Phys 123: Classical Mechanics - Reference Sheet

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Contents

1 Vectors

Defn 1 (Vector)**.** A *vector* is a way to show both magnitude of displacement and direction of displacement. Vectors are drawn as rays.

Remark 1.1. Vectors and Scalars may seem similar, but are different.

Defn 2 (Scalar)**.** A *scalar* is a way to show ***ONLY*** the magnitude of a displacement, without any direction information.

1.1 Vector Properties

(i) $\vec{A} + \vec{B} = \vec{C}$
(ii) $\vec{0} = \langle 0, 0, 0, \ldots, 0 \rangle$
(iii) $\vec{A} + \vec{0} = \vec{A}$
(iv) $\vec{A} + -\vec{A} = \vec{0}$
(v) $\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$
(vi) $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
(vii) Magnitude of vector: $\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

1.1.1 Getting Components

Getting the components of a vector involves solving the imaginary pythagorean triangle around the vector.

For a 2-dimensional vector, $\vec{V}$, you have the components $\langle V_x, V_y \rangle$. You find their values with this equation:

$$\begin{aligned} V_x &= V \cos\theta \\ V_y &= V \sin\theta \end{aligned} \tag{1.1}$$

1.1.2 3D Unit Vectors

3-dimensional vectors shouldn't be any too crazy by this point. They are just another variable that can be thrown around in the vector. However, the three 3D Unit Vectors are special. You can also use these to describe any lower-dimensional vector as well.

$$\begin{aligned} \hat{\imath} &= \langle 1, 0, 0 \rangle \\ \hat{\jmath} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle \end{aligned} \tag{1.2}$$

1.1.3 Addition

Vectors are additive, and are done from head-to-tail. This means that

$$\vec{A} + \vec{B} = \vec{C} \tag{1.3}$$

This means that in 3-dimensional vectors, they are added like this:

$$\begin{aligned} \vec{A} &= \langle A_x, A_y, A_z \rangle \\ \vec{B} &= \langle B_x, B_y, B_z \rangle \\ \vec{A} + \vec{B} &= \langle A_x + B_x, A_y + B_y, A_z + B_z \rangle \end{aligned} \tag{1.4}$$

1.1.4 Scalar Multiplication

When applying multiplication between a scalar and a vector, you perform Scalar Multiplication.

$$2 \times \vec{V} = 2\langle V_x, V_y \rangle = \langle 2V_x, 2V_y, 2V_z \rangle \tag{1.5}$$

This means that you do ***NOT*** modify the direction of the vector, you only change its magnitude.

1.1.5 Scalar (Dot) Product

The Scalar (Dot) Product is the first of two ways to multiply 2 vectors. The other is the Vector (Cross) Product. There are 2 ways to calculate the Scalar (Dot) Product.

The first involves using the magnitudes of each vector and multiplying those by the cosine of the angle between them.

$$\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\| \cos(\theta) \tag{1.6}$$

The second is done by adding the product of each component of each vector.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.7}$$

Remark. This means that when you apply the Scalar (Dot) Product to 2 vectors, you return a Scalar.

Properties of Scalar (Dot) Product

(i) $(\vec{A})^2 = \vec{A} \cdot \vec{A}$

(ii) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

1.1.6 Vector (Cross) Product

The Vector (Cross) Product is the second of two ways to multiply 2 vectors. The other is the Scalar (Dot) Product. There are 2 ways to calculate the Vector (Cross) Product.

The first involves using the magnitudes of each vector and multiplying those by the sine of the angle between them.

$$\vec{A} \times \vec{B} = \|\vec{A}\|\|\vec{B}\| \sin\theta \tag{1.8}$$

The second is done by taking the determinant of a 2×2 or 3×3 matrix.

$$\begin{aligned}
\vec{A} \times \vec{B} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= (A_y B_z - A_z B_y)\,\hat{\imath} - (A_x B_z - A_z B_x)\,\hat{\jmath} + (A_x B_y - A_y B_x)\,\hat{k} \\
&= \langle A_y B_z - A_z B_y, -(A_x B_z + A_z B_x), A_x B_y - A_y B_x \rangle
\end{aligned} \tag{1.9}$$

Remark. This means that when you apply the Vector (Cross) Product to 2 vectors, you return a Vector.

Properties of Vector (Cross) Product

(i) $\vec{A} \times \vec{A} = \vec{0}$

(ii) $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$

(iii) $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$

(iv) $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C}\left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)$

2 Kinematics

Defn 3 (Kinematics)**.** *Kinematics* is a way to describe macroscopic motion with equations. This includes anything moving, falling, thrown, shot, launched, etc. This forms the fundamental basis for all of classical mechanics.

2.1 1-D Kinematics

Defn 4 (1-D Displacement)**.** *One dimensional displacement* is calculated based on the change in position of the 'thing.'

$$s = x_2 - x_1 \tag{2.1}$$

Remark 4.1. *Displacement is different than path!* Displacement is the change in position of an object. Path is the length of the path takes between its starting and end point.

