Math 252: Introduction to Differential Equations — Reference Sheet
Illinois Institute of Technology

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Contents

1 Introduction

This section will introduce the basic terminology and definitions for solving ordinary differential equations.

1.1 Definitions and Terminology

Defn 1 (Differential Equation)**.** A *differential equation (DE)* is an equation with 1 or more derivatives.

Remark 1.1. The highest differential determines the order of the differential equation. This means that the differential equation below is of order 2.

$$y'' + y = 0$$
$$\frac{d^2y}{dx^2} + y = 0$$

Defn 2. Initial Value Problem A differential equation with one or more initial conditions is called an *initial value problem (IVP).*

Remark 2.1. To solve an initial value problem, you must have the same number of initial conditions as the order of the differential equation.

Remark 2.2 (Existence of Unique Solution). R is a rectangular region on the xy-plane $a \le x \le b$, $c \le y \le d$ that contains (x_0, y_0) interior. If $f(x, y)$ and $\frac{df}{dy}$ are continuous on R, then an interval exists I_0 such that $(x_0 - h, x_0 + h)$ where $h > 0$, on the interval $[a, b]$, and a unique function $y(x)$, defined on I_0 that is a solution of the initial value problem.

1.2 Confirm If Differential Equation

You can confirm if the solution $y(x)$ found for a differential equation $y(x)'$ is the solution by differentiating the solution and putting that in the solved differential equation and verfiying that the equation holds true. This is shown in Example 1.1.

Example 1.1: Confirm Differential Solution.

Given the differential equation, $2y' + y = 0$, is $y = e^{\frac{-x}{2}}$ a solution?

$$y' = \frac{-1}{2} e^{\frac{-x}{2}}$$
$$2\left(\frac{-1}{2} e^{\frac{-x}{2}}\right) + \left(e^{\frac{-x}{2}}\right) = 0$$
$$-e^{\frac{-x}{2}} + e^{\frac{-x}{2}} = 0$$
$$0 = 0 \checkmark$$

1.3 Separable Differential Equation

Defn 3 (Separable)**.** A *separable* differential equation allows you to move various elements around to solve the equation. For example,

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P} dP = kdt$$
$$\ln(P) = kt + C$$
$$P = Ce^{kt}$$

Remark 3.1. These are used extensively in modelling phenomena with differential equations. These include: Population Growth, Radioactive Decay, Newton's Law of Cooling/Heating, and Spread of Disease.

1.4 Modeling with Differential Equations

1.4.1 Population Growth

Defn 4 (Population Growth)**.** *Population growth* can be modelled with a separable differential equation. Namely,

$$\frac{dP}{dt} = kP \tag{1.1}$$

Remark 4.1 (Population Growth Parameters)*.* The parameters for the Population Growth equation are given below.

- $k > 0$
- $P > 0$

1.4.2 Radioactive Decay

Defn 5 (Radioactive Decay)**.** *Radioactive decay* is the process that some particularly heave atoms undergo to become lighter, more stable atoms.

Defn 6 (Half-Life)**.** The *half-life* is the usual reported metric, and is defined as the amount of time required for an element to half its mass through Radioactive Decay.

$$\frac{1}{2}A_0 = A_0 e^{kt} \tag{1.2}$$

Remark 6.1 (Radioactive Decay Parameters)*.* The parameters for the Radioactive Decay equation are given below.

- $k < 0$
- $A > 0$

1.4.3 Newton's Law of Cooling/Heating

Defn 7 (Newton's Law of Cooling/Heating)**.** *Newton's Law of Cooling/Heating* is the same equation, but some of the parameters change. This equation is defined as:

$$\frac{dT}{dt} = k\left(T - T_m\right) \tag{1.3}$$

Remark 7.1*.* The parameters for the Newton's Law of Cooling/Heating equation are given below.

- $\frac{dT}{dt}$; The rate of change of temperature in the object per unit time.
- $k < 0$; The cooling constant and is unique to every object.
- T; The starting temperature.
- T_m; The temperature of the surrounding medium.

1.4.4 Spread of Disease

Defn 8 (Spread of Disease)**.** This is used to model the spread of something throughout a society or group of people.

$$\frac{dx}{dt} = kxy \tag{1.4}$$

Remark 8.1*.* The parameters for the Spread of Disease equation are given below.

