MMAE 320: Thermofluid Dynamics — Reference Material
Illinois Institute of Technology

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Contents

List of Theorems

1 Introduction

Defn 1 (Energy). *Energy* is the stuff in the universe that can cause changes in material states. A change in energy is the ability to do Work. Energy can never be negative, but changes in energy can.

Some examples are:

- Thermal Energy
- Electrical Energy
- Mechanical Energy

One fundamental thing about Energy that was discovered was the Law of Conservation of Energy.

Defn 2 (Law of Conservation of Energy). The *law of conservation of energy* states that energy cannot be created or destroyed; it can only change form. This means that the total amount of energy in an interaction is constant.

Expressed as an equation, this is recognized as

$$\sum_{i \in I} E_i = C \tag{1.1}$$

where

I: The set of all interactions of interest.
E: The energy of this particular interaction, i.
C: The total energy of the system, a constant.

Law of Conservation of Energy closely mirrors the other major fundamental physical law, the Law of Conservation of Mass.

Defn 3 (Law of Conservation of Mass). The *law of conservation of mass* states that a Process cannot create nor destroy any matter; it can only change form/state.

The Law of Conservation of Energy leads to Equation (1.2), shown below.

$$E_{in} - E_{out} = \Delta E \tag{1.2}$$

In a typical use, we are concerned with Heat as the form of Energy.

Defn 4 (Heat). *Heat* is the form of Energy that can be transferred from one System to another as a result of temperature difference. The variable letter for the energy in a system is Q.

Defn 5 (Thermodynamics). *Thermodynamics* is the science of equilibrium states and changes between these states.

Pure thermodynamic analysis will show the toal amount of Energy moving through the system, but not the rate at which it occurs. This is where Heat Transfer comes in.

Defn 6 (Heat Transfer). *Heat transfer* is the science that deals with the determination of the *rates* of Heat-based Energy transfers. This science deals with systems that lack a thermal equilibrium, meaning it cannot be based on only principles of Thermodynamics.

For every flow of energy in a system, there **MUST** be a Driving Force.

Defn 7 (Driving Force). *Driving force* are the conditions of the system for the transfer of Energy to occur. There are any number of driving forces in the universe, some are listed here:

- Heat Transfer requires a temperature difference.
- Electric current flow requires a voltage difference.
- Fluid flow requires a pressure difference.

For Heat Transfer, the Temperature Gradient determines the rate ofthe transfer.

Defn 8 (Temperature Gradient). *Temperature gradient* is the temperature difference per unit length, or the rate of change of temperature.

Remark 8.1 (Extension). The use of gradients extends to other Driving Forces as well.

1.1 Fluid Mechanics

Defn 9 (Stress). *Stress* is defined as force per unit area.

$$\tau = \frac{F}{A} \tag{1.3}$$

Obeying Newton's third law, typically, there is a normal force per unit area, called the **normal stress**. In a fluid at rest, the normal stress is called **pressure**.

1.2   Units/Dimensions

1.2.1   SI System

There are 7 fundamental dimensions and units in the SI system, shown in Table 1.1.

Dimension	Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Temperature	kelvin (K)
Electric Current	ampere (A)
Amount of Light	candela (cd)
Amount of Matter	mole (mol)

Table 1.1: The 7 Fundamental Dimensions and Units

1.2.2   English System

The English system uses a very different set of units to describe these dimensions.

1.2.2.1   Mass   The English unit of mass is the pound-mass.

$$\text{lbm}$$

1.2.2.2   Force   The English unit of mass is the pound-force.

$$1\,\text{lbf} = 32.174\,\text{lbm ft/s}^2$$

1.2.2.3   Energy   The English unit of energy is the British Thermal Unit (BTU). 1 btu raises the temperature of 1 lbm of water at 68°F by 1°F.

1.2.2.4   Work

Defn 10 (Work). *Work* is defined to be force multiplied by the path distance the force was applied over.

$$W = \vec{F}d \ \text{btu} \tag{1.4}$$

1.2.2.5   Power

Defn 11 (Power). *Power* is defined as energy per unit time.

$$P = \frac{E}{t} \ \text{hp} \tag{1.5}$$

2   Basic Concepts

2.1   Systems

Defn 12 (System). A *system* is defined as a quantity of matter or a region in space chosen for study.

The mass or region outside the System is the **surroundings**. The surface that separates the System from the surroundings is the **boundary**.

2.1.1   Type of Systems

There are 2 types of systems:

1. Open System
2. Closed System

Defn 13 (Open System)**.** An *open system* is a System in which the mass of the system is **not** constant. Thus, mass and Energy can flow from the system to the surroundings. Energy can flow by either Work, or it can be Heat.

Remark 13.1 (Control Volume)*.* Sometimes an Open System is called a *control volume*, because the volume of the system is constant.

Remark 13.2*.* Flow through these devices is typically easier by selecting the region based on volume, rather than mass.

Defn 14 (Closed System)**.** A *closed system* is a System where the mass of the system **is** constant, but Energy can move between the system and the surroundings. This type of system can either be Adiabatic or non-adiabatic. For each case:

Adiabatic There is an Energy transfer by Heat **and** Work.
Non-adiabatic The energy transfer is done by Work **only**.

