EITF75: Systems and Signals - Reference Sheet

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Contents

1 Sinusoids

There are several ways to characterize Sinusoids. The first is by dimension:

1. Multidimensional/Multichannel Signals
2. Monodimensional/Monochannel Signals

You can also classify sinusoids by their independent variable (usually time) and the values they take.

1. Continuous-Time Signals or Analog Signals
2. Discrete-Time Signals
3. There is a third way to classify sinusoids and their signals: Digital Signals

Defn 1 (Continuous-Time Signals)**.** *Continuous-time signals* or *Analog signals* are defined for every value of time and they take on values in the continuous interval (a, b), where a can be $-\infty$ and b can be ∞. Mathematically, these signals can be described by functions of a continuous variable.

For example,

$$x_1(t) = \cos \pi t,\ x_2(t) = e^{-|t|},\ -\infty < t < \infty$$

Defn 2 (Discrete-Time Signals)**.** *Discrete-time signals* are defined only at certain specifed values of time. These time instants ***need not*** be equidistant, but in practice, they are usually taken at equally speced intervals for computation convenience and mathematical tractability.

For example,

$$x(t_n) = e^{-|t_n|},\ n = 0, \pm 1, \pm 2, \ldots$$

A Discrete-Time Signals can be represented mathematically by a sequence of real or complex numbers.

Remark 2.1. To emphasize the discrete-time nature of the signal, we shall denote the signal as $x(n)$, rather than $x(t)$.

Remark 2.2. If the time instants t_n are equally spaced (i.e., $t_n = nT$), the notation $x(nT)$ is also used.

1.1 Continuous-Time Signals

1.1.1 Frequency in Continuous-Time Signals

A simple harmonic oscillation is mathematically described by Equation (1.1).

$$x_a(t) = A\cos\left(\Omega t + \theta\right),\ -\infty < t < \infty \tag{1.1}$$

Remark. The subscript a is used with $x(t)$ to denote an analog signal.

This signal is completely characterized by three parameters:

1. A, the *amplitude* of the sinusoid
2. Ω, the *frequency* in radians per second (rad/s)
3. θ, the *phase* in radians.

Instead of Ω, the frequency F in cycles per second or hertz (Hz) is used.

$$\Omega = 2\pi F \tag{1.2}$$

Plugging (1.2) into (1.1), yields

$$x_a(t) = A\cos\left(2\pi F t + \theta\right),\ -\infty < t < \infty \tag{1.3}$$

1.1.2 Properties of Continuous-Time Sinusoidal Signals

The analog sinusoidal signal in equation (1.3) is characterized by the following properties:

(i) For every fixed value of the frequency F, $x_a(t)$ is periodic.

$$x_a(t + T_p) = x_a(t)$$

where $T_p = \frac{1}{F}$ is the fundamental period.

(ii) Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.

(iii) Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in the given time interval.

1.2 Discrete-Time Signals

1.2.1 Frequency in Discrete-Time Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta)\,,\; n \in \mathbb{Z}\,,\; -\infty < n < \infty \tag{1.4}$$

The signal is characterized by these parameters:

1. n, the sample number. MUST be an integer.
2. A, the *amplitude* of the sinusoid
3. ω, the *angular frequency* in radians per sample
4. θ, is the *phase*, in radians.

Instead of ω, we use the frequency variable f defined by

$$\omega \equiv 2\pi f \tag{1.5}$$

Using (1.4) and (1.5) yields

$$x(n) = A\cos(2\pi f n + \theta)\,,\; n \in \mathbb{Z}\,,\; -\infty < n < \infty \tag{1.6}$$

1.2.2 Properties of Discrete-Time Sinusoidal Signals

(i) A discrete-time sinusoid is periodic ***ONLY*** if its frequency is a rational number.

(ii) Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical. This leads us to the idea of a Frequency Alias.

(iii) The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pm\pi$ or, equivalently, $f = \pm\frac{1}{2}$.

