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# 1 Introduction

## 1.1 Basic Chemistry Things

**Defn 1** (Chemistry). *Chemistry* is the study of Matter and its changes.

*Remark 1.1.* We tend to use the macroscopic world to visualize the microscopic world.

**Defn 2** (Matter). *Matter* is “stuff” that has both mass and volume.

**Defn 3** (Scientific Method). The *scientific method* is a systematic approach to research that utilizes qualitative or quantitative measurements.

**Defn 4** (Hypothesis). A *hypothesis* is a tentative explanation that will be tested using the Scientific Method.

**Defn 5** (Law). A *law* is a statement of a relationship between phenomena that is always the same, under the same conditions.

*Remark 5.1.* These tend to be drawn from large amounts of data.

<b>Example 1.1: Law 1.</b>
Chlorine (Cl) is a highly reactive gas.

<b>Example 1.2: Law 2.</b>
Matter is neither created nor destroyed.

**Defn 6** (Theory). A *theory* is a unifying principle that explains a body of facts based on facts and laws. These are constantly tested for validity.

*Remark 6.1.* A Hypothesis can turn into a Theory with enough experimentation and acceptance.

<b>Example 1.3: Theory 1.</b>
Reactivity of elements depends on the element's electron ( $e^-$ ) configuration.

<b>Example 1.4: Theory 2.</b>
All matter is made up of tiny, indestructible particles, called atoms.

## 1.2 Matter

As Definition 2 said, matter must have both volume and mass. Matter can have several Matter States.

**Defn 7** (Matter State). A *matter state* or *state of matter* is just the configuration of atoms in a particular material. There are 3 common states:

1. Solid
2. Liquid
3. Gas

But, Matter can be categorized in different ways as well. The Matter Tree is one way to categorize them.

Figure 1: Matter Tree

Others include:

- Atomic Weight
- Chemical Properties
- Physical Properties
- The Periodic Table
- And many others

## 1.3 Significant Figures

**Defn 8** (Significant Figures). *Significant Figures* or *Sig Figs* are ways to handle uncertainty in our measurements. In general, we treat the data that we receive as inexact numbers, thus we must confirm our suspicions several times. Additionally, Precision and Accuracy are used interchangeably when they shouldn't.

**Defn 9** (Precision). *Precision* is defined as the closeness of data points to each other. If you think about a dartboard, this would be all the darts landing right next to each other.

**Defn 10** (Accuracy). *Accuracy* is defined as how close your data is to the predicted true real value.

*Remark 10.1.* Generally, this must be done with a minimum of 3 trials, but more will yield more accurate data.

### 1.3.1 Rules for Significant Figures

1. 0s between any non-zero digit is significant (100, both 0s are significant)
2. 0s at the beginning of an integer are not significant (010 = 10)
3. 0s at the end are significant if the number is a decimal (0.003050 has 4 sig figs)
4. 0s at the end, if there is no decimal/fractional portion, are not significant (16000 has 2 sig figs)

#### Example 1.5: Addition and Subtraction of Significant Figures.

Add 20.3056, 1.34, and 54.2 and keeping in mind significant figures.

You want to find the least precise number first, in this case it is 54.2 because it only has one decimal place. This also determines how many decimal places to go past on the solution. Adding these 3 together gives 75.8456, but because of 54.2, it becomes 75.8.

#### Example 1.6: Multiplication and Division of Significant Figures.

Multiply 3.4456 and 2.15 keeping in mind significant figures.

You find the number with the least number of significant figures and use that. So, 2.15 has 3 sig figs, that's the same amount your answer must have.

$$3.4456 \times 2.15 = 7.40804$$

But because we can only have 3 sig figs in our answer, 7.41 is our solution.

## 2 History of Chemistry

### 2.1 Dalton

Dalton created the first meaningful definition of an atom. He made several claims:

1. Atoms are very small
2. The same element's atoms are identical, but different elements have different atoms.
3. Atoms are neither created, nor destroyed (Law of Conservation of Matter)
4. Compounds are 2 or more elements together.