Remark 14.1 (Control Mass)*.* Sometimes a Closed System is called a *control mass*, because mass is constant.

There is also a third type of system, the Isolated System, which is a special case of the Closed System.

Defn 15 (Isolated System)**.** An *isolated system* is a System where **neither** mass or Energy can move from the system to the surroundings. This system has the properties of being Adiabatic **and** there is no Work done.

2.1.2 Properties of Systems

Defn 16 (Property)**.** A *property* of a System is a characteristic of the system.

Some common properties are:

- P, Pressure
- T, Temperature
- V, Volume
- m, Mass

There are 2 types of properties:

1. Intensive Property
2. Extensive Property

Defn 17 (Intensive Property)**.** *Intensive properties* are properties that are **independent** of the mass of the system. These include:

- P, Pressure
- T, Temperature
- ρ, Density

Defn 18 (Extensive Property)**.** *Extensive properties* are properties that are **dependent** on the size or extent of the system. These include:

- Total mass
- Total volume
- Total momentum

Some extensive properties can be used to form a Specific Property.

Defn 19 (Specific Property)**.** A *specific property* is an Extensive Property per unit mass. Some examples are:

- Specific Volume $\nu = \frac{V}{m}$
- Specific Total Energy $e = \frac{E}{m}$

2.1.3 Generalizations

We make some generalizations about the fluids we are working with to make calculations easier. For example, we treat fluids as a **continuum**, rather than the true atomic nature of the substance. This allows us to:

- Treat properties as point functions
- Assume properties vary continually in space with no discontinuities

This assumption is valid so long as the size of the System is relatively large compared to the space between the component molecules.

2.2 Density

Defn 20 (Density)**.** *Density* is defined as mass per unit volume.

$$\rho = \frac{m}{V} \ \text{kg/m}^3 \tag{2.1}$$

For substances that lack a uniform mass and volume, density can also be realized by Equation (2.2).

$$\rho = \frac{dm}{dV} \tag{2.2}$$

Sometimes the density of a substance is given relative to another substance, usually water. This is Specific Gravity.

Defn 21 (Specific Gravity)**.** *Specific gravity* is the Density of a substance as a ratio to another substance, usually water. This is expressed as

$$\text{SG} = \frac{\rho}{\rho_{\text{H}_2\text{O}}} \tag{2.3}$$

Defn 22 (Specific Weight)**.** *Specific weight* is the weight of a unit volume of a substance. It is expressed as:

$$\gamma_s = \rho g \ \text{N/m}^3 \tag{2.4}$$

2.3 States

Defn 23 (State)**.** The *state* of the system includes all properties that can be measured or calculated which can completely describe the condition of the system.

In any given State, all the properties of a system will have fixed values. If any single property changes, all the others will change in accordance. We are typically interested in Equilibrium states.

Defn 24 (Equilibrium)**.** *Equilibrium* is a State of a System which does not change when isolated from it surroundings. A system will only leave equilibrium when disturbed by an outside Energy, or an unbalanced Driving Force.

There are several different types of Equilibrium:

Thermal Equilibrium If the temperature throughout the entire System is constant.
Mechanical Equilibrium If there is no change in pressure at any point of the System with regards to time. However, pressure can vary within the system, but for most of our concerns, this isn't a problem.
Phase Equilibrium If there is more than one phase of matter, this type of equilibrium is reached with each phase reaches its equilibrium level and stays there.
Chemical Equilibrium If the chemical composition of a System does not change with time, i.e. no chemical reactions could occur.

2.3.1 The State Postulate

Defn 25 (State Postulate)**.** The *state postulate* states: the State of a simple compressible system is **completely** specific by 2 independent, intensive properties.

Remark 25.1 (Simple Compressible System)*.* A system is called a simple compressible system in the absence of electrical, magnetic, gravitational, motion, and surface tension effects.

Properties are **independent** if one property can be varied and the other one remains constant.

2.4 Processes

Defn 26 (Process)**.** A *process* is the change in a System from one Equilibrium state to another. The **path** of the process is the set of Quasi-equilibrium processes.

It is easiest to deal with paths where if one of the variables is changed just a small amount, the change becomes linear. This is similar to the concept of a derivative, because if we sample at a small/short enough rate, the changes introduced to the System will be linear.

Defn 27 (Quasi-equilibrium)**.** A *quasi-equilibrium* process is one that is sufficiently slow so that the System can adjust itself internally so that properties change constantly **throughout** the system.

Lastly, if the Process is a Cycle, then the initial and final States are the same.

Defn 28 (Cycle)**.** A *cycle* is a Process that returns to its initital state at the end of the process.

2.4.1 *iso-* Processes

Some Processes are special in that one Property remains constant. There are several of these:

- Isothermal
- Isobaric
- Isometric

Defn 29 (Isothermal). *Isothermal* Processes are ones where the temperature T remains constant.

Defn 30 (Isobaric). *Isobaric* Processes are ones where the pressure P remains constant.

Defn 31 (Isometric). *Isometric* (*Isochoric*) Processes are ones where the volume V remains constant.

2.5 Temperature

Defn 32 (0th Law of Thermodynamics). The *0th law of thermodynamics* states that if two bodies are in thermal Equilibrium with a third body, they are also in thermal equilibrium with each other.

