Phys 123: Classical Mechanics - Reference Sheet

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Contents

1 Vectors

Defn 1 (Vector)**.** A *vector* is a way to show both magnitude of displacement and direction of displacement. Vectors are drawn as rays.

Remark 1.1. Vectors and Scalars may seem similar, but are different.

Defn 2 (Scalar)**.** A *scalar* is a way to show ***ONLY*** the magnitude of a displacement, without any direction information.

1.1 Vector Properties

(i) $\vec{A} + \vec{B} = \vec{C}$
(ii) $\vec{0} = \langle 0, 0, 0, \dots, 0 \rangle$
(iii) $\vec{A} + \vec{0} = \vec{A}$
(iv) $\vec{A} + -\vec{A} = \vec{0}$
(v) $\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$
(vi) $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
(vii) Magnitude of vector: $\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

1.1.1 Getting Components

Getting the components of a vector involves solving the imaginary pythagorean triangle around the vector.

For a 2-dimensional vector, $\vec{V}$, you have the components $\langle V_x, V_y \rangle$. You find their values with this equation:

$$\begin{aligned} V_x &= V \cos\theta \\ V_y &= V \sin\theta \end{aligned} \tag{1.1}$$

1.1.2 3D Unit Vectors

3-dimensional vectors shouldn't be any too crazy by this point. They are just another variable that can be thrown around in the vector. However, the three 3D Unit Vectors are special. You can also use these to describe any lower-dimensional vector as well.

$$\begin{aligned} \hat{\imath} &= \langle 1, 0, 0 \rangle \\ \hat{\jmath} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle \end{aligned} \tag{1.2}$$

1.1.3 Addition

Vectors are additive, and are done from head-to-tail. This means that

$$\vec{A} + \vec{B} = \vec{C} \tag{1.3}$$

This means that in 3-dimensional vectors, they are added like this:

$$\begin{aligned} \vec{A} &= \langle A_x, A_y, A_z \rangle \\ \vec{B} &= \langle B_x, B_y, B_z \rangle \\ \vec{A} + \vec{B} &= \langle A_x + B_x, A_y + B_y, A_z + B_z \rangle \end{aligned} \tag{1.4}$$

1.1.4 Scalar Multiplication

When applying multiplication between a scalar and a vector, you perform Scalar Multiplication.

$$2 \times \vec{V} = 2\langle V_x, V_y \rangle = \langle 2V_x, 2V_y, 2V_z \rangle \tag{1.5}$$

This means that you do ***NOT*** modify the direction of the vector, you only change its magnitude.

1.1.5 Scalar (Dot) Product

The Scalar (Dot) Product is the first of two ways to multiply 2 vectors. The other is the Vector (Cross) Product. There are 2 ways to calculate the Scalar (Dot) Product.

The first involves using the magnitudes of each vector and multiplying those by the cosine of the angle between them.

$$\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\| \cos(\theta) \tag{1.6}$$

The second is done by adding the product of each component of each vector.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.7}$$

Remark. This means that when you apply the Scalar (Dot) Product to 2 vectors, you return a Scalar.

Properties of Scalar (Dot) Product

(i) $(\vec{A})^2 = \vec{A} \cdot \vec{A}$

(ii) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

1.1.6 Vector (Cross) Product

The Vector (Cross) Product is the second of two ways to multiply 2 vectors. The other is the Scalar (Dot) Product. There are 2 ways to calculate the Vector (Cross) Product.

The first involves using the magnitudes of each vector and multiplying those by the sine of the angle between them.

$$\vec{A} \times \vec{B} = \|\vec{A}\|\|\vec{B}\| \sin\theta \tag{1.8}$$

The second is done by taking the determinant of a 2×2 or 3×3 matrix.

$$\begin{aligned}
\vec{A} \times \vec{B} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= (A_y B_z - A_z B_y)\,\hat{\imath} - (A_x B_z - A_z B_x)\,\hat{\jmath} + (A_x B_y - A_y B_x)\,\hat{k} \\
&= \langle A_y B_z - A_z B_y, -(A_x B_z + A_z B_x), A_x B_y - A_y B_x \rangle
\end{aligned} \tag{1.9}$$

Remark. This means that when you apply the Vector (Cross) Product to 2 vectors, you return a Vector.

