ECE 311: Engineering Electronics — Reference Sheet
Illinois Institute of Technology

Karl Hallsby

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Contents

List of Theorems

1 Operational Amplifiers

Defn 1 (Op-Amp)**.** An *op-amp*, (*operational amplifier*), is a active circuit element that is a 2-port network element. An circuit symbol for an op-amp is shown in Figure 1.1.

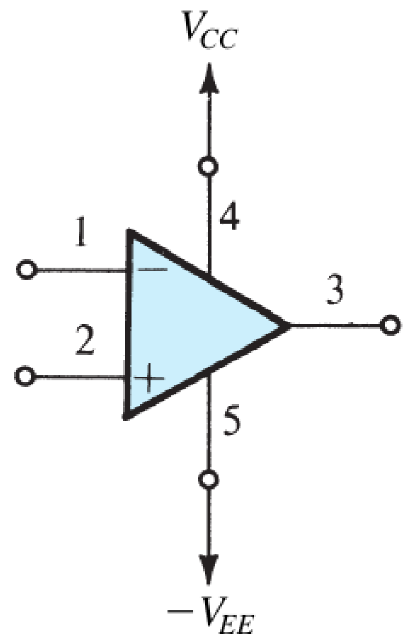


Figure 1.1: Complete Circuit Symbol for an Op-Amp (Sedra et al. 2015, p. 60)

Typically, terminals 4 and 5 are **not** included in circuit schematics, as they are implied to exist, because the Op-Amp requires input power to operate.

Defn 2 (Open-Loop Gain)**.** The *open-loop gain* of an Op-Amp, typically represented with A is the gain in voltage at the output given an input voltage. The equation for the open-loop gain is given in Equation (1.1).

$$A = \frac{V_O}{V_+ - V_-} \tag{1.1}$$

1.1 The Ideal Op-Amp

The ideal Op-Amp is one that has:

1. Infinite input impedance
2. Zero output impedance
3. Zero common-mode gain/infinite common-mode rejection
4. Infinite Open-Loop Gain
5. Infinite bandwidth (There is no distortion of the output signal due to amplification.)

This can be summarized using just two equations, shown in Equations (1.2a) and (1.2b).

$$V_+ = V_- \tag{1.2a}$$

$$I_+ = I_- = 0 \tag{1.2b}$$

When connecting an Op-Amp, even an ideal one, to a circuit, the Closed-Loop Gain becomes a factor.

Defn 3 (Closed-Loop Gain)**.** The *closed-loop gain* occurs when an Op-Amp is connected to a surrounding circuit. It's defining equation is shown in Equation (1.3).

$$G = \frac{V_O}{V_I} \tag{1.3}$$

1.2 Non-Ideal Op-Amps

We have only considered ideal Op-Amps throughout this text. However, in the real world, like everything, op-amps do not behave ideally. In the subsections below, the various non-ideal properties are discussed.

1.2.1 Offset Voltage

The first major non-ideal behavior Op-Amps have is that even when the differential input voltage is zero ($V_+ - V_- = 0$), there is a finite output voltage. This is because of an internal voltage called the Input Offset Voltage.

Defn 4 (Input Offset Voltage)**.** The *input offset voltage* is a non-ideal behavior of Op-Amps. The typical symbol is V_{OS}, and typically has the units µV/°C. For most Op-Amps, V_{OS} is typically in the range $1\,\text{mV} \leq V_{OS} \leq 5\,\text{mV}$.

A way to model a non-ideal Op-Amp that is being affected by an Input Offset Voltage is shown in Figure 1.2.

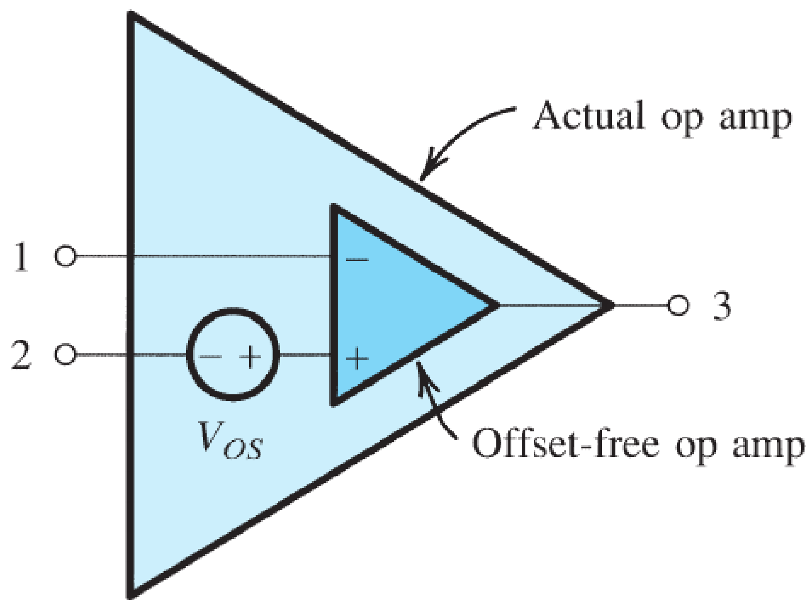


Figure 1.2: Circuit Model for Op-Amp with Input Offset Voltage V_{OS} (Sedra et al. 2015, p. 97)

1.2.2 Input Bias and Offset Currents

The other major non-ideal behavior Op-Amps have is that they have Input Bias Currents and Input Offset Current.

Defn 5 (Input Bias Current)**.** The *input bias current*s are due to the fact that op-amps require DC input currents. These are independent of the fact that a real Op-Amp has finite input resistance. The two input bias currents are: $I_{B,1}$ and $I_{B,2}$. The average value of the input bias current is usually provided on component datasheets. The equation for the average value is given in Equation (1.4).

$$I_B = \frac{I_{B,1} + I_{B,2}}{2} \tag{1.4}$$

Defn 6 (Input Offset Current)**.** The *input offset current* is the differenceen the Input Bias Currents. The mathematical symbol for this current is I_{OS}

$$I_{OS} = |I_{B,1} - I_{B,2}| \tag{1.5}$$

A figure depicting the setup of an Op-Amp that has bias currents is shown in Figure 1.3.

2 Semiconductors

2.1 Intrinsic Semiconductors

Defn 7 (Semiconductor)**.** *Semiconductor*s are materials whose conductivity is somewhere between that of true conductors, like copper, and insulators, such as glass. Because semiconductors are somewhere between conductors and insulators, they have electrical properties that are easily manipulated through Doping.

Defn 8 (Electron)**.** An *electron* in this scenario is a **free electron**. This means the electron is not bound to any particular atomic nucleus. Such an electron is free to conduct electric current if an electric field is applied.

If an atom is missing electrons due to an electron being free, a Hole can be thought of in its place.

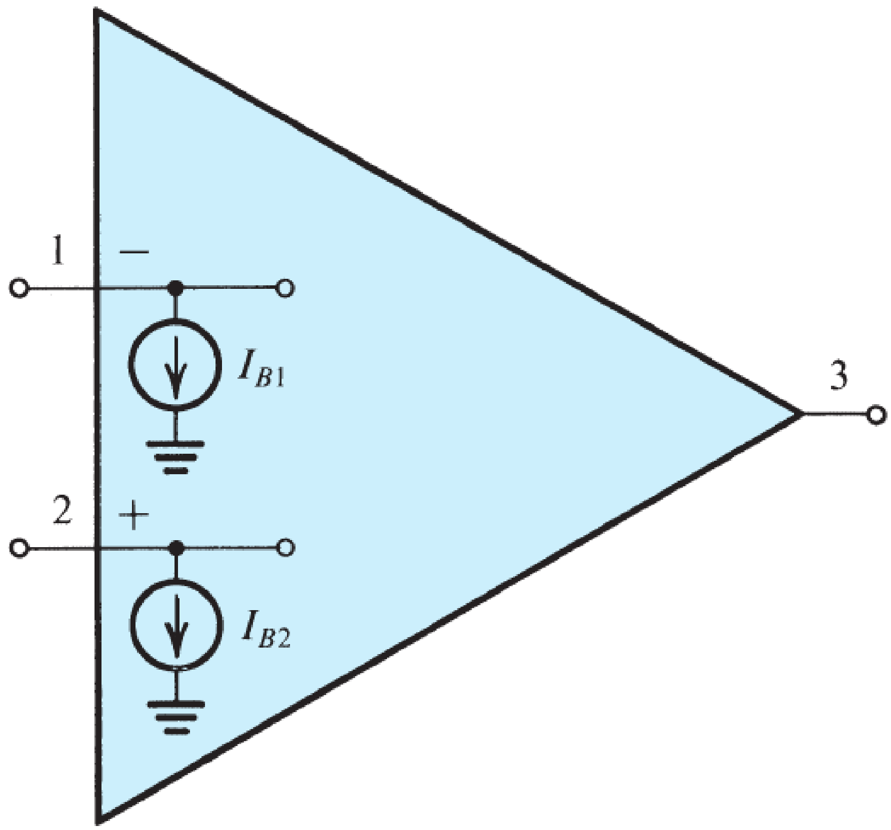


Figure 1.3: Circuit Model for Op-Amp with Input Bias Current and Input Offset Current (Sedra et al. 2015, p. 101)

The concentration of free Electrons in a material is given the symbol shown in Equation (2.1).

$$n \tag{2.1}$$

Defn 9 (Hole)**.** A *hole* is the lack of an Electron being attached to an atom. These have the same, but opposite charge of an electron.

These are NOT particles in any physical sense. Holes are useful abstractions to use when thinking about current flow in a semiconductor's crystal lattice.

The concentration of free Holes in a material is given the symbol shown in Equation (2.2).

$$p \tag{2.2}$$

Defn 10 (Intrinsic)**.** An *intrinsic* material is one that is pure. It has a regular lattice structure, where atoms are held in place by covalent bonds.

In an Intrinsic material, the concentration of free Electrons and Holes are equal. This is represented by the relation in Equation (2.3).

$$n = p = n_i \tag{2.3}$$

Typically, we express the product of Hole and free Electron concentrations as a product, shown in Equation (2.4).

$$pn = n_i^2 \tag{2.4}$$

Semiconductor physics tells us that n_i is defined by Equation (2.5).

$$n_i = BT^{\frac{3}{2}} e^{\frac{-E_g}{2kT}} \tag{2.5}$$

where

B is a material-dependent parameter.
T is the temperature
k is Boltzmann's constant.

2.2 Doped Semiconductors

Defn 11 (Doping)**.** *Doping* is the process of deliberately adding atomic impurities to alter the electrical/conductivity characteristics of Semiconductors. This is done by substantially increasing the concentration of Electrons or Holes, but without changing the crystal properties of the original semiconductor.

There are two kinds of doping:

1. n Type. This is done by doping with an element with 5 valence electrons, typically phosphorus.
2. p Type. This is done by doping with an element with 3 valence electrons, typically boron.

If the concentration of donor atoms in an n-type doped Semiconductor is N_D, where N_D is much greater than n_i ($N_D \gg p$), then the concentration of **free Electrons** is:

$$n_n \simeq N_D \tag{2.6}$$

By substituting Equation (2.6) into Equation (2.5), we can find the Hole concentration for an n-type Semiconductor.

$$p_n \simeq \frac{n_i^2}{N_D} \tag{2.7}$$

In an n-type doped Semiconductor, the free Electrons have a significantly larger concentration, and are said to be the **majority charge carriers**. The Holes are the **minority charge carriers**.

The complete opposite of this holds true for p-type doped Semiconductors. Meaning,

$$p_p \simeq N_A$$

$$n_p \simeq \frac{n_i^2}{N_A}$$

Similarly oppositely, in an p-type doped Semiconductor, the free Holes have a significantly larger concentration, and are said to be the **majority charge carriers**. The free Electrons are the **minority charge carriers**.

2.3 Current Flow in Semiconductors

2.3.1 Drift Current

Defn 12 (Drift Current)**.** *Drift current* arises when an electric field $\vec{E}$ is applied to a Semiconductor. The Holes are accelerated **in** the direction of $\vec{E}$, and the free Electrons are accelerated in the **opposite** direction of $\vec{E}$.

In the presence of an electric field, because the drift current is made of of Electrons and Holes moving due to a force field, they have a velocity. This velocity is the Drift Velocity.

Defn 13 (Drift Velocity)**.** *Drift velocity* is the velocity that Holes or Electrons gain when an electric field (voltage) is applied to the Semiconductor. There are two separate equations for drift velocity, one for Holes and one for Electrons, shown in Equations (2.8) and (2.9), respectively.

$$v_{p-\text{drift}} = \mu_p \vec{E} \tag{2.8}$$

$$v_{n-\text{drift}} = -\mu_n \vec{E} \tag{2.9}$$

Equation (2.9) is negative because Electrons move in the opposite direction of the electric field $\vec{E}$.
In Equations (2.8) and (2.9), the constants μ_p and μ_n are used. μ_p is the Hole Mobility. μ_n is the Electron Mobility.

Defn 14 (Hole Mobility)**.** *Hole mobility*, μ_p is a value representing how "easy" it is for Holes to move through the Semiconductor's crystal structure in response to an applied electric field, $\vec{E}$.
The hole mobility for Intrinsic silicon is a known constant, and is shown in Equation (2.10).

$$\mu_p = 480\,\text{cm}^2/(\text{V s}) \tag{2.10}$$

Defn 15 (Electron Mobility)**.** *Electron mobility*, μ_n is a value representing how "easy" it is for Electrons to move through the Semiconductor's crystal structure in response to an applied electric field, $\vec{E}$.
The electron mobility for Intrinsic silicon is a known constant, and is shown in Equation (2.11).

$$\mu_n = 1350\,\text{cm}^2/(\text{V s}) \tag{2.11}$$

Remark 15.1. Electrons move through a semiconductor's crystal lattice much more easily than Holes do. This can be seen by $\mu_n \approx 2.5\mu_p$.

We are interested in the current flowing through an object due to an applied electric field, so we use:

$$I_P = Aqpv_{p-\text{drift}} \tag{2.12}$$

If we substitute $v_{p-\text{drift}}$ with our knowledge from Equation (2.8), then we have the equation below.

$$I_P = Aqp\mu_p \vec{E}$$

If we divide this equation by the cross-sectional area, A, then we have Equation (2.13).

$$J_p = \frac{I_P}{A} = qp\mu_p \vec{E} \tag{2.13}$$

Similarly, we can find the drift current equation for I_N and the current density equation.

$$I_N = -Aqnv_{p-\text{drift}} \tag{2.14}$$

$$J_n = qn\mu_n \vec{E} \tag{2.15}$$

Then the total drift current is just the addition of the two separate drift currents.

$$J = J_p + J_n \tag{2.16}$$

By factoring out the constant terms, we end up with:

$$\begin{aligned}
J &= J_p + J_n \\
&= qp\mu_p \vec{E} + qn\mu_n \vec{E} \\
&= q(p\mu_p + n\mu_n)\vec{E} \\
J &= \sigma \vec{E}
\end{aligned}$$

This leads to two important equations, Equations (2.17a) and (2.17b).

$$J = \sigma \vec{E} \tag{2.17a}$$

$$J = \frac{\vec{E}}{\rho} \tag{2.17b}$$

Defn 16 (Conductivity)**.** *Conductivity*, typically represented with σ, is how good a conductor an object is.

$$\sigma = q(p\mu_p + n\mu_n) \tag{2.18}$$

Defn 17 (Resistivity)**.** *Resistivity*, typically represented with ρ, is how good an object is at preventing the flow of current.

$$\rho \equiv \frac{1}{\sigma} = \frac{1}{q(p\mu_p + n\mu_n)}\ \Omega\,\text{cm} \tag{2.19}$$

2.3.2 Diffusion Current

Defn 18 (Diffusion Current)**.** *Diffusion current* arises due to concentration differences in Electrons and Holes in a semiconductor. It travels from the p side to the n side.

