

MMAE 320: Thermofluid Dynamics — Reference Material

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1 Introduction

Defn 1 (Energy). *Energy* is the stuff in the universe that can cause changes in material states. Some examples are:

- Thermal Energy
- Electrical Energy
- Mechanical Energy

One fundamental thing about Energy that was discovered was the Law of Conservation of Energy.

Defn 2 (Law of Conservation of Energy). The *law of conservation of energy* states that energy cannot be created or destroyed; it can only change form. This means that the total amount of energy in an interaction is constant.

Expressed as an equation, this is recognized as

$$\sum_{i \in I} E_i = C \quad (1.1)$$

where

I : The set of all interactions of interest.

E : The energy of this particular interaction, i .

C : The total energy of the system, a constant.

The Law of Conservation of Energy leads to Equation (1.2), shown below.

$$E_{in} - E_{out} = \Delta E \quad (1.2)$$

In a typical use, we are concerned with Heat as the form of Energy.

Defn 3 (Heat). *Heat* is the form of Energy that can be transferred from one system to another as a result of temperature difference.

Defn 4 (Thermodynamics). *Thermodynamics* is the science of equilibrium states and changes between these states.

Pure thermodynamic analysis will show the total amount of Energy moving through the system, but not the rate at which it occurs. This is where Heat Transfer comes in.

Defn 5 (Heat Transfer). *Heat transfer* is the science that deals with the determination of the *rates* of Heat-based Energy transfers. This science deals with systems that lack a thermal equilibrium, meaning it cannot be based on only principles of Thermodynamics.

For every flow of energy in a system, there **MUST** be a Driving Force.

Defn 6 (Driving Force). *Driving force* are the conditions of the system for the transfer of Energy to occur. There are any number of driving forces in the universe, some are listed here:

- Heat Transfer requires a temperature difference.
- Electric current flow requires a voltage difference.
- Fluid flow requires a pressure difference.

For Heat Transfer, the Temperature Gradient determines the rate of the transfer.

Defn 7 (Temperature Gradient). *Temperature gradient* is the temperature difference per unit length, or the rate of change of temperature.

Remark 7.1 (Extension). The use of gradients extends to other Driving Forces as well.

1.1 Fluid Mechanics

Defn 8 (Stress). *Stress* is defined as force per unit area.

$$\tau = \frac{F}{A} \quad (1.3)$$

Obedying Newton's third law, typically, there is a normal force per unit area, called the **normal stress**. In a fluid at rest, the normal stress is called **pressure**.

1.2 Units/Dimensions

1.2.1 SI System

There are 7 fundamental dimensions and units in the SI system, shown in Table 1.1.

Dimension	Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Temperature	kelvin (K)
Electric Current	ampere (A)
Amount of Light	candela (cd)
Amount of Matter	mole (mol)

Table 1.1: The 7 Fundamental Dimensions and Units

1.2.2 English System

The English system uses a very different set of units to describe these dimensions.

1.2.2.1 Mass The English unit of mass is the pound-mass.

$$\text{lbm}$$

1.2.2.2 Force The English unit of mass is the pound-force.

$$1 \text{ lbf} = 32.174 \text{ lbm ft/s}^2$$

1.2.2.3 Energy The English unit of energy is the British Thermal Unit (BTU). 1 btu raises the temperature of 1 lbm of water at 68°F by 1°F.

1.2.2.4 Work The English unit of work (energy per time) is also the watt (W).

1.2.2.5 Power The unit for power is typically the horsepower.

2 Basic Concepts

2.1 Systems

Defn 9 (System). A *system* is defined as a quantity of matter or a region in space chosen for study.

The mass or region outside the System is the **surroundings**. The surface that separates the System from the surroundings is the **boundary**.

2.1.1 Type of Systems

There are 2 types of systems:

1. Open System
2. Closed System

Defn 10 (Open System). An *open system* is a System in which the mass of the system is **not** constant. Thus, mass and Energy can flow from the system to the surroundings.

Remark 10.1 (Control Volume). Sometimes an Open System is called a *control volume*, because the volume of the system is constant.

Remark 10.2. Flow through these devices is typically easier by selecting the region based on volume, rather than mass.