1.2.3 Frequency Aliases

The concept of a Frequency Alias is drawn from the idea that discrete-time sinusoids whose frequencies are separated by an integer mutliple of 2π are identical and that frequencies $|f| > \frac{1}{2}$ are identical. (Properties **(ii)** and **(iii)**)

Defn 3 (Frequency Alias)**.** A *frequency alias* is a sinusoid having a frequency $|\omega| > \pi$ or $|f| > \frac{1}{2}$. This is because this sinusoid is *indistinguishable* (*identical*) to one with frequency $|\omega| < \pi$ or $|f| < \frac{1}{2}$.

A *frequency alias* is a sequence resulting from the following assertion based on the sinusoid $\cos(\omega_0 n + \theta)$. It follows that

$$\cos\left[(\omega_0 + 2\pi)\, n + \theta\right] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta$$

As a result, all sinusoidal sequences

$$x_k(n) = A\cos(\omega_k n + \theta),\; k = 0, 1, 2, \ldots$$

where

$$\omega k = \omega_0 + 2k\pi,\; -\pi \le \omega_0 \le \pi$$

are *indistinguishable* (i.e., *identical*).

Because of this, we regard frequencies in the range of $-\pi \le \omega \le \pi$ or $-\frac{1}{2} \le f \le \frac{1}{2}$ as unique, and all frequencies that fall outside of these ranges as aliases.

Remark 3.1. It should be noted that there is a difference between discrete-time sinusoids and continuous-time sinusoids here. Continuous-time sinusoids have distinct signals for Ω or F in the entire range $-\infty < \Omega < \infty$ or $-\infty < F < \infty$.

1.3 Sampling Rates and Sampling Frequency

Most signals of interest are analog. To process these signals, they must be collected and converted to a digital form, that is, to convert them to a sequence of numbers having finite precision. This is called *analog-to-digital (A/D) conversion*. Conceptually, we view this conversion as a 3-step process.

1. Sampling
2. Quantization
3. Coding

1.3.1 Nyquist Rate

1.3.2 Nyquist Frequency

1.4 Digital Signals

Defn 4 (Digital Signals)**.** *Digital signals* are a subset of Discrete-Time Signals. In this case, not only are the values being measured occuring at fixed points in time, the values themselves can only take certain, fixed values.

1.4.1 Quantization

Defn 5 (Quantization)**.** This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-value (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the Quantization Error.

1.4.1.1 Quantization Levels

1.4.1.2 Quantization Error

Defn 6 (Quantization Error)**.** The *quantization error* of something.

1.4.1.3 Bit Requirements

1.4.1.4 Bit Rate

2 Discrete-Time Systems

As discussed in Section 1.2, $x(n)$ is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is *not defined* at instants between the samples. Also, if n is not an integer, $x(n)$ is not defined.

Besides graphical representation of a discrete-time system, there are 3 ways to represent a discrete-time signal.

1. Functional Representation
2. Tabular Representation
3. Sequence Representation

2.1 Representing Discrete-Time Systems

2.1.1 Functional Representation

This representation of a discrete-time system is done as a mathematical function.

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1}$$

2.1.2 Tabular Representation

This representation of a discrete-time sysem is done as a table of corresponding values.

n	...	-2	-1	0	1	2	3	4	5	...
$x(n)$	...	0	0	0	1	4	1	0	0	...

2.1.3 Sequence Representation

There are 2 methods of representation for this. The first includes all values for $-\infty < n < \infty$. In all cases, $n = 0$ is marked in the sequence, somehow. I will do this with an underline.

$$x(n) = \{\dots, 0, \underline{0}, 1, 4, 1, 0, 0, \dots\} \tag{2.2}$$

The second only works if all $x(n)$ values for $n < 0$ are 0.

$$x(n) = \{\underline{0}, 1, 4, 1, 0, 0, \dots\} \tag{2.3}$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underline{-2}, 5, 0, 4, -1\} \tag{2.4}$$

This is identified as a seven-point sequence.