The Diffusion Current density is proportional to the slope of the concentration gradient at any given point in the Semiconductor. For example, if a Semiconductor has **holes** added to it such that there is a larger amount of holes at one side of a block than the other, we end up with Equation (2.20).

$$J_p = -qD_p\frac{dp(x)}{dx} \tag{2.20}$$

Similarly, for a free Electron gradient, we have Equation (2.21).

$$J_n = qD_n\frac{dn(x)}{dx} \tag{2.21}$$

Defn 19 (Hole Diffusivity)**.** *Hole diffusivity* or the *hole diffusion constant* is a constant representing how easy it is for a Hole to diffuse through the substrate's crystal lattice. It is given the symbol D_p.

Defn 20 (Electron Diffusivity)**.** *Electron diffusivity* or the *electron diffusion constant* is a constant representing how easy it is for a free Electron to diffuse through the substrate's crystal lattice. It is given the symbol D_n.

There exists a relationship between the diffusivity constants (Definitions 19 and 20) and the mobility constants (Definitions 14 and 15), as seen in Equation (2.22).

$$\frac{D_n}{\mu_n} = \frac{D_p}{\mu_p} = V_T \tag{2.22}$$

The term, V_T is the Thermal Voltage.

Defn 21 (Thermal Voltage)**.** The *thermal voltage*, V_T is the voltage produced within the pn-Junction due to temperature.

$$V_T = \frac{kT}{q} \tag{2.23}$$

where

k Boltzmann's Constant
T Temperature
q Charge of an electron

2.4 The pn-Junction Junction

A simplified image of the physical structure of a pn-Junction is shown in Figure 2.1.

In a pn-Junction, the p's majority carriers are Holes. They are typically drawn as "+". Similarly, the n's majority carriers are Electrons, and are typically drawn as "-".

For the rest of this part of the discussion about pn-Junctions, we will assume the junction is **not** connected to anything.

There are several "steps" the junction goes through if we imagine the junction starting with a perfect separation of majority carriers. Each of the steps is discussed in a subsection below.

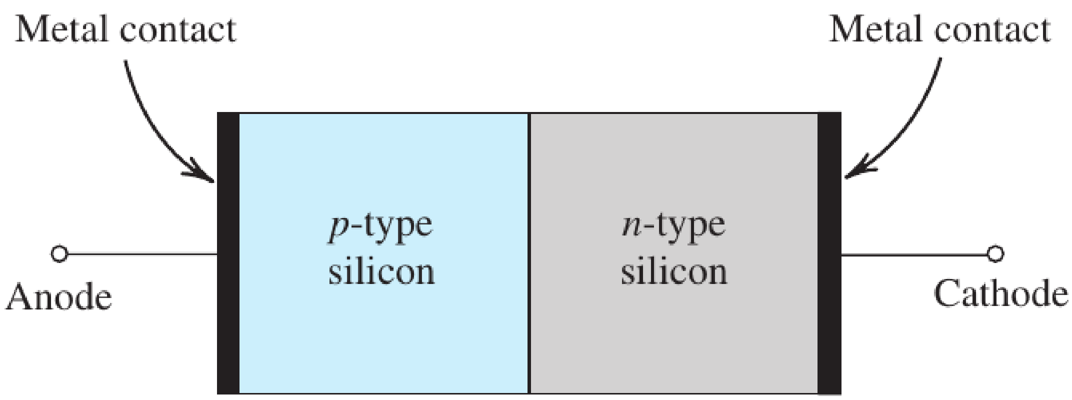


Figure 2.1: The pn-Junction (Sedra et al. 2015, p. 149)

2.4.1 Diffusion Current, I_D

Because of the concentration gradients of the majority carriers on one side compared ot the other side, the majority carriers diffuse over, creating a Diffusion Current. For example, in a pn-Junction, if we look at the n side, then the majority carrier is the Electron. Because the other, p side's concentration of electrons (which are its minority carrier), the electrons from the n diffuse over to the p side.

2.4.2 Depletion Layer

Defn 22 (Depletion Layer)**.** The *depletion layer* is the location in the pn-Junction where the two differently-doped sides meet. Here, there is a barrier of the opposing carrier on each side. This is visualized in Figure 2.2.

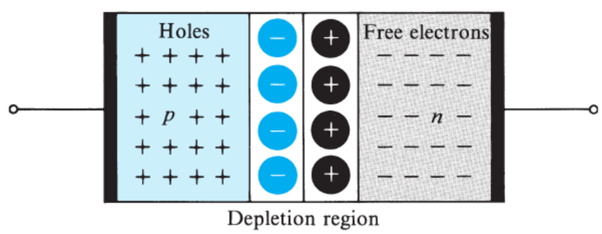


Figure 2.2: Depletion Layer (Sedra et al. 2015, p. 150)

Over time, as the Depletion Layer has more majority carriers recombine within that region, it becomes devoid of that side's majority carrier. This means that the depletion layer for the n side is now actually p. This difference between the depletion layer and that respective side of the junction means an electric field, $\vec{E}$ is created. This electric field gives rise to a Drift Current.

On top of all of this, the Depletion Layer acts like a barrier that one side's majority carrier must overcome for diffusion to happen. Eventually, the barrier becomes too strong for the majority carriers to cross the barrier without an external electric field being applied. This barrier voltage, V_0, is strongly dependent on the Diffusion Current, I_D.

2.4.3 The Drift Current, I_S

Due to the Depletion Layer's creation, an internal electric field is created. This accelerates the **minority carriers** from each side towards the other. This is the **drift current** of the pn-Junction.

In open circuit conditions, like we are discussing, the drift current is equal to the diffusion current.

$$I_D = I_S$$

2.4.4 Junction Built-In Voltage

Remember when we discussed the barrier voltage that must be overcome for a majority carrier to cross the Depletion Layer, in Section 2.4.1? Well, now we have defined in Equation (2.24).

$$V_0 = V_T \ln\left(\frac{N_A N_D}{n_i^2}\right) \tag{2.24}$$

2.4.5 Depletion Layer Width

We can find the width of the Depletion Layer using Equation (2.25).

$$W = \sqrt{\frac{2\epsilon_{\text{Si}}}{q}\left(\frac{1}{N_A} + \frac{1}{N_D}\right) V_0} \tag{2.25}$$

The Depletion Layer "bleeds" into each side of the *pn*-Junction. We can find the distance the depletion layer falls into each side with Equations (2.26a) and (2.26b).

$$x_n = W\left(\frac{N_A}{N_A + N_D}\right) \tag{2.26a}$$

$$x_p = W\left(\frac{N_D}{N_A + N_D}\right) \tag{2.26b}$$

Lastly, the sum of the "bleed" in both directions is equal to the width of the entire Depletion Layer.

$$W = x_n + x_p \tag{2.27}$$

2.4.6 Depletion Layer Charge

By performing some derivation that will not be included in this document, we can find the charge stored in each part of the Depletion Layer. Typically we are not interested in the particular sign of the charge, but their magnitude. Thus, we end up with Equations (2.28a) and (2.28b).

$$|Q_+| = qAx_pN_D \tag{2.28a}$$

$$|Q_-| = qAx_nN_A \tag{2.28b}$$

We know that these charges must be equal and opposite, to ensure we follow energy conservation. Thus, if we set $|Q_+| = |Q_-|$, we end up with an incredibly important equation, Equation (2.29).

$$\frac{x_n}{x_p} = \frac{N_A}{N_D} \tag{2.29}$$

Equation (2.29) can be used to construct Equations (2.25), (2.26a) and (2.26b).

Lastly, we can use W to obtain the total charge stored in the Depletion Layer, shown in Equation (2.30).

$$Q_J = A\sqrt{2\epsilon_{\text{Si}} q\left(\frac{N_A N_D}{N_A + N_D}\right) V_0} \tag{2.30}$$

2.5 The *pn*-Junction Junction with Applied Voltage

There are three possible cases for a *pn*-Junction to have a voltage applied, shown in Figures 2.3a to 2.3c.

With voltages applied, the Depletion Layer changes a bit, as seen by Equations (2.31) and (2.32).

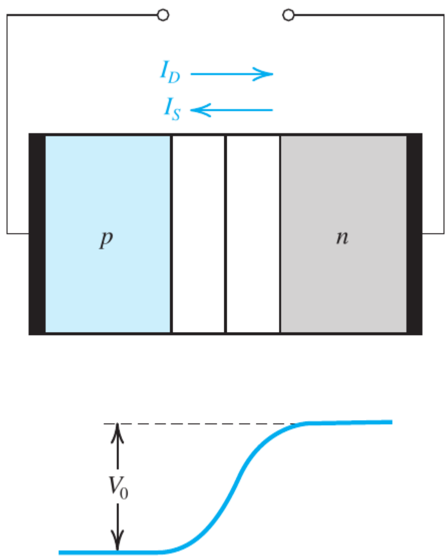
$$W = \sqrt{\frac{2\epsilon_{\text{Si}}}{q}\left(\frac{1}{N_A} + \frac{1}{N_D}\right)(V_0 + V_R)} \tag{2.31}$$

$$Q_J = A\sqrt{2\epsilon_{\text{Si}} q\left(\frac{N_A N_D}{N_A + N_D}\right)(V_0 + V_R)} \tag{2.32}$$

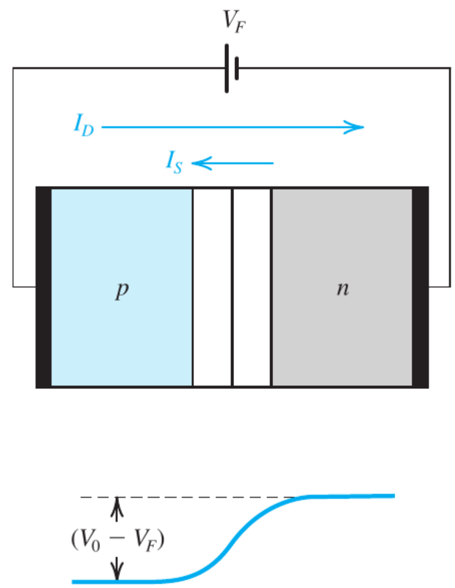
The current I in the external circuit is the difference between I_S and I_D.

> To conclude, the *pn*-Junction can conduct substantial current in the forward-bias region and the current that is allowed to flow is mostly a Diffusion Current whose value is determined by the forward-bias voltage.

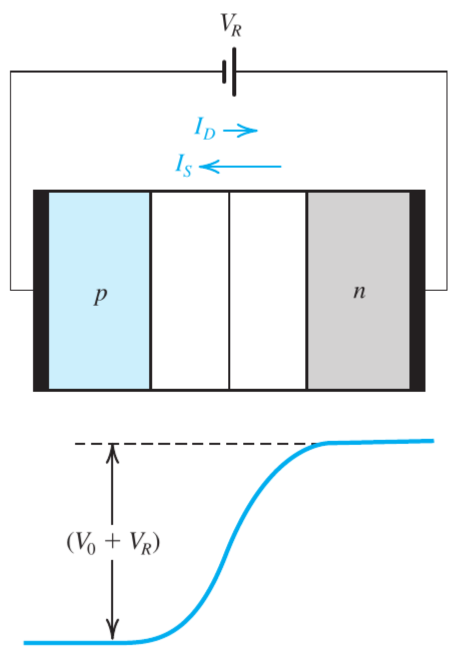
After performing some derivation that will not be included in this document (refer to the textbook, Sedra et al. 2015, for the relevant derivation), we end up with the equation for the current through a *pn*-Junction when a voltage is applied.



(a) *pn*-Junction in Various States when Voltage Applied



(b) *pn*-Junction when Forward Voltage Applied for Forward bias



(c) *pn*-Junction when Reverse Voltage Applied for Reverse bias

Figure 2.3: *pn*-Junction in Various States when Voltage Applied (Sedra et al. 2015, p. 156)

$$I = Aqn_i^2 \left(\frac{D_p}{L_p N_D} + \frac{D_n}{L_n N_A} \right) \left(e^{\frac{V}{V_T}} - 1 \right) \tag{2.33}$$

Equation (2.33) is usually simplified down to Equation (2.34).

$$I = I_S \left(e^{\frac{V}{V_T}} - 1 \right) \tag{2.34}$$

Equation (2.33) forms an exponential figure, as seen in Figure 2.4.

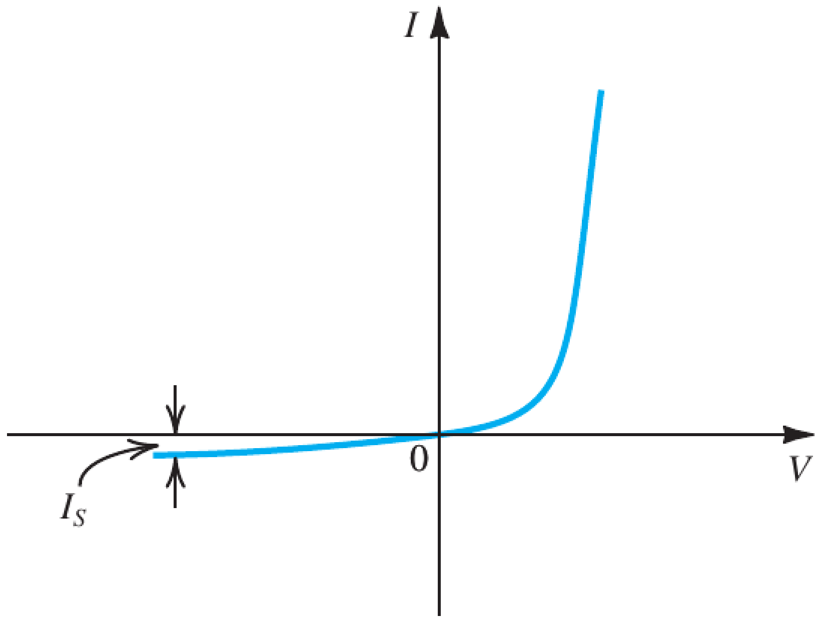


Figure 2.4: *pn*-Junction Current-Voltage (*I*-*V*) Characteristic (Sedra et al. 2015, p. 160)

2.6 Values of Constants and Parameters for Intrinsic Silicon

Refer to Sedra et al. 2015, Table 3.1 for all of the details about these constants and parameters.

The only parameter I have chosen to change in this document is ϵ_{Si}. In the textbook, it is provided as ϵ_s.

3 Diodes

A Diode is one of the basic building blocks for almost all analog circuit systems.

Defn 23 (Diode)**.** A *diode* is a 2-port non-linear circuit element. Its circuit symbol is shown in Figure 3.1.

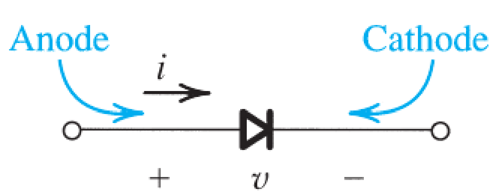


Figure 3.1: Diode Circuit Symbol (Sedra et al. 2015, p. 177)

An **ideal diode** is one that behaves as a short-circuit when a voltage applied is forward-biased, and acts as an open-circuit with the voltage is reverse-biased. The current-voltage characteristic is shown in Figure 3.2.