2.5.1 Subjective Temperature Scales

There are 2 subjective temperature scales:

1. Celsius (°C)
2. Fahrenheit (°F)

These are based off of physical temperatures at certain points on Earth regarding water freezing and water boiling. These are useful for regular use, but not as well-suited for thermodynamic use.

$$\frac{5}{9}(T(\text{°F}) - 32) = T(\text{°C}) \tag{2.5}$$

2.5.2 Objective Temperature Scales

Objective temperature scales are ones that are based off of physical, universal constants. There are 2 objective temperature scales:

1. Kelvin (K)
2. Rankine (R)

Both of these have placed 0 at the point where all molecular motion stops.
The equations to convertion between Celsius and Kelvin is shown in Equation (2.6).

$$T(\text{K}) = T(\text{°C}) + 273.15 \tag{2.6}$$

The equations to convertion between Fahrenheit and Rankine is shown in Equation (2.7).

$$T(\text{R}) = T(\text{°F}) + 459.67 \tag{2.7}$$

2.6 Pressure

Defn 33 (Pressure). *Pressure* is the normal **scalar** force exerted per unit area. Because it is a scalar, it has no dependency on the direction of the normal force. It can be expressed using Density, gravity, and depth/length.

$$P = \rho g d \tag{2.8}$$

The SI unit of pressure is the pascal:

$$\text{Pa} = \frac{\text{N}}{\text{m}^2} \tag{2.9}$$

The bar is also used:

$$1\,\text{bar} = 100\,\text{kPa}$$

The English unit of pressure is the pound-force per square inch.

$$\text{lbf/in}^2 = \text{psi} \tag{2.10}$$

We exist, roughly, at one atmosphere of pressure, or 1 atm. The equivalents of this are:

$$1 \, \text{atm} = 101\,325 \, \text{kPa}$$
$$1 \, \text{atm} = 14.696 \, \text{psi}$$

Using this, we can measure pressure, typically with either Gage Pressure or Vacuum Pressure.

Defn 34 (Gage Pressure). *Gage pressure* is the Pressure that a gage reads after being calibrated to some standard pressure, typically one atmosphere. Thus, the actual pressure of something read by the gage is represented by Equation (2.11) below.

$$P_{\text{abs}} = P_{\text{atm}} + P_{\text{gage}} \tag{2.11}$$

Defn 35 (Vacuum Pressure). *Vacuum pressure* is the Pressure that a gage reads after being calibrated to some standard pressure, typically one atmosphere. Thus, the actual pressure of something read by the gage is represented by Equation (2.12) below.

$$P_{\text{abs}} = P_{\text{atm}} - P_{\text{vacuum}} \tag{2.12}$$

2.6.1 Pressure Inside a Fluid

Pressure inside a fluid increases linearly with depth. This is seen in Equation (2.13), where Δz is the distance between the two measured points.

$$\begin{aligned} P_{\text{below}} &= P_{\text{above}} + \rho g |\Delta z| \\ &= P_{\text{above}} + \gamma_s |\Delta z| \end{aligned} \tag{2.13}$$

ρ: The density of the fluid (kg/m^3)

Defn 36 (Pascal's Law). *Pascal's Law* states that the Pressure along any horizontal plane in the same fluid is the same pressure. This leads to Equation (2.14).

$$\begin{aligned} P_1 &= P_2 \\ \frac{\vec{F}_1}{A_1} &= \frac{\vec{F}_2}{A_2} \end{aligned} \tag{2.14}$$

3 Energy, and Energy Transfer

Defn 37 (Macroscopic Energy Form). *Macroscopic energy form*s are typically ones that have to deal with objects on a macroscopic level. These energies are:

1. Kinetic
2. Potential

Remark 37.1. This definition is included because the textbook makes use of it.

Defn 38 (Microscopic Energy Form). *Microscopic energy forms* are energies that act on non-macroscopic levels. Namely, they affect their systems on microscopic levels. These energies include:

1. Sensible
   - Heat
   - Kinetic energy of molecules
2. Latent
   - Phase Changes
3. Chemical
   - Combustion
4. Nuclear

Remark 38.1. This definition is included because the textbook makes use of it.

Defn 39 (Internal Energy). *Internal energy* is equivalent to Microscopic Energy Forms. It means the Energy that the object in question inherently has at that point in time.

3.1 Energy Quality

Energy has quality!

- Macroscopic Energy Form
  - Structured
  - Moves as a single unit
- Microscopic Energy Form
  - **Not** structures
  - Does **not** move as a single unit

These differences mean that we measure the efficiency of each type of energy form differently.

3.2 Energy and Flows

When moving a fluid through a pipe, we can find the amount of work done by the fluid flowing, called teh Flow Energy.

$$P = \frac{F}{\text{Area}}$$
$$V_{\text{Cylinder}} = \ell \cdot \text{Area}$$
$$W = F \cdot \text{Distance}$$

If we substitute for the common terms in the formula for work, then we end up with Equation (3.1).