Defn 11 (Closed System). A *closed system* is a System where the mass of the system **is** constant, but Energy can move between the system and the surroundings.

Remark 11.1 (Control Mass). Sometimes a Closed System is called a *control mass*, because mass is constant.

There is also a third type of system, the Isolated System, which is a special case of the Closed System.

Defn 12 (Isolated System). An *isolated system* is a System where **neither** mass or Energy can move from the system to the surroundings.

2.1.2 Properties of Systems

Defn 13 (Property). A *property* of a System is a characteristic of the system.

Some common properties are:

- P , Pressure
- T , Temperature
- V , Volume
- m , Mass

There are 2 types of properties:

1. Intensive Property
2. Extensive Property

Defn 14 (Intensive Property). *Intensive properties* are properties that are **independent** of the mass of the system. These include:

- P , Pressure
- T , Temperature
- ρ , Density

Defn 15 (Extensive Property). *Extensive properties* are properties that are **dependent** on the size or extent of the system. These include:

- Total mass
- Total volume
- Total momentum

Some extensive properties can be used to form a Specific Property.

Defn 16 (Specific Property). A *specific property* is an Extensive Property per unit mass. Some examples are:

- Specific Volume $v_s = \frac{V}{m}$
- Specific Total Energy $e_s = \frac{E}{m}$

2.1.3 Generalizations

We make some generalizations about the fluids we are working with to make calculations easier. For example, we treat fluids as a **continuum**, rather than the true atomic nature of the substance. This allows us to:

- Treat properties as point functions
- Assume properties vary continually in space with no discontinuities

This assumption is valid so long as the size of the System is relatively large compared to the space between the component molecules.

2.2 Density

Defn 17 (Density). *Density* is defined as mass per unit volume.

$$\rho = \frac{m}{V} \text{ kg/m}^3 \quad (2.1)$$

Defn 18 (Specific Volume). *Specific volume* is the reciprocal of Density, and is the amount of volume per unit mass.

$$v_s = \frac{V}{m} = \frac{1}{\rho} \quad (2.2)$$

For substances that lack a uniform mass and volume, density can also be realized by Equation (2.3).

$$\rho = \frac{dm}{dV} \quad (2.3)$$

Sometimes the density of a substance is given relative to another substance, usually water. This is Specific Gravity.

Defn 19 (Specific Gravity). *Specific gravity* is the Density of a substance as a ratio to another substance, usually water. This is expressed as

$$\text{SG} = \frac{\rho}{\rho_{\text{H}_2\text{O}}} \quad (2.4)$$

Defn 20 (Specific Weight). *Specific weight* is the weight of a unit volume of a substance. It is expressed as:

$$\gamma_s = \rho g \text{ N/m}^3 \quad (2.5)$$

2.3 States

Defn 21 (State). The *state* of the system includes all properties that can be measured or calculated which can completely describe the condition of the system.

In any given State, all the properties of a system will have fixed values. If any single property changes, all the others will change in accordance. We are typically interested in Equilibrium states.

Defn 22 (Equilibrium). *Equilibrium* is a State of a System which does not change when isolated from its surroundings. A system will only leave equilibrium when disturbed by an outside Energy, or an unbalanced Driving Force.

There are several different types of Equilibrium:

Thermal Equilibrium If the temperature throughout the entire System is constant.

Mechanical Equilibrium If there is no change in pressure at any point of the System with regards to time. However, pressure can vary within the system, but for most of our concerns, this isn't a problem.

Phase Equilibrium If there is more than one phase of matter, this type of equilibrium is reached with each phase reaches its equilibrium level and stays there.

Chemical Equilibrium If the chemical composition of a System does not change with time, i.e. no chemical reactions could occur.

2.3.1 The State Postulate

Defn 23 (State Postulate). The *state postulate* states: the State of a simple compressible system is **completely** specific by 2 independent, intensive properties.

Remark 23.1 (Simple Compressible System). A system is called a simple compressible system in the absence of electrical, magnetic, gravitational, motion, and surface tension effects.

Properties are **independent** if one property can be varied and the other one remains constant.

2.4 Processes

Defn 24 (Process). A *process* is the change in a System from one Equilibrium state to another. The **path** of the process is the set of Quasi-equilibrium processes.

