ETSN10: Network Architecture and Performance - Reference Sheet

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List of Theorems

1 Networking Review

This course assumes that you already know about basic networking terms and models. However, this section is meant as a refresher for people who have already taken that particular set of courses, or as a quick introduction for those who have not taken those courses yet.

1.1 Networking Stack

1.2 Physical Layer

1.3 Data-Link Layer

1.4 Networking Layer

Defn 1 (Networking Layer)**. TODO!**

Defn 2 (Routing)**.** *Routing* is the process of determining where to forward packets onto next.

There are algorithms designed to discover the other participants in the network by analyzing the grpah that is created when each participant is considered a node in this graph.

Defn 3 (Routing Protocol)**.** A *routing protocol* is used to construct a Routing Table in each node.

Defn 4 (Routing Table)**.** The *routing table* is a table that contains entries on the shortest distances through the current network graph. This is used to help determine where to next forward packets onto.

Each node has a routing table that contains the cost to the destination and the next hop for each destination. For a valid routing table, we need to have:

1. Which destination the packets are going to?
2. What is the next hop after the destination?
3. The "metric" or weight of that particular path through the network graph?
    - We need this term so we can update the routing table as we discover new routes.
    - This way if a new route with a lower cost presents itself, we can replace the higher-cost one with the lower one.

Defn 5 (Forwarding)**.** *Forwarding* is the process of moving a packet from one place to the next based on the Routing Table present in this node and where the packet would like to go next.

1.4.1 Discovering Routing Information

1.4.1.1 Dijkstra's Algorithm There are 2 main problems with Dijkstra's Algorithm:

1. The weights along the edges of the graph cannot be negative.
2. We **must** know the entire network's topology before beginning the algorithm.
    - Typically, this is not possible on vast networks, like the Internet.

Algorithm 1.1: Dijkstra's Algorithm

Input : A graph with weights on each edge representing "distance" between the nodes.
Output: The path through the graph with the least weight from some starting node.

TODO!
From Networking Review Lecture Video 2.

1.4.1.2 Bellman-Ford Algorithm The Bellman-Ford algorithm works with graphs that have negative weight values on edges. It also **does not** require us to know the entire network's topology before beginning. However, it is much slower than Dijkstra's Algorithm.

Algorithm 1.2: Bellman-Ford Algorithm

Input : A graph with weights on each edge representing "distance" between the nodes.
Output: The path through the graph with the least weight from some starting node.

TODO!
From Networking Review Lecture Video 2.

1.4.2 Routing Information

There are 2 ways to collect the routing information necessary to properly route packets through a network. These 2 routing tables are:

1. Static Routing Tables
2. Dynamic Routing Tables

1.4.2.1 Static Routing Tables

The static routing tables are manually configured by a user or a program before the system begins routing the information. This means that there is not overhead when trying to determine where the packet should next be forwarded to.

1.4.2.2 Dynamic Routing Tables

To use a dynamic Routing Table, we must use a Routing Protocol to build the table as we go, rather than have a completely finished table when starting. These are built using either:

1. **??**
2. **??**

Defn 6 (Distance Vector Routing Protocol)**.** The *distance vector routing algorithm* is a Routing Protocol that uses Bellman-Ford Algorithm to construct a Routing Table. **TODO!**

Defn 7 (Link State Routing Protocol)**.** The *link state routing protocol* is a Routing Protocol that uses Dijkstra's Algorithm to construct a Routing Table. Thus, we need to know what the whole network looks like. **TODO!**

However, the problem with both Distance Vector Routing Protocol and Link State Routing Protocol is that they do not scale well. However, to help with this, both protocols are contained within a single Autonomous System.

Defn 8 (Autonomous System)**.** An *autonomous system* is a smaller network, like a business or home, that performs a **??** upon itself. It is done this way because both the Distance Vector Routing Protocol and Link State Routing Protocol do not scale to Internet-sized networks well.

To combat this, each of the autonomous systems has a speaker node that speaks to the outside world. This speaker node is used in the Path Vector Routing Protocol.