Defn 40 (Flow Energy). *Flow energy* is the energy that a fluid flowing through a long, straight pipe has.

$$W = PV$$
$$\text{FE} = PV \tag{3.1}$$

Remark 40.1 (Energy Form). Typically, Flow Energy is categorized with the Macroscopic Energy Forms, because it behaves more like those and can be nearly as efficient as them. This is true even though this is technically an application of microscopic energies. This is because we are not worried about the internal energy of the fluid in the pipe, but are instead interested in the mechanical movement of it.

3.3 Divisions of Energy

We are always interested in the change in energy that occurs due to something. This is seen as Equation (3.2).

$$\Delta E = \Delta U + \Delta \text{KE} + \Delta \text{PE} + \Delta \text{FE} \tag{3.2}$$

But, we are typically more interested in dealing with Equation (3.2) using intensive properties of the system, so we can reduce with that, which produces the change in total Specific Energy.

- The internal energy cannot be completely converted into work.
- Mechanical energy is typically defined to be these types of energies. These can be completely converted into work by an ideal machine.
  - Kinetic energy (KE)
  - Potential energy (PE)
  - Flow energy (FE)

We are also interested in the Specific Energy of the system.

Defn 41 (Specific Energy). *Specific energy* is an Intensive Property of a system. It is the total energy of a system divided by the total mass of the system.

$$e = \frac{E}{m}$$
$$= \frac{U}{m} + \frac{1}{2}v^2 + gh + \frac{PV}{m} \tag{3.3}$$
$$= \frac{U}{m} + \frac{1}{2}v^2 + gh + \frac{P}{\rho}$$

$$\Delta e = \frac{U_2 - U_1}{m} + \frac{v_2^2 - v_1^2}{2} + g(h_2 - h_1) + \frac{P_2 - P_1}{\rho} \tag{3.4}$$

The Specific Energy of a system can be used to find the change in energy per unit time, or the Power.

$$\Delta \dot{E} = \dot{m}\Delta e$$
$$P = \dot{m}\Delta e$$

(3.5)

where $\dot{m}$ is the mass flowrate, seen by Equation (3.6)

$$\dot{m} = \frac{m}{t}$$

(3.6)

3.4 Energy Flow and Systems

These systems have Energy interactions that cross the boundary of a System. This only has 2 options:

1. Heat transfer into or out of the System.
2. Work done on the system by the surroundings.

Example 3.1: Power of Water. Lecture 4, Problem 3.12

Consider a river flowing towards a lake at an average velocity of $v_{\text{River}} = 3\,\text{m/s}$ and a volume flow rate of $\dot{V} = 500\,\text{m}^3/\text{s}$. The $500\,\text{m}^3/\text{s}$ is at a location $90\,\text{m}$ above the lake's surface. Determine the total mechanical energy in the river water per unit mass and the power generation potential of the entire river?

- -

The question is really asking us to find the Specific Energy and the power generation.
Assumption: Assume the flow is constant, that we are in steady flow.
Concepts and Explore:
There is no state change, likely meaning $\Delta U = 0$. The velocity of the water within the lake should be $v_{\text{Lake}} = 0$ ($\Delta \text{KE} \neq 0$). There is a potential energy change, $\Delta \text{PE} \neq 0$. There is a flow, but the only pressure involved is P_{atm}, and the height difference is so small that the change in pressure is negligible.

$$P = \dot{m}\Delta e$$

To find $\dot{m}$, we can multiply the density of water with the volume flow rate of the river, to find the mass flow rate.

$$\dot{m} = \rho_{H_2O}\dot{V}$$

Plan:

1. Solve for Δe using $\Delta \text{KE} + \Delta \text{PE}$.
2. Solve for mass flow rate, $\dot{m} = \rho_{H_2O}\dot{V}$.
3. Solve for $P = \dot{m}\Delta e$.

Solve:

$$\Delta e = \Delta \text{KE} + \Delta \text{PE}$$
$$= \frac{(3\,\text{m/s})^2 + (0\,\text{m/s})^2}{2} + 9.81\,\text{m/s}^2(90\,\text{m} - 0\,\text{m})$$
$$= 4.5\,\text{m}^2/\text{s}^2 + 882\,\text{m}^2/\text{s}^2 = 886.5\,\text{J/kg}$$

$$\dot{m} = \rho\dot{V}$$
$$= 1000\,\text{kg/m}^3(500\,\text{m}^3/\text{s})$$
$$= 500\,000\,\text{kg/s}$$

$$P = \dot{m}\Delta e$$
$$= 886.5\,\text{J/kg}(500\,000\,\text{kg/s})$$
$$= 444\,000\,000\,\text{W}$$
$$= 444\,\text{MW}$$

Generalize: Most of the energy in this problem came from the water falling in height. Overall, the pressure change and the velocity of the water made very little impact on the total energy in the system, in comparison to the change in potential energy.

Example 3.2: Mechanical Energy of Air. Lecture 4, Problem 3.14

Wind is blowing steadily at $v = 10\,\text{m/s}$. Determine the mechanical energy of the air per unit mass and the power generation potential of a wind turbine with $d = 60\,\text{m}$ diameter blades at that location? Take $\rho_{\text{air}} = 1.25\,\text{kg/m}^3$.