Properties of Vector (Cross) Product

(i) $\vec{A} \times \vec{A} = \vec{0}$

(ii) $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$

(iii) $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$

(iv) $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C}\left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)$

2 Kinematics

Defn 3 (Kinematics)**.** *Kinematics* is a way to describe macroscopic motion with equations. This includes anything moving, falling, thrown, shot, launched, etc. This forms the fundamental basis for all of classical mechanics.

2.1 1-D Kinematics

Defn 4 (1-D Displacement)**.** *One dimensional displacement* is calculated based on the change in position of the 'thing.'

$$s = x_2 - x_1 \tag{2.1}$$

Remark 4.1. *Displacement is different than path!* Displacement is the change in position of an object. Path is the length of the path takes between its starting and end point.

Defn 5 (1-D Velocity)**.** *One dimensional velocity* is calculated as the displacement per unit time. There is instantaneous velocity and average velocity. Average velocity is calculated with Equation (2.2).

$$v = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.2}$$

Instantaneous velocity is calculated by reducing the time interval Δt to 0. This can be summarized in Equation (2.3).

$$\begin{aligned} v &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \\ &= \frac{dx}{dt} \end{aligned} \tag{2.3}$$

Defn 6 (Acceleration)**.** *One dimesional acceleration* is the change in velocity over time. Again, there is average acceleration and instantaneous acceleration. Average acceleration is calculated with Equation (2.4)

$$a = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \tag{2.4}$$

Instantaneous acceleration is calculated by reducing the time interval Δt to 0. This can be summarized by Equation (2.5).

$$\begin{aligned} a &= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \\ &= \frac{dv}{dt} = \frac{d^2x}{dt^2} \end{aligned} \tag{2.5}$$

2.2 Multi-Dimensional Kinematics

Because we can represent a two-dimensional and three-dimensional space in sets, and movement through this space as their respectively dimensioned vectors, we can construct multi-dimensional problems with multi-dimensional vectors! This is a massive simplification, because instead of solving for one equation with three variables, we can solve three equations for one variable each!!

For the following definitions, I have assumed that we are in a 3-dimensional space (x, y, z).

Defn 7 (Multi-Dimensional Position)**.** *Position* in multiple dimensions is done by referring to each of the consituent dimensions.

$$\vec{s} = (x(t), y(t), z(t)) \tag{2.6}$$

Defn 8 (Multi-Dimensional Displacement)**.** *Displacement* in multiple dimensions can be broken down into several 1-D Displacements. Since 1-D Displacement is calculated as the differnce between the start and end position, the same is true for th emulti-dimensional case.

$$\begin{aligned} \vec{r} &= \Delta \vec{s} = \vec{s}_2 - \vec{s}_1 \\ &= \langle x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t) \\ &= \langle r_x(t), r_y(t), r_z(t) \end{aligned} \tag{2.7}$$

Defn 9 (Multi-Dimensional Velocity)**.** *Velocity* in multiple dimensions is described in much the same way as 1-D Velocity.

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \left\langle \frac{dr_x(t)}{dt}, \frac{dr_y(t)}{dt}, \frac{dr_z(t)}{dt} \right\rangle \\ &= \left\langle r'_x(t), r'_y(t), r'_z(t) \right\rangle \end{aligned} \tag{2.8}$$

Defn 10 (Multi-Dimensional Acceleration)**.** *Acceleration* in multiple dimensions is described in much the same way as Acceleration.

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt} \\ &= \left\langle \frac{dv_x(t)}{dt}, \frac{dv_y(t)}{dt}, \frac{dv_z(t)}{dt} \right\rangle = \left\langle \frac{d^2r_x(t)}{dt}, \frac{d^2r_y(t)}{dt}, \frac{d^2r_z(t)}{dt} \right\rangle \\ &= \left\langle v'_x(t), v'_y(t), v'_z(t) \right\rangle = \left\langle r''_x(t), r''_y(t), r''_z(t) \right\rangle \end{aligned} \tag{2.9}$$

2.3 Projectile Motion

Defn 11 (Projectile)**.** A *projectile* is any body given an initial velocity that then follows a path determined by gravity and air resistance.

Remark 11.1. However, for most of our calculations, we will neglect air resistance. Air resistance can be a difficult thing to calcultate for, especially in the variable cases that we will have.

There are a few things to keep in mind with projectiles in motion.

