac{1}{2j}[X(z) - X^*(z^*)]$                                 | Includes ROC   |
| $\mathcal{Z}$ -Domain Differentiation              | $nx(n)$                                    | $-z \frac{dX(z)}{dz}$   | $r_2 <  z  r_1$  |
| $\mathcal{Z}$ -Domain Convolutions                 | $x_1 * x_2$                                | $X_1(z)X_2(z)$  | At least, the intersection of ROC <sub>1</sub> and ROC <sub>2</sub>    |
| $\mathcal{Z}$ -Transform 2 Sequence Correlation    | $r_{x_1x_2}(l) = x_1(l) * x_2(-l)$         | $R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$                             | At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$        |
| Initial Value Theorem for $\mathcal{Z}$ -Transform | If $x(n)$ causal                           | $x(0) = \lim_{z \rightarrow \infty} X(z)$                       |  |
| $\mathcal{Z}$ -Transform 2 Sequence Multiplication | $x_1(n)x_2(n)$                             | $\frac{1}{2\pi j} \oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$       | At least, $r_{1l}r_{2l} <  a  < r_{1u}r_{2u}$                          |
| Parsevals Relation for $\mathcal{Z}$ -Transform    | $\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$ | $= \frac{1}{2\pi j} \oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$ |  |

Table 4.2: Properties of the  $\mathcal{Z}$ -Transform

#### 4.3.1 $\mathcal{Z}$ -Transform Linearity

If

$$\begin{aligned} x_1(n) &\xrightarrow{z} X_1(z) \\ x_2(n) &\xrightarrow{z} X_2(z) \end{aligned}$$

then

$$x(n) = a_1x_1(n) + a_2x_2(n) \xrightarrow{z} X(z) = a_1X_1(z) + a_2X_2(z) \quad (4.8)$$

for any constants  $a_1$  and  $a_2$ .

The linearity property can be generalized to an arbitrary number of signals.

**Example 4.2: Simple Z-Transform Linearity Problem. Example 3.2.1**

Determine the  $z$ -transform and the ROC of the signal

$$x(n) = [3(2^n) - 4(3^n)]\mathcal{U}(n)$$

Solution on Page 158.

**Example 4.3: Z-Transform Linearity on Trig Functions. Example 3.2.2**

Determine the  $z$ -transform of the signals

(a)  $x(n) = (\cos \omega_0 n)\mathcal{U}(n)$

(b)  $x(n) = (\sin \omega_0 n)\mathcal{U}(n)$

Solution on Pages 158-159.

**4.3.2 Z-Transform Time Shifting**

If

$$x(n) \xleftrightarrow{z} X(z)$$

then

$$x(n-k) \xleftrightarrow{z} z^{-k}X(z) \quad (4.9)$$

The ROC of  $z^{-k}X(z)$  is the same as that of  $X(z)$  except for  $z=0$  if  $k>0$  and  $z=\infty$  if  $k<0$ .

**Example 4.4: Z-Transform Time Shifting. Example 3.2.3**

By applying the time-shifting property, determine the  $z$ -transform of the signals

$$x_1(n) = \{1, 2, \underline{5}, 7, 0, 1\}$$

$$x_2(n) = \{\underline{0}, 0, 1, 2, 5, 7, 0, 1\}$$

from the  $z$ -transform of

$$x_0(n) = \{1, 2, 5, 7, 0, 1\}$$

$$X_0(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}, \text{ROC : entire } z\text{-plane except } z=0$$

Solution on Page 160.

**4.3.3 Z-Domain Scaling**

If

$$x(n) \xleftrightarrow{z} X(z), \text{ROC : } r_1 < |z| < r_2$$

then

$$a^n x(n) \xleftrightarrow{z} X(a^{-1}z), \text{ROC : } |a|r_1 < |z| < |a|r_2 \quad (4.10)$$

**4.3.4 Z-Transform Time Reversal**

If

$$x(n) \xleftrightarrow{z} X(z), \text{ROC : } r_1 < |z| < r_2$$

then

$$x(-n) \xleftrightarrow{z} X(z^{-1}), \text{ROC : } \frac{1}{r_2} < |z| < \frac{1}{r_1} \quad (4.11)$$

**Example 4.5: Z-Transform Time Reversal. Example 3.2.6**

Determine the  $z$ -transform of the signal

$$x(n) = \mathcal{U}(-n)$$

The transform for  $\mathcal{U}(n)$  is given in Table 4.3.

$$\mathcal{U}(n) \xleftrightarrow{z} \frac{1}{1 - x^{-1}}, \text{ ROC : } |z| > 1$$