- -

Concepts and Explore:

- The air is flowing steadily.
- There is no state change, so $\Delta U = 0$.
- There is a change in the blades' velocity, $\Delta KE \neq 0$.
- There is no change in the potential energy of the air, so $\Delta PE = 0$.
- There is no pressure change on the different sides of the turbine, so $\Delta FE = 0$.

$$P = \dot{m}\Delta e$$
$$\dot{m} = \rho_{\text{air}}\dot{V}$$
$$\dot{V} = v\left(\pi\left(\frac{d}{2}\right)^2\right)$$

Plan:

1. Solve for Δe using KE only.
2. Simplify $\dot{m}$.
3. Solve for $P = \dot{m}\Delta e$.

Solve:

$$\Delta e = \frac{1}{2}(10\,\text{m/s})^2$$
$$= 50\,\text{m}^2/\text{s}^2$$
$$= 50\,\text{J/kg}$$

$$\dot{m} = \rho_{\text{air}}v\left(\pi\left(\frac{d}{2}\right)^2\right)$$
$$= 1.25\,\text{kg/m}^3(10\,\text{m/s})\left(\frac{\pi 60^2}{4}\right)$$
$$= 35\,343\,\text{kg/s}$$

$$P = \dot{m}\Delta e$$
$$= 35\,343\,\text{kg/s}(50\,\text{J/kg})$$
$$= 1\,767\,150\,\text{J/s}$$
$$= 1767\,\text{kW}$$

Validate: Since we had a straightforward application of the equations, it is likely that the answers make sense. In addition, we did perform some dimensional analysis to figure out the way the units should be put together, which further reinforces the likelihood of this solution being the right one.

Generalize: Like Example 3.1, the value we received only holds true when the air is flowing steadily. If steady flow were not happening, then the received value will fluctuate.

Most of the power generated through this turbine is done because of the area of the blades that the air is moving through. This happens despite the very low density of air in this problem.

When looking through the equations, the velocity the air is moving becomes cubed, meaning a small change in velocity will have drastic changes in the power generated.

3.5 Adiabatic Processes

Heat is a form of energy transfer. However, there are two types of processes:

1. Processes that have **no** heat transfer (Adiabatic), $Q = 0$.
2. Processes that **have** heat transfer (Non-Adiabatic), $Q \neq 0$.

Defn 42 (Adiabatic)**.** A process is *adiabatic* when there is **no** transfer of Heat whatsoever. This can be achieved by a system that is heavily insulated, preventing a temperature difference from causing a heat transfer. Adiabatic processes yield the expression below:

$$\Delta Q = 0 \tag{3.7}$$

Remark 42.1. An Adiabatic process does **not** say anything about Work in the System. A process can be both adiabatic and have work done on it, meaning there is still a change in energy in the system.

Defn 43 (1st Law of Thermodynamics)**.** The *1st law of thermodynamics* states that the total energy of a system **cannot** be created or destroyed during a Process; it can only change **forms**. Thus, the total change in energy of all energy forms must sum to zero.

Symbolically, this is represented as Equation (3.8).

$$\Delta E = 0 \tag{3.8}$$

If this law were to be violated in a problem, that means there is an energy form that is not being measured.

Using Equation (3.2), we can "derive" a new equation that is useful.

$$\Delta E = \Delta Q + \Delta W_{\text{Mech}}$$
$$E_{\text{In}} - E_{\text{Out}} = (W_{\text{In}} - W_{\text{Out}}) + (Q_{\text{In}} - Q_{\text{Out}})$$

ΔQ is equivalent to ΔU.
ΔW_{Mech} is equivalent to $\Delta \text{KE} + \Delta \text{PE} + \Delta \text{FE}$.

Remark 43.1 (Change Energy Flowrate). By the definition of the 1st Law of Thermodynamics, the change in energy flowrate (energy per unit time) is also defined to be zero, because the amount of energy in the system is constant.

$$\Delta \dot{E} = 0 \tag{3.9}$$

3.6 Energy Efficiency

Typically, we are concerned with how efficient a Process is at using the Energy it is provided to achieve a certain goal. This is typically defined as Equation (3.10).

$$\eta = \frac{\text{Desired Output}}{\text{Required Input}} \times 100 \tag{3.10}$$

3.6.1 Combustion

$$\eta_{\text{Combustion}} = \frac{\dot{Q}_{\text{Released}}}{\frac{\dot{Q}_{\text{Available}}}{m} \dot{m}} \tag{3.11}$$

3.6.2 Pumps

$$\eta_{\text{Pump}} = \frac{\text{Energy Increase of Fluid}}{\text{Energy Input}} \tag{3.12}$$

3.6.3 Turbines

$$\eta_{\text{Turbine}} = \frac{\text{Mechanical Energy Output}}{\text{Total Energy Decrease of Fluid}} \tag{3.13}$$

4 Properties of Pure Substances

Many of the substances that we study in Thermodynamics are what we call Pure Substances.

Defn 44 (Pure Substance)**.** A *pure substance* is one whose chemical composition is fixed throughout. Properties of these substances are completely known, and these properties are completely uniform throughout the substance.