- $\frac{dx}{dt}$; Change in the number of infected per unit time.
- $k < 0$; Transmission Constant
- x; Number of Infected
- y; Number of non-infected, y is really a function of x
  - $y = n + 1 - x$

1.4.5 Chemical Reactions

Defn 9 (Chemical Reactions)**.** These model how molecules interact in certain proportions to achieve some resultant molecule.

$$\frac{dx}{dt} = k(\alpha - x)(\beta - x) \tag{1.5}$$

Remark 9.1. The parameters for the Chemical Reactions equation are given below.

- x; Amount of resultant chemical
- k; Reaction rate, must be greater than 0, $k > 0$
- $\frac{dx}{dt}$; Rate of creation of resultant molecule per unit time
- α; Initial amount of Chemical "A"
- β; Initial amount of Chemical "B"
- $x(0) = 0$; Initial amount of resultant molecule must be 0 at the start

1.4.6 Tank Mixture

Defn 10 (Tank Mixture)**.** A well-mixed dissolved influent "thing" is brought into a tank and drained at some rate. What is the change in the amount of dissolved "thing" at any point in time?

$$\frac{dA}{dt} = R_{\text{in}} - R_{\text{out}} \tag{1.6}$$

Remark 10.1. The parameters for the Tank Mixture equation are given below.

- A; The amount of dissolved "thing"
- t; The time of time the tank has taken
- R_{in}; The rate of dissolved "thing" into the tank
- R_{out}; The rate of dissolved "thing" out of the tank

1.4.7 Torricelli's Law

Defn 11 (Torricelli's Law)**.** This equation relates the rate the volume in a tank changes to the height of the water to the hole in the tank.

$$\frac{dV}{dt} = -A_h\sqrt{2gh} \tag{1.7}$$

Remark 11.1. The parameters for the Torricelli's Law equation are given below.

- $V = A_w h$; The volume of water above the hole
- $\frac{dV}{dt} = A_w \frac{dh}{dt}$; The change in the volume of the water above the hole
- h; Height of the water
- A_h; Width of the hole
- A_w; Cross-sectional area of the tank

1.4.8 LRC Circuits

Defn 12 (LRC Circuits)**.** An *LRC Circuit* is analyzed in terms of the energy moving through the circuit. There is a unique relationship for the energy in each element:

$$E(t) = \frac{q}{C} \tag{1.8}$$

$$E(t) = RI = R\frac{dq}{dt} \tag{1.9}$$

$$E(t) = L\frac{dI}{dt} = L\frac{d^2q}{dt^2} \tag{1.10}$$

Remark 12.1. Depending on the circuit given, you might use a combination of these, but you ***must*** have at least one capacitor or inductor, otherwise it is not a differential equation.

Remark 12.2. These equations *add* together when the entire circuit is in series, i.e. the elements are put together back-to-back.

1.5 Linear and Non-Linear Differential Equations

Defn 13 (Linear Differential Equation)**.** A *linear differential equation* is one that satisfies one of the following equations below.

$$\begin{aligned} a_1(x)\frac{dy}{dx} + a_0(x) &= g(x) \\ a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x) &= g(x) \end{aligned} \tag{1.11}$$

Remark 13.1. The equations in Equation (1.11) can be generalized to the nth order as shown below.

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x) = g(x) \tag{1.12}$$

Defn 14 (Non-Linear)**.** A *non-linear* differential equation is one that does not satisfy the definition of a Linear Differential Equation. It does not obey Equation (1.12).

2 Solving First Degree Differential Equations

This section shows various ways to solve first-degree ordinary differential equations.

2.1 Solution Curves without a Solution

These differential equations are ones that do not have a solution. Instead, they can be categorized by Direction Fields

Defn 15 (Direction Fields)**.** *Direction fields* are similar to vector fields, in that they show how potential solutions could satisfy the equation.

Remark 15.1. It is important to note that this only applies to first order differential equations.

2.2 Separable Ordinary Differential Equations

These are some of the simplest ordinary differential equations to solve.

Defn 16 (Separable Ordinary Differential Equations)**.**

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \int \frac{1}{h(y)}dy &= \int g(x)\,dx \end{aligned} \tag{2.1}$$

Remark 16.1. To be *separable*, all functions of respective variables must be on the same side.