By using (4.11), we obtain

$$\mathcal{U}(-n) \xleftrightarrow{z} \frac{1}{1 - z}, \text{ ROC : } |z| < 1$$

**4.3.5 Z-Domain Differentiation**

If

$$x(n) \xleftrightarrow{z} X(z)$$

then

$$nx(n) \xleftrightarrow{z} -z \frac{dX(z)}{dz} \quad (4.12)$$

**Example 4.6: Z-Domain Differentiation. Example 3.2.7**

Determine the  $z$ -transform of the signal

$$x(n) = na^n \mathcal{U}(n)$$

The signal  $x(n)$  can be expressed as  $nx_1(n)$ , where  $x_1(n) = a^n \mathcal{U}(n)$ . By passing this through the  $z$ -transform, we have

$$x_1(n) = a^n \mathcal{U}(n) \xleftrightarrow{z} X_1(z) = \frac{1}{1 - az^{-1}}, \text{ ROC : } |z| > |a|$$

Then by using (4.12), we obtain

$$na^n \mathcal{U}(n) \xleftrightarrow{z} X(z) = -z \frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1 - az^{-1})^2}$$

**4.3.6 Z-Domain Convolutions**

If

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X(z) = X_1(z)X_2(z) \quad (4.13)$$

The ROC of  $X(z)$  is, at least, the intersection of that for  $X_1(z)$  and  $X_2(z)$ .

**Example 4.7: Z-Domain Convolutions. Example 3.2.9**

Compute the convolution  $x(n)$  of the signals

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \leq n \leq 6 \\ 0, & \text{elsewhere} \end{cases}$$

When

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

According to (4.13) we carry out the multiplication of  $X_1(z)$  and  $X_2(z)$ . Thus

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underline{1}, -1, 0, 0, 0, 0, -1, 1\}$$

**4.3.7 Z-Transform 2 Sequence Correlation**

If

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \xleftrightarrow{z} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) \quad (4.14)$$

**Example 4.8: Z-Transform 2 Sequence Correlation. Example 3.2.10**

Determine the autocorrelation of the signal

$$x(n) = a^n \mathcal{U}(n), \quad -1 < a < 1$$

Solution on Page 166.

**4.3.8 Z-Transform 2 Sequence Multiplication**

If

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

then

$$x(n) = x_1(n)x_2(n) \xleftrightarrow{z} X_z = \frac{1}{2\pi j} \oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv \quad (4.15)$$

where  $C$  is a closed contour that encloses the origin and lies within the region of convergence common to both  $X_1(v)$  and  $X_2(\frac{1}{v})$ .

**4.3.9 Parsevals Relation for Z-Transform**

If  $x_1(n)$  and  $x_2(n)$  are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv \quad (4.16)$$

#### 4.3.10 Initial Value Theorem for $\mathcal{Z}$ -Transform

If  $x(n)$  is *causal* [i.e.,  $x(n) = 0$  for  $n < 0$ ], then

$$x(0) = \lim_{z \rightarrow \infty} X(z) \quad (4.17)$$

### 4.4 Properties of the One-Sided $\mathcal{Z}$ -Transform

(i)

### 4.5 Rational $\mathcal{Z}$ -Transforms

An important family of  $z$ -transforms are those for which  $X(z)$  is a rational function, a ratio of two polynomials in  $z^{-1}$  (or  $z$ ).

#### 4.5.1 Poles and Zeros of a $\mathcal{Z}$ -Transform

**Defn 18** (Zeros). The *zeros* of a  $z$ -transform  $X(z)$  are the values of  $z$  for which  $X(z) = 0$ .

This is analogous to “setting the numerator equal to zero.”

**Defn 19** (Poles). The *poles* of a  $z$  transform  $X(z)$  are the values of  $z$  for which  $X(z) = \infty$ .

This is analogous to “setting the denominator equal to zero.”

If  $X(z)$  is a rational function, then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

If  $a_0 \neq 0$  and  $b_0 \neq 0$ , we can avoid negative powers of  $z$  by factoring out the terms  $z^{-M}$  and  $z^{-N}$ .