A finite-duration sequence where $x(n) = 0$ for all $n < 0$ is represented as

$$x(n) = \{\underline{0}, 1, 4, 1\} \tag{2.5}$$

This is identified as a four-point sequence.

2.2 Elementary Discrete-Time Signals

The following signals are basic signals that appear often and play an important role in signal processing.

2.2.1 Unit Impulse Signal

Defn 7 (Unit Impulse Signal)**.** The *unit impulse signal* or *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv= \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.6}$$

This function is a signal that is zero everywhere, except at $n = 0$, where its value is 1.

Remark 7.1. This signal is different that the analog signal $\delta(t)$, which is also called a unit impulse, and is defined to be 0 everywhere except $t = 0$. The discrete unit impulse sequence is much less mathematically complicated.

2.2.2 Unit Step Signal

Defn 8 (Unit Step Signal)**.** The *unit step signal* is denoted as $u(n)$ and is defined as

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.7}$$

2.2.3 Unit Ramp Signal

Defn 9 (Unit Ramp Signal)**.** The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.8}$$

2.2.4 Exponential Signal

Defn 10 (Exponential Signal)**.** The *exponential signal* is a sequence of the form

$$x(n) = a^n \text{ for all } n \tag{2.9}$$

If a is real, then $x(n)$ is a real signal. When a is complex valued ($a \equiv b \pm cj$), it can be expressed as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n \left(\cos\theta n + j \sin\theta n\right) \end{aligned} \tag{2.10}$$

This can be expressed by graphing the real and imaginary parts

$$\begin{aligned} x_R(n) &\equiv r^n \cos\theta n \\ x_I(n) &\equiv r^n j \sin\theta n \end{aligned} \tag{2.11}$$

or by graphing the amplitude function and phase function.

$$\begin{aligned} |x(n)| &= A(n) \equiv r^n \\ \angle x(n) &= \phi(n) \equiv \theta n \end{aligned} \tag{2.12}$$

2.3 Classification of Discrete-Time Signals

In order to apply some mathematical methods to discrete-time signals, we must characterize these signals.

2.3.1 Energy Signal

Defn 11 (Energy Signal)**.** The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.13}$$

The energy of a signal can be finite or infinite. If E is finite ($0 < E < \infty$), then $x(n)$ is called an *energy signal.*

2.3.2 Power Signal

Defn 12 (Power Signal)**.** The average power of a discrete time signal $x(n)$ is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \tag{2.14}$$

This means that there are 2 potential outcomes:

1. If E is finite, $P = 0$
2. If E is infinite, P may be either finite or infinite

If P is finite and nonzero, the signal is called a *power signal.*

2.3.3 Periodic and Aperiodic Signals

A signal $x(n)$ is periodic with period N ($N > 0$) if and only if

$$x(n+N) = x(n)\text{for all } n \tag{2.15}$$

The smallest value of N for which (2.15) holds is called the fundamental period. If there is no value of N that satisfies (2.15), the signal is called *nonperiodic* or *aperiodic.*

2.3.4 Symmetric and Antisymmetric Signals

A real-valued signal $x(n)$ is called *symmetric* or *even* if

$$x(n) = x(-n) \tag{2.16}$$

On the other hand, a signal $x(n)$ is called *antisymmetric* or *odd* if

$$x(n) = -x(-n) \tag{2.17}$$

2.4 Discrete-Time Signal Manipulations

2.4.1 Transformation of the Independent Variable (Time)

It is important to note that Shifting in Time and Folding are not commutative. For example,

$$\mathrm{TD_k}\{\mathrm{FD}\left[x(n)\right]\} = \mathrm{TD_k}\left[x(-n)\right] = x(-n+k) \tag{2.18}$$

whereas

$$\mathrm{FD}\{\mathrm{TD_k}\left[x(n)\right]\} = \mathrm{FD}\left[x(n-k)\right] = x(-n-k) \tag{2.19}$$

2.4.1.1 Shifting in Time A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time (moves left). If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units of time (moves right).