**Defn 11** (Law of Conservation of Matter). Matter is neither created, nor destroyed. It can *only* change forms.

### 2.2 Thomson

Thomson made several discoveries about atoms and their constituent particles. For his experimentation, he used a cathode ray (a beam of positively charged ions) and magnets. He discovered the Electron.

**Defn 12** (Electron). The *electron* is one of 3 particles that make up an atom. Electrons are negatively charged particles that are contained *outside* of the nucleus. An electron's position and velocity can not be known simultaneously. This is known as the Heisenberg's Uncertainty Principle

The cathode ray deflected from the "negative" magnetic plate to the "positive". From this he calculated the Magnetic Deflection.

**Defn 13** (Magnetic Deflection). When Thomson deflected his cathode ray with magnets, he measured how far it deviated from the starting line.

$$1.76 \times 10^8 \text{C/g} \tag{2.1}$$

## 2.3 Millikan

Millikan made 2 significant contributions to the model of the atom. He discovered the charge of a single Electron and the mass of a single Electron.

$$1.602 \times 10^{-19} \text{C} \quad (2.2)$$

$$9.10938 \times 10^{-28} \text{g} \quad (2.3)$$

Both the Millikan and Millikan were drawn from Thomson's work with Magnetic Deflection.

## 2.4 Becquerel

Becquerel did his work with high energy radiation caused by radioactivity. He found that there were 3 types of particles released by radioactive decay.

1. Alpha Particles ( $\alpha$ ) - Positively charged particles that are charged helium atoms.
2. Beta Particles ( $\beta$ ) - Negatively charged particles that are essentially high speed electrons.
3. Gamma particles ( $\gamma$ ) - Uncharged particles that have next to no mass and are quite energetic.

## 2.5 Johnson

Johnson developed one of the first models for single atoms. This was called the Plum Pudding Model.

**Defn 14** (Plum Pudding Model). The *Plum Pudding Model* is a visualization of an atom. It is based off the plum pudding desert, which was one of Johnson's favorites. The positive charges were held together in a "soft" shell. The electrons were evenly distributed on the outer surface of the positively charged "plum." One of the hallmarks of this model was that the entire atom was *not* empty space.

## 2.6 Rutherford

Rutherford performed experiments with  $\alpha$ -particles. He "shot" these particles at a piece of gold foil and observed what happened with after the particles passed through.

**Defn 15** (Rutherford Model). This is, more or less, the next model of the atom. Rutherford challenged Johnson's Plum Pudding Model of the atom. When Rutherford sent the beam of alpha particles through the gold foil, he found most of them didn't deflect, i.e. hit any thing. However some did, and were scattered in all directions. Rutherford proved that atoms are mostly empty space, with the positive charges being held in a small dense area he called the *nucleus*.

*Remark 15.1.* One thing to note about the *nucleus* in the Rutherford Model is that there was no concept of the neutron yet. Since neutrons are uncharged particles, they were not discovered until much later.

## 2.7 Atomic Mass Units, u

Eventually the neutron was discovered and the current understanding of the fundamental particles in atoms was completed. These include the:

- Proton ( $p^+$ )
- Neutron ( $N^0$ )
- Electron ( $e^-$ )

The mass of each of these particles was found and the Atomic Mass Unit was developed to make calculations easier.

$$1\text{u} = 1.66054 \times 10^{-24} \text{g} \quad (2.4)$$

Thus, the atomic mass for each particles is as follows:

- Proton ( $p^+$ ) -  $1.672623 \times 10^{-24} \text{g} = 1.0074\text{u}$
- Neutron ( $N^0$ ) -  $1.674927 \times 10^{-24} \text{g} = 1.0087\text{u}$
- Electron ( $e^-$ ) -  $9.109383 \times 10^{-28} \text{g} = 5.486 \times 10^{-4}\text{u}$

## 3 The Periodic Table

The Periodic Table was developed as a way to categorize the many chemical elements in the world. Most of the elements on the table are naturally occurring, but some of the heaviest elements have been synthesized in laboratories.

**Defn 16** (The Periodic Table). The *Periodic Table* is an arrangement of elements by their atomic number,  $Z$ , or the number of protons in the nucleus.