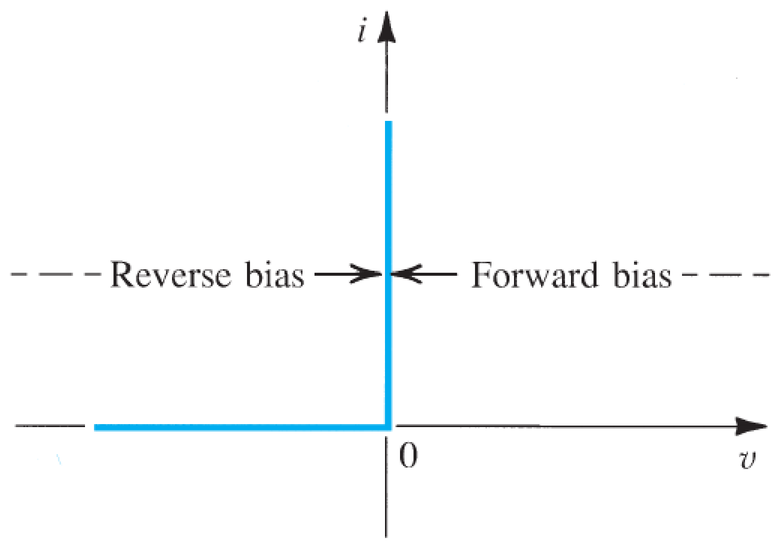


Figure 3.2: Ideal Diode I-V Characteristic (Sedra et al. 2015, p. 177)

3.1 Terminal Characteristics of Junction Diodes

A **junction diode** is a Diode built using a pn-Junction. In this case, there are three distinct regions in the characteristic curve:

1. The Forward-Bias Region, where $v > 0$.
2. The Reverse-Bias Region, where $v < 0$.
3. The Breakdown Region, where $v < -V_{ZK}$.

3.1.1 The Forward-Bias Region

The forward-bias region is the one where the terminal voltage v is positive.

In the forward region, we can approximate the current-voltage relationship by using Equation (2.34), with minor modifications, as seen in Equation (3.1).

$$i = I_S \left(e^{\frac{v}{V_T}} - 1 \right) \tag{3.1}$$

There is a slightly **more** modified version, in Equation (3.2)

$$i = I_S \left(e^{\frac{v}{nV_T}} - 1 \right) \tag{3.2}$$

n has a value between 1 and 2. n is a number that depends on the material and physical structure of the diode. From here-on-out, we assume that $n = 1$. However, there may be cases when you should use a different v value instead.

If the diode has $i \gg I_S$, then we can approximate Equation (3.1) using Equation (3.3).

$$i \simeq I_S e^{\frac{v}{V_T}} \tag{3.3}$$

The relationship in Equation (3.3) can be expressed in terms of currents instead of voltages as well.

$$v = V_T \ln \left(\frac{i}{I_S} \right) \tag{3.4}$$

3.1.2 The Reverse-Bias Region

In the reverse-bias region, the voltage applied to the Diode is in the reverse direction. This causes the diode to behave *like* an open-circuit. However, this is not technically true, because a small amount of current does "leak" out, as seen by Equation (3.5).

$$i \simeq -I_S \tag{3.5}$$

3.1.3 The Breakdown Region

Lastly, the breakdown region is where the voltage applied in the reverse-bias direction is so great, that the Diode starts working in reverse, conducting current in the reverse direction. This region will be discussed much more in the section devoted to Zener Diodes.

3.2 Modeling the Diode's Forward Characteristic

There are several different models, appropriate at different times of analysis.

3.2.1 The Exponential Model

In the exponential model of a Diode, you use the diode's current-voltage characteristic and another equation containing the current through the diode and the voltage across the diode to solve for I_D and V_D.

The downside to this model is that it can require a lot of knowledge about the Diode and its properties. In addition, because an exponential function $I_D = I_S e^{\frac{V_D}{V_T}}$ and a linear function are being used to find a solution, finding that solution can be computationally difficult. Due to this, it is only used in the final stages of circuit development/analysis and is not much discussed in either this document or this course. Instead, we focus on ways to more quickly model a Diode and its characteristics.

3.2.2 The Constant-Voltage-Drop Model

In the constant-voltage-drop model, we assume the Diode is semi-ideal. Meaning that we approximate the real exponential function, $I_D = I_S e^{\frac{V_D}{V_T}}$ to a piecewise linear one. The constant-voltage-drop model for a silicon-based diode is shown in Figure 3.3.

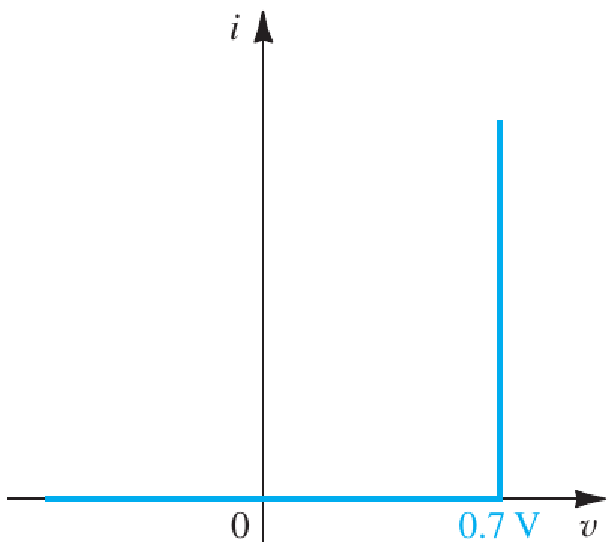


Figure 3.3: Silicon Diode Constant-Voltage-Drop Model (Sedra et al. 2015, p. 193)

Essentially, this simplifies a Diode down to a DC voltage source, where the source's positive terminal is on the anode of the diode. This source can then be used when performing circuit analysis to get reasonably accurate answers for the amount of work required.

3.2.3 The Ideal Diode Model

In the ideal diode model, it is assumed that the Diode in question **is** ideal. This means that if the voltage across the diode is greater than zero, the diode acts as a short-circuit. This can be seen graphically in Figure 3.2.

3.2.4 The Small-Signal Model

In the small-signal model, we are interested in how small changes across the Diode affect its properties. For example, say we increase the original source voltage v_S by Δv_{DD}, we would be interested in the change in the diode's current and voltage.

To solve this, we actually use the exponential model, but we are only concerned with the highest one or two orders in the infinite summation.

Remember that Euler's exponential can be represented using an infinite series, seen below:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned}$$

If we are working with a small enough change ($\Delta v_{DD} \ll V_T$), then the exponential curve is dominated by the $1 + x$ term in the infinite series.

To simplify the explanations here, we will pretend that the Δv_{DD} is actually an AC voltage, $v_d(t)$.

Then, by the theory of superposition, the voltage across the diode is also a time-varying function shown below.

$$v_d(t) = V_D + v_d(t)$$

Knowing how a Diode works, we can say,

$$i_d(t) = I_S e^{\frac{v_d}{V_T}}$$

Substitute for $v_d(t)$

$$\begin{aligned} &= I_S e^{\frac{V_D + v_d(t)}{V_T}} \\ &= I_S e^{\frac{V_D}{V_T}} e^{\frac{v_d(t)}{V_T}} \end{aligned}$$

Let $I_D = I_S e^{\frac{V_D}{V_T}}$.

$$i_d(t) = I_D e^{\frac{v_d(t)}{V_T}}$$

If we keep the amplitude of the change we introduce remain small, meaning the $e^{\frac{v_d(t)}{V_T}} \approx 1$, then we can solve this. Such a constraint means that $\frac{v_d(t)}{V_T} \ll 1$. Expanding the exponential using its infinite series and truncating after two terms yields a reasonable approximation. In that case:

$$i_d(t) \simeq I_D \left(1 + \frac{v_d(t)}{V_T} \right)$$

This last derivation is important, because it is the **small-signal approximation**. I have restated it here to demonstrate just *how* important it is.

$$i_d(t) \simeq I_D \left(1 + \frac{v_d(t)}{V_T} \right) \tag{3.6}$$

By continuing to solve for this, we can find the Incremental Resistance.

Defn 24 (Incremental Resistance)**.** *Incremental resistance* is the resistance in a Diode. These are found when performing small-signal modeling on a diode. Typically these values are very small, as a diode with a high resistance would be somewhat pointless. In the The Small-Signal Model, the incremental resistance can be defined as shown in Equation (3.7).

$$r_d = \frac{V_T}{I_D} \tag{3.7}$$

What all of this derivation means is that when using a small-signal model on a Diode, finding its Incremental Resistance will allow you to replace the diode with an equivalent resistor. Using this resistor, one can then solve for any term in the circuit they wish for by using this resistor approximation. The end result is visualized in Figure 3.4.

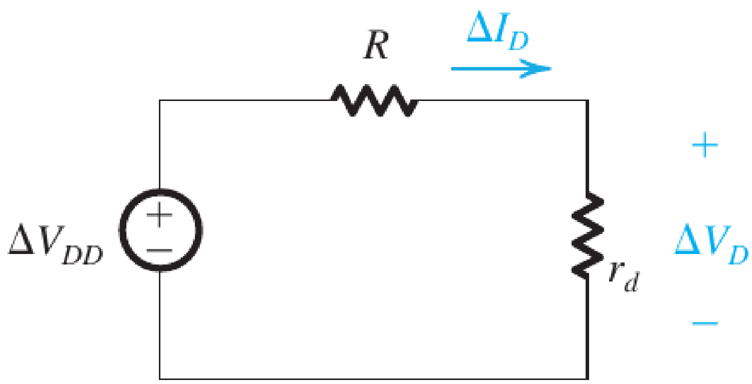


Figure 3.4: Resistor-Equivalent Diode Small-Signal Model (Sedra et al. 2015, p. 198)

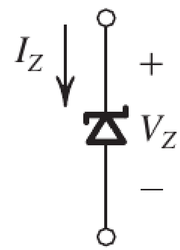


Figure 3.5: Zener Diode Circuit Symbol (Sedra et al. 2015, p. 202)

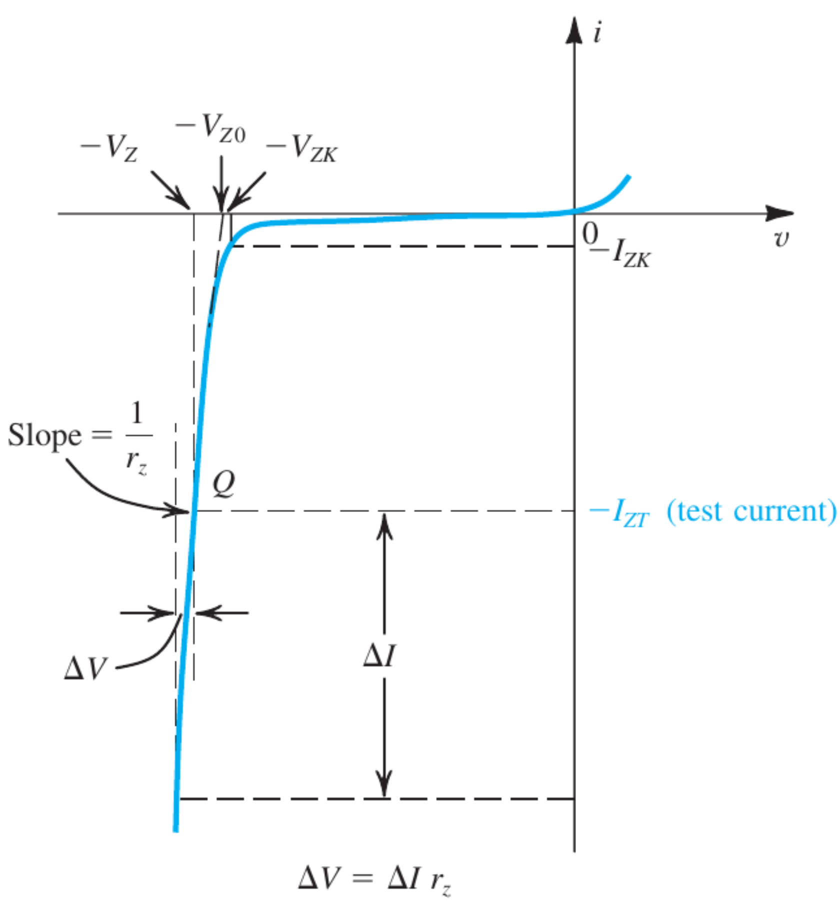


Figure 3.6: Zener Diode Current-Voltage Characteristics (Sedra et al. 2015, p. 203)

3.3 Zener Diodes

Defn 25 (Zener Diode)**.** A *zener diode* is a specially constructed Diode with a particularly low reverse breakdown voltage, V_{ZK}. The circuit symbol for a zener diode is shown in Figure 3.5.

The Zener Diode has normal current-voltage characteristics in the forward-biased region, so we skip that part of this discussion. It behaves uniquely in the reverse-bias region, so we study it in more depth here. Figure 3.6 shows this.

As seen in Figure 3.6, the once inside the reverse breakdown region, the diode's characteristic is nearly linear. So, we choose to model it as a voltage source and a resistor in series, as denoted in Equation (3.8) and seen in Figure 3.7.

$$V_Z = V_{Z,0} + r_z I_Z \tag{3.8}$$

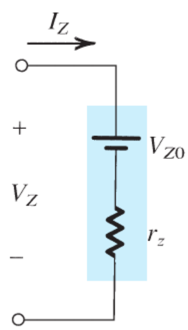


Figure 3.7: Zener Diode Model (Sedra et al. 2015, p. 204)

3.4 Applications

There are no end to the applications for Diodes, both regular and Zener Diodes. They can be used to build Rectifiers, voltage doublers, clamped capacitors, digital logic creators, etc.

Defn 26 (Rectifier)**.** A *rectifier* takes in AC power and converts it to DC.

There are numeroud Rectifiers that can be built with Diodes. This document may yet grow to support discussing that material, but at the time of writing, the author was short of time and has to leave that particular discussion out of this document.

4 MOS Field-Effect Transistors

In this section, we start studying Transistors.

Defn 27 (Transistor)**.** A *transistor* is a Semiconductor device used to amplify or switch electronic signals and electrical power. Transistors are one of the basic building blocks of modern electronics. It is composed of semiconductor material usually with at least three terminals for connection to an external circuit.

The MOSFET is the first Transistor we will be studying in this course. It is the second oldest transistor, but is the most frequently used one today, particularly for digital applications.

Defn 28 (MOSFET)**.** *MOSFET*, short for *Metal-Oxide-Semiconductor Field-Effect Transistor*, is an insulated-gate field-effect transistor.

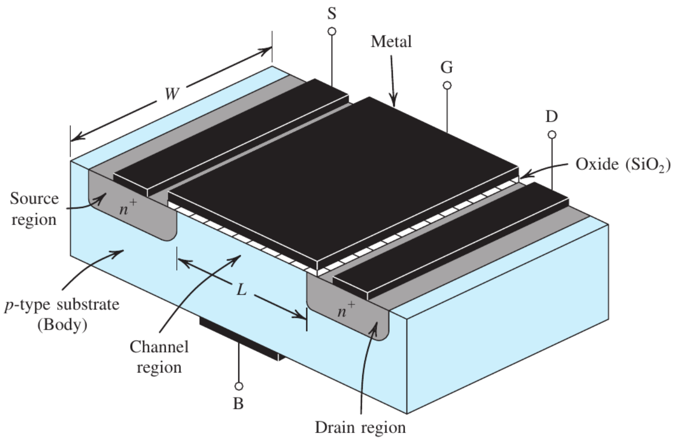
The insulated-gate portion of its name implies that the gate is completely isolated from the rest of the circuit. This is achieved with the metal oxide layer (seen in Figure 4.1b) acting as an insulator, preventing any current from entering the transistor from that terminal.