This could be denoted by

$$\mathrm{TD_k}\left[x(n)\right] = x(n-k) \tag{2.20}$$

You cannot advance a signal that is being generated in real-time. Because that would involve signal samples that haven't been generated yet. So, you can only advance a signal that is stored on something. However, you can always introduce a delay to a signal.

2.4.1.2 Folding

Another useful modification of the time base is to replace n with $-n$. The result is a *folding* or *reflection* of the original signal around $n = 0$.

This could be denoted by

$$\mathrm{FD}\left[x(n)\right] = x(-n) \tag{2.21}$$

2.4.2 Addition, Multiplication, and Scaling

Amplitude modifications include Addition, Multiplication, and Amplitude Scaling.

2.4.2.1 Addition

The *sum* of 2 signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the sum of the values of these two signals at that instant.

$$y(n) = x_1(n) + x_2(n), \ -\infty < n < \infty \tag{2.22}$$

2.4.2.2 Multiplication

The *product* of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the product of the values of these two signals at that instant.

$$y(n) = x_1(n)x_2(n), \ -\infty < n < \infty \tag{2.23}$$

2.4.2.3 Amplitude Scaling

Amplitude scaling of a signal by a constant A is accomplished by multiplying every signal smaple by A. Consequently, we obtain

$$y(n) = Ax(n), \ -\infty < n < \infty \tag{2.24}$$

3 Convolutions

Defn 13 (Convolution)**.** The *convolution* operator.

$$y(t) = \sum_{k=-\infty}^{\infty} x(k) * h(n-k) \tag{3.1}$$

4 The Z-Transform

The Z-Transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace Transform does in the analysis of continuous time-signals and LTI systems.

4.1 The Z-Transform

Defn 14 (Z-Transform)**.** The *z-transform* is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

4.1.1 The One-Sided Z-Transform

4.2 The Inverse Z-Transform

4.2.1 The Inverse Z-Transform by Contour Integration

4.2.2 The Inverse Z-Transform by Power Series Expansion

4.2.3 The Inverse Z-Transform by Partial-Fraction Expansion

4.3 Properties of the Z-Transform

(i)

4.4 Rational Z-Transforms

4.4.1 Decomposition of Rational Z-Transforms

4.5 Analysis of LTI Systems in the Z-Domain

A Trigonometry

A.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm\imath\alpha} = \cos(\alpha) \pm \imath\sin(\alpha) \tag{A.3}$$

$$\sin(x) = \frac{e^{\imath x} + e^{-\imath x}}{2} \tag{A.4}$$

$$\cos(x) = \frac{e^{\imath x} - e^{-\imath x}}{2\imath} \tag{A.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.7}$$

A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{A.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{A.9}$$

A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{A.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{A.11}$$

A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{A.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{A.13}$$

A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{A.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{A.15}$$

A.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{A.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{A.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{A.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{A.19}$$

A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{A.20}$$

$$\cos(\alpha) + \cos(\alpha) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{A.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{A.22}$$

A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{A.23}$$

A.10 Rectangular to Polar

$$a + \imath b = \sqrt{a^2 + b^2}e^{\imath\theta} = re^{\imath\theta} \tag{A.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{A.25}$$

A.11 Polar to Rectangular

$$re^{\imath\theta} = r\cos(\theta) + \imath r\sin(\theta) \tag{A.26}$$

B Calculus

B.1 Fundamental Theorems of Calculus

Defn B.1.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{B.1}$$

Defn B.1.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{B.2}$$

Defn B.1.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\underset{x}{\operatorname{argmax}}$$

B.2 Rules of Calculus

B.2.1 Chain Rule

Defn B.2.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{B.3}$$

C Laplace Transform

Defn C.0.1 (Laplace Transform)**.** The *laplace transformation* is ...