Defn 5 (1-D Velocity)**.** *One dimensional velocity* is calculated as the displacement per unit time. There is instantaneous velocity and average velocity. Average velocity is calculated with Equation (2.2).

$$v = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.2}$$

Instantaneous velocity is calculated by reducing the time interval Δt to 0. This can be summarized in Equation (2.3).

$$\begin{aligned} v &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \\ &= \frac{dx}{dt} \end{aligned} \tag{2.3}$$

Defn 6 (Acceleration)**.** *One dimesional acceleration* is the change in velocity over time. Again, there is average acceleration and instantaneous acceleration. Average acceleration is calculated with Equation (2.4)

$$a = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \tag{2.4}$$

Instantaneous acceleration is calculated by reducing the time interval Δt to 0. This can be summarized by Equation (2.5).

$$\begin{aligned} a &= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \\ &= \frac{dv}{dt} = \frac{d^2 x}{dt^2} \end{aligned} \tag{2.5}$$

2.2 Multi-Dimensional Kinematics

Because we can represent a two-dimensional and three-dimensional space in sets, and movement through this space as their respectively dimensioned vectors, we can construct multi-dimensional problems with multi-dimensional vectors! This is a massive simplification, because instead of solving for one equation with three variables, we can solve three equations for one variable each!!

For the following definitions, I have assumed that we are in a 3-dimensional space (x, y, z)**.**

Defn 7 (Multi-Dimensional Position)**.** *Position* in multiple dimensions is done by referring to each of the consituent dimensions.

$$\vec{s} = (x(t), y(t), z(t)) \tag{2.6}$$

Defn 8 (Multi-Dimensional Displacement)**.** *Displacement* in multiple dimensions can be broken down into several 1-D Displacements. Since 1-D Displacement is calculated as the differnce between the start and end position, the same is true for th emulti-dimensional case.

$$\begin{aligned} \vec{r} = \Delta \vec{s} &= \vec{s}_2 - \vec{s}_1 \\ &= \langle x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t) \\ &= \langle r_x(t), r_y(t), r_z(t) \end{aligned} \tag{2.7}$$

Defn 9 (Multi-Dimensional Velocity)**.** *Velocity* in multiple dimensions is described in much the same way as 1-D Velocity.

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \left\langle \frac{dr_x(t)}{dt}, \frac{dr_y(t)}{dt}, \frac{dr_z(t)}{dt} \right\rangle \\ &= \left\langle r'_x(t), r'_y(t), r'_z(t) \right\rangle \end{aligned} \tag{2.8}$$

Defn 10 (Multi-Dimensional Acceleration)**.** *Acceleration* in multiple dimensions is described in much the same way as Acceleration.

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt} \\ &= \left\langle \frac{dv_x(t)}{dt}, \frac{dv_y(t)}{dt}, \frac{dv_z(t)}{dt} \right\rangle = \left\langle \frac{d^2 r_x(t)}{dt}, \frac{d^2 r_y(t)}{dt}, \frac{d^2 r_z(t)}{dt} \right\rangle \\ &= \left\langle v'_x(t), v'_y(t), v'_z(t) \right\rangle = \left\langle r''_x(t), r''_y(t), r''_z(t) \right\rangle \end{aligned} \tag{2.9}$$

2.3 Projectile Motion

Defn 11 (Projectile)**.** A *projectile* is any body given an initial velocity that then follows a path determined by gravity and air resistance.

Remark 11.1. However, for most of our calculations, we will neglect air resistance. Air resistance can be a difficult thing to calcultate for, especially in the variable cases that we will have.

There are a few things to keep in mind with projectiles in motion.

1. Origin is where the projectile starts from
2. The x-axis is the *distance* that the projectile travels. This is its displacement.
3. The y-axis is the *height* that the projectile travels.
4. The end point (landing point) is the only thing that may change on the x-axis.
5. The acceleration vector is as follows: $\langle 0, -g \rangle$.
6. *Trajectory* depends on $\vec{v}_0$ and $\vec{a}$ *ONLY*.
7. The two components of the projectile's initial velocity are *independent* $(v_{0,x}, v_{0,y})$.

2.3.1 Projectile Motion Equations

The following equations are used to solve for various questions that could be asked about projectile motion.

Initial Velocity Components

$$v_{0,x} = v_0 \cos(\theta) \quad v_{0,y} = v_0 \sin(\theta) \tag{2.10}$$

Velocity Components

$$v_x = v_{0,x} \cos(\theta) \quad v_y = v_{0,y} \sin(\theta) - gt \tag{2.11}$$

Projectile Position

$$x = v_0 t \cos(\theta) \quad y = v_0 t \sin(\theta) - \frac{1}{2} g t^2 \tag{2.12}$$

Projectile Time

$$t = \frac{x}{v_0 \cos(\theta)} \quad t = \frac{v_0 \sin(\theta)}{g} \tag{2.13}$$

Projectile Range

$$R = \frac{v_0}{g} \cos(\theta) \sin(\theta) \tag{2.14}$$

Projectile Maximum Range

$$R_{\text{Max}} = \frac{v_0^2}{g} \tag{2.15}$$

This means that the θ in Equation (2.14) is 45°.