1. Origin is where the projectile starts from
2. The x-axis is the *distance* that the projectile travels. This is its displacement.
3. The y-axis is the *height* that the projectile travels.
4. The end point (landing point) is the only thing that may change on the x-axis.
5. The acceleration vector is as follows: $\langle 0, -g \rangle$.
6. *Trajectory* depends on $\vec{v}_0$ and $\vec{a}$ *ONLY.*
7. The two components of the projectile's initial velocity are *independent* $(v_{0,x}, v_{0,y})$.

2.3.1 Projectile Motion Equations

The following equations are used to solve for various questions that could be asked about projectile motion.

Initial Velocity Components

$$v_{0,x} = v_0 \cos(\theta) \quad v_{0,y} = v_0 \sin(\theta) \tag{2.10}$$

Velocity Components

$$v_x = v_{0,x} \cos(\theta) \quad v_y = v_{0,y} \sin(\theta) - gt \tag{2.11}$$

Projectile Position

$$x = v_0 t \cos(\theta) \quad y = v_0 t \sin(\theta) - \frac{1}{2} g t^2 \tag{2.12}$$

Projectile Time

$$t = \frac{x}{v_0 \cos(\theta)} \quad t = \frac{v_0 \sin(\theta)}{g} \tag{2.13}$$

Projectile Range

$$R = \frac{v_0}{g} \cos(\theta) \sin(\theta) \tag{2.14}$$

Projectile Maximum Range

$$R_{\text{Max}} = \frac{v_0^2}{g} \tag{2.15}$$

This means that the θ in Equation (2.14) is 45°.

Projectile Height

$$h = \frac{v_0^2}{2g} \sin^2(\theta) \tag{2.16}$$

Projectile Maximum Height

$$h = \frac{v_0^2}{2g} \tag{2.17}$$

The lack of $\sin^2(\theta)$ from Equation (2.16) means that there is ***NO*** y-component to the velocity, meaning the projectile is at its instant of maximum height.

3 Uniform Circular Motion

Defn 12 (Uniform Circular Motion)**.** *Uniform circular motion* is when an object of moving in a perpetual circular motion. There is no outside source of acceleration changing the state of the system.

Remark 12.1. This does *not* happen in real life. However, it is useful for modelling things under ideal conditions that do happen in real life.

3.1 Angular Part of Circular Motion

During Uniform Circular Motion, your terminology changes a little bit.

Defn 13 (Angular Position)**.** *Angular position* is determined with radians around a circle. It is denoted with

$$\vec{\theta}$$

Defn 14 (Angular Velocity)**.** *Angular velocity* is orthogonal to the flat 2-D plane that the object is traveling in. It is the dervative of the Angular Position.

$$\begin{aligned}\vec{\omega} &= \frac{d\theta}{dt} \\ &= \langle 0, 0, \omega \rangle\end{aligned} \tag{3.1}$$

Defn 15 (Angular Acceleration)**.** *Angular acceleration* is the derivative of the Angular Velocity.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt} \tag{3.2}$$

Remark 15.1. Note that under Uniform Circular Motion, by its very definition, there cannot be any acceleration on the object. Therefore, when an object is in uniform circular motion, $\vec{\alpha} = 0$.

However, when the object is ***NOT*** in Uniform Circular Motion the object is undergoing Linear Acceleration.

3.2 Linear Part of Circular Motion

Defn 16 (Linear Position)**.** *Linear position* relates the position of an object from the cartesian coordinate plane to the polar. This means that:

$$x = r\cos(\theta) \quad y = r\sin(\theta) \tag{3.3}$$

Defn 17 (Linear Velocity)**.** *Linear velocity* relates the velocity of an object in a line to its Angular Velocity.

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ \frac{d\vec{r}}{dt} &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, 0 \right\rangle = \left\langle \frac{d}{dt} r\cos(\theta), \frac{d}{dt} r\sin(\theta), 0 \right\rangle \\ &= \Big\langle -r\sin(\theta)\omega, r\cos(\theta)\omega, 0 \Big\rangle \\ &= \vec{\omega} \times \vec{r}\end{aligned} \tag{3.4}$$

Defn 18 (Linear Acceleration)**.** *Linear acceleration* is the derivative of Linear Velocity. It relates the acceleration of an object in a line is relative to its Angular Acceleration

$$\begin{aligned}\frac{d\vec{a}}{dt} &= \frac{d\vec{v}}{dt} \\ &= \langle -r\omega^2\cos(\theta), -r\omega^2\sin(\theta), 0 \rangle = \omega^2 \langle -r\cos(\theta), -r\sin(\theta), 0 \rangle \; () \\ &= -\omega^2\vec{r}\end{aligned} \tag{3.5}$$

3.3 Relation Between the Angular Part of Circular Motion and the Linear Part of Circular Motion

There are a few equations that relate both the Angular Part of Circular Motion and the Linear Part of Circular Motion.