Defn 9 (Path Vector Routing Protocol)**.**

1.4.3 IP Addresses

Defn 10 (IP Address)**. TODO!**

Defn 11 (Subnet)**.** All hosts that are in the subnet will have IP Addresses that match the number of bits that are present in the subnet.

For example, in `192.168.2.0/24`, the `24` means the first 24 bits of the subnet are 1's. So, the first 3 IP Address blocks `192.168.2` will remain the same for every client in that subnet. **TODO!**

Defn 12 (Classless Inter-Domain Routing)**. TODO!**

1.5 Transport Layer

Defn 13 (Transport Layer)**.** The *transport layer* is built where the **??** has delivered all the data to the end host. This means that only the source and destination hosts have this layer. So, routers and switches do not have this, but your phone, laptop, and the server you're connecting to do.

This layer may be responsible for:

- Connection-Oriented Communication
- Multiplexing different data flows
- Reliable data delivery
- Data flow control
- Data congestion control

Some implementation of this layer do not handle all of these functions, but all **must** handle the multiplexing of different data flows.

There are 2 main transport layers in-use today:

1. Transmission Control Protocol
2. User Datagram Protocol

Defn 14 (Connection-Oriented Communication)**.** A *connection-oriented communication* system means that a connection must be established before any data is transferred. In addition, this connection must be maintained throughout the transmission of data.

Remark 14.1 (Connectionless Communication). In a *connectionless communication* system, hosts can send data at any time without prior connection being made.

Defn 15 (Transmission Control Protocol)**. TODO!**

Defn 16 (User Datagram Protocol)**. TODO!**

1.6 Application Layer

2 Probability Review

This section is meant to quick review and introduce the equations that will be used throughout this course. It is not meant to be comprehensive and/or in-depth. For more information about the topic of probability and statistics, refer to the Math 374 - Probability and Statistics document.

2.1 Axioms of Probability

Defn 17 (Sample Space)**.** The *sample space* is the set of all possible outcomes in a random experiment. It is denoted with the capital Greek omega.

$$\Omega \tag{2.1}$$

Defn 18 (Event)**.** An *event* is a subset of the Sample Space that we are interested in. These are generally denoted with capital letters.

$$A \subseteq \Omega \tag{2.2}$$

Defn 19 (Mutually Exclusive)**.** Any two Events are *mutually exclusive* if the equation below holds.

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) \tag{2.3}$$

Laws that follow from the above definitions (Definitions 17 to 19).

1. The conjugate of the Event occurring, i.e. the Event **not** occurring is:

$$\mathrm{P}(\bar{A}) = 1 - \mathrm{P}(A) \tag{2.4}$$

2. The probability of the union of 2 Events is:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) \tag{2.5}$$

   - If A and B are Mutually Exclusive, then $\mathrm{P}(A \cup B) = 0$.

2.2 Conditional Probability

Defn 20 (Conditional Probability)**.** *Conditional probability* is the probability of an Event occurring when it is known that another Event occurred.

$$\mathrm{P}(A \,|\, B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \tag{2.6}$$

Defn 21 (Independent)**.** Events are *independent* if the probability of the events' intersection is the same as their probabilities multipled together.

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B) \tag{2.7}$$

Remark 21.1 (Conditional Probability and Independent Events). If A and B are Events and are Independent, then

$$\begin{aligned} \mathrm{P}(A \,|\, B) &= \mathrm{P}(A) \\ \mathrm{P}(B \,|\, A) &= \mathrm{P}(B) \end{aligned} \tag{2.8}$$

2.3 Random Variables

Defn 22 (Random Variable)**.** A *random variable* is a mapping from an Event's outcome to a real number.

There are 2 types of random variables, based on what the mapping ends up with:

1. Discrete Random Variables are mapped to integers, $\mathbb{Z}$.
2. Continuous Random Variables are mapped to the real numbers, $\mathbb{R}$.

