1 Introduction

Example 1. *Confirm Differential Solution*
Test example.

1.1 Definitions and Terminology

Defn 1 (Differential Equation)**.** A *differential equation (DE)* is an equation with 1 or more derivatives.

Remark 1.1. The highest differential determines the order of the differential equation. This means that the differential equation below is of order 2.

$$y'' + y = 0$$
$$\frac{d^2y}{dx^2} + y = 0$$

Defn 2. Initial Value Problem A differential equation with one or more initial conditions is called an *initial value problem (IVP)*.

Remark 2.1. To solve an initial value problem, you must have the same number of initial conditions as the order of the differential equation.

Remark 2.2 (Existence of Unique Solution). R is a rectangular region on the xy-plane $a \le x \le b$, $c \le y \le d$ that contains (x_0, y_0) interior. If $f(x, y)$ and $\frac{df}{dy}$ are continuous on R, then an interval exists I_0 such that $(x_0 - h, x_0 + h)$ where $h > 0$, on the interval $[a, b]$, and a unique function $y(x)$, defined on I_0 that is a solution of the initial value problem.

1.2 Confirm If Differential Equation

You can confirm if the solution $y(x)$ found for a differential equation $y(x)'$ is the solution by differentiating the solution and putting that in the solved differential equation and verfiying that the equation holds true. This is shown in Example 2

Example 2.

Given the differential equation, $2y' + y = 0$, is $y = e^{\frac{-x}{2}}$ a solution?

$$y'$$

1.3 Separable Differential Equation

Defn 3 (Separable)**.** A *separable* differential equation allows you to move various elements around to solve the equation. For example,

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P}dP = kdt$$
$$\ln(P) = kt + C$$
$$P = Ce^{kt}$$

Remark 3.1. These are used extensively in modelling phenomena with differential equations. These include: Population Growth, Radioactive Decay, Newton's Law of Cooling/Heating, and Spread of Disease.

1.4 Modeling with Differential Equations

1.4.1 Population Growth

Defn 4 (Population Growth)**.** *Population growth* can be modelled with a separable differential equation. Namely,

$$\frac{dP}{dt} = kP \tag{1}$$

Remark 4.1 (Population Growth Parameters). The parameters for the Population Growth equation are given below.

- $k > 0$
- $P > 0$

1.4.2 Radioactive Decay

Defn 5 (Radioactive Decay)**.** *Radioactive decay* is the process that some particularly heave atoms undergo.

Defn 6 (Half-Life)**.** The *half-life* is the usual reported metric, and is defined as the amount of time required for an element to half its mass through Radioactive Decay.

$$\frac{1}{2}A_0 = A_0 e^{kt} \tag{2}$$

Remark 6.1 (Radioactive Decay Parameters)*.* The parameters for the Radioactive Decay equation are given below.

- $k < 0$
- $A > 0$

1.4.3 Newton's Law of Cooling/Heating

Defn 7 (Newton's Law of Cooling/Heating)**.** *Newton's Law of Cooling/Heating* is the same equation, but some of the parameters change. This equation is defined as:

$$\frac{dT}{dt} = k\left(T - T_m\right) \tag{3}$$

Remark 7.1*.* The parameters for the Newton's Law of Cooling/Heating equation are given below.

- $\frac{dT}{dt}$; The rate of change of temperature in the object per unit time.
- $k < 0$; The cooling constant and is unique to every object.
- T; The starting temperature.
- T_m; The temperature of the surrounding medium.

1.4.4 Spread of Disease

Defn 8 (Spread of Disease)**.** This is used to model the spread of something throughout a society or group of people.

$$\frac{dx}{dt} = kxy \tag{4}$$

Remark 8.1*.* The parameters for the Spread of Disease equation are given below.

- $\frac{dx}{dt}$; Change in the number of infected per unit time.
- $k < 0$; Transmission Constant
- x; Number of Infected
- y; Number of non-infected, y is really a function of x
  - $y = n + 1 - x$

1.4.5 Chemical Reactions

1.4.6 Tank Mixture

1.4.7 Torricelli's Law

1.4.8 LRC Circuits

Defn 9 (LRC Circuits)**.** An *LRC Circuit* is analyzed in terms of the energy moving through the circuit. There is a unique relationship for the energy in each element:

$$E\left(t\right) = \frac{q}{C} \tag{5}$$

$$E\left(t\right) = RI = R\frac{dq}{dt} \tag{6}$$

$$E\left(t\right) = L\frac{dI}{dt} = L\frac{d^2q}{dt^2} \tag{7}$$

Remark 9.1*.* Depending on the circuit given, you might use a combination of these, but you ***must*** have at least one capacitor or inductor, otherwise it is not a differential equation.

Remark 9.2*.* These equations *add* together when the entire circuit is in series, i.e. the elements are put together back-to-back.

1.5 Linear and Non-Linear Differential Equations

Defn 10 (Linear Differential Equation)**.** A *linear differential equation* is one that satisfies one of the following equations below.

$$\begin{aligned} a_1(x)\frac{dy}{dx} + a_0(x) &= g(x) \\ a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x) &= g(x) \end{aligned} \tag{8}$$

Remark 10.1*.* The equations in Equation (8) can be generalized to the nth order as shown below.

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x) = g(x) \tag{9}$$

Defn 11 (Non-Linear)**.** A *non-linear* differential equation is one that does not satisfy the definition of a Linear Differential Equation. It does not obey Equation (9).

A Reference Material

A.1 Trigonometry

A.1.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.1.2 Euler Equivalents of Trigonometric Functions

$$\sin(x) = \frac{e^{\imath x} + e^{-\imath x}}{2} \tag{A.3}$$

$$\cos(x) = \frac{e^{\imath x} - e^{-\imath x}}{2\imath} \tag{A.4}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.5}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.6}$$

A.2 Calculus

A.2.1 Fundamental Theorems of Calculus

Defn 12 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{A.7}$$

Defn 13 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{A.8}$$