Velocity and Angular Velocity

$$v = \omega r \tag{3.6}$$

Acceleration and Angular Acceleration

$$\begin{aligned} a &= \omega^2 r \\ &= \frac{v^2}{r} \end{aligned} \tag{3.7}$$

4 Reference Frames

Defn 19 (Reference Frames)**.** An *inertial reference frame* is a frame for the world. It is easiest to think of of an inertial reference frame with an example. For instance, when you're in a car going 40 mph and you see someone going 45, you can only tell they're going 5 mph than you.

Defn 20 (Galileo Transformation/Relativity Principle)**.** This is a transformation that happens when you're calculating in one Reference Frames and the event is happening in a different Reference Frames.

$$\begin{aligned} \frac{d\vec{R}}{dt} &= \frac{d\vec{r}}{dt} + \frac{d\vec{r}'}{dt} \\ \vec{V} &= \vec{v} + \vec{v}' \end{aligned} \tag{4.1}$$

5 Newton's Laws

There are 3 fundamental laws of classical mechanics.

1. An object in motion/at reast stays as such, unless acted upon by an outide force(s).
2. Force is equal to the change in momentum ($p = mv$) per change in time. For a constant mass, force equals mass times acceleration ($\vec{F} = m\vec{a}$).
3. For every force there is an equal and opposite force.

There are a few forces that are fundamental in our universe.

- Gravitational Force
- Magnetic Force
- Electric Force
- Frictional Force
- Normal Force
- Tension Force

Remark. ***REMEMBER*** that forces are vectors!!

To solve any force problem, you should construct a free-body diagram.

5.1 Newton's Laws in 1-D

Example 5.1: Atwood Machine.

Find the acceleration that is the result of dropping one side of the machine? Neglect any friction that may occur because of the pulley.

$$\begin{aligned} m_2\vec{a}_2 &= m_2\vec{g} + \vec{T}_2 \\ m_1\vec{a}_1 &= m_1\vec{g} + \vec{T}_1 \end{aligned}$$

One thing to remember is that both masses will have the same acceleration, because the pulley has no friction. So because $\vec{a}_2 = \vec{a}_1$, we can solve for a single $\vec{a}$ variable.

$$\begin{aligned} m_2\vec{a} &= m_2\vec{g} + \vec{T}_2 \\ m_1\vec{a} &= m_1\vec{g} + \vec{T}_1 \\ \vec{a}\left(m_2 + m_1\right) &= \vec{g}\left(m_2 - m_1\right) \\ \vec{a} &= \frac{g\left(m_2 - m_1\right)}{m_2 + m_1} \end{aligned}$$

5.2 Newton's Laws in Multi-D

For a multi-dimensional situation, you break the free-body problem's vectors down into their components.

Example 5.2: Sliding Block.

If a block of mass m slides down a ramp of angle θ, what is the coefficient of static and kinetic friction?

5.3 Common Forces

This section will discuss various forces in greater detail.

5.3.1 Gravitational Force

Defn 21 (Gravitational Force)**.** *Gravitational force* is a force that arises from Gravity. Gravity will be covered more in Section 10, Gravitation. This force arises when one object is being pulled towards another due to Gravity. Gravitational Force is generally the acceleration vector due to Earth's gravity.

$$\vec{g} = 9.81\text{m/s}^2 \tag{5.1}$$

5.3.2 Magnetic Force

This force is discussed in much greater detail in the Physics 221: Electromagnetics and Optics course.

5.3.3 Electric Force

This force is discussed in much greater detail in the Physics 221: Electromagnetics and Optics course.

5.3.4 Frictional Force

Frictional force is the common culprit for the "Equal and Opposite Force" in classical mechanics.