It is easiest to deal with paths where if one of the variables is changed just a small amount, the change becomes linear. This is similar to the concept of a derivative, because if we sample at a small/short enough rate, the changes introduced to the System will be linear.

Defn 25 (Quasi-equilibrium). A *quasi-equilibrium* process is one that is sufficiently slow so that the System can adjust itself internally so that properties change constantly **throughout** the system.

Lastly, if the Process is a Cycle, then the initial and final States are the same.

Defn 26 (Cycle). A *cycle* is a Process that returns to its initial state at the end of the process.

2.4.1 *iso*- Processes

Some Processes are special in that one Property remains constant. There are several of these:

- Isothermal
- Isobaric
- Isometric

Defn 27 (Isothermal). *Isothermal* Processes are ones where the temperature T remains constant.

Defn 28 (Isobaric). *Isobaric* Processes are ones where the pressure P remains constant.

Defn 29 (Isometric). *Isometric* (*Isochoric*) Processes are ones where the volume V remains constant.

2.5 Temperature

Defn 30 (0th Law of Thermodynamics). The *0th law of thermodynamics* states that if two bodies are in thermal Equilibrium with a third body, they are also in thermal equilibrium with each other.

2.5.1 Subjective Temperature Scales

There are 2 subjective temperature scales:

1. Celsius ($^{\circ}\text{C}$)
2. Fahrenheit ($^{\circ}\text{F}$)

These are based off of physical temperatures at certain points on Earth regarding water freezing and water boiling. These are useful for regular use, but not as well-suited for thermodynamic use.

$$\frac{5}{9}(T(^{\circ}\text{F}) - 32) = T(^{\circ}\text{C}) \quad (2.6)$$

2.5.2 Objective Temperature Scales

Objective temperature scales are ones that are based off of physical, universal constants. There are 2 objective temperature scales:

1. Kelvin (K)
2. Rankine (R)

Both of these have placed 0 at the point where all molecular motion stops.

The equations to conversion between Celsius and Kelvin is shown in Equation (2.7).

$$T(\text{K}) = T(^{\circ}\text{C}) + 273.15 \quad (2.7)$$

The equations to conversion between Fahrenheit and Rankine is shown in Equation (2.8).

$$T(\text{R}) = T(^{\circ}\text{F}) + 459.67 \quad (2.8)$$

2.6 Pressure

Defn 31 (Pressure). *Pressure* is the normal **scalar** force exerted per unit area. Because it is a scalar, it has no dependency on the direction of the normal force. It can be expressed using Density, gravity, and depth/length.

$$P = \rho g d \quad (2.9)$$

The SI unit of pressure is the pascal:

$$\text{Pa} = \frac{\text{N}}{\text{m}^2} \quad (2.10)$$

The bar is also used:

$$1 \text{ bar} = 100 \text{ kPa}$$

The English unit of pressure is the pound-force per square inch.

$$\text{lbf/in}^2 = \text{psi} \quad (2.11)$$

We exist, roughly, at one atmosphere of pressure, or 1 atm. The equivalents of this are:

$$1 \text{ atm} = 101\,325 \text{ kPa}$$

$$1 \text{ atm} = 14.696 \text{ psi}$$

Using this, we can measure pressure, typically with either Gage Pressure or Vacuum Pressure.

Defn 32 (Gage Pressure). *Gage pressure* is the Pressure that a gage reads after being calibrated to some standard pressure, typically one atmosphere. Thus, the actual pressure of something read by the gage is represented by Equation (2.12) below.

$$P_{\text{abs}} = P_{\text{atm}} + P_{\text{gage}} \quad (2.12)$$

Defn 33 (Vacuum Pressure). *Vacuum pressure* is the Pressure that a gage reads after being calibrated to some standard pressure, typically one atmosphere. Thus, the actual pressure of something read by the gage is represented by Equation (2.13) below.

$$P_{\text{abs}} = P_{\text{atm}} - P_{\text{vacuum}} \quad (2.13)$$

2.6.1 Pressure Inside a Fluid

Pressure inside a fluid increases linearly with depth. This is seen in Equation (2.14), where Δz is the distance between the two measured points.

$$\begin{aligned}P_{\text{below}} &= P_{\text{above}} + \rho g |\Delta z| \\ &= P_{\text{above}} + \gamma_s |\Delta z|\end{aligned}\tag{2.14}$$