**Remark 16.1.** The number of protons is the *ONLY* thing that determines what an element is and where it is located on The Periodic Table. If an element has a different number of neutrons, that is an Isotope. If an element has a different number of electrons, that is an Ion.

**Remark 16.2.** On The Periodic Table, the elements are always in their electroneutral form. This means they have the same number of protons and electrons.

A single element drawn out of The Periodic Table will look like Figure.

Figure 2: Example Element from Periodic Table

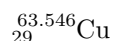
There are 4 parts of each element in The Periodic Table.

1. The element's name
2. The element's atomic number,  $Z$ , which is also the number of protons in the element
3. The element's symbol
4. The element's *AVERAGE* atomic mass (g/mol),  $A$

An element is usually written as such:  ${}^A_ZX$ .

**Example 3.1: Element Notation.**

What is Copper's (Cu) notation in text?



### 3.1 Variations of Atoms

**Defn 17** (Isotope). An *isotope* is the same element, but with a different number of neutrons. While this does *NOT* change the element, it may change some of the properties of this element.

**Example 3.2: Number of Neutrons.**

What is the number of neutrons in the average isotope of these elements:  ${}^{28}_{14}\text{Si}$ ,  ${}^{29}_{14}\text{Si}$ ,  ${}^{30}_{14}\text{Si}$ ?

Since Silicon, Si has 14 protons, so subtract 14 from each of the total nuclei weight.

1.  ${}^{28}_{14}\text{Si}$  has 14 neutrons
2.  ${}^{29}_{14}\text{Si}$  has 15 neutrons
3.  ${}^{30}_{14}\text{Si}$  has 16 neutrons

**Defn 18** (Ion). An *ion* is the same element as the non-ionic form, however, the number of electrons and protons is different. This means that an ion can be positively or negatively charged.

**Remark 18.1.** The number of electrons in relation to protons determines the type of ion it is.

- If there are more electrons than protons, it is a negatively charged ion.
- If there are more protons than electrons, it is a positively charged ion.

**Remark 18.2.** In general:

- Metals are usually cations
- Non-Metals are usually anions

**Example 3.3: What is that Element?.**

Given that an element has 38 protons, 50 neutrons, and 36 electrons, what is the element?

Since there are 38 protons, that makes it Strontium (Sr). The atomic weight will be  $38 + 50 = 88$ . The ion is  $38 - 36 = +2$ . Therefore, the written atom would be  ${}^{88}_{38}\text{Sr}^{+2}$ .

**Example 3.4: An Ion's Protons Neutrons and Electrons.**

Given the element  ${}^{75}_{33}\text{As}^{-3}$ , how many protons, neutrons and electrons are there?

Astatine has 33 protons. The number of neutrons is  $75 - 33 = 42$ . Since this is Astatine, the number of electrons is  $|-33 - -3| = 36$ .

## 3.2 Types of Materials

**Defn 19** (Metal). A *metal* is a type of material that:

- Conducts heat well
- Conducts electricity well by freely losing electrons
- Their nuclei are congregated and their electrons form an outer electron cloud

*Remark 19.1.* For Metals, there are variations of some metals. For example, Iron (II) and Iron (RomanNumeral3). These are *NOT* different ions. Rather, they are Iron in different Oxidation States.

**Defn 20** (Non-Metal). A *non-metal* is a type of material that:

- Do not conduct heat well
- Do not conduct electricity well
- Gain electrons easily due to high electronegativity

**Defn 21** (Metalloid). A *metalloid* is a type of material that has properties of both Metal and Non-Metal. These elements on right on the edge between Metals and Non-Metals on the periodic table. These include:

- Boron, B
- Silicon, Si
- Germanium, Ge
- Astatine, As
- Antimony, Sb
- Tellurium, Te

**Example 3.5: Metal Nonmetal or Metalloid?.**

Are these elements Metals, Non-Metals, or Metalloids?