2.3.1 Discrete Random Variables

Defn 23 (Discrete Random Variable)**.** A *Discrete Random Variable* is one whose values are mapped from an Event's outcome to the integer numbers ($\mathbb{Z}$). These Random Variables are drawn from outcomes that are finite (sides on a die) or countably infinite.

The probability of a single value of the discrete random variable is denoted differently here than in the course material. The subscript refers to which discrete random variable we are working with (in this case X) and the variable in parentheses is the value we are calculating for (in this case $x \in X$).

$$p_X(x) \tag{2.9}$$

The sum of all probabilities for values that the discrete random variable can take **must** sum to 1.

$$\sum_{x \in X} p_X(x) = 1 \tag{2.10}$$

The mean or expected value of a discrete random variables is shown below:

$$\begin{aligned} \mu &= \sum_{x \in X} x p_X(x) \\ \mathbb{E}[X] &= \sum_{x \in X} x p_X(x) \end{aligned} \tag{2.11}$$

The variance of a discrete random variable is how "off" a value from the random variable is from the mean/expected value.

$$\begin{aligned} \sigma^2 &= \sum_{x \in X} (x - \mu)^2 p_X(x) \\ \mathrm{VAR}[X] &= \sum_{x \in X} \left(x - \mathbb{E}[X]\right)^2 p_X(x) \end{aligned} \tag{2.12}$$

The standard deviation is the square root of the variance.

$$\begin{aligned} \sigma = \sqrt{\sigma^2} &= \sqrt{\sum_{x \in X} (x - \mu)^2 p_X(x)} \\ \mathrm{STD}[X] = \sqrt{\mathrm{VAR}[X]} &= \sqrt{\sum_{x \in X} \left(x - \mathbb{E}[X]\right)^2 p_X(x)} \end{aligned} \tag{2.13}$$

There are 5 different Discrete Random Variable distributions that we will be heavily utilizing in this course.

2.3.1.1 Uniform Random Variable

Defn 24 (Uniform Random Variable)**.** The *uniform random variable* is a Discrete Random Variable whose probabilities for each outcome is equal.

For a Discrete Random Variable X, which has $|X|$ possible values,

$$p_X(x) = \frac{1}{|X|} \tag{2.14}$$

Example 2.1: Uniform Random Variable. Lecture 1

For example, the roll of a die is typically modelled as a uniform random variable. Find the probability distribution function, the expected value, and the variance.

Let's assume this is a 6-sided die. And let's map each side's number to a value in the range of $X \in [1, 6]$.
Using Equation (2.14), we can find the probability distribution easily.

$$p_X(x) = \begin{cases} \frac{1}{6} & x = 1 \\ \frac{1}{6} & x = 2 \\ \frac{1}{6} & x = 3 \\ \frac{1}{6} & x = 4 \\ \frac{1}{6} & x = 5 \\ \frac{1}{6} & x = 6 \end{cases}$$

Using Equation (2.11), we can find the the expected value/mean.

$$\begin{aligned} \mu = \mathbb{E}[X] &= \sum_{x=1}^{6} x p_X(x) \\ &= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} \\ &= \frac{1+2+3+4+5+6}{6} \\ &= \frac{21}{6} = 3.5 \end{aligned}$$

Using Equation (2.12), we can find the variance.

$$\begin{aligned} \sigma^2 = \mathrm{VAR}[X] &= \sum_{x=1}^{6} \left(x - \mathbb{E}[X]\right)^2 p_X(x) \\ &= \sum_{x=1}^{6} (x - 3.5)^2 \left(\frac{1}{6}\right) \\ &= 2.91667 \end{aligned}$$

Using Equation (2.13), we can find the standard deviation.

$$\begin{aligned} \sigma = \mathrm{STD}[X] &= \sqrt{\sum_{x=1}^{6} \left(x - \mathbb{E}[X]\right)^2 p_X(x)} \\ &= \sqrt{\sum_{x=1}^{6} (x - 3.5)^2 \left(\frac{1}{6}\right)} \\ &= \sqrt{2.91667} \\ &= 1.70783 \end{aligned}$$

2.3.1.2 Bernoulli Random Variable

Defn 25 (Bernoulli Random Variable)**.** The *Bernoulli random variable* is one where **only one** test occurs, and there are only 2 outcomes.