In general, the pressure of a substance is an exponential function that is dependent on the temperature of the water. In addition, the pressure of the substance depends on its altitude. As you increase in altitude, this means that the temperature required to boil the substance at decreases. All of these are visualized in Figure 4.1.
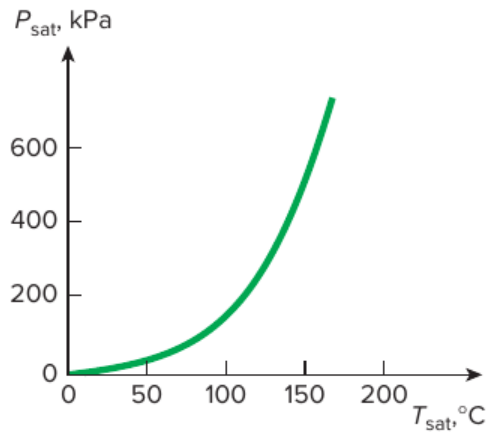
Figure 4.1: Temperature-Pressure (T-P) Liquid-Vapor Curve (Çengel, Cimbala, and Turner 2017, pg. 106)

4.1 Phase Changes

Phase changes in materials are unique, because they maintain a constant temperature **until** all the material undergoes the phase change, however, the volume increases **drastically**.

Defn 45 (Phase Change). A *phase change* in a substance is when it changes from one state of matter to another. This means a substance goes from solid to liquid, liquid to gas, gas to solid, solid to gas, etc. Each of these "directions" has a name:

Solid to Liquid: Melting/Fusion
Liquid to Solid: Solidification/Crystallization
Liquid to Gas: Evaporation/Vaporization
Gas to Liquid: Condensation
Solid to Gas: Sublimation
Gas to Solid: Desposition

This is visualized as the line between points A and B on the graph in Figure 4.2.
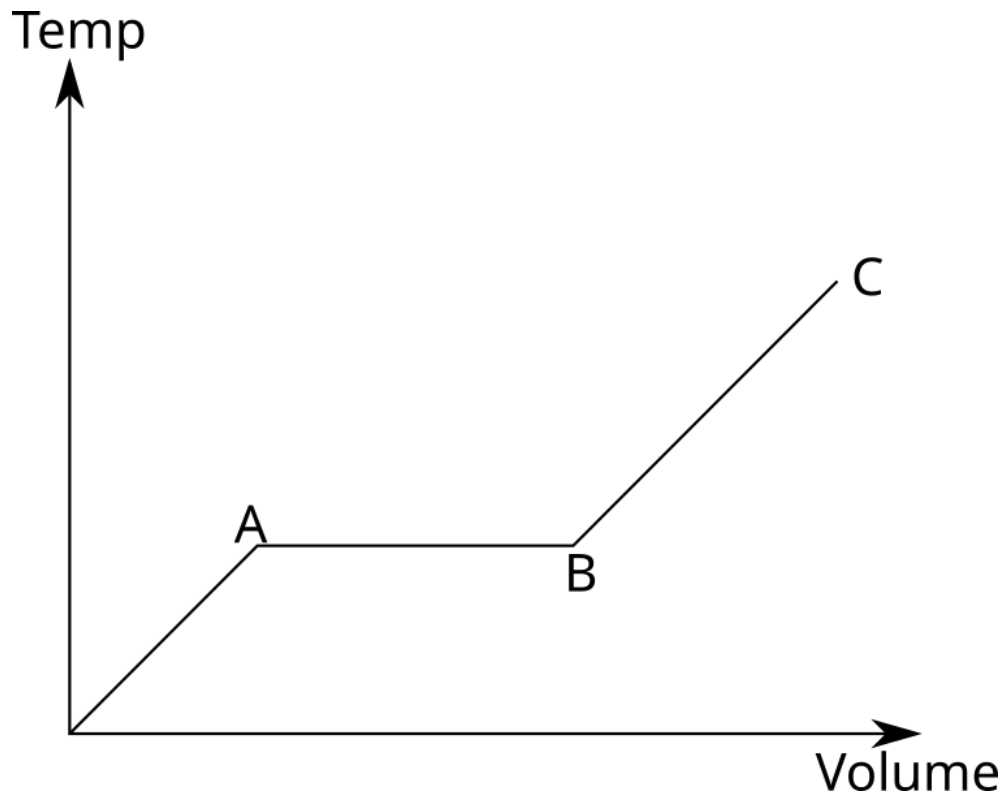
Figure 4.2: Phase Change Graph

The line drawn in Figure 4.2 assumes that the material undergoing the phase change is under Isobaric conditions, meaning the pressure is **not** changing. However, different pressures (under their own Isobaric conditions) will yield different, but similarly shaped, lines as well.

Remark. Figure 4.2 is also **NOT** drawn to proportion, as the angle on the first portion of the line is typically **very** sharp. Obviously, Figure 4.2 is an idealized graph for the temperature-volume graph of a substance and its respective phase change. Even though the graph is idealized, it is fine for our needs, and for most needs in Thermodynamics.

There are several notable points on the graph in Figure 4.2:

1. Between the origin and point A, the substance is called a Compressed Liquid. Table A.7 in the textbook has data points for some materials in this state, but the actual values vary very little.
2. At point A, the solid begins to boil, becoming a Saturated Liquid, and is denoted ν_f.
3. Between points A and B, the substance is called a Saturated Mixture. Here, there are parts of the substance in the liquid phase and the other parts of the substance are in the vapor phase. In the textbook, Tables A.4 and A.5 are used here.
4. At point C, we have a Saturated Vapor and is denoted ν_g. In the textbook, Tables A.4 and A.5 are used here.
5. Beyond point C, the substance is a Superheated Vapor. Table A.6 is used in the textbook for substances in this state.