The field-effect portion of the name implies that the electric field of the gate is the driving force in this circuit. Because no current is allowed through the gate terminal of the transistor, this only has a voltage applied, causing an electric field to form.

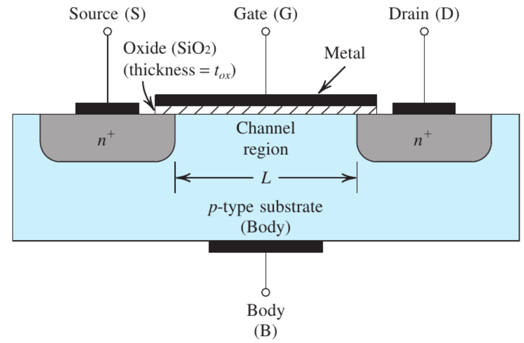
The typical circuit symbol (for NMOS) is shown in Figure 4.3.

The physical structure of a MOSFET is shown in Figures 4.1a and 4.1b.

As can be seen in Figure 4.1a, the drain and source both form a *pn*-Junction with the base.



(a) Perspective View (Sedra et al. 2015, p. 249)



(b) Cross-Sectional View (Sedra et al. 2015, p. 249)

Figure 4.1: Physical Structure of Enhancement-type NMOS Transistor

4.1 No Gate Voltage

When the gate has **no** voltage applied, the *pn*-Junctions of the source and drain to the substrate forms a diode-like relationship, where the resistance is very high (of the order $10^{12}\Omega$) This prevents nearly all current from the drain from flowing.

4.2 Gate Voltage Applied

When the gate receives a positive voltage (in relation to the source), then an electric field is formed on the gate, causing a Depletion Layer to form in the substrate. Then, there is a channel that is formed due to the application of the electric field that allows current to flow between the drain and the source. Figure 4.2 shows this phenomena for NMOS Transistors.

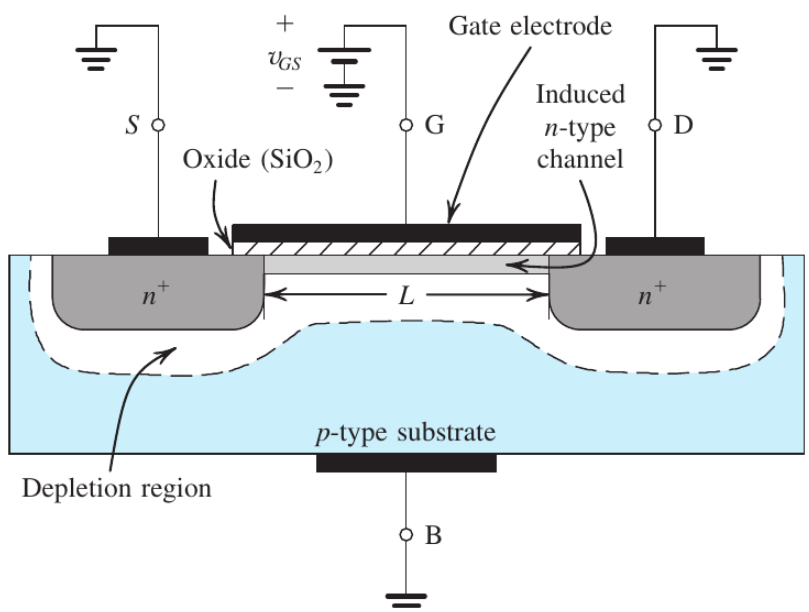


Figure 4.2: NMOS MOSFET with Positive Voltage Applied to Gate (Sedra et al. 2015, p. 251)

Defn 29 (NMOS)**.** *NMOS*, short for *n-channel Metal-Oxide Semiconductor*. An n-channel MOSFET is formed in a p substrate, and the induced channel is of p. The channel is created by inverting the substrate surface from p to n. Hence the induced channel is also called an *inversion layer*.

The circuit symbol for an NMOS transistor is shown in Figure 4.3.

Defn 30 (PMOS)**.** *PMOS*, short for *p-channel Metal-Oxide Semiconductor*. An p-channel MOSFET is formed in a n substrate, and the induced channel is of p. The channel is created by inverting the substrate surface from n to p. Hence the induced channel is also called an *inversion layer*.

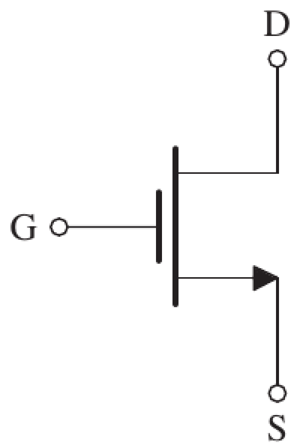


Figure 4.3: NMOS MOSFET Circuit Symbol (Sedra et al. 2015, p. 265)

The circuit symbol for a PMOS transistor is shown in Figure 4.4. The physical structure of the PMOS transistor (Figure 4.5) is similar to the NMOS, which we have studied.

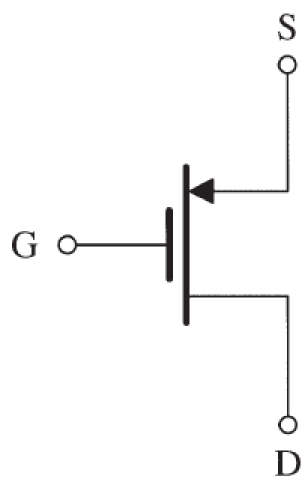


Figure 4.4: PMOS MOSFET Circuit Symbols (Sedra et al. 2015, p.274)

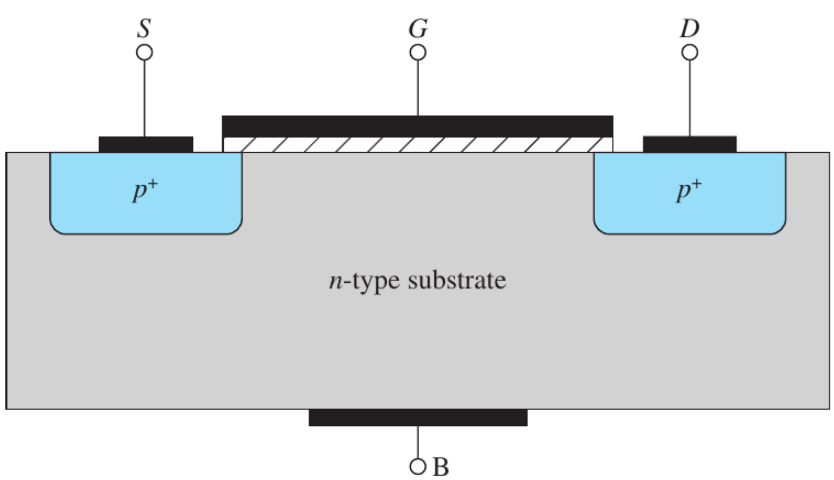


Figure 4.5: PMOS Physical Structure (Sedra et al. 2015, p. 262)

For such a channel to form, the voltage between the gate and the source, v_{gs}, **must** be greater than some Threshold

Voltage.

Defn 31 (Threshold Voltage)**.** The *threshold voltage* is a specific value for a MOSFET that determines the minimum required forltage that must be aplied at the gate for the transistor to operate. In this text, the threshold voltage is denoted as V_t.

Remark 31.1 (Notation). Some materials use V_T as the Threshold Voltage. We use V_t, to distinguish this value from the Thermal Voltage.

If the gate-source voltage (v_{gs}) exceeds the Threshold Voltage, we define a new term called Overdrive Voltage.

Defn 32 (Overdrive Voltage)**.** The *overdrive voltage* is the difference in voltage applied to the gate and the Threshold Voltage. Its defining equation is given in Equation (4.1).

$$V_{OV} = v_{gs} - V_t \tag{4.1}$$

If we leave the voltage at the drain (in relation to the source) equal to zero, then the channel has uniform depth in the MOSFET.

Because the gate and substrate are two parallel plates, separated by a dielectric, the plates function as a capacitor. The capacitivity of the oxide is given by Equation (4.2).

$$C_{OX} = \frac{\epsilon_{OX}}{t_{OX}} \text{ F/m}^2 \tag{4.2}$$

The magnitude of the total electron charge in the channel can be found using Equation (4.3).

$$\|Q\| = C_{OX} W L V_{OV} \tag{4.3}$$

To simplify many of our calculations, we use the terms shown in Equations (4.4) and (4.5).

$$k'_n = \mu_n C_{OX} \tag{4.4}$$

$$k = k'_n \left(\frac{W}{L}\right) \tag{4.5}$$

Defn 33 (CMOS)**.** *CMOS*, or *Complementary Metal-Oxide Semiconductor* is a MOSFET element that combines an NMOS and PMOS into a single package.

Figure 4.6 shows the physical structure of the CMOS transistor.

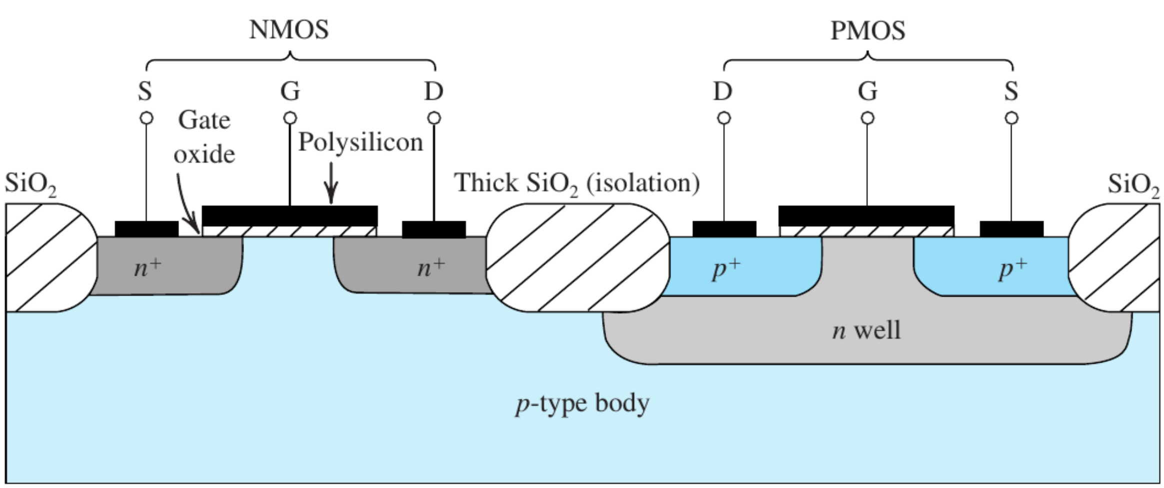


Figure 4.6: CMOS Transistor Physical Structure (Sedra et al. 2015, p. 264)

4.3 Operating Regions

MOSFETs operate in one of three regions, based on two different voltages and their relationship.

1. Cutoff, $V_{GS} < V_t$
2. Triode, $V_{GS} \geq V_t$ **and** $V_{DS} < V_{OV}$
3. Saturation, $V_{GS} \geq V_t$ **and** $V_{DS} \geq V_{OV}$

The current-voltage characteristic of the NMOS MOSFET is shown in Figure 4.7.

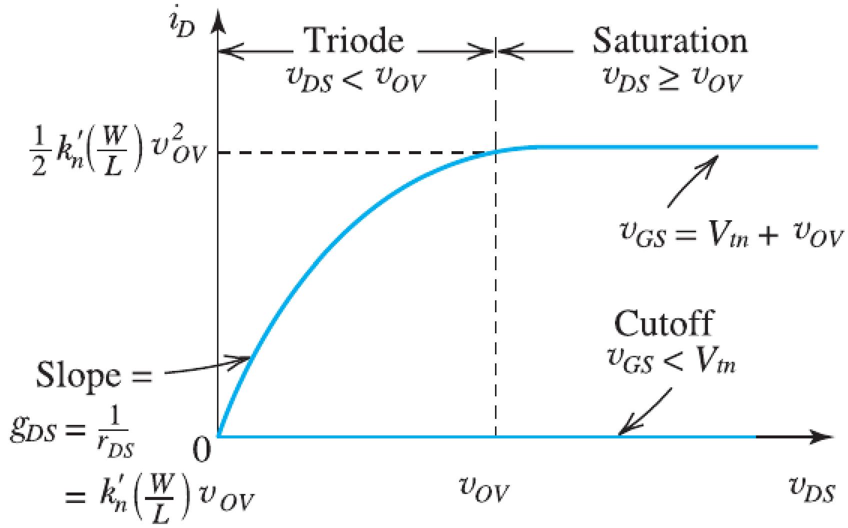


Figure 4.7: i_d-v_{ds} NMOS MOSFET Characteristic (Sedra et al. 2015, p. 266)

4.3.1 Cutoff

The cutoff region is when the voltage applied at the gate is not enough to make a channel form in the substrate of the MOSFET, meaning $V_{GS} < V_t$. Because no channel is formed, no current flows, thus the current is as defined in Equation (4.6).

$$I_D = 0 \tag{4.6}$$

4.3.2 Triode

The triode region is formed when the MOSFET is turned on ($v_{gs} \geq V_t$), and when the channel has not reached its pinchoff point yet ($v_{ds} < V_{OV}$).

In short, the two requirements are:

1. $v_{gs} \geq V_t$
2. $v_{ds} < V_{OV}$

$$i_d = k_n \left(V_{OV} - \frac{1}{2}v_{ds}\right) v_{ds} \tag{4.7}$$

The Triode has a special case when the drain-source voltage drop is less than the Overdrive Voltage, called the Linear.

4.3.2.1 Linear The linear region of operation for a MOSFET is actually a special case of the Triode operation region. A MOSFET will operate in the *linear* region when the input voltage is quite low. Namely, $v_{ds} \ll V_{OV}$, meaning $v_{ds} \ll V_t$.

The linear region is technically a part of the Triode region, but because V_{DS} is so small, the squared term becomes negligible in Equation (4.7), leaving us with Equation (4.8).

$$i_d = k_n V_{OV} v_{ds} \tag{4.8}$$

4.3.3 Saturation

The saturation region is the main operating region of the MOSFET. Its two requirements are:

1. $V_{GS} \geq V_t$
2. $V_{DS} \geq V_{OV}$

Sometimes, the Saturation is referred to as the location where channel *pinchoff* occurs. This is because the channel finishes its triangle and the drain's end of the triangle pinches off, see Figure 4.8.

When the MOSFET is in this operating region, it has a current relationship of Equation (4.9).

$$i_d = \frac{1}{2}k_n V_{OV}^2 \tag{4.9}$$

An equivalent circuit can be constructed for the operation of a MOSFET in the Saturation, given in Figure 4.9.