The probability of success is denoted

$$p_X(\text{success}) = p \tag{2.15}$$

The probability of failure is denoted

$$p_X(\text{failure}) = 1 - p \tag{2.16}$$

The mean/expected value is:

$$\mu = \mathbb{E}[X] = p \tag{2.17}$$

The variance is:

$$\sigma^2 = \mathrm{VAR}[X] = (1 - p)p \tag{2.18}$$

2.3.1.3 Binomial Random Variable

Defn 26 (Binomial Random Variable)**.** The *binomial random variable* is one where n trials are run with no stops for a success, where the Random Variable in each run is a Bernoulli Random Variable.

The probability of k successes with n trials is

$$\binom{n}{k}p^k(1-p)^{n-k} = \frac{n!}{k!(n-k)!}(1-p)^{n-k} \tag{2.19}$$

The mean/expected value after n trials is

$$\mu = \mathbb{E}[X] = np \tag{2.20}$$

The variance after n trials is

$$\sigma^2 = \mathrm{VAR}[X] = np(1-p) \tag{2.21}$$

2.3.1.4 Geometric Random Variable

Defn 27 (Geometric Random Variable)**.** The *geometric random variable* is one where n trials are run, where the nth trial is a success, meaning there are $n-1$ previous failures. The Random Variable in each run is a Bernoulli Random Variable.

This means **each trial** has a probability of success of

$$p_X(\text{success}) = p \tag{2.22}$$

And **each trial** has a probability of failure of

$$p_X(\text{failure}) = 1-p \tag{2.23}$$

The mean/expected value is

$$\mu = \mathbb{E}[X] = \frac{1}{p} \tag{2.24}$$

The variance is

$$\sigma^2 = \mathrm{VAR}[X] = \frac{1-p}{p^2} \tag{2.25}$$

2.3.1.5 Poisson Random Variable

Defn 28 (Poisson Random Variable)**.** The *Poisson random variable* is used to model the number of independent events that occur over a given period of time.

The Poisson random variable has one parameter,

$$\lambda \tag{2.26}$$

λ is the average number of events per unit of time.

The probability function for the value of $x \in X$ of this random variable is

$$p_X(x) = e^{-\lambda}\left(\frac{\lambda^x}{k!}\right) \tag{2.27}$$

The mean/expected value is

$$\mu = \mathbb{E}[X] = \lambda \tag{2.28}$$

The variance is

$$\sigma^2 = \mathrm{VAR}[X] = \lambda \tag{2.29}$$

2.3.2 Continuous Random Variables

Defn 29 (Continuous Random Variable)**.** A *Continuous Random Variable* is one whose values are mapped from an Event's outcome to the real numbers ($\mathbb{R}$).

Continuous random variables cannot be calculated at single points, but must be integrated over a range. This is called the Cumulative Distribution Function (CDF). The subscript refers to the random variable we are using, and x is the value being calculated for.

$$F_X(x) = \mathrm{P}(X \leq x) \tag{2.30}$$

Continuous random variables have a Probability Density Function (PDF), which is the derivative of the CDF.

$$f_X(x) = \frac{d}{dx}F_X(x) \tag{2.31}$$

This PDF must integrate to 1

$$\int\limits_{x \in X} f_X(x) = 1 \tag{2.32}$$

The expected value/mean is

$$\begin{aligned} \mu &= \int\limits_{x \in X} x f_X(x)\, dx \\ \mathbb{E}[X] &= \int\limits_{x \in X} x f_X(x)\, dx \end{aligned} \tag{2.33}$$

The variance is

$$\begin{aligned} \sigma^2 &= \int\limits_{x \in X} (x - \mu)^2 f_X(x)\, dx \\ \text{VAR}[X] &= \int\limits_{x \in X} \left(x - \mathbb{E}[X]\right)^2 f_X(x)\, dx \end{aligned} \tag{2.34}$$

2.3.2.1 Negative Exponential Random Variable

Defn 30 (Negative Exponential Random Variable)**.** The *negative exponential random variable*, sometimes shortened to exponential random variable, is a Continuous Random Variable that has 1 parameter μ, the rate of decay.