ρ : The density of the fluid (kg/m^3)

Defn 34 (Pascal's Law). *Pascal's Law* states that the Pressure along any horizontal plane in the same fluid is the same pressure. This leads to Equation (2.15).

$$\begin{aligned}P_1 &= P_2 \\ \frac{\vec{F}_1}{A_1} &= \frac{\vec{F}_2}{A_2}\end{aligned}\tag{2.15}$$

A Complex Numbers

Complex numbers are numbers that have both a real part and an imaginary part.

$$z = a \pm bi \quad (\text{A.1})$$

where

$$i = \sqrt{-1} \quad (\text{A.2})$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j . Since this is an appendix section, I will denote complex numbers with i , to make it more general. However, electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instead.

$$Ae^{-ix} = A [\cos(x) + i \sin(x)] \quad (\text{A.3})$$

A.1 Complex Conjugates

If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\bar{z} = a \mp bi \quad (\text{A.4})$$

Defn A.1.1 (Complex Conjugate). The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign.

The complex conjugate can also be denoted with an asterisk (*). This is generally done for complex functions, rather than single variables.

$$z^* = \bar{z} \quad (\text{A.5})$$

A.1.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\bar{z}} \quad (\text{A.6})$$

$$\overline{\log(z)} = \log(\bar{z}) \quad (\text{A.7})$$

A.1.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned} \overline{\cos(x)} &= \cos(x) \\ &= \frac{1}{2} (e^{ix} + e^{-ix}) \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} \overline{\sin(x)} &= \sin(x) \\ &= \frac{1}{2i} (e^{ix} - e^{-ix}) \end{aligned} \quad (\text{A.9})$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{B.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{B.2})$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j \sin(\alpha) \quad (\text{B.3})$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \quad (\text{B.4})$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \quad (\text{B.5})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{B.6})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{B.7})$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \quad (\text{B.8})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \quad (\text{B.9})$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad (\text{B.10})$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (\text{B.11})$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \quad (\text{B.12})$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \quad (\text{B.13})$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = (\sin(\alpha))^2 = \frac{1 - \cos(2\alpha)}{2} \quad (\text{B.14})$$

$$\cos^2(\alpha) = (\cos(\alpha))^2 = \frac{1 + \cos(2\alpha)}{2} \quad (\text{B.15})$$

B.7 Product-to-Sum Identities

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad (\text{B.16})$$

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad (\text{B.17})$$

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (\text{B.18})$$

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (\text{B.19})$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{B.20})$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{B.21})$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (\text{B.22})$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \quad (\text{B.23})$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2} e^{j\theta} = r e^{j\theta} \quad (\text{B.24})$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \quad (\text{B.25})$$

B.11 Polar to Rectangular

$$r e^{j\theta} = r \cos(\theta) + j r \sin(\theta) \quad (\text{B.26})$$

C Calculus

C.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcilable results.

Lemma C.0.1 (L'Hopital's Rule). *If the equation*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) holds.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad (\text{C.1})$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{C.2})$$

Defn C.2.2 (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I , and states that if F is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{C.3})$$

Defn C.2.3 (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule). The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot h(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \quad (\text{C.4})$$

C.4 Useful Integrals

$$\int \cos(x) dx = \sin(x) \quad (\text{C.5})$$

$$\int \sin(x) dx = -\cos(x) \quad (\text{C.6})$$

$$\int x \cos(x) dx = \cos(x) + x \sin(x) \quad (\text{C.7})$$

Equation (C.7) simplified with Integration by Parts.