Defn 22 (Frictional Force)**.** Frictional force is defined as

$$\vec{F}_f = \mu\vec{N} \tag{5.2}$$

- μ - The coefficient of the type of friction you are dealing with (Either the Coefficient of Static Friction or Coefficient of Kinetic Friction)
- $\vec{N}$ - The Normal Force

There are 2 types of frictional force:

1. Static Friction
2. Kinetic Friction

Defn 23 (Static Friction)**.** *Static friction* is friction that arises when an object is starting from a static position, and is being moved. This force tends to be stronger than Kinetic Friction because of electrostatic bonds between the object and it's supporting surface, along with other reasons. However, static friction is drawn from the Coefficient of Static Friction. This is a Scalar number that represents how much the object *does not* want to move.

$$\vec{F}_{f,s} = \mu_s\vec{N} \tag{5.3}$$

- μ_s - The Coefficient of Static Friction
- $\vec{N}$ - The Normal Force

Defn 24 (Coefficient of Static Friction)**.** *Coefficient of static friction* is a Scalar number that represents how much the object does not want to **START** moving. This value tends to be greater than the Coefficient of Kinetic Friction. This is because there are additional bonds and forces at play that resist the start of an objects motion. The Coefficient of Kinetic Friction is denoted as such:

$$\mu_s \tag{5.4}$$

Defn 25 (Kinetic Friction)**.** *Kinetic friction* is a frction that arises when an object is in motion. This force is weaker than Static Friction. The value we may use for kinetic friction is drawn from the Coefficient of Kinetic Friction.

$$\vec{F}_{f,k} = -\mu_k\vec{N} \tag{5.5}$$

- μ_k - The Coefficient of Kinetic Friction
- $\vec{N}$ - The Normal Force that is going ***IN*** the direction of motion. You might need to break the normal force's vector down into its components.

Defn 26 (Coefficient of Kinetic Friction)**.** *Coefficient of kinetic friction* is Scalar number that represents how much the object does not want to **CONTINUE** moving. This value tends to be smaller than its Coefficient of Static Friction counterpart. The Coefficient of Kinetic Friction is denoted as such:

$$\mu_k \tag{5.6}$$

5.3.5 Normal Force

Defn 27 (Normal Force)**.** The *normal force* is, as the name implies, a normalizing force. This does not necessarily make it special, as both Kinetic Friction and Static Friction can be considered normalizing forces as well. However, the normal force is generally considered whenever something is being held against something. For example, a book on a table. The table is exerting a Normal Force on the book to prevent it from falling through the table. Likewise, the ground is exerting a Normal Force on the table to prevent it from falling through the ground.

This normal force is defined as

$$\vec{N} = -m\vec{g} \tag{5.7}$$

6 Dynamics of Circular Motion

There are three cases when the Dynamics of Circular Motion are easily visible. Each of these is illustrated with an example.

1. Turning on a flat curve, neglecting friction (Example 6.1)
2. Turning on an angled curve, neglecting friction (Example 6.2)
3. Turning on an angled curve, with friction (Example 6.3)

Example 6.1: Car Turning on a Flat Curve.

A car of mass m is turning on a flat curve. What is its maximum velocity, v_{Max}?

Solution from Dynamics of Circular Motion Notes Page.

Example 6.2: Car Turning on Angled Curve with No Friction.

A car of mass m is turning on a curve with an angle θ from the horizontal. What is its maximum velocity, v_{Max}?

Solution from Dynamics of Circular Motion Notes Page.

Example 6.3: Car Turning on Angled Curve with Friction.

A car of mass m is turning on a curve with an angle θ from the horizontal. There is friction, with the coefficient of static friction being μ_s and coefficient of kinetic friction being μ_k. What is its maximum velocity, v_{Max}?

Solution from Dynamics of Circular Motion Notes Page.

7 Springs

Springs are useful things. We can model many things as a spring for Statics

7.1 Hooke's Law

Defn 28 (Hooke's Law)**.** *Hooke's law* relates the distance a spring is pulled from equilibrium to the force it will exert as it returns to equilibrium. Because of Newton's Laws, this will also tell us the force required to move a spring from its equilibrium position.

$$\vec{F} = -k\Delta x \tag{7.1}$$

- k - The spring constant. A unique value for the "springyness" of any spring

- Δx - The distance displaced from the equilibrium position

Remark 28.1. Hooke's Law only works for distances where Δx are relatively small. If Δx were to become too large, your spring would cease to be a spring and become a straight piece of metal.