Defn 46 (Compressed Liquid). A *compressed liquid* is a substance, in its liquid state, as it is being heated and gaining volume, until right before it starts to boil. Table A.7 in the textbook has data points for some materials in this state, but the actual values vary very little. This is because Pressure does not change volume much. However, temperature **does** change the volume, therefore, treat compressed liquids as Saturated Liquid at temperature T, so use Table A.4.

Defn 47 (Saturated Liquid). A *saturated liquid* is a substance, starting its Phase Change state between liquid and gas. The Specific Volume where a Compressed Liquid begins to boil is denoted ν_f. Tables A.4, and A.5 in the textbook are used here.

Defn 48 (Saturated Mixture). A *saturated mixture* is a substance in the middle of its Phase Change state betwen a liquid and a gas. Parts of the substance in the liquid phase and the other parts of the substance are in the vapor phase. In the textbook, Tables A.4 and A.5 are used here.

Defn 49 (Saturated Vapor). A *saturated vapor* is a substance that has **JUST** completely undergone its Phase Change to fully become a gas. The Specific Volume where this occurs is denoted ν_g. Table A.6 in the textbook contains data points for these vapors.

Defn 50 (Superheated Vapor). A *superheated vapor* is a substance that has undergone a complete Phase Change to become a gas. Here, even if heat is removed, the substance remains a gas until it cools enough or increases in pressure enough to become a Saturated Vapor, then a Saturated Mixture again. Table A.6 in the textbook is used to find values for substances in this state.

Figure 4.2 can be overlayed with a curve that intersets the different isobars of that Pure Substance based on their Saturated Liquid and Saturated Vapor points.

Defn 51 (Interpolate). Many of the values in the tables in the textbook are useful. However, sometimes the value we are interested in is *not* in these tables, but between some known entries in the table. So, we must *interpolate* a result to use.

$$y = y_0 + (x - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right) \tag{4.1}$$

y The value we intend to interpolate for.
y_0 The "earlier" value to use. Typically, this is the known data point below our value of interest.
y_1 The "later" value to use. Typically, this is the known data point above our value of interest.
x The independent value that we are interested in interpolating *at*.
x_0 The "earlier" value to use. Typically, this is the known data point below our value of interest.
x_1 The "later" value to use. Typically, this is the known data point above our value of interest.

Example 4.1: Properties of Water.

Given water at temperature $T = 75\,°C$ and a pressure $P = 100\,kPa$. What is the water's current state?

- -

We can start by looking in Table A.4 in the textbook.
At $75\,°C$, water boils at $38\,kPa$. This water will **not** boil at that temperature and pressure. Now going to Table A.5, which says that at our pressure, water boils at $99.61\,°C$.
This means that the water is in its liquid state, and is not in its Saturated Mixture state yet.
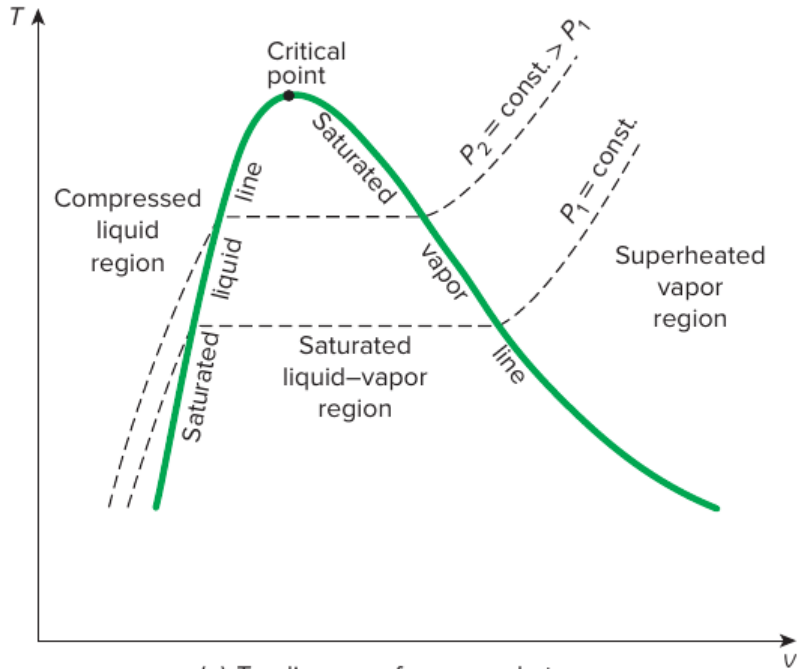
Figure 4.3: Temperature-Volume (T-V) Diagram (Çengel, Cimbala, and Turner 2017, pg. 110)
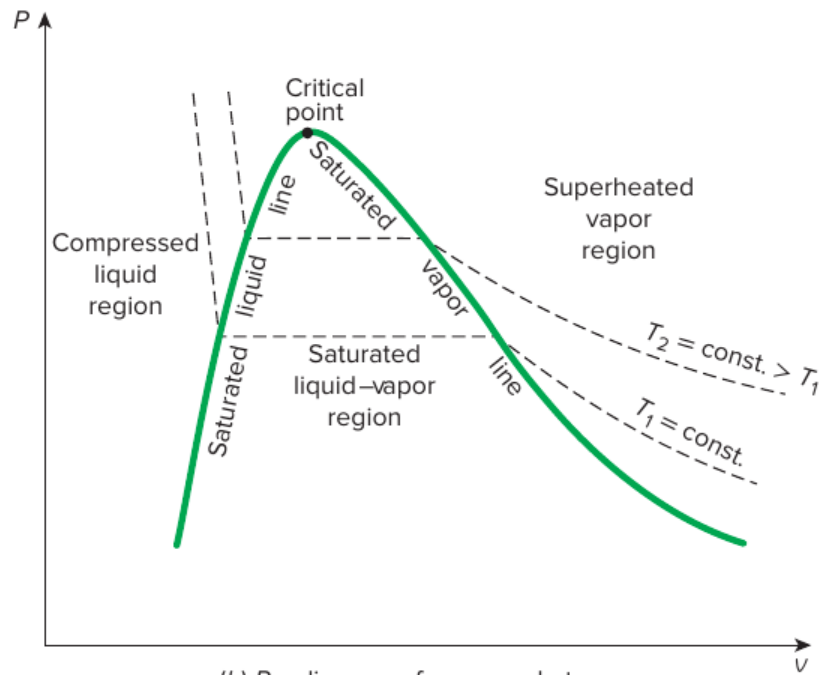
Figure 4.4: Pressure-Volume (P-V) Diagram (Çengel, Cimbala, and Turner 2017, pg. 110)