Projectile Height

$$h = \frac{v_0^2}{2g} \sin^2(\theta) \tag{2.16}$$

Projectile Maximum Height

$$h = \frac{v_0^2}{2g} \tag{2.17}$$

The lack of $\sin^2(\theta)$ from Equation (2.16) means that there is ***NO*** y-component to the velocity, meaning the projectile is at its instant of maximum height.

3 Uniform Circular Motion

Defn 12 (Uniform Circular Motion)**.** *Uniform circular motion* is when an object of moving in a perpetual circular motion. There is no outside source of acceleration changing the state of the system.

Remark 12.1. This does *not* happen in real life. However, it is useful for modelling things under ideal conditions that do happen in real life.

During Uniform Circular Motion, your terminology changes a little bit.

Defn 13 (Angular Position)**.** *Angular position* is determined with radians around a circle. It is denoted with

$$\vec{\theta}$$

Defn 14 (Angular Velocity)**.** *Angular velocity* is orthogonal to the flat 2-D plane that the object is traveling in.

$$\begin{aligned}\vec{\omega} &= \frac{d\theta}{dt} \\ &= \langle 0, 0, \omega \rangle\end{aligned} \tag{3.1}$$

Defn 15 (Angular Acceleration)**.** *Angular acceleration* is the derivative of the angular velocity.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt} \tag{3.2}$$

Remark 15.1. Note that under Uniform Circular Motion, by its very definition, there cannot be any acceleration on the object. Therefore, when an object is in uniform circular motion, $\vec{\alpha} = 0$.

However, when the object is ***NOT*** in Uniform Circular Motion the object is undergoing Linear Acceleration.

Defn 16 (Linear Position)**.** *Linear position* relates the position of an object from the cartesian coordinate plane to the polar. This means that:

$$x = r\cos(\theta) \quad y = r\sin(\theta) \tag{3.3}$$

Defn 17 (Linear Velocity)**.** *Linear velocity* relates the velocity of an object in a line to its Angular Velocity.

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ \frac{d\vec{r}}{dt} &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, 0 \right\rangle = \left\langle \frac{d}{dt} r\cos(\theta), \frac{d}{dt} r\sin(\theta), 0 \right\rangle \\ &= \left\langle -r\sin(\theta)\omega, r\cos(\theta)\omega, 0 \right\rangle \\ &= \vec{\omega} \times \vec{r}\end{aligned} \tag{3.4}$$

Defn 18 (Linear Acceleration)**.** *Linear acceleration* is the derivative of Linear Velocity. It relates the acceleration of an object in a line is relative to its Angular Acceleration

$$\begin{aligned}\frac{d\vec{a}}{dt} &= \frac{d\vec{v}}{dt} \\ &= \langle -r\omega^2\cos(\theta), -r\omega^2\sin(\theta), 0 \rangle = \omega^2 \langle -r\cos(\theta), -r\sin(\theta), 0 \rangle \; () \\ &= -\omega^2\vec{r}\end{aligned} \tag{3.5}$$

A Physical Constants

Constant Name	**Variable Letter**	**Value**
Boltzmann Constant	R	8.314J/mol K
Universal Gravitational	G	$6.67408 \times 10^{-11}\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$
Planck's Constant	h	$6.62607004 \times 10^{-34}\text{mkg/s} = 4.163 \times 10^{-15}\text{eV s}$
Speed of Light	c	$299792458\text{m/s} = 2.998 \times 10^8\ \text{m/s}$
Charge of Electron	e	$1.602 \times 10^{-19}\text{C}$
Mass of Electron	m_{e^-}	$9.11 \times 10^{-31}\text{kg}$
Mass of Neutron	m_{n^0}	$1.67 \times 10^{-31}\text{kg}$
Mass of Earth	m_{Earth}	$5.972 \times 10^{24}\text{kg}$
Diameter of Earth	d_{Earth}	12742km

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm\imath\alpha} = \cos(\alpha) \pm \imath\sin(\alpha) \tag{B.3}$$

$$\sin(x) = \frac{e^{\imath x} + e^{-\imath x}}{2} \tag{B.4}$$

$$\cos(x) = \frac{e^{\imath x} - e^{-\imath x}}{2\imath} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\alpha) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

B.10 Rectangular to Polar

$$a + \imath b = \sqrt{a^2 + b^2}e^{\imath\theta} = re^{\imath\theta} \tag{B.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.25}$$

B.11 Polar to Rectangular

$$re^{\imath\theta} = r\cos(\theta) + \imath r\sin(\theta) \tag{B.26}$$

C Calculus

C.1 Fundamental Theorems of Calculus

Defn C.1.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.1}$$

Defn C.1.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.2}$$

Defn C.1.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

D Complex Numbers

$$Ae^{-ix} = A\left[\cos(x) + i\sin(x)\right] \tag{D.1}$$