1. Phosphorus, P
2. Osmium, Os
3. Selenium, Se
4. Thallium, Tl
5. Nickel, Ni
6. Argon, Ar
7. Tellurium, Te

1. Phosphorus, P is a Non-Metal
2. Osmium, Os is a Metal
3. Selenium, Se is a Non-Metal
4. Thallium, Tl is a Metal
5. Nickel, Ni is a Metal
6. Argon, Ar is a Non-Metal
7. Tellurium, Te is a Metalloid

## 4 Naming Compounds

There is a specific way to name compounds, however, it depends on the type of elements in the compound.

### 4.1 Naming Inorganic Compounds

Inorganic compounds are non-carbon-based compounds. There are a 2 rules for naming these compounds:

1. If there are only 2 elements in the compound and the first element only has 1 atom, then *DO NOT USE A PREFIX*

2. Otherwise, use the numerical prefix for the number of atoms present
  - (a) 1 atom  $\rightarrow$  Mono-
  - (b) 2 atoms  $\rightarrow$  Di-
  - (c) 3 atoms  $\rightarrow$  Tri-
  - (d) 4 atoms  $\rightarrow$  Tetra-
  - (e) so on and so forth

## 4.2 Naming Metallic Compounds

Metallic compounds are ones that have Metals in them. The rules for naming metallic compounds are similar to the inorganic ones. The only difference is naming the metal itself. You use the roman numerals when writing the compound and say the number out loud. For example, rust is  $\text{Fe}_2\text{O}_3$ , is written Iron (III) Oxide.

## 5 Chemical Equations

Chemical Equations form one of the key pillars of chemistry.

**Defn 22** (Chemical Equations). Just like in mathematical equations, *chemical equations* define the way certain chemical reactions take place. Just like in mathematics, there are some rules that are followed.

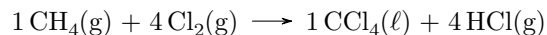
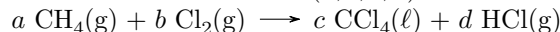
1. You need to have the same elements on each side of the reaction
2. There must be reactants (input) and products (output)
3. There must be a *Species* on each element/compound
4. There must be an arrow,  $\rightarrow$  or  $\leftrightarrow$  showing the direction of reaction
  - $\leftrightarrow$  is for reactions that can occur in both directions. Nearly all do not follow this.
5. Elements **MUST** follow the Law of Conservation of Matter

**Defn 23** (Species). An element's or compound's *species* is the current state of matter. It can only be one of the following:

- Solid - (s)
- Liquid - ( $\ell$ )
- Gas - (g)
- Aqueous - (aq)

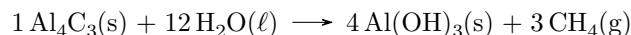
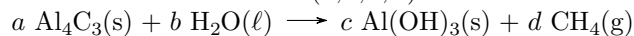
### Example 5.1: Chemical Equation 1.

What are the coefficients ( $a, b, c, d$ ) for this chemical equation?



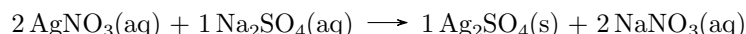
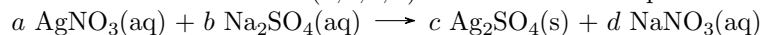
### Example 5.2: Chemical Equation 2.

What are the coefficients ( $a, b, c, d$ ) for this chemical equation?



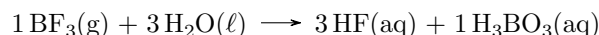
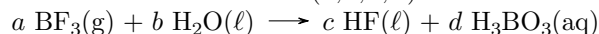
### Example 5.3: Chemical Equation 3.

What are the coefficients ( $a, b, c, d$ ) for this chemical equation?



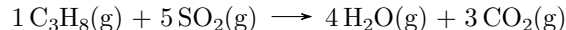
### Example 5.4: Chemical Equation 4.

What are the coefficients ( $a, b, c, d$ ) for this chemical equation?