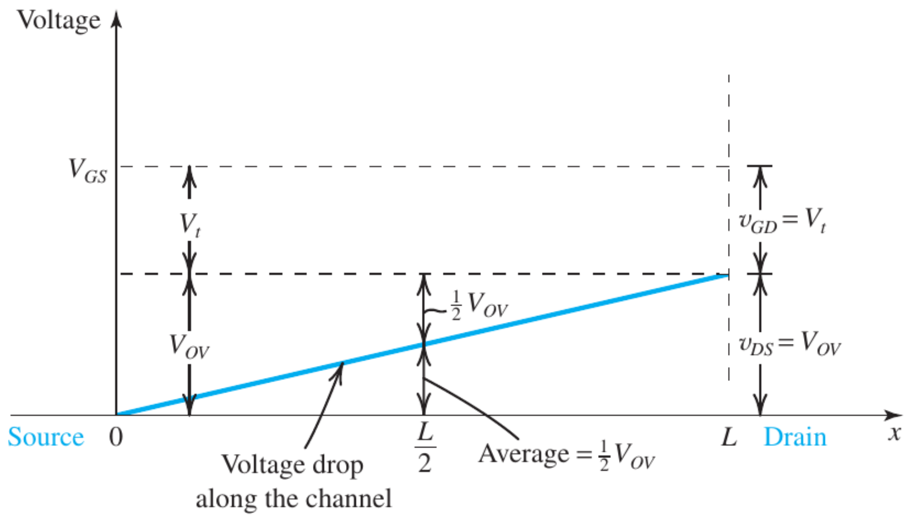


Figure 4.8: MOSFET in Pinchoff (Saturation Region) (Sedra et al. 2015, p. 259)

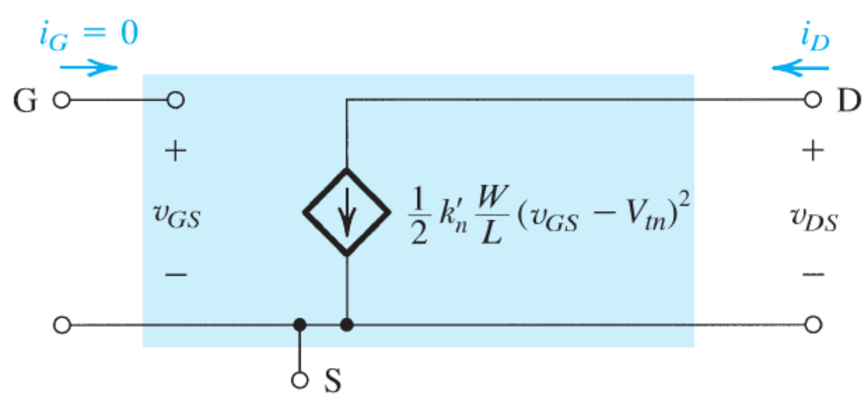


Figure 4.9: Large-Signal MOSFET Equivalent Circuit (Sedra et al. 2015, p. 268)

4.3.4 The Early Effect

The Early effect, named after James M. Early, is an imperfection of Transistors. The effect can only occur in the Saturation region, and allows some current to flow through the device even when the gate voltage is less than zero.

Defn 34 (Channel Length Modulation Factor)**.** The *channel length modulation factor*, or λ, is a constant that makes Transistors behave in non-ideal ways.

$$V_A = \frac{1}{\lambda} \tag{4.10}$$

$$i_d = \frac{1}{2} k_n V_{OV}^2 (1 + \lambda v_{ds}) \tag{4.11}$$

4.4 Steps to Solve

1. Find voltage at:
   - The Gate, G
   - The Drain, D
   - The Source S
2. Remember that the voltage at any location, $v_{xy} = v_x - v_y$, where v_x and v_y are the voltages at nodes x and y respectively, measured in comparison to ground.
3. Find the gate-source voltage, v_{gs}, and determine if it is greater than V_t.
   - If $v_{gs} \geq V_t$, then the MOSFET is turned on, and is either in the Saturation or Triode region.
   - If $v_{gs} < V_t$, then the MOSFET is in the Cutoff, and $i_d = 0$.
4. Find the voltage at the drain, compared to the source (v_{ds}).
   - If $v_{ds} \geq V_{OV}$, then the MOSFET is in the Saturation, and follows Equation (4.9) for i_d.
   - If $v_{ds} < V_{OV}$, then the MOSFET is in the Triode, and follows Equation (4.7) for i_d.

5 Binary Junction Transistors

Defn 35 (BJT)**.** The *BJT*, short for *Binary Junction Transistor*, is another important Transistor.
The physical structure of both *npn*-Transistors and *pnp*-Transistors is shown in Figures 5.1 and 5.2.

Remark 35.1 (Emitter-Collector Notation). The emitter of the BJT is named so because of the way we define current. We actually define current to be the reverse flow of electrons (not technically the holes). Thus, the emitter emits "negative electrons".

The collector collects the electrons that have been injected. Because we consider current to be the opposite direction of the flow of electrons in a circuit, the collector's current is denoted as moving inwards, even when the electrons are flowing outwards.

For this entire section, we discuss *only npn*-Transistors. However, *pnp*-Transistors behave nearly identically, with the only major difference being that every instance of V_{BE} should be replaced with V_{EB}. This is because a *pnp*-Transistor behaves like a "reversed" *npn*-Transistor.

5.1 Operating Regions

There are four operating regions for a BJT.

1. Cutoff. When both terminals are reverse biased. For an *npn*-Transistor, this happens when both the emitter and collector are positively biased.
2. Active When the emitter is negatively biased (negative terminal of source on emitter and positive on base), and the collector is positively biased (positive terminal of source on collector, negative on base).
3. Reverse Active Region. When the emitter is positively biased and the collector is negatively biased. This behaves the same way as the active region, for the most part.
4. Saturation When both terminals are negatively biased (the negative terminal of each source is on the collector and emitter, and their positive terminal is on the base).

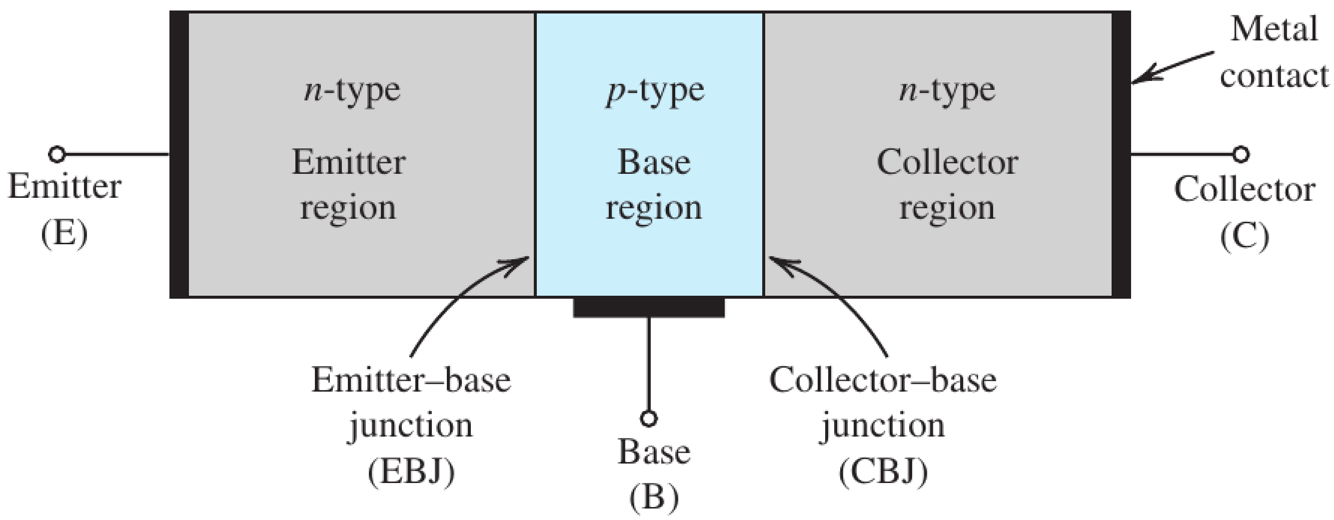


Figure 5.1: *npn*-Transistor Physical Structure (Sedra et al. 2015, p. 307)

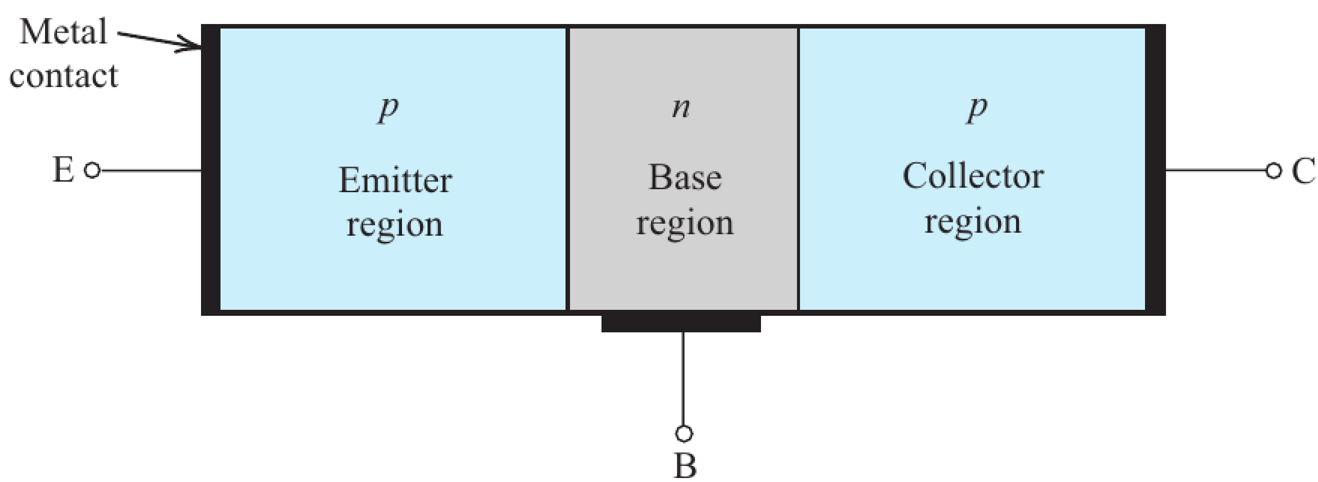


Figure 5.2: *pnp*-Transistor Physical Structure (Sedra et al. 2015, p. 307)

5.1.1 Active

This is the most important region to us right now. When the emitter is negatively biased (negative terminal of source on emitter and positive on base), and the collector is positively biased (positive terminal of source on collector, negative on base).

The two criteria that must be met for a BJT to be in the Active are given below:

1. $V_{EB} > 0$
2. $V_{CB} < 0$

Remark. The operation of a BJT in the active region is like a MOSFET operating in its Saturation.

Figure 5.3 shows all the currents and voltages present in the component when in this region.

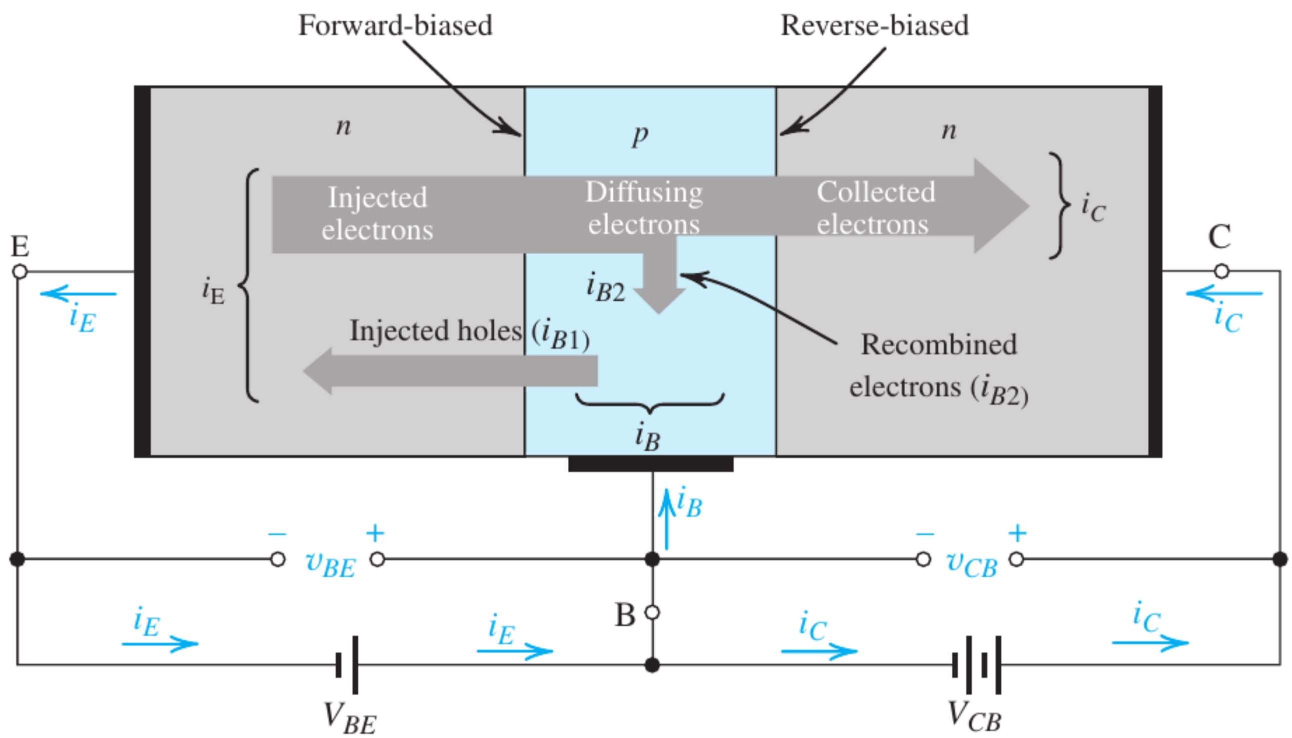


Figure 5.3: BJT Current Flow in Active (Sedra et al. 2015, p. 308)

By treating the collector and base interface as if it were a *pn*-Junction, then we can find its current to be defined by the equation in Equation (5.1).

$$i_c = I_S e^{\frac{v_{be}}{V_T}} \tag{5.1}$$

Then, the base current in terms of the collector is defined in Equation (5.7). The emitter current in terms of the collector is defined in Equation (5.6).

It should also be noted that KCL can be applied across the entire element, allowing us to say:

$$i_e = i_c + i_b \tag{5.2}$$

Using all of the information we know, we can define four separate large-signal equivalent-circuit models for the *npn*-Transistor, operating in the Active. They are given in Figures 5.4a to 5.4d

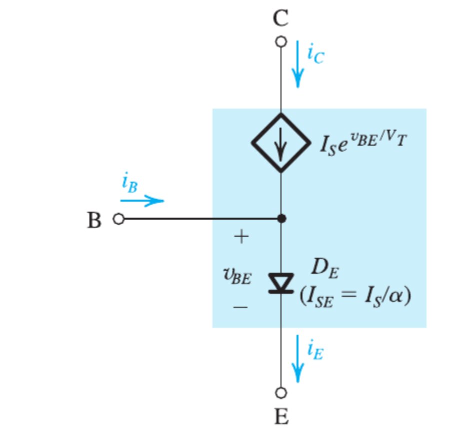
5.1.2 Saturation

When both terminals are negatively biased (the negative terminal of each source is on the collector and emitter, and their positive terminal is on the base).