8 Energy

There is a way to rewrite Newton's Laws such that we get the Work-Kinetic Energy Theorem.

Defn 29 (Work)**.** *Work* is defined as the amount of force done over a distance. This is summarized with its equation.

$$\mathrm{W} = \vec{F} \cdot \vec{s} \tag{8.1}$$

- $\vec{F}$ - The force applied
- $\vec{s}$ - The distance the thing travelled while under influence of the force

Remark 29.1. One outcome of Equation (8.1) is that if a force is acting orthogonally to the direction of motion, the force is doing ***NO*** Work.

Defn 30 (Work-Kinetic Energy Theorem)**.** The *work-energy theorem* rewrites Newton's Second law.

$$\vec{F}_{\text{Net}} = m\frac{d\vec{v}}{dt}$$

This differential equation can be solved for.

$$\begin{aligned} m\left(\vec{v}\right) d\vec{v} &= \vec{F}_{\text{Net}}\left(\vec{v}\, dt\right) \\ \int m\vec{v}\, d\vec{v} &= \int \vec{F}_{\text{Net}} \cdot d\vec{r} \\ \frac{1}{2}m\vec{v}^2 &= \mathrm{W} \end{aligned}$$

8.1 Kinetic Energy

Defn 31 (Kinetic Energy)**.** Kinetic Energy is the energy an object has due to its velocity. Using the Work-Kinetic Energy Theorem, we can define the *Kinetic Energy* of an object as:

$$\mathrm{K} = \frac{1}{2}m\vec{v}^2 \tag{8.2}$$

- m - Mass of the object
- $\vec{v}$ - Velocity of the object
- K - The total kinetic energy of the object

Example 8.1: Kinetic Energy of Rolling Downhill.

If something is moving downhill with a force of friction $F_f = 71\text{N}$, mass of $m = 58\text{kg}$ an initial velocity of $v_0 = 3.6\text{m/s}$ over a distance of $s = 57\text{m}$, what is its final velocity?

Solution on Work/Kinetic Energy note page.

8.2 Potential Energy

Defn 32 (Potential Energy)**.** *Potential energy* is the energy an object has because of a change in height. It is derived as shown.

$$\begin{aligned} \vec{F}_h &= -m\vec{g} \\ \mathrm{W} &= \int_{h_1}^{h_2} \vec{F}_h\, dh \\ \mathrm{W} &= \int_{h_1}^{h_2} -m\vec{g}\, dh \\ \mathrm{W} &= -m\vec{g}\left(h_2 - h_1\right) \end{aligned}$$

This means that the Potential Energy of something is defined as

$$\mathrm{U} = m\vec{g}h \tag{8.3}$$

- m - Mass of the object
- $\vec{g}$ - Gravity of Earth
- h - Height of the object above some reference height that we call "0"
- U - The Potential Energy of the object

Example 8.2: Potential Energy of Skiier.

If a skiier is on super slick ice such that there is no friction , starting from a height of $h_0 = 200\text{m}$, is starting from rest, $v_0 = 0$, what is their final velocity, v?

To start this off, lets use the Law of Conservation of Energy

$$\mathrm{K}_0 + \mathrm{U}_0 = \mathrm{K} + \mathrm{U}$$
$$\frac{1}{2}mv_0^2 + m\vec{g}h_0 = \frac{1}{2}mv^2 + m\vec{g}h$$

To simplify this, we can say that $h = 0$, meaning that the final height is our reference height. This means that our final potential energy will be 0, $\mathrm{U} = 0$. Also, since the skiier starts at rest, their initial kinetic energy, $\mathrm{K}_0 = 0$.

$$\begin{aligned} 0 + m\vec{g}h_0 &= \frac{1}{2}mv^2 + 0 \\ \vec{g}h_0 &= \frac{1}{2}v^2 \\ v^2 &= 2\vec{g}h_0 \\ v &= \sqrt{2\vec{g}h_0} \\ v &= \sqrt{2(9.81)(200)} \\ v &= 62.94\text{m/s} \end{aligned}$$

So, the final velocity of the skiier after going down the hill is 62.94 m/s.