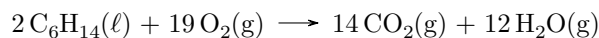
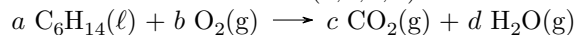


**Example 5.5: Chemical Equation 5.**

What are the coefficients  $(a, b, c, d)$  for this chemical equation?

**Example 5.6: Chemical Equation 6.**

What are the coefficients  $(a, b, c, d)$  for this chemical equation?



## 6 Reduction-Oxidation Reactions

These reactions are somewhat unique because the electrons *ONLY* move around but do not change the output of the reaction.

**Defn 24** (Reduction-Oxidation Reactions). *Reduction-Oxidation Reactions* or *Redox Reactions* are a type of chemical reaction. These reactions take place due to differences in electronegativity and Oxidation States.

**Defn 25** (Oxidation States). An *oxidation state* is the ...

## 7 Quantum Chemistry

This section is a brief introduction to how we have discovered certain properties of atoms due to quantum mechanics and physics. One of the biggest ideas in quantum mechanics is Heisenberg's Uncertainty Principle.

**Defn 26** (Heisenberg's Uncertainty Principle). *Heisenberg's Uncertainty Principle* states that it is impossible to accurately know both the position and velocity of a particle in a system.

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi} \quad (7.1)$$

*Remark 26.1.* The parameters for Equation (7.1) are below.

- $\Delta x$  is the change in position of the “thing”
- $\Delta p$  is the change in the momentum of the “thing”
- $h$  is Planck's Constant.



## A Physical Constants

Constant Name	Variable Letter	Value
Boltzmann Constant	$R$	8.314J/mol K
Universal Gravitational	$G$	$6.67408 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$
Planck's Constant	$h$	$6.62607004 \times 10^{-34} \text{mkg/s} = 4.163 \times 10^{-15} \text{eV s}$
Speed of Light	$c$	$299792458 \text{m/s} = 2.998 \times 10^8 \text{ m/s}$
Charge of Electron	$e$	$1.602 \times 10^{-19} \text{C}$
Mass of Electron	$m_{e-}$	$9.11 \times 10^{-31} \text{kg}$
Mass of Neutron	$m_{n^0}$	$1.67 \times 10^{-31} \text{kg}$
Mass of Earth	$m_{Earth}$	$5.972 \times 10^{24} \text{kg}$
Diameter of Earth	$d_{Earth}$	12742km

## B Trigonometry

### B.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{B.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{B.2})$$

### B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm i\alpha} = \cos(\alpha) \pm i \sin(\alpha) \quad (\text{B.3})$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad (\text{B.4})$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (\text{B.5})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{B.6})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{B.7})$$

### B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \quad (\text{B.8})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \quad (\text{B.9})$$

### B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad (\text{B.10})$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (\text{B.11})$$

### B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \quad (\text{B.12})$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \quad (\text{B.13})$$

### B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2} \quad (\text{B.14})$$

$$\cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2} \quad (\text{B.15})$$

### B.7 Product-to-Sum Identities

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad (\text{B.16})$$

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad (\text{B.17})$$

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (\text{B.18})$$

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (\text{B.19})$$

## B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{B.20})$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{B.21})$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (\text{B.22})$$

## B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \quad (\text{B.23})$$

## B.10 Rectangular to Polar

$$a + ib = \sqrt{a^2 + b^2} e^{i\theta} = r e^{i\theta} \quad (\text{B.24})$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \quad (\text{B.25})$$

## B.11 Polar to Rectangular

$$r e^{i\theta} = r \cos(\theta) + ir \sin(\theta) \quad (\text{B.26})$$

## C Calculus

### C.1 Fundamental Theorems of Calculus

**Defn C.1.1** (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if  $f$  is continuous on the closed interval  $[a, b]$  and  $F$  is the indefinite integral of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{C.1})$$

**Defn C.1.2** (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for  $f$  a continuous function on an open interval  $I$  and  $a$  any point in  $I$ , and states that if  $F$  is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{C.2})$$

**Defn C.1.3** (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname{argmax}_x$$

## D Complex Numbers

$$Ae^{-ix} = A [\cos (x) + i \sin (x)] \tag{D.1}$$