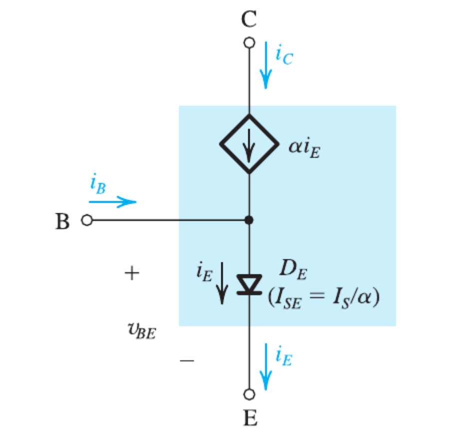
The two criteria that must be met for a BJT to be in the Saturation are given below:

1. $V_{EB} > 0$
2. $V_{CB} > 0$

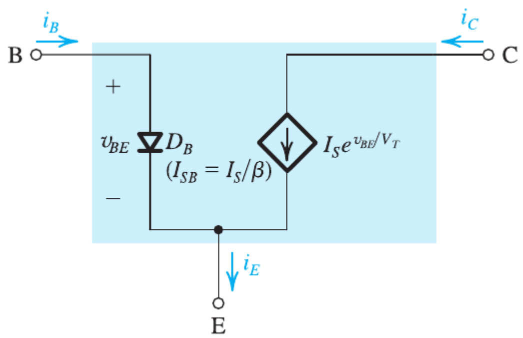
Remark. The operation of a BJT in the saturation region is similar to the operation of a MOSFET in its Triode. This is visualized in Figure 5.5.



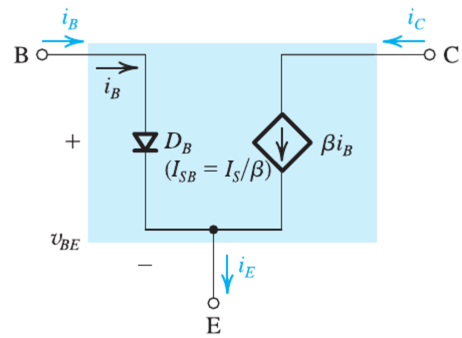
(a) Equivalent Circuit using α and v_{be}



(b) Equivalent Circuit using α and v_e



(c) Equivalent Circuit using β and v_{be}



(d) Equivalent Circuit using β and v_b

Figure 5.4: Large-Signal Equivalent-Circuit models of *npn*-Transistor (Sedra et al. 2015, p. 313)

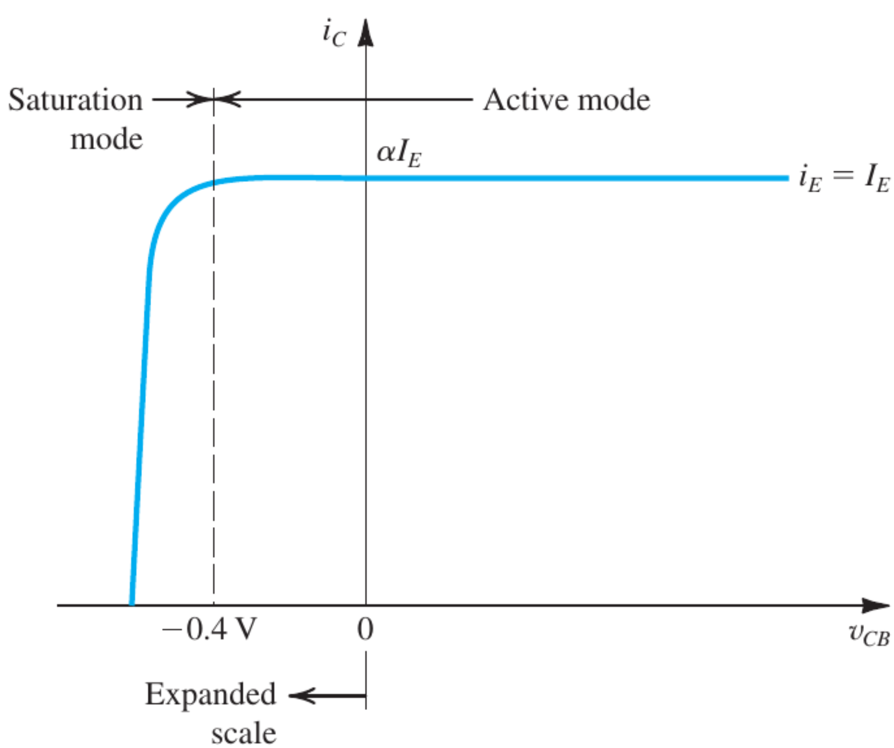


Figure 5.5: i_c-v_{cb} Characteristic (Sedra et al. 2015, p. 317)

When in the saturation region, the BJT has the following unique property:

$$V_{CE,\text{SAT}} = 0.3\,\text{V} \tag{5.3}$$

However, when **very** deep in the saturation region, the BJT's collector-emitter voltage drop is even lower.

$$V_{CE,\text{SAT}} = 0.2\,\text{V} \tag{5.4}$$

When this deep in the saturation region, the β value changes quite dramatically too.

$$\beta_{\text{Forced}} = \left.\frac{i_c}{i_b}\right|_{\text{Saturation}} \leq \beta \tag{5.5}$$

5.2 α and β Parameters

Defn 36 (Common-Base Current Gain)**.** The α term for BJTs, is also called the *common-base current gain*. The α parameter is typically in the range of 0.98 to 1. Its defining equation is given in Equation (5.6).

$$I_C = \alpha I_E \tag{5.6}$$

Defn 37 (Common-Emitter Current Gain)**.** The β term for BJTs, is also called the *common-emitter current gain*. The β parameter is typically in the range of 50 to 200, but can be as high as 1000. Its defining equation is given in Equation (5.7).

$$I_C = \beta I_B \tag{5.7}$$

$$\alpha = \frac{\beta}{\beta + 1} \tag{5.8}$$

$$\beta = \frac{\alpha}{1 - \alpha} \tag{5.9}$$

5.3 Circuit Symbols

The circuit symbol for an *npn*-Transistor BJT is given in Figure 5.6.

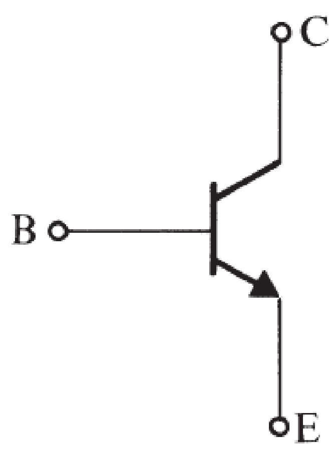


Figure 5.6: BJT *npn*-Transistor Circuit Symbol (Sedra et al. 2015, p. 321)

Although we do not cover the *pnp*-Transistor, for completion, its circuit symbol is included in Figure 5.7.

6 Transistor Amplifiers

Now, we are going to apply *both* AC and DC signals to a Transistor, and watch the outputs. There is some important terminology to be aware of here:

DC Uses all uppercase letters, such as V_B or I_D.

AC Uses all lowercase letters, such as v_{gs} or i_d.

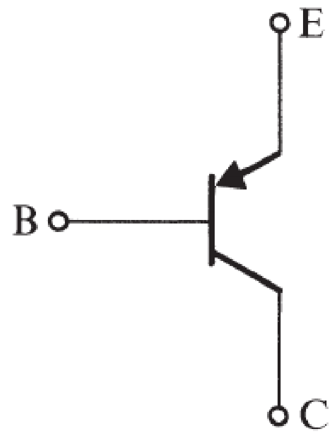


Figure 5.7: BJT *pnp*-Transistor Circuit Symbol (Sedra et al. 2015, p. 321)

Mixed Uses lowercase letters, with capital subscripts, such as v_{GS}. These are typically defined as $v_{GS} = v_{gs} + V_{GS}$.

To solve a transistor amplifier problem, there are a few steps:

1. Find the DC bias point of the Transistor. This involves performing DC analysis of the transistor, like was done in Section 4 and Section 5. Zero *all* AC sources and treat all capacitors as open circuits.
   (a) When solving, assume the Transistor is in the useful region for amplification.
2. Find the transconductance gain, g_m of the circuit (Equations (6.1) and (6.6)).
3. Find the AC operation of the Transistor. This involves performing AC analysis of the transistor. Zero *all* DC sources, and treat all capacitors are short circuits.
4. Make use of a small-signal equivalent circuit.

6.1 Biasing

Biasing of a Transistor is required so that it is operating in its active region (Saturation for MOSFETs, and Active for BJTs).

If we view the v_{gs}-v_{ds} voltage transfer characteristics of a MOSFET (Figure 6.1) or the v_{be}-v_{ce} voltage transfer characteristic of a BJT (Figure 6.2), we can see why this is necessary.

For a Transistor to properly amplify a given input signal, it must have a non-constant voltage-transfer characteristic, otherwise the amplifier does nothing. Thus, the Transistor **must** be in its active region for anything to happen. If the transistor were to enter either its cutoff or saturation modes, then the input signal would become distorted at the output. See Figure 6.3 for an example/visualization of how this works.

This means that Transistors **must** be biased with a DC voltage to operate in the appropriate region, but they must not be too close to any edge. If they are too close to an edge, the input AC signal will make the bias point leave the active region of the Transistor, causing the output signal to be distorted.

Defn 38 (Quiescent Point)**.** The *quiescent point* is a given biasing point in the transfer function of a voltage transfer characteristic. If you calculate a bias point, and it is *not* the maximum bias point for that signal, then it is called a quiescent point. The quiescent point is denoted with Q.

Quiescent Points are denoted with Q, and the maximum bias point is B.

6.2 MOSFET Amplifiers

The Transconductance Gain is defined in Definition 39, and its defining equations are shown in Equation (6.1).

Defn 39 (Transconductance Gain)**.** The *(open-circuit) transconductance gain*, g_m is the transconductance that the Transistor experiences when operating in its DC mode. It is defined to two different equations depending on the Transistor used.

MOSFET Equation (6.1)
BJT Equation (6.6)

Equations (6.1a) to (6.1c) provide all the possible equations one can use to find the Transconductance Gain of a MOSFET.

$$g_m = \mu_n C_{OX} \frac{W}{L} V_{OV} \tag{6.1a}$$

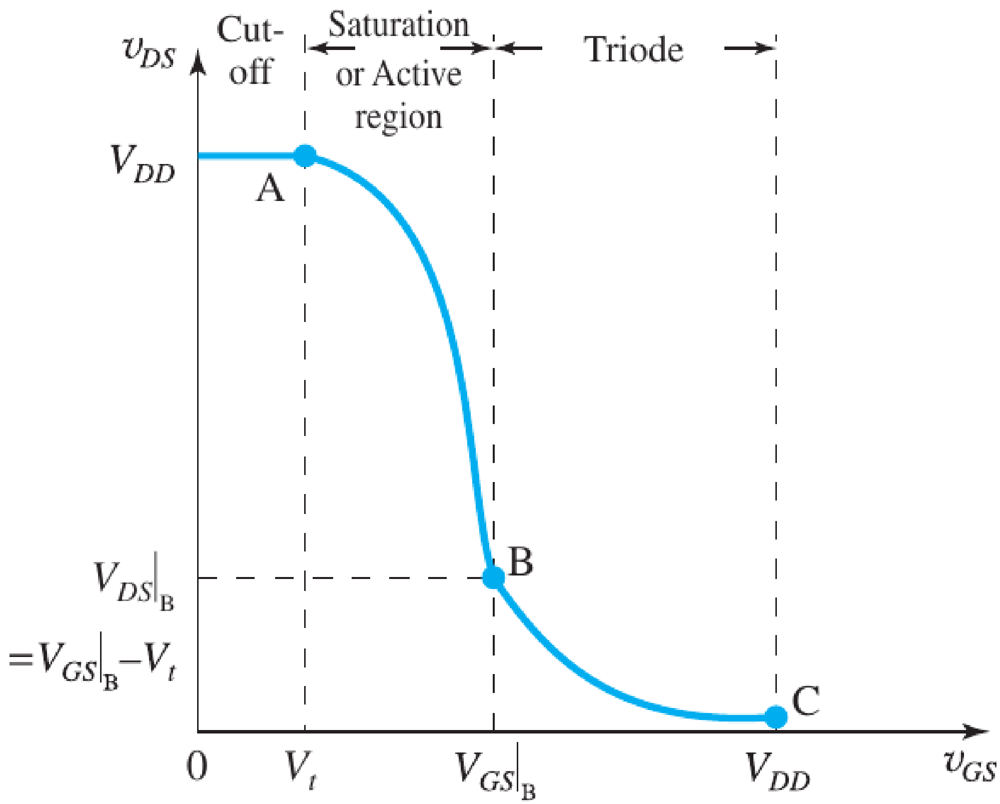


Figure 6.1: MOSFET Voltage Transfer Characteristic

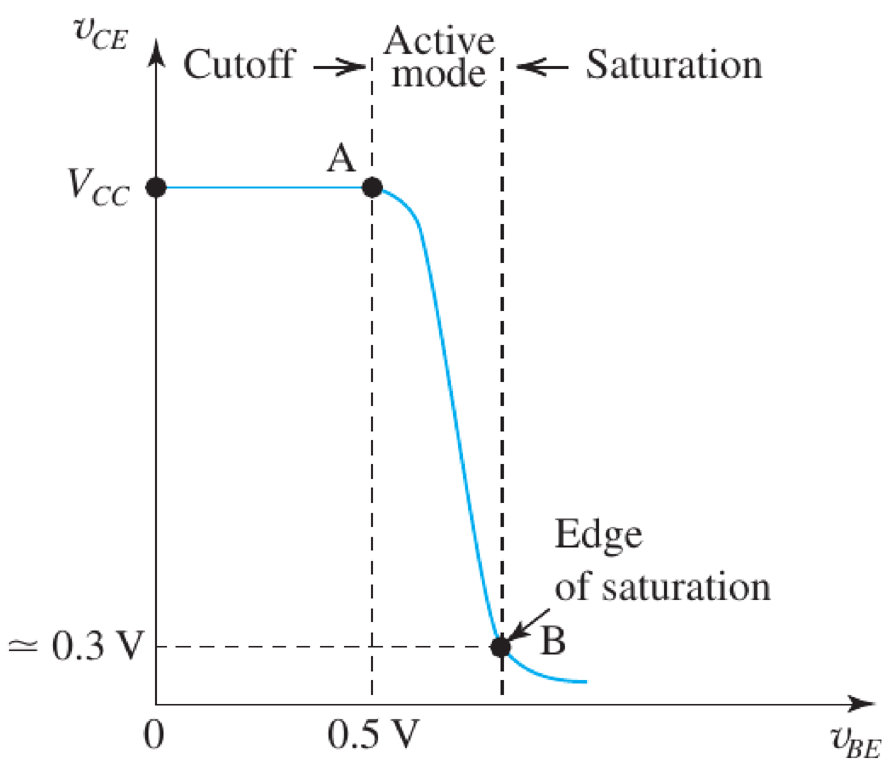


Figure 6.2: BJT Voltage Transfer Characteristic

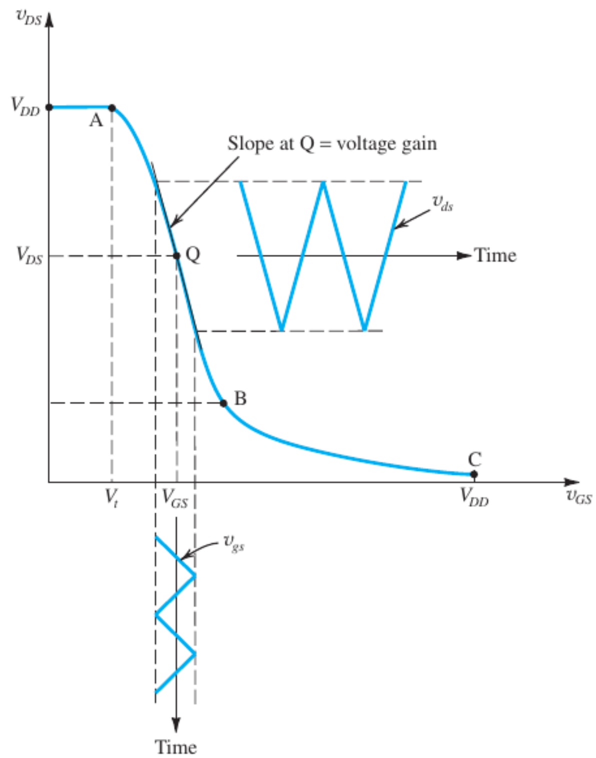


Figure 6.3: MOSFET Voltage Transfer Characteristic Superimposed with Input Signal

$$g_m = \sqrt{2\mu_n C_{OX} \frac{W}{L} I_D} \tag{6.1b}$$

$$g_m = \frac{2I_D}{V_{OV}} \tag{6.1c}$$

The small-signal open circuit voltage gain of a MOSFET is given in Equation (6.2), and the maximum value is found in Equation (6.3).

$$A_v = -\frac{V_{DD} - V_{DS}}{\frac{V_{OV}}{2}} \tag{6.2}$$

$$|A_{v,Max}| = \frac{V_{DD} - V_{OV}|_B}{\frac{V_{OV}|_B}{2}} \tag{6.3}$$

In addition to Equation (6.2), the small-signal amplification of a MOSFET is also dependent on several factors of the circuit, as shown in Equation (6.4).

$$A_v = -k_n R_D V_{OV} \tag{6.4}$$

The resistance at the output *due to the inefficiencies of the transistor* is defined by Equation (6.5).

$$r_o = \frac{V_A}{I_D} \tag{6.5}$$

This resistance goes in *parallel* with the voltage-dependent current source that the Transistor can be modeled as, see Figure 6.5.