8.2.1 Hooke's Law and Potential Energy

Hooke's Law can be applied to the concept of potential energy as well.

$$\begin{aligned} F_{\mathrm{H}} &= -k\Delta x \\ \mathrm{U} &= -\int F_{\mathrm{H}}\, dx \\ \mathrm{U} &= -\int -k\Delta x\, dx \\ \mathrm{U} &= \frac{1}{2}k\left(\Delta x\right)^2 \end{aligned}$$

$$\mathrm{U} = \frac{1}{2}k\left(\Delta x\right)^2 \tag{8.4}$$

8.3 Conservation of Energy

Defn 33 (Law of Conservation of Energy)**.** The *law of conservation of energy* states that energy can never be created, nor destroyed. Energy can only change forms.

$$\sum \mathrm{E} = \text{Constant} \tag{8.5}$$

In this classical mechanical context, it means that kinetic energy and potential energy are always going to have to add up and be equal between the start and end of an experiment.

$$
\begin{aligned}
\mathrm{K} + \mathrm{U} &= \text{Constant} \\
\mathrm{K}_0 + \mathrm{U}_0 &= \mathrm{K} + \mathrm{U}
\end{aligned} \tag{8.6}
$$

Remark 33.1. This law always hold true. However, if you do not make your experiment a closed system, it might seem like energy was created or destroyed. When really is was provided by or lost to the environment. Types of systems are discussed much more in Physics 224: Modern Physics.

Example 8.3: Conservation of Energy.

If you drop a pendulum, where its initial conditions were:

- $\ell = 2\text{m}$ - Length of the string the mass is attached to
- $\theta = 30°$ - Angle from equilibrium pendulum was pulled to

What is the maximum velocity that the pendulum achieves v_{Max}?

Let's start by using Equation (8.6).

$$
\begin{aligned}
\mathrm{K}_0 + \mathrm{U}_0 &= \mathrm{K} + \mathrm{U} \\
\frac{1}{2}mv_0^2 + m\vec{g}h_0 &= \frac{1}{2}mv^2 + m\vec{g}h
\end{aligned}
$$

We can start by assuming that the velocity when it is dropped is $v_0 = 0$. This makes $\mathrm{K}_0 = 0$. The height that the pendulum is dropped from will be $h_0 = \ell - (\ell \cos(\theta))$. The point where the pendulum has the greatest velocity is right when it reaches the vertex of its swing, at the very bottom. This means that we want to know about $h = 0$.

$$
\begin{aligned}
0 + m\vec{g}h_0 &= \frac{1}{2}mv_{\text{Max}}^2 + m\vec{g}h \\
0 + m\vec{g}\left(\ell - \ell\cos(\theta)\right) &= \frac{1}{2}mv_{\text{Max}}^2 + 0 \\
\vec{g}\ell\left(1 - \cos(\theta)\right) &= \frac{1}{2}v_{\text{Max}}^2 \\
v_{\text{Max}}^2 &= 2\vec{g}\ell\left(1 - \cos(\theta)\right) \\
v_{\text{Max}} &= \sqrt{2\vec{g}\ell\left(1 - \cos(\theta)\right)} \\
v_{\text{Max}} &= \sqrt{2(9.81)(2)\left(1 - \cos(30°)\right)} \\
v_{\text{Max}} &= 2.4\text{m/s}
\end{aligned}
$$

So, the maximum velocity of the pendulum is 2.4 m/s.

9 Systems of Particles

Up until now we have only been considering the objects that we work with to be single, equally distributed masses. However, in the real world, we have strangely shaped things and objects that have multiple materials inside of them for balance and strength.

To account for oddities in these objects, we calculate something called the Center of Mass.

9.1 Center of Mass

Defn 34 (Center of Mass)**.** The *center of mass* of an object is a weighted average of all particles in a system. Center of mass takes the mass of a point and the distance from the center the point is into account, then normalizes by mass.

$$
\vec{R} = \frac{\sum\limits_{i=1}^{n} m_i \vec{r}_i}{\sum\limits_{i=1}^{n} m_i} \tag{9.1}
$$

Example 9.1: Center of Mass of Atomic Bond.

Take a carbon monoxide molecule, CO. Carbon has a mass of 12 u and Oxygen has a mass of 16 u. The bond between them is a long. What is the center of mass of this system?

We need to start by having a reference point, which we can define to be the center of the carbon atom. So, $r_C = 0$ and $r_O = a$.

$$\vec{R} = \frac{m_C(0) + m_O(a)}{m_C + m_O}$$
$$\vec{R} = \frac{12(0) + 16(a)}{12 + 16}$$
$$\vec{R} = \frac{16}{28}a$$
$$\vec{R} = \frac{4}{7}a$$

So, the center of mass of this carbon monoxide molecule is $\frac{4}{7}$ of the way towards the Oxygen atom.