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M} b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^{-N} a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

Since  $B(z)$  and  $A(z)$  are polynomials in  $z$ , they can be expressed in factored form as

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M} (z - z_1)(z - z_2) \cdots (z - z_M)}{z^{-N} (z - p_1)(z - p_2) \cdots (z - p_N)} \quad (4.18)$$

Thus,  $X(z)$  has  $M$  finite Zeros at  $z = z_1, z_2, \dots, z_M$  (the roots of the numerator polynomial),  $N$  finite Poles at  $z = p_1, p_2, \dots, p_N$  (the roots of the denominator polynomial), and  $|N - M|$  zeros (if  $N > M$ ) or poles (if  $N < M$ ) at the origin  $z = 0$ . Poles and zeroes may occur at  $z = \infty$ . A zero exists at  $z = \infty$  if  $X(\infty) = 0$  and a pole exists at  $z = \infty$  if  $X(\infty) = \infty$ .

**Defn 20** (Pole-Zero Plot). Poles and Zeros of a  $z$ -transform can be shown graphically by a *pole-zero plot* in the complex plane, which shows the location of poles by crosses ( $\times$ ) and the location of zeros by circles. Multiplicity is shown by a number close to the corresponding cross or circle. The ROC of a  $z$ -transform should not contain any poles, by definition.

#### 4.5.2 Decomposition of Rational $\mathcal{Z}$ -Transforms

### 4.6 Analysis of LTI Systems in the $\mathcal{Z}$ -Domain

### 4.7 Common $\mathcal{Z}$ -Transforms

## 5 The Fourier Transform and Fourier Series

When a signal is decomposed with either the Fourier Transform or the Fourier Series, you receive either sinusoids or complex-valued exponentials. This decomposition is said to be represented in the *frequency domain*.

**Defn 21** (Fourier Transform). When decomposing the class of signals with finite energy, you perform a *Fourier transform*. This is generally shown as the function

$$c_k = F\{x(t)\}$$

There are 2 possible equations for the Fourier Transform, depending of the function is continuous-time or discrete-time.

1. Continuous-Time: Equation (5.1)
2. Discrete-Time: Equation (5.2)

| Signal, $x(n)$                         | $z$ -Transform, $X(z)$  | ROC         |
|--|---|-------------|
| $\delta(n)$                            | 1   | All $z$     |
| $\mathcal{U}(n)$                       | $\frac{1}{1-z^{-1}}$  | $ z  > 1$   |
| $a^n \mathcal{U}(n)$                   | $\frac{1}{1-az^{-1}}$   | $ z  >  a $ |
| $na^n \mathcal{U}(n)$                  | $\frac{az^{-1}}{(1-az^{-1})^2}$   | $ z  >  a $ |
| $-a^n \mathcal{U}(-n-1)$               | $\frac{1}{1-az^{-1}}$   | $ z  <  a $ |
| $-na^n \mathcal{U}(-n-1)$              | $\frac{az^{-z}}{(1-az^{-1})^2}$   | $ z  <  a $ |
| $(\cos \omega_0 n) \mathcal{U}(n)$     | $\frac{1-z^{-1} \cos \omega_0}{1-2z^{-1} \cos \omega_0 + z^{-2}}$       | $ z  > 1$   |
| $(\sin \omega_0 n) \mathcal{U}(n)$     | $\frac{z^{-1} \sin \omega_0}{1-2z^{-1} \cos \omega_0 + z^{-2}}$         | $ z  > 1$   |
| $(a^n \cos \omega_0 n) \mathcal{U}(n)$ | $\frac{1-az^{-1} \cos \omega_0}{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}}$ | $ z  >  a $ |
| $(a^n \sin \omega_0 n) \mathcal{U}(n)$ | $\frac{az^{-1} \sin \omega_0}{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}}$   | $ z  >  a $ |

Table 4.3: Common  $\mathcal{Z}$ -Transforms

The Fourier Transform is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt \quad (5.1)$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad (5.2)$$

*Remark 21.1.* Sometimes  $X(F)$  and  $X(f)$  will be denoted with  $\Omega$  and  $\omega$  ( $X(\Omega)$  and  $X(\omega)$ ) respectively. In both cases,  $\Omega$  and  $\omega$  mean something similar.

$$\begin{aligned} \Omega &= 2\pi F \\ \omega &= 2\pi f \end{aligned}$$

This means that we can rewrite Equations (5.1) to (5.2) as

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \quad (5.3)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad (5.4)$$

*Remark 21.2.* Generally, when people say the Fourier Transform, they are referring to the transform on Continuous-Time Signals. There is a distinction that occurs with the *DTFT* or *Discrete-Time Fourier Transform*.

This document explains them side-by-side, but will primarily focus on the Discrete-Time Fourier Transform.

**Defn 22** (Fourier Series). When decomposing the class of periodic signals, you are returned a *Fourier series*. This is generally shown as the function

$$X(F) = F\{x(t)\}$$

**Defn 23** (Discrete-Time Fourier Transform). The *Discrete-Time Fourier Transform*, *DTFT* is a special case of the Fourier Transform that occurs when the input function  $x(n)$  is a case of Discrete-Time Signals.