Example 2.1: Separable Ordinary Differential Equation-Example 1.

Solve

$$x\frac{dy}{dx} = 4y$$

$$\begin{aligned} \frac{1}{y}dy &= \frac{4}{x}dx \\ \ln|y| &= (4\ln|x| + C) \\ |y| &= x^4 \cdot e^C \\ y &= \pm e^C x^4 \\ y &= Cx^4 \end{aligned}$$

Now we have to check our answer.

$$\begin{aligned}\frac{dy}{dx} &= 4Cx^3\\ x\left(4Cx^3\right) &= 4y\\ 4x^4 &= 4y,\, C=1\end{aligned}$$

Example 2.2: Separable Ordinary Differential Equation-Example 2.

Solve

$$\frac{dP}{dt} = P\left(1-P\right)$$

$$\begin{aligned}\frac{1}{P\left(1-P\right)}dP &= dt\\ \int \frac{1}{P} + \frac{1}{1-P}dP &= dt\\ \ln\left(P\right) - \ln\left(1-P\right) &= t + C\\ \ln\left(\frac{P}{1-P}\right) &= e^{t+C}\\ \frac{P}{1-P} &= Ce^t\\ P &= Ce^t\left(1-P\right)\\ P + PCe^t &= Ce^t\\ P\left(1+Ce^t\right) &= Ce^t\\ P\left(t\right) &= \frac{Ce^t}{1+Ce^t}\end{aligned}$$

Example 2.3: Application of Newton's Law of Cooling/Heating.

Find the function for the constant for Newton's Law of Cooling/Heating, where $k = -2$ and the temperature of the surrounding medium is $T_m = 70$.

$$\begin{aligned}\frac{dT}{dt} &= k\left(T - T_m\right)\\ \frac{dT}{dt} &= -2\left(T-70\right)\\ \frac{1}{T-70}dT &= -2dt\\ \ln|T-70| &= -2t + C\\ |T-70| &= e^Ce^{-2t}\\ T-70 &= \pm e^Ce^{-2t}\\ T-70 &= Ce^{-2t}\\ T\left(t\right) &= Ce^{-2t}+70\\ T\left(0\right) &= Ce^0 + 70\\ T\left(0\right) &= C+70\\ C &= T\left(0\right) - 70\end{aligned}$$

2.3 Linear Differential Equations

Defn 17 (Linear Differential Equation)**.** A *linear differential equation* is one that satisfies the below equation.

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \\ a_1(x)y' + a_0(x)y = g(x) \tag{2.2}$$

This can be "simplified" down to:

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$
$$y' + P(x)y = f(x)$$

It will have an integrating factor of:

$$I(x) = e^{\int \frac{a_0(x)}{a_1(x)}\,dx} \tag{2.3}$$

If we plug the integrating factor into Equation (2.2), the we have:

$$I(x)\frac{dy}{dx} + I(x)\frac{a_0(x)}{a_1(x)}y = I(x)f(x)$$

But, using the Chain Rule you get

$$(I(x)y)' = I'(x)P(x) + I(x)P'(x)$$
$$I'(x)y = I(x)P(x)y + I(x)y'$$
$$I'(x) = I(x)P(x)$$

This means that the solution to a linear differential equation is simplified to

$$(I(x)y)' = I(x)f(x) \tag{2.4}$$

Remark 17.1. Each of the terms in Equation (2.2) can be functions that accept one parameter.

Example 2.4: Linear Differential Equation-Example 1.

Solve the differential equation:

$$\frac{dy}{dx} = 0.2xy$$

Example 2.5: Linear Differential Equation-Example 2.

Solve the differential equation:

$$\frac{dA}{dt} = 6 - \frac{1}{100}A$$

Example 2.6: Linear Differential Equation-Example 3.

Solve the differential equation:

$$y' + 3x^2y = x^2$$

2.4 Exact Differential Equations

Defn 18 (Exact Differential Equation)**.** An *exact differential equation* is one defined Equation (2.5).

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.5}$$

For a differential equation to be *exact* there are 2 criteria:

1. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. There exists a function of $f(x,y)$ such that:
   (a) $\frac{\partial f}{\partial x} = M(x,y) \rightarrow f(x,y) = \int M(x,y)\,dx$
   (b) $\frac{\partial f}{\partial y} = N(x,y) \rightarrow f(x,y) = \int N(x,y)\,dy$

3 Solving Differential Equations with Laplace Transforms

Example 3.1: Solve Differential Equation with Laplace Transform 1.