The negative exponential random variable has a PDF of

$$f_X(x) = \mu e^{-\mu x} \tag{2.35}$$

The negative exponential random variable has a CDF of

$$F_X(x) = 1 - e^{-\mu x} \tag{2.36}$$

Its mean/expected value is

$$\begin{aligned} \mu &= \frac{1}{\mu} \\ \mathbb{E}[X] &= \frac{1}{\mu} \end{aligned} \tag{2.37}$$

Its variance is

$$\begin{aligned} \sigma^2 &= \left(\frac{1}{\mu}\right)^2 \\ \text{VAR}[X] &= \left(\frac{1}{\mu}\right)^2 \end{aligned} \tag{2.38}$$

Its standard deviation is

$$\begin{aligned} \sigma &= \sqrt{\sigma^2} = \frac{1}{\mu} \\ \text{STD}[X] &= \sqrt{\text{VAR}[X]} = \frac{1}{\mu} \end{aligned} \tag{2.39}$$

2.3.2.2 Gaussian Random Variable

Defn 31 (Gaussian Random Variable)**.** The *Gaussian random variable*, sometimes called the normal random variable, has 2 parameters, μ and σ.

The Probability Density Function (PDF) is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x-\mu^2}{2\sigma^2}} \tag{2.40}$$

The mean/expected value is

$$\begin{aligned} \mu &= \mu \\ \mathbb{E}[X] &= \mu \end{aligned} \tag{2.41}$$

The variance is

$$\begin{aligned} \sigma^2 &= \sigma^2 \\ \text{VAR}[X] &= \sigma^2 \end{aligned} \tag{2.42}$$

The standard deviation is

$$\begin{aligned} \sigma &= \sqrt{\sigma^2} = \sigma \\ \text{STD}[X] &= \sqrt{\text{VAR}[X]} = \sigma \end{aligned} \tag{2.43}$$

2.4 Multiple Random Variables

Defn 32 (Joint Probability Function)**.** If X, Y are Discrete Random Variables, the probability of the joint event occurring is

$$p_{X,Y}(x, y) \tag{2.44}$$

where the subscripted X, Y refers to the Discrete Random Variables in use and x, y refers to the values being calculated for.

$$\mathrm{P}(x, y) = \mathrm{P}(X = x, Y = y) \tag{2.45}$$

Defn 33 (Joint Cumulative Distribution Function)**.** If X, Y are Continuous Random Variables, then their *joint cumulative distribution function* is

$$F_{X,Y}(x, y) = \mathrm{P}(X \leq x, Y \leq y) \tag{2.46}$$

Remark 33.1 (Independent Multiple Random Variables)*.* X, Y are independent if and only if

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \tag{2.47}$$

Defn 34 (Joint Probability Density Function)**.** If X, Y are Continuous Random Variables, then their *joint probability density function* is

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) \tag{2.48}$$

2.5 Properties of Expected Value

(i) Expected Value obeys the laws of linearity.

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b \tag{2.49}$$

(ii) For any Random Variables X, Y

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] \tag{2.50}$$

(iii) For any Independent Multiple Random Variables X, Y,

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] \tag{2.51}$$

2.6 Properties of Variance

(i) Variance does not obey the laws of linearity.

$$\mathrm{VAR}[aX + b] = a^2\,\mathrm{VAR}[X] \tag{2.52}$$

(ii) For any Independent Multiple Random Variables X, Y,

$$\mathrm{VAR}[X + Y] = \mathrm{VAR}[X] + \mathrm{VAR}[Y] \tag{2.53}$$

2.7 Covariance

Defn 35 (Covariance)**.** The *covariance* of two Random Variables is defined as

$$\begin{aligned} \mathrm{Cov}[X, Y] &= \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)\big(Y - \mathbb{E}[Y]\big)\Big] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] \end{aligned} \tag{2.54}$$

Remark 35.1 (Relationship Between Covariance and Variance)*.* Variance is actually a case of Covariance of a Random Variable with itself.

$$\mathrm{VAR}[X] = \mathrm{Cov}[X, X] \tag{2.55}$$

The Covariance of two Random Variables can have 3 possible values:

1. Positive: If one Random Variable increases, the other does too.
2. Negative: If one Random Variable increases, the other decreases.
3. Zero: If one Random Variable increases, the other does nothing.