The transformation (analysis) equations are:

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad (5.5a)$$

$$\omega = 2\pi f$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad (5.5b)$$



The reverse (synthesis) equations are:

$$x(n) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f n} df \quad (5.6a)$$

$$x(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega n} d\omega \quad (5.6b)$$

These equations are expanded more upon in Section 5.2, The Inverse Fourier Transform.

## 5.1 Fourier Transform Relations

Each of these relations is just a side-note, the only relation of real importance is Equation (5.7). The Fourier Transform is just a special case in each of these scenarios. The Fourier Transform is evaluated around the unit circle on the real-imaginary plane.

### 5.1.1 Laplace Transform Fourier Transform Relation

There is a correlation between the Laplace Transform and the Fourier Transform. The Fourier Transform is a more specific case of the Laplace Transform, when

$$e^{-st} = e^{-j2\pi ft}$$

### 5.1.2 Z-Transform Discrete-Time Fourier Transform Relation

There is a relationship between the Z-Transform and the Discrete-Time Fourier Transform.

$$\begin{aligned} z &= e^{2\pi f} \\ z &= e^{2\pi n} \end{aligned} \quad (5.7)$$

## 5.2 The Inverse Fourier Transform

**Defn 24** (Inverse Fourier Transform). Since the Fourier Transform is a “lossless” function (the definition of a transformation), the *inverse fourier transform* is just the opposite setup of Equations (5.1) to (5.2).

In both cases, a Continuous-Time signal and a Discrete-Time signal, you use the below synthesis equations (Equations (5.8) to (5.9)).

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF \\ x(n) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi fn} df \end{aligned} \quad (5.8)$$

If you're calculating with  $\Omega$  or  $\omega$  instead of  $F$  or  $f$ , then use these synthesis equations.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \\ x(n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega n} d\omega \end{aligned} \quad (5.9)$$

## 5.3 Fourier Transform Properties for Discrete-Time Signals

One thing to keep in mind with all of these properties is that  $\omega = 2\pi f$ .

### 5.3.1 Linearity

If

$$\begin{aligned} x_1(n) &\xleftrightarrow{F} X_1(f) \\ x_2(n) &\xleftrightarrow{F} X_2(f) \end{aligned}$$

then

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{F} a_1 X_1(f) + a_2 X_2(f) \quad (5.10)$$

| Property                            | Time Domain<br>$x(n)$                      | Frequency Domain<br>$X(f)$ or $X(\omega)$  |
|-------------------------------------|--|--|
|                                     | $x(n)$                                     | $X(\omega)$  |
| Notation                            | $x_1(n)$                                   | $X_1(\omega)$  |
|                                     | $x_2(n)$                                   | $X_2(\omega)$  |
| Linearity                           | $a_1x_1(n) + a_2x_2(n)$                    | $a_1X_1(\omega) + a_2X_2(\omega)$  |
| Time Shifting                       | $x(n-k)$                                   | $e^{-j\omega k}X(\omega)$  |
| Time Reversal                       | $x(-n)$                                    | $X(-\omega)$   |
| Convolution                         | $x_1(n) * x_2(n)$                          | $X_1(\omega)X_2(\omega)$   |
| Correlation                         | $r_{x_1, x_2}(l) = x_1(l) * x_2(-l)$       | $S_{x_1, x_2}(\omega) = X_1(\omega)X_2(\omega)$<br>$= X_1(\omega)X_2^*(\omega)$<br>[if $x_2(n)$ is real] |
| Wiener-Khintchine Theorem           | $r_{xx}(l)$                                | $S_{xx}(\omega)$   |
| Frequency Shifting                  | $e^{j\omega_0 n}x(n)$                      | $X(\omega - \omega_0)$   |
| Modulation                          | $x(n) \cos(\omega_0 n)$                    | $\frac{1}{2}X(\omega + \omega_0) + \frac{1}{2}X(\omega - \omega_0)$                                      |
| Multiplication in Time Domain       | $x_1(n)x_2(n)$                             | $\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)x_2(\omega - \lambda)d\lambda$                             |
| Differentiation in Frequency Domain | $nx(n)$                                    | $j \frac{dX(\omega)}{d\omega}$   |
| Conjugation                         | $x^*(n)$                                   | $X^*(-\omega)$   |
| Parseval's Theorem                  | $\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$ | $= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$                                     |