Given the ODE

$$y''(t) + 4y(t) = \delta(t-\pi)\ y(0) = 0,\ y'(0) = 0$$

$$\begin{aligned}
\mathcal{L}\{y''(t)\} + \mathcal{L}\{4y(t)\} &= \mathcal{L}\{\delta(t-\pi)\\
\left(-y'(0) - sy(0) + s^2Y(s)\right) + 4Y(s) &= e^{-s\pi}\\
\left(s^2Y(s) - 0 - s(0)\right) + 4Y(s) &= e^{-s\pi}\\
Y(s)\left(s^2+4\right) &= e^{-s\pi}\\
Y(s) &= \frac{e^{-s\pi}}{s^2+4}\\
y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s\pi}}{s^2+4}\right\} &= \left\{\frac{e^{-s\pi}}{s^2+4} + \frac{0}{s^2+4}\right\}
\end{aligned}$$

Now using our Laplace Transform table we receive our answer.

$$y(t) = 0 + \mathcal{U}(t-\pi)\frac{1}{2}\sin\left(2\left(t-\pi\right)\right)$$

Example 3.2: Solve Differential Equation with Laplace Transform 2.

Given the ODE

$$y''(t) + 2y'(t) + 2y(t) = \delta(t-\pi)\ y(0) = 1,\ y'(0) = 0$$

$$\begin{aligned}
\mathcal{L}\{y''(t)\} + \mathcal{L}\{2y'(t)\} + \mathcal{L}\{2y(t)\} &= \mathcal{L}\{\delta(t-\pi)\\
\mathcal{L}\{y''(t)\} + 2\,\mathcal{L}\{y'(t)\} + 2\,\mathcal{L}\{y(t)\} &= \mathcal{L}\{\delta(t-\pi)\\
\left(-y'(0) - sy(0) + s^2Y(s)\right) + 2\left(-y(0) + sY(s)\right) + 2Y(s) &= e^{-s\pi}\\
\left(-0 - 1s + s^2Y(s)\right) + 2\left(-1 + sY(s)\right) + 2Y(s) &= e^{-s\pi}\\
\left(-s + s^2Y(s)\right) + -2 + 2sY(s) + 2Y(s) &= e^{-s\pi}\\
s^2Y(s) + 2sY(s) + 2Y(s) &= e^{-s\pi} + s + 2\\
Y(s)\left(s^2 + 2s + 2\right) &= e^{-s\pi} + s + 2\\
Y(s) &= \frac{e^{-s\pi} + s + 2}{s^2 + 2s + 2}\\
Y(s) &= \frac{e^{-s\pi} + s + 2}{\left(s+1\right)^2 + 1}
\end{aligned}$$

Now we perform partial fraction decomposition and receive

$$\begin{aligned}
Y(s) &= \frac{e^{-s\pi}}{\left(s+1\right)^2+1} + \frac{s+1}{\left(s+1\right)^2+1} + \frac{1}{\left(s+1\right)^2+1}\\
y(t) &= \mathcal{L}\left\{\frac{e^{-s\pi}}{\left(s+1\right)^2+1}\right\} + \mathcal{L}\left\{\frac{s+1}{\left(s+1\right)^2+1}\right\} + \mathcal{L}\left\{\frac{1}{\left(s+1\right)^2+1}\right\}
\end{aligned}$$

Now, using our handy-dandy Laplace Transform table, we receive our answer.

$$y(t) = \mathcal{U}(t-\pi)e^{-(t-\pi)}\sin(t-\pi) + e^{-t}\cos(t-\pi) + e^{-t}\sin(t-\pi)$$

A Trigonometry

A.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{A.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{A.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{A.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.7}$$

A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{A.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{A.9}$$

A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{A.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{A.11}$$

A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{A.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{A.13}$$

A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1-\cos(2\alpha)}{2} \tag{A.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1+\cos(2\alpha)}{2} \tag{A.15}$$

A.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{A.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{A.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{A.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{A.19}$$

A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{A.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{A.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{A.22}$$

A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{A.23}$$

A.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{A.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{A.25}$$

A.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{A.26}$$

B Calculus

B.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcialable results.