2.8 Correlation Coefficient

Defn 36 (Correlation)**.** The *correlation of X and Y* is defined as the $1,1$ moment.

$$\mathbb{E}\left[X^1Y^1\right] \tag{2.56}$$

Defn 37 (Correlation Coefficient)**.** The *correlation coefficient of X and Y* is a measure of the **LINEAR relationship** between X and Y. It does not say anything about nonlinear dependence. It is defined as

$$\begin{aligned}\rho_{X,Y} &= \frac{\operatorname{Cov}[X,Y]}{\sigma_X\sigma_Y} \\ &= \frac{\operatorname{Cov}[X,Y]}{\operatorname{STD}[X]\operatorname{STD}[Y]}\end{aligned} \tag{2.57}$$

Remark 37.1. $\rho_{X,Y}$ only ranges from $-1 \leq \rho_{X,Y} \leq 1$.

Remark 37.2. The closer $\rho_{X,Y}$ are to $+1$, the closer X and Y are to having a positive linear relationship. The closer $\rho_{X,Y}$ are to -1, the closer X and Y are to having a negative linear relationship. The closer $\rho_{X,Y}$ are to 0, the closer X and Y are to being *uncorrelated*.

If $\rho_{X,Y} = 0$, then

- If X, Y are Independent Multiple Random Variables, then they are uncorrelated.
- If X, Y are uncorrelated ($\rho_{X,Y} = 0$), then they may still **not** be independent.

2.9 Stochastic Processes

Defn 38 (Stochastic Process)**.** A *stochastic process* or random process $\mathbf{x}(t)$ has 2 meanings:

1. For every time instant, $x(t)$ is a Random Variable.
2. For every point (sample) in an outcome space Ω, $x(t)$ is a real-valued function of time.

There are 4 different possible types:

1. Discrete-Time and discrete-value
2. Discrete-Time and continuous-value
3. Continuous-Time and discrete-value
    - Packet arrival to destination over time.
4. Continuous-Time and continuous-value
    - End-to-end delay in a network.

Defn 39 (Stationary Process)**.** A Stochastic Process $x(t)$ is called (weakly) stationary if:

- $\mathbb{E}[x(t)]$ is a constant, it is independent of t.
- The Correlation or Covariance of the process at 2 points in time $x(t_1)$ and $x(t_2)$, is a function of the difference $t_2 - t_1$ **only**.

2.10 The Poisson Process

Defn 40 (The Poisson Process)**.** This is a continuous-time, discrete-value Stochastic Process. It is also a Stationary Process. This process in **memoryless**, the previous time instant's values do not affect this time instant's value.

This is very commonly used to describe the arrivals into a queueing system or network. There is a parameter λ that is the average rate (packets per second, in this case).

There are 3 different definitions for this process:

1. Behaviour for a very small interval of time.
    - Approximately a Bernoulli Random Variable.

- The time interval is small enough such that only 1 event occurs with probability $\lambda\Delta t + o(\Delta t)$, which becomes $\lambda\Delta t$ for small Δt.
- 0 events occur with probability $1 - \lambda\Delta t + o(\Delta t)$, which becomes $1 - \lambda\Delta t$ for small Δt.
- Probabilities between non-overlapping intervals are independent

2. Behaviour over a longer period of time.
   - Receive multiple events, k.
   - Probability of k events is $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$
3. Behaviour between events.
   - Approximately a Negative Exponential Random Variable.
   - Time t between events, $p(t) = \lambda e^{-\lambda t}$

Poisson Process Definition 1/2 Equivalence. **TODO!** ■

Poisson Process Definition 2/3 Equivalence. **TODO!** ■

Poisson Process Definition 3/1 Equivalence. **TODO!** ■

2.10.1 Sums of the Poisson Processes

The superposition of several different Poisson processes.