9.1.1 Center of Mass in Multiple Dimensions

Center of Mass can be applied to systems in multiple dimensions. All that must be done is that the distances be calculated for each dimension separately.

Example 9.2: Center of Mass of Atomic Bond in 2-D.

Given a water atom, H_20, what is its center of mass?

- Mass of Hydrogen is 1 u
- Mass of Oxygen is 16 u
- Distance between hydrogen bonds is a

We can start by placing the hydrogen atoms on the x-axis of an xy-plane. The oxygen atom will sit on the y-axis. The variable h stands for the distance between the oxygen atom and the hydrogen bond location.

The center of mass, $\vec{R}$ will be broken down into its components. The x portion of the center of mass will be given the variable $\vec{X}$. The y portion of the center of mass will be given the variable $\vec{Y}$.

$$\vec{X} = \frac{m_H\left(\frac{-a}{2}\right) + m_H\left(\frac{-a}{2}\right) + m_O(0)}{m_H + m_H + m_O} = \frac{0}{2m_H + m_O}$$
$$\vec{X} = 0$$

Since the oxygen atom is at $x = 0$, it does not contribute to the center of mass in the x direction.

$$\vec{Y} = \frac{m_O(h) + m_H(0) + m_H(0)}{m_O + m_H + m_H}$$
$$\vec{Y} = \frac{m_O}{m_O + 2m_H}h$$
$$\vec{Y} = \frac{16}{16 + 2(1)}h = \frac{16}{18}h$$
$$\vec{Y} = \frac{8}{9}h$$

Since the hydrogen atoms are on $y = 0$, they do not contribute to the center of mass in the y direction.
Combining both the x and y solutions, we come up with

$$\vec{R} = \left\langle 0, \frac{8}{9}h \right\rangle$$

10 Gravitation

Defn 35 (Gravity). *Gravity* is one of The 4 Fundamental Interactions in the Universe. It is also one of the least understood. However, it it the thing that is keeping us “glued” to this plant.

11 Statics

12 The 4 Fundamental Interactions in the Universe

There are 4 fundamental interactions present in our universe, as we know it right now.

1. Gravity
2. Electromagnetic
3. “Weak” (Neutrinos and the like)
4. “Strong” (Holds atomic nucleus together)

A Physical Constants

Constant Name	**Variable Letter**	**Value**
Boltzmann Constant	R	8.314J/mol K
Universal Gravitational	G	$6.67408 \times 10^{-11}\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$
Planck's Constant	h	$6.62607004 \times 10^{-34}\text{mkg/s} = 4.163 \times 10^{-15}\text{eV s}$
Speed of Light	c	$299792458\text{m/s} = 2.998 \times 10^8\ \text{m/s}$
Charge of Electron	e	$1.602 \times 10^{-19}\text{C}$
Mass of Electron	m_{e^-}	$9.11 \times 10^{-31}\text{kg}$
Mass of Neutron	m_{n^0}	$1.67 \times 10^{-31}\text{kg}$
Mass of Earth	m_{Earth}	$5.972 \times 10^{24}\text{kg}$
Diameter of Earth	d_{Earth}	12742km

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm\imath\alpha} = \cos(\alpha) \pm \imath\sin(\alpha) \tag{B.3}$$

$$\sin(x) = \frac{e^{\imath x} + e^{-\imath x}}{2} \tag{B.4}$$

$$\cos(x) = \frac{e^{\imath x} - e^{-\imath x}}{2\imath} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\alpha) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

B.10 Rectangular to Polar

$$a + \imath b = \sqrt{a^2 + b^2}e^{\imath\theta} = re^{\imath\theta} \tag{B.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.25}$$

B.11 Polar to Rectangular

$$re^{\imath\theta} = r\cos(\theta) + \imath r\sin(\theta) \tag{B.26}$$

C Calculus

C.1 Fundamental Theorems of Calculus

Defn C.1.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{C.1}$$

Defn C.1.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.2}$$

Defn C.1.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

D Complex Numbers

$$Ae^{-ix} = A\left[\cos\left(x\right) + i\sin\left(x\right)\right] \tag{D.1}$$