6.2.1 Small-Signal Models

Figure 6.4 is the simplest model we can use. In it, we neglect the dependence of i_d on v_{ds}, The Early Effect.

Figure 6.5 is a more accurate model of a MOSFET. It includes the resistance due to the Transistor's channel length modulation, The Early Effect. Equation (6.5) is the value of r_o.

An alternative set of models drawn from the first two small-signal models (Figures 6.4 and 6.5) is the T-Model, Figures 6.6 and 6.7.

Remark (Why T-Model?). The T-models are preferred when there is a resistor attached to the source of the MOSFET.

6.2.2 Configurations

There are three possible configurations for a MOSFET, as seen in Figures 6.8a to 6.8c.

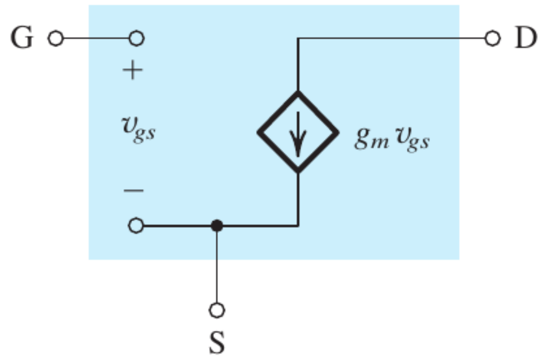


Figure 6.4: Small Signal Model of MOSFET (Sedra et al. 2015, p. 387)

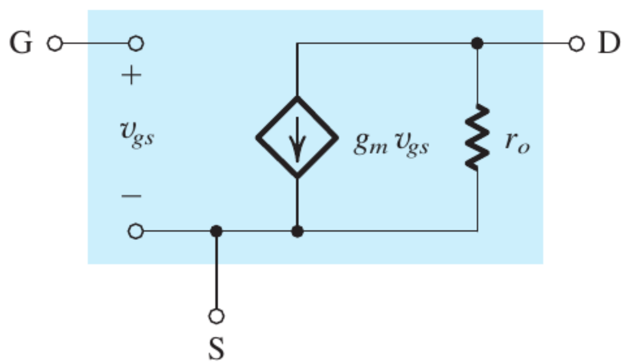


Figure 6.5: Small Signal Model of MOSFET, including Channel Modulation Resistance (Sedra et al. 2015, p. 387)

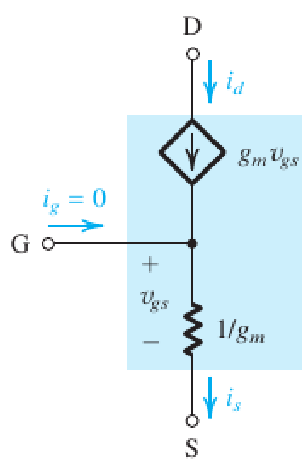


Figure 6.6: T-Model Equivalent of MOSFET (Sedra et al. 2015, p. 394)

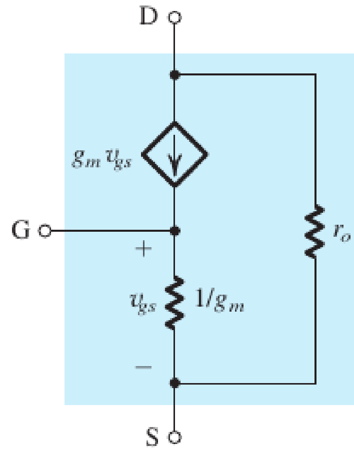
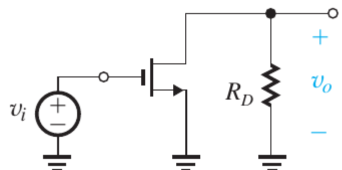
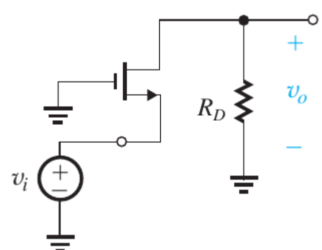


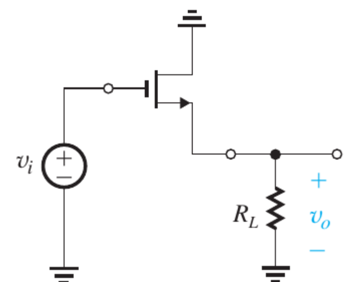
Figure 6.7: T-Model Equivalent of MOSFET, including Channel Modulation Resistance (Sedra et al. 2015, p. 395)



(a) Common Source Configuration (Sedra et al. 2015, p. 424)



(b) Common Gate Configuration (Sedra et al. 2015, p. 424)



(c) Common Drain Configuration (Sedra et al. 2015, p. 424)

Figure 6.8: MOSFET Configurations

6.3 BJT Amplifiers

The Transconductance Gain of a BJT is given by Equation (6.6).

$$g_m = \frac{I_C}{V_T} \tag{6.6}$$

Then, the resistance due to the channel modulation resistance is given by Equation (6.9).

$$r_o = \frac{|V_A|}{I_C} \tag{6.7}$$

Lastly, the resistances from the BJT's construction is Equations (6.7) and (6.8).

$$r_e = \frac{V_T}{I_E} = \alpha \frac{V_T}{I_C} \tag{6.8}$$

$$r_\pi = \frac{V_T}{I_B} = \beta \frac{V_T}{I_C} \tag{6.9}$$

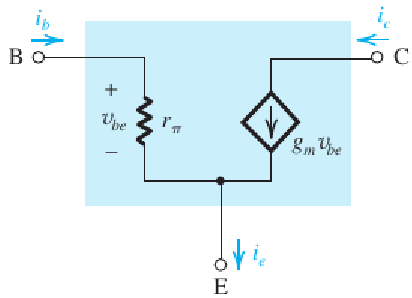
The small-signal voltage gain of a BJT is shown in Equation (6.10).

$$A_V = -\frac{I_C R_C}{V_T} \tag{6.10}$$

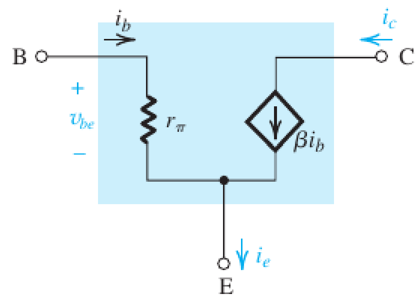
6.3.1 Small-Signal Models

Figures 6.9a to 6.9d are small-signal equivalents for a BJT. This is for an ideal BJT, but a realistic one includes the channel modulation resistance, r_o. Figures 6.10a to 6.10d are the equivalent circuits with the channel length resistance included.

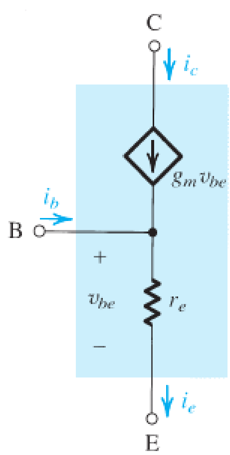
Remark (Why T-Model?). The T-models are preferred when there is a resistor attached to the emitter of the BJT.



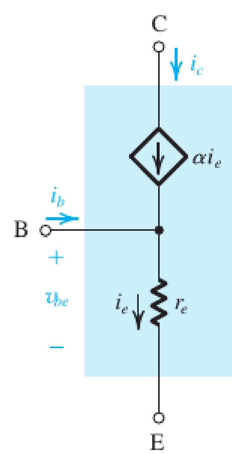
(a) Hybrid-π Model Small-Signal Model, using v_π (Sedra et al. 2015, p. 407)



(b) Hybrid-π Model Small-Signal Model, using i_b (Sedra et al. 2015, p. 407)

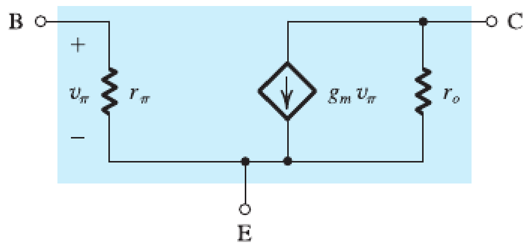


(c) T-Model Small-Signal Model, using v_π (Sedra et al. 2015, p. 409)

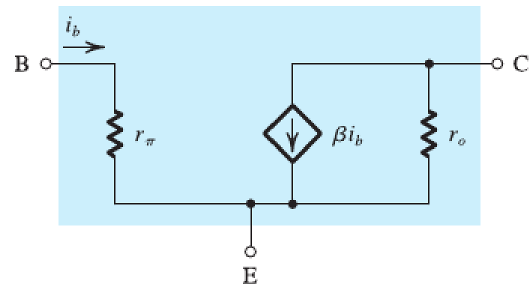


(d) T-Model Small-Signal Model, using i_b (Sedra et al. 2015, p. 409)

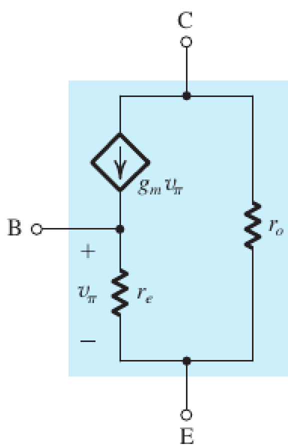
Figure 6.9: Small-Signal Models of a BJT



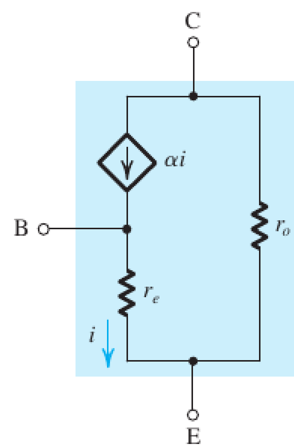
(a) Hybrid-π Model Small-Signal Model, using v_π (Sedra et al. 2015, p. 408)



(b) Hybrid-π Model Small-Signal Model, using i_b (Sedra et al. 2015, p. 408)

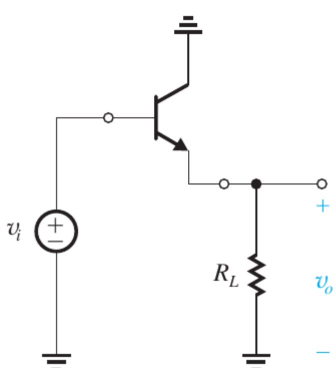


(c) T-Model Small-Signal Model, using v_π (Sedra et al. 2015, p. 410)

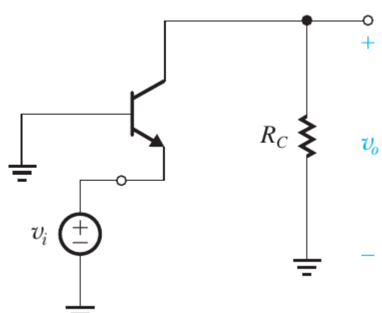


(d) T-Model Small-Signal Model, using i_b (Sedra et al. 2015, p. 410)

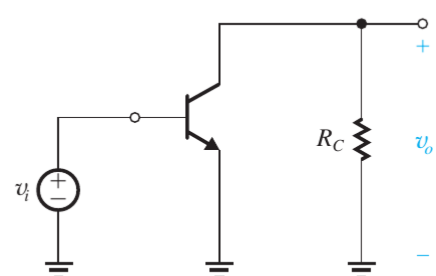
Figure 6.10: Small-Signal Models of a BJT, including channel modulation resistance



(a) Common Emitter Configuration (Sedra et al. 2015, p. 424)



(b) Common Base Configuration (Sedra et al. 2015, p. 424)



(c) Common Collector Configuration (Sedra et al. 2015, p. 424)

Figure 6.11: BJT Configurations

6.3.2 Configurations

The three possible configurations for a BJT are shown in Figures 6.11a to 6.11c.

6.4 Amplification

The *amplification* of a circuit is the ratio of the measured output against its input. Equation (6.11) illustrates this. **Remember that v_{out} MUST NOT include the load resistance** This is also known as the open-circuit gain and/or amplification.

$$A = \frac{v_{out}}{v_{in}} \tag{6.11}$$

6.5 Gain

The *gain* of a circuit is determined by the ratio of the output signal when a load is attached to the input signal. This is distinctly different than Amplification because a load resistance has been added, which alters the output signal.

Equation (6.12) gives a general equation for the gain of a circuit. **Remember that v_{out} MUST include the load resistance!**

$$G = \frac{v_{out}}{v_{in}} \tag{6.12}$$

The gain of a circuit can be calculated by multiplying the amplification of various subcomponents in the circuit with each other. Thus, Equation (6.13) can be put together.

$$G = A_{1\to 2}A_{2\to 3}\cdots A_{n-1\to n} \tag{6.13}$$

Remark (Gain vs. Amplification). Gain and amplification seem to be identical. This is due to the technicality based on what components are included in a circuit when calculating the gain/amplification. In addition, gain tends to be used when the amplification is from the input to the output of the amplifier. Sub-components within the amplifier have amplifications, which when multiplied together, form the gain.

A Complex Numbers

Defn A.0.1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted $\mathbb{C}$.