Table 5.1: Properties of the Fourier Transform for Discrete-Time Signals

### 5.3.2 Time Shifting

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$x(n-k) \xleftrightarrow{F} e^{-j\omega k}X(f) \quad (5.11)$$

### 5.3.3 Time Reversal

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$x(-n) \xleftrightarrow{F} X(-f) \quad (5.12)$$

### 5.3.4 Convolution

If

$$x_1(n) \xleftrightarrow{F} X_1(f)$$

$$x_2(n) \xleftrightarrow{F} X_2(f)$$

then

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{F} X(f) = X_1(f)X_2(f) \quad (5.13)$$

*Remark.* There is one thing to note here. Both  $x_1(n)$  and  $x_2(n)$  must be reasonably well-behaved and have be BIBO-stable for this relation to hold.

### 5.3.5 Correlation

If

$$x_1(n) \xleftrightarrow{F} X_1(f)$$

$$x_2(n) \xleftrightarrow{F} X_2(f)$$

then

$$r_{x_1 x_2}(m) \xleftrightarrow{F} S_{x_1 x_2}(f) = X_1(f)X_2(-f) \quad (5.14)$$

### 5.3.6 Wiener-Khintchine Theorem

Let  $x(n)$  be a real signal. Then

$$r_{xx}(l) \xleftrightarrow{F} S_{xx}(f) \quad (5.15)$$

That is, the energy spectral density of an energy signal is the Fourier Transform of its autocorrelation sequence. This is a special case of Equation (5.14).

### 5.3.7 Frequency Shifting

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$e^{-i2\pi f_0 n} x(n) \xleftrightarrow{F} X(f - f_0) \quad (5.16)$$

### 5.3.8 Modulation

If

$$x(n) \xleftrightarrow{F} X(f)$$

### 5.3.9 Multiplication in Time Domain

This is also called the *Windowing Theorem*.

If

$$x_1(n) \xleftrightarrow{F} X_1(f)$$

$$x_2(n) \xleftrightarrow{F} X_2(f)$$

then ...

### 5.3.10 Differentiation in Frequency Domain

If

$$x(n) \xleftrightarrow{F} X(f)$$

### 5.3.11 Parseval's Theorem

If

$$x_1(n) \xleftrightarrow{F} X_1(f)$$

$$x_2(n) \xleftrightarrow{F} X_2(f)$$

then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \int_{-\pi}^{\pi} X_1(f)X_2^*(f)df \quad (5.17)$$

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega \quad (5.18)$$

Both Equations (5.17) to (5.18) can be expressed in another format.

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \int_{-\pi}^{\pi} |X_1(f)|^2 df \quad (5.19)$$

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X_1(\omega)|^2 d\omega \quad (5.20)$$

## A Trigonometry

### A.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{A.2})$$

### A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j \sin(\alpha) \quad (\text{A.3})$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \quad (\text{A.4})$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \quad (\text{A.5})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{A.6})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{A.7})$$

### A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \quad (\text{A.8})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \quad (\text{A.9})$$

### A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad (\text{A.10})$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (\text{A.11})$$

### A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \quad (\text{A.12})$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \quad (\text{A.13})$$

### A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2} \quad (\text{A.14})$$

$$\cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2} \quad (\text{A.15})$$

### A.7 Product-to-Sum Identities

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad (\text{A.16})$$

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad (\text{A.17})$$

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (\text{A.18})$$

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (\text{A.19})$$

## A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{A.20})$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.21})$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.22})$$

## A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \quad (\text{A.23})$$

## A.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2} e^{j\theta} = r e^{j\theta} \quad (\text{A.24})$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \quad (\text{A.25})$$

## A.11 Polar to Rectangular

$$r e^{j\theta} = r \cos(\theta) + j r \sin(\theta) \quad (\text{A.26})$$

## B Calculus

### B.1 Fundamental Theorems of Calculus

**Defn B.1.1** (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if  $f$  is continuous on the closed interval  $[a, b]$  and  $F$  is the indefinite integral of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{B.1})$$

**Defn B.1.2** (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for  $f$  a continuous function on an open interval  $I$  and  $a$  any point in  $I$ , and states that if  $F$  is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{B.2})$$

**Defn B.1.3** (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname{argmax}_x$$

### B.2 Rules of Calculus

#### B.2.1 Chain Rule

**Defn B.2.1** (Chain Rule). The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot h(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \quad (\text{B.3})$$

## C Laplace Transform

**Defn C.0.1** (Laplace Transform). The *Laplace transformation* operation is denoted as  $\mathcal{L}\{x(t)\}$  and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{C.1}$$