- There are m independent Poisson Processes, with rates λ_i for $i = 1, 2, \ldots, m$
- Sum is also a Poisson Process: $\lambda = \sum_{i=1}^{m} \lambda_i$

3 Performance Evaluation

We do this to:

- Evaluate existing systems
- Design new network systems
- Predict system behaviours under different conditions

3.1 Performance Measures

How do we measure the performance of a large complex network?

- Data transfer speed
- Reliability:
  - Guaranteed throughput
  - Guarantee of any other performance measurement
  - Integrity of data
  - Predictability of errors
  - Uptime/Downtime/Availability
- Security
- User satisfaction
- Sustainability
- Maintainability
- Throughput/Goodput
- Delay/Latency
- Energy Efficiency
- Jitter (Delay variance)
- Packet Loss

3.2 Performance Evaluation

How can we evaluate the performance of a large complex network?

- Analysis: Mathematical modelling, calculations.
- Simulation: Software implementation of system model.
- Real-World Experimentation: Testing the actual system.

Analysis	Simulation	Experimentation
— Requires detailed understanding of system properties	+ Only requires modelling the environment with a straightforward implementation	++ No modelling or understanding of how the system required
— Usually requires approximations and simplifying assumptions.	+ Possible to implement complex details of system without approximation	++ Captures complete behaviour of system and environment without approximation.
++ Allows for deep insight for a broad range of scenarios.	+ Allows insight to broad range of scenarios.	— Requires deployment of every scenarios tested and may be difficult to reproduce.
+ Rare events and boundary cases are included.	+ Study of rare events is tricky, but possible.	— Rare events may be impossible to study.

Table 3.1: Performance Evaluation Pros and Cons

3.3 Statistical Data Analysis

Only analysis produces exact results. Simulation and experimentation produce samples from some underlying random distribution. This means we need to perform statistical analysis of these resuilts.

3.3.1 Sampling

We assume a random variable Z with an unknown probability distribution, but we can assume a distribution to start with. We estimate the key distribution metrics:

- Mean (1st moment)
- Variance (2nd moment)
- Variance of the variance (3rd moment)

We obtain n **independent** samples, $z_1, z_2, \ldots, z_n$. To estimate the mean/expected value, we use the equation below.

$$\bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i \tag{3.1}$$

Where $\bar{z}$ is also a Random Variable. So, we can perform an expected value calculation on $\bar{z}$.

$$\mathbb{E}[\bar{z}] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} z_i\right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[z_i] \tag{3.2}$$

So, as $n \to \infty$, $\mathbb{E}[\bar{z}] \to \mu$

A Complex Numbers

Complex numbers are numbers that have both a real part and an imaginary part.

$$z = a \pm bi \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Since this is an appendix section, I will denote complex numbers with i, to make it more general. However, electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

$$Ae^{-ix} = A\left[\cos\left(x\right) + i\sin\left(x\right)\right] \tag{A.3}$$

A.1 Complex Conjugates

If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{A.4}$$

Defn A.1.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign.

The complex conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{A.5}$$

A.1.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.6}$$

$$\overline{\log(z)} = \log(\overline{z}) \tag{A.7}$$

A.1.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned} \overline{\cos(x)} &= \cos(x) \\ &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \end{aligned} \tag{A.8}$$

$$\begin{aligned} \overline{\sin(x)} &= \sin(x) \\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \end{aligned} \tag{A.9}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.25}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.26}$$

C Calculus

C.1 Fundamental Theorems of Calculus

Defn C.1.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{C.1}$$

Defn C.1.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.2}$$

Defn C.1.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

C.2 Rules of Calculus

C.2.1 Chain Rule

Defn C.2.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.3}$$

D Laplace Transform

Defn D.0.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$