A complex number is written, in Cartesian form, as shown in Equation (A.1) below.

$$z = a \pm ib \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

A.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn A.1.1 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\text{Re}\{z\} = x \tag{A.3}$$

Remark A.1.1.1 (Alternative Notation). The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re\mathfrak{e}$$

Defn A.1.2 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\text{Im}\{z\} = y \tag{A.4}$$

Remark A.1.2.1 (Alternative Notation). The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im\mathfrak{m}$$

A.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$z = x_1 + iy_1$$
$$w = x_2 + iy_2$$

A.2.1 Addition

The addition operation, still denoted with the + symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w \coloneqq (x_1 + x_2) + i(y_1 + y_2) \tag{A.5}$$

A.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$\begin{aligned} zw &:= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\ &= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned} \tag{A.6}$$

A.3 Complex Conjugates

Defn A.3.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{A.7}$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{A.8}$$

A.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$
2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - xyi + xyi - i^2y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2 \end{aligned}$$

Thus,

$$z\overline{z} = x^2 + y^2 \tag{A.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}$$

Using our solution from Equation (A.9):

$$\begin{aligned} &= \frac{(x + iy)^2}{x^2 + y^2} \\ &= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2} \end{aligned}$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$\begin{aligned}&= \frac{\left(x^2 - y^2\right) + 2xyi}{x^2 + y^2} \\ &= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2}\end{aligned}$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{A.10}$$

A.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.
First, start by defining some expressions so that we can prove some of these properties:

$$\begin{aligned}z &= x + iy \\ \overline{z} &= x - iy\end{aligned}$$

(i) The conjugation operation is commutative.
(ii) The conjugation operation can be distributed over addition and multiplication.

$$\begin{aligned}\overline{z + w} &= \overline{z} + \overline{w} \\ \overline{zw} &= \overline{z}\overline{w}\end{aligned}$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers ($z, w \in \mathbb{C}$) where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.
We start by simplifying the left-hand side of the equation ($\overline{z + w}$).

$$\begin{aligned}\overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2)\end{aligned}$$

Now, we simplify the other side ($\overline{z} + \overline{w}$).

$$\begin{aligned}\overline{z} + \overline{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= (x_1 + x_2) - i(y_1 + y_2)\end{aligned}$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

■

A.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn A.4.1 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in \mathbb{C}$. This is taken from Descarte's definition of a completely real plane.
The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

A.4.1 Modulus of a Complex Number

Defn A.4.2 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\overline{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{A.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in \mathbb{C}$). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (A.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\overline{z}\overline{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \end{aligned}$$

Note how we never needed to define z or w, so this is as general a result as possible. ■

A.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right) \\ &= |z|^2|w|^2 \end{aligned}$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

A.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

A.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (A.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{A.12}$$

Example A.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$\begin{aligned}
|5iz + 10| &= 7 \\
|5i(z + \frac{10}{5i})| &= 7 \\
|5i(z + \frac{2}{i})| &= 7 \\
|5i(z + \frac{2}{i}\frac{-i}{-i})| &= 7 \\
|5i(z - 2i)| &= 7
\end{aligned}$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$\begin{aligned}
|5i||z - 2i| &= 7 \\
5|z - 2i| &= 7 \\
|z - 2i| &= \frac{7}{5}
\end{aligned}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

A.5.1 Annulus

Defn A.5.1 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (A.13).

$$r_1 \leq |z - a| \leq r_2 \tag{A.13}$$

In Equation (A.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

A.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (A.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{A.14}$$

Remark. Note that in the definition of polar form (Equation (A.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$\begin{aligned}
r &= |z| \\
\theta &= \arg(z)
\end{aligned}$$

Example A.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\text{Re}\{z\}^2 + \text{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned} \cos(\theta) &= \frac{-\sqrt{3}}{2} \\ \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned} r &= 2 \\ \theta &= \frac{5\pi}{6} \\ \arg(z) &= \frac{5\pi}{6} + 2\pi k \end{aligned}$$

A.6.1 Converting Between Cartesian and Polar Forms

Using Equation (A.14) and Equation (A.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{A.15}$$

A.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

A.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \tag{A.16}$$

A.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big) \tag{A.17}$$

A.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph A.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn A.6.1 (de Moiver's Law)**.** Given a complex number z, $z \in \mathbb{C}$ and a rational number n, $n \in \mathbb{Q}$, the exponentiation of z^n is defined as Equation (A.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{A.18}$$

A.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{A.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (A.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{A.20}$$

A w that satisfies Equation (A.20) is an nth root of z.

Example A.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example A.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (A.19), we can find the general form of the roots.

$$\begin{aligned}
z &= 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)\\
z^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right)\\
&= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right)
\end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \ldots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

k = 2
$$\sqrt[3]{2}\left(\cos\left(\frac{\pi+12\pi(2)}{18}\right)+i\sin\left(\frac{\pi+12\pi(2)}{18}\right)\right)=\sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right)+i\sin\left(\frac{25\pi}{18}\right)\right)$$
k = 3
$$\begin{aligned}\sqrt[3]{2}\left(\cos\left(\frac{\pi+12\pi(3)}{18}\right)+i\sin\left(\frac{\pi+12\pi(3)}{18}\right)\right)&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}+\frac{36\pi}{18}\right)+i\sin\left(\frac{\pi}{18}+\frac{36\pi}{18}\right)\right)\\&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}+2\pi\right)+i\sin\left(\frac{\pi}{18}+2\pi\right)\right)\\&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right)+i\sin\left(\frac{\pi}{18}\right)\right)\end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned}z_1^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right)+i\sin\left(\frac{\pi}{18}\right)\right)\\z_2^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right)+i\sin\left(\frac{13\pi}{18}\right)\right)\\z_3^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right)+i\sin\left(\frac{25\pi}{18}\right)\right)\end{aligned}$$

A.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn A.8.1 (Argument)**.** The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example A.4: Argument of Complex Number. Lecture 3, Example 1

If $z=-1-i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned}z&=-1-i\\\arg(z)=\tan(\theta)&=\frac{-i}{-1}\\&=\frac{\pi}{4}\end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$=\frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z)=\frac{5\pi}{4}+2\pi k$$

Thus, the argument of $z=-1-i$ is $\arg(z)=\frac{5\pi}{4}+2\pi k$.

Defn A.8.2 (Principal Argument)**.** The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi<\operatorname{Arg}(z)\leq\pi$.

Example A.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) &= \tan(\theta) = \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\operatorname{Arg}(z) = \frac{5\pi}{4}$.

A.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (A.21).

$$e^z = e^{x+iy} = e^x \big(\cos(y) + i\sin(y)\big) \tag{A.21}$$

If instead of e as the base, we have some value a, then we have Equation (A.22).

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ &= e^{\operatorname{Re}\{z\ln(a)\}}\Big(\cos\big(\operatorname{Im}\{z\ln(a)\}\big) + i\sin\big(\operatorname{Im}\{z\ln(a)\}\big)\Big) \end{aligned} \tag{A.22}$$

In the case of Equation (A.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example A.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$\begin{aligned} x &= -1 \\ y &= \sqrt{3} \end{aligned}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\begin{aligned} \left|e^{-1+i\sqrt{3}}\right| &= \left|e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right| \\ &= e^{-1} \\ \arg\left(e^{-1+i\sqrt{3}}\right) &= \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} + 2\pi k \\ \operatorname{Arg}\left(e^{-1+i\sqrt{3}}\right) &= \operatorname{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} \end{aligned}$$

Example A.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^{0}\big(\cos(-1) + i\sin(-1)\big) \\ &= 1\big(\cos(-1) + i\sin(-1)\big) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1\big(\cos(-1)+i\sin(-1)\big)} \\ &= e^{-1\big(\cos(1)-i\sin(1)\big)} \\ &= e^{-\cos(1)+i\sin(1)} \end{aligned}$$

If we refer back to Equation (A.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big) \end{aligned}$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} \left|e^{-e^{-i}}\right| &= \left|e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right| \\ &= e^{-\cos(1)} \\ \arg\left(e^{-e^{-i}}\right) &= \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) + 2\pi k \\ \operatorname{Arg}\left(e^{-e^{-i}}\right) &= \operatorname{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) \end{aligned}$$

Example A.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^{i}$?

Use Equation (A.22) to simplify this to a base of e, where we can use the usual Equation (A.21) to solve this.

$$\begin{aligned} a^{z} &= e^{z\ln(a)} \\ 1^{i} &= e^{i\ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^{2}} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

A.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.23}$$

A.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn A.10.1 (Complex Logarithm)**.** The *complex logarithm* is defined in Equation (A.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e |w| + i \arg(w) \end{aligned} \tag{A.24}$$

Remark A.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example A.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (A.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e |z| + i \arg(z) \\ &= \log_e |-1| + i \arg(-1) \\ &= \log_e(1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \ldots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example A.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (A.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e |w| + i \arg(w) \\ \ln(1) &= \log_e |1| + i \arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e |2\pi k i| + i \arg(2\pi k i) \\ &= \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases} \right) \end{aligned}$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases}\right)$.

A.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\overline{z}) \tag{A.25}$$

A.11 Complex Trigonometry

For the equations below, $z \in \mathbb{C}$. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{A.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{A.27}$$

Example A.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

We start by using the defintion of complex cosine Equation (A.26).

$$\begin{aligned}
\cos(z) &= 5 \\
\frac{e^{iz} + e^{-iz}}{2} &= 5 \\
e^{iz} + e^{-iz} &= 10 \\
e^{iz}\left(e^{iz} + e^{-iz}\right) &= e^{iz}(10) \\
{e^{iz}}^2 + 1 &= 10e^{iz} \\
{e^{iz}}^2 - 10e^{iz} + 1 = 0
\end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned}
e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} \\
&= \frac{10 \pm \sqrt{100 - 4}}{2} \\
&= \frac{10 \pm \sqrt{96}}{2} \\
&= \frac{10 \pm 4\sqrt{6}}{2} \\
&= 5 \pm 2\sqrt{6}
\end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned}
iz &= \ln(5 \pm 2\sqrt{6}) \\
&= \log_e\left|5 \pm 2\sqrt{6}\right| + i\arg(5 \pm 2\sqrt{6}) \\
&= \log_e\left|5 \pm 2\sqrt{6}\right| + i(0 + 2\pi k) \\
&= \log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi k i \\
z &= \frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi k i\right) \\
&= \frac{-i}{-i}\frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right|\right) + 2\pi k \\
&= 2\pi k - i\log_e\left|5 \pm 2\sqrt{6}\right|
\end{aligned}$$

Thus, $z = 2\pi k - i \log_e \left|5 \pm 2\sqrt{6}\right|$.

A.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x \pm iy) = \big(\cos(x)\cosh(y)\big) \mp i\big(\sin(x)\sinh(y)\big) \tag{A.28}$$

$$\sin(x \pm iy) = \big(\sin(x)\cosh(y)\big) \pm i\big(\cos(x)\sinh(y)\big) \tag{A.29}$$

Example A.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (A.28).

$$\begin{aligned}\cos(x + iy) &= \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big)\\ \cos(2 + 3i) &= \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)\end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (A.21), we can solve this.

$$\begin{aligned}e^z = e^{x+iy} &= e^x\big(\cos(y) + i\sin(y)\big)\\ x &= \cos(2)\cosh(3)\\ y &= -\sin(2)\sinh(3)\\ e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} &= e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)\end{aligned}$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned}|z| &= \left|e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)\right|\\ &= e^{\cos(2)\cosh(3)}\\ \arg(z) &= \arg(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3) + 2\pi k\\ \operatorname{Arg}(z) &= \operatorname{Arg}(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3)\end{aligned}$$

A.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned}\overline{\cos(x)} &= \cos(x)\\ &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\end{aligned} \tag{A.30}$$

$$\begin{aligned}\overline{\sin(x)} &= \sin(x)\\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\end{aligned} \tag{A.31}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1-\cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1+\cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{B.24}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.26}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.27}$$

C Calculus

C.1 L'Hôpital's Rule

L'Hôpital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcilable results.

Lemma C.0.1 (L'Hôpital's Rule)**.** *If the equation*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) *holds.*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.3}$$

Defn C.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Quotient Rule

If

$$f(x) = \frac{g(x)}{h(x)}$$

then,

$$f'(x) = \frac{\frac{d\,g(x)}{dx}h(x) - g(x)\frac{d\,h(x)}{dx}}{\big(h(x)\big)^2} \tag{C.4}$$

C.3.2 Chain Rule

Defn C.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.5}$$

C.4 Useful Integrals

$$\int \cos(x)\,dx = \sin(x) \tag{C.6}$$

$$\int \sin(x)\,dx = -\cos(x) \tag{C.7}$$

$$\int x\cos(x)\,dx = \cos(x) + x\sin(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x\sin(x)\,dx = \sin(x) - x\cos(x) \tag{C.9}$$

Equation (C.9) simplified with Integration by Parts.

$$\int x^2\cos(x)\,dx = 2x\cos(x) + (x^2-2)\sin(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int x^2\sin(x)\,dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{C.11}$$

Equation (C.11) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\,dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x}\sin(\beta x)\,dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.13}$$

$$\int e^{\alpha x}\,dx = \frac{e^{\alpha x}}{\alpha} \tag{C.14}$$

$$\int xe^{\alpha x}\,dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{C.15}$$

Equation (C.15) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha+\beta x} = \int \frac{1}{\alpha+\beta x}\,dx = \frac{1}{\beta}\ln(\alpha+\beta x) \tag{C.16}$$

$$\int \frac{dx}{\alpha^2+\beta^2x^2} = \int \frac{1}{\alpha^2+\beta^2x^2}\,dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{C.17}$$

$$\int \alpha^x\,dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.18}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{C.19}$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\big[b(t),t\big]\,\frac{db(t)}{dt} - f\big[a(t),t\big]\,\frac{da(t)}{dt} \tag{C.20}$$

C.6 Laplace's Equation

Laplace's Equation is used to define a harmonic equation. These functions are twice continuously differentiable $f : U \to R$, where U is an open subset of $\mathbb{R}^n$, that satisfies Equation (C.21).

$$\frac{\partial^2 f}{\partial {x_1}^2} + \frac{\partial^2 f}{\partial {x_2}^2} + \cdots + \frac{\partial^2 f}{\partial {x_n}^2} = 0 \tag{C.21}$$

This is usually simplified down to

$$\nabla^2 f = 0 \tag{C.22}$$

D Laplace Transform

D.1 Laplace Transform

Defn D.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int\limits_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$

D.2 Inverse Laplace Transform

Defn D.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\, ds \tag{D.2}$$

D.3 Properties of the Laplace Transform

D.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (D.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{D.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{D.3b}$$

D.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (D.4).

$$\mathcal{L}\big\{x(\alpha t)\big\} = \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \tag{D.4}$$

D.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (D.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\big\{x(t-a)\big\} = X(s)e^{-as} \tag{D.5}$$

D.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\big\{X(s-a)\big\} = x(t)e^{at} \tag{D.6}$$

D.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\, d\lambda\right\} = \frac{1}{s}X(s) \tag{D.7}$$

D.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\big\{x(t) * v(t)\big\} = X(s)V(s) \tag{D.8}$$

D.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (D.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \tag{D.9}$$

D.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem D.1 (Intial Value Theorem)**.** *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Theorem D.2 (Final Value Theorem)**.** *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

D.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t-T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n\,\mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t-T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at}\,\mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at}\,\mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)\,\mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)\,\mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)\,\mathcal{U}(t)$	$\begin{cases} a: & \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: & \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb}+\frac{re^{-j\theta}}{s+a+jb}\right) \\ c: & \frac{As+B}{s^2+2as+c} \begin{cases} r & = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta & = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$
$e^{-at}\left(A\cos(\sqrt{c-a^2}t)+\frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

D.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0)-X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0)+X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n\frac{d^n}{ds^n}X(s)\ n\in\mathbb{N}$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s)-\sum_{0}^{n-1}s^{n-1-i}\frac{d^i}{dt^i}x(t)\vert_{t=0^-}\ n\in\mathbb{N}$

References

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