$$\int x \sin(x) dx = \sin(x) - x \cos(x) \quad (\text{C.8})$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2 \cos(x) dx = 2x \cos(x) + (x^2 - 2) \sin(x) \quad (\text{C.9})$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2 \sin(x) dx = 2x \sin(x) - (x^2 - 2) \cos(x) \quad (\text{C.10})$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x} \cos(\beta x) dx = \frac{e^{\alpha x} (\alpha \cos(\beta x) + \beta \sin(\beta x))}{\alpha^2 + \beta^2} + C \quad (\text{C.11})$$

$$\int e^{\alpha x} \sin(\beta x) dx = \frac{e^{\alpha x} (\alpha \sin(\beta x) - \beta \cos(\beta x))}{\alpha^2 + \beta^2} + C \quad (\text{C.12})$$

$$\int e^{\alpha x} dx = \frac{e^{\alpha x}}{\alpha} \quad (\text{C.13})$$

$$\int x e^{\alpha x} dx = e^{\alpha x} \left(\frac{x}{\alpha} - \frac{1}{\alpha^2} \right) \quad (\text{C.14})$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha + \beta x} = \int \frac{1}{\alpha + \beta x} dx = \frac{1}{\beta} \ln(\alpha + \beta x) \quad (\text{C.15})$$

$$\int \frac{dx}{\alpha^2 + \beta^2 x^2} = \int \frac{1}{\alpha^2 + \beta^2 x^2} dx = \frac{1}{\alpha \beta} \arctan \left(\frac{\beta x}{\alpha} \right) \quad (\text{C.16})$$

$$\int \alpha^x dx = \frac{\alpha^x}{\ln(\alpha)} \quad (\text{C.17})$$

$$\frac{d}{dx} \alpha^x = \frac{d\alpha^x}{dx} = \alpha^x \ln(\alpha) \quad (\text{C.18})$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule). *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x, t) dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x, t)}{\partial t}$ continuous in both t and x , then

$$\frac{d}{dt} g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx + f[b(t), t] \frac{db(t)}{dt} - f[a(t), t] \frac{da(t)}{dt} \quad (\text{C.19})$$

D Laplace Transform

D.1 Laplace Transform

Defn D.1.1 (Laplace Transform). The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \quad (\text{D.1})$$

D.2 Inverse Laplace Transform

Defn D.2.1 (Inverse Laplace Transform). The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st} ds \quad (\text{D.2})$$

D.3 Properties of the Laplace Transform

D.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (D.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad (\text{D.3a})$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \quad (\text{D.3b})$$

D.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (D.4).

$$\mathcal{L}\{x(\alpha t)\} = \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \quad (\text{D.4})$$

D.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (D.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\{x(t-a)\} = X(s)e^{-as} \quad (\text{D.5})$$

D.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\{X(s-a)\} = x(t)e^{at} \quad (\text{D.6})$$

D.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda) d\lambda\right\} = \frac{1}{s} X(s) \quad (\text{D.7})$$

D.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\{x(t) * v(t)\} = X(s)V(s) \quad (\text{D.8})$$

D.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (D.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \quad (\text{D.9})$$

D.4 Theorems

There are 2 theorems that are most useful here:

1. Initial Value Theorem
2. Final Value Theorem

Theorem D.1 (Initial Value Theorem). *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$$

Theorem D.2 (Final Value Theorem). *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$$

D.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t - T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n \mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t - T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at} \mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at} \mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt) \mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt) \mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at} \cos(bt) \mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at} \sin(bt) \mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at} \cos(bt + \theta) \mathcal{U}(t)$	$\begin{cases} a : \frac{sr \cos(\theta) + ar \cos(\theta) - br \sin(\theta)}{s^2 + 2as + (a^2 + b^2)} \\ b : \frac{1}{2} \left(\frac{re^{j\theta}}{s+a-jb} + \frac{re^{-j\theta}}{s+a+jb} \right) \\ c : \frac{As+B}{s^2+2as+c} \begin{cases} r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$
$e^{-at} \left(A \cos(\sqrt{c-a^2}t) + \frac{B-Aa}{\sqrt{c-a^2}} \sin(\sqrt{c-a^2}t) \right) \mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

D.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t) \sin(\omega_0 t)$	$\frac{j}{2} (X(s + j\omega_0) - X(s - j\omega_0))$
$x(t) \cos(\omega_0 t)$	$\frac{1}{2} (X(s + j\omega_0) + X(s - j\omega_0))$
$t^n x(t)$	$(-1)^n \frac{d^n}{ds^n} X(s) \quad n \in \mathbb{N}$
$\frac{d^n}{dt^n} x(t)$	$s^n X(s) - \sum_{i=0}^{n-1} s^{n-1-i} \frac{d^i}{dt^i} x(t) _{t=0^-} \quad n \in \mathbb{N}$