Lemma B.0.1 (L'Hopital's Rule)**.** *If the equation*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (B.1) *holds.*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{B.1}$$

B.2 Fundamental Theorems of Calculus

Defn B.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{B.2}$$

Defn B.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{B.3}$$

Defn B.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

B.3 Rules of Calculus

B.3.1 Chain Rule

Defn B.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.
If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{B.4}$$

B.4 Useful Integrals

$$\int \cos(x)\, dx = \sin(x) \tag{B.5}$$

$$\int \sin(x)\, dx = -\cos(x) \tag{B.6}$$

$$\int x\cos(x)\, dx = \cos(x) + x\sin(x) \tag{B.7}$$

Equation (B.7) simplified with Integration by Parts.

$$\int x \sin(x)\; dx = \sin(x) - x\cos(x) \tag{B.8}$$

Equation (B.8) simplified with Integration by Parts.

$$\int x^2 \cos(x)\; dx = 2x\cos(x) + (x^2 - 2)\sin(x) \tag{B.9}$$

Equation (B.9) simplified by using Integration by Parts twice.

$$\int x^2 \sin(x)\; dx = 2x\sin(x) - (x^2 - 2)\cos(x) \tag{B.10}$$

Equation (B.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\; dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2 + \beta^2} + C \tag{B.11}$$

$$\int e^{\alpha x}\sin(\beta x)\; dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2 + \beta^2} + C \tag{B.12}$$

$$\int e^{\alpha x}\; dx = \frac{e^{\alpha x}}{\alpha} \tag{B.13}$$

$$\int x e^{\alpha x}\; dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{B.14}$$

Equation (B.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha + \beta x} = \int \frac{1}{\alpha + \beta x}\; dx = \frac{1}{\beta}\ln(\alpha + \beta x) \tag{B.15}$$

$$\int \frac{dx}{\alpha^2 + \beta^2 x^2} = \int \frac{1}{\alpha^2 + \beta^2 x^2}\; dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{B.16}$$

$$\int \alpha^x\; dx = \frac{\alpha^x}{\ln(\alpha)} \tag{B.17}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{B.18}$$

B.5 Leibnitz's Rule

Lemma B.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\, dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\, dx + f\big[b(t),t\big]\,\frac{db(t)}{dt} - f\big[a(t),t\big]\,\frac{da(t)}{dt} \tag{B.19}$$

C Laplace Transform

C.1 Laplace Transform

Defn C.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int\limits_{-\infty}^{\infty} x(t)e^{-st}dt \tag{C.1}$$

C.2 Inverse Laplace Transform

Defn C.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\, ds \tag{C.2}$$

C.3 Properties of the Laplace Transform

C.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (C.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{C.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{C.3b}$$

C.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (C.4).

$$\mathcal{L}\big\{x(\alpha t)\big\} = \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \tag{C.4}$$

C.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (C.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\big\{x(t-a)\big\} = X(s)e^{-as} \tag{C.5}$$

C.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\big\{X(s-a)\big\} = x(t)e^{at} \tag{C.6}$$

C.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\, d\lambda\right\} = \frac{1}{s} X(s) \tag{C.7}$$

C.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\big\{x(t) * v(t)\big\} = X(s)V(s) \tag{C.8}$$

C.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (C.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \tag{C.9}$$

C.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem C.1 (Intial Value Theorem)**.** *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Theorem C.2 (Final Value Theorem)**.** *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

C.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t-T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n\,\mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t-T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at}\,\mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at}\,\mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)\,\mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)\,\mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)\,\mathcal{U}(t)$	$\begin{cases} a: & \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: & \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb}+\frac{re^{-j\theta}}{s+a+jb}\right) \\ c: & \frac{As+B}{s^2+2as+c}\ \begin{cases} r & = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta & = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$
$e^{-at}\left(A\cos(\sqrt{c-a^2}t)+\frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

C.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0)-X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0)+X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n\frac{d^n}{ds^n}X(s)\ n\in\mathbb{N}$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s)-\sum_0^{n-1}s^{n-1-i}\frac{d^i}{dt^i}x(t)\rvert_{t=0^-}\ n\in\mathbb{N}$