Defn 52 (Enthalpy of Vaporization). The *Specific Enthalpy of vaporization* (or *latent heat of vaporization*) represents the amount of Energy required to vaporize a unit mass of Saturated Liquid at a given temperature or pressure. This value **decreases** as the temperature and/or pressure increase and becomes 0 at the critical point. This value is denoted using h_{fg}, and is composed using Equation (4.2) below.

$$h_{fg} = h_f - h_g \tag{4.2}$$

4.2 Mixtures

In the real world, we come across Mixtures all the time. They do not have the same properties as their Pure Substance counterparts.

Defn 53 (Mixture). A *mixture* is a substance or set of substances that contain at least more than one different Saturated Liquid and Saturated Vapor at a time.

When a Saturated Liquid is boiling to become a Saturated Vapor, (Points A and B in Figure 4.2), all of the liquid **must** turn to gas first. This Process has a Quality.

Defn 54 (Quality). *Quality* is the ratio of a Mixture that is a Saturated Vapor to Saturated Liquid.

$$\text{Quality} \equiv \frac{\text{Mass of Saturated Vapor}}{\text{Total mass of Mixture}}$$
$$x = \frac{m_{Vapor}}{m_{Liquid} + m_{Vapor}} \tag{4.3}$$

Equation (4.3) is always valued $0 \le x \le 1$. This is like Energy Efficiency, η.

Defn 55 (Specific Volume). *Specific volume* is the volume per unit mass of a substance, making it the reciprocal of Density. This is typically well-defined for Saturated Mixtures, because the system is in 2 different phases. If we are interested in the specific volume of the Mixture, we have Equation (4.4).

$$\nu = \frac{\nu_g}{\nu_f} \tag{4.4}$$

ν: The Specific Volume at any point in time during a Process.
ν_g: The Saturated Liquid volume.
ν_f: The Saturated Vapor volume.

$$x = \frac{V_{Gas}}{V_{Total}}$$
$$\nu = (1 - x)\nu_f + x\nu_g \tag{4.5}$$
$$= \nu_f + x(\nu_g - \nu_f)$$

From Equation (4.4), it is clear that to get the total volume of **one phase of the system**, we use Equation (4.6)

$$\nu m_{Phase} = V_{Phase} \tag{4.6}$$

Now, Equation (4.6) only finds the volume for a single phase in the system, but because volume is additive, we can use Equation (4.7).

$$\nu m_{Total} = m_{Liquid}\nu_f + m_{Gas}\nu_g \tag{4.7}$$

If using Mixtures, we can define Specific Enthalpy using Equation (4.8).

$$h = h_{Liquid} + x h_{Gas-Liquid} \tag{4.8}$$

where $h_{Gas-Liquid} = h_{Gas} - h_{Liquid}$
If using Mixtures, we can define Specific Energy using Equation (4.9).

$$u = u_{Liquid} + x u_{Gas-Liquid} \tag{4.9}$$

where $u_{Gas-Liquid} = u_{Gas} - u_{Liquid}$

In fact, Equation (4.7) can be re-expressed using Quality, which gives us Equation (4.5).

$$\nu m_{Total} = m_{Liquid}\nu_f + m_{Gas}\nu_g$$
$$\nu m_{Total} = (m_{Total} - m_{Gas})\nu_f + m_{Gas}\nu_g$$
$$\nu = \frac{1}{m_{Total}}\left((m_{Total} - m_{Gas})\nu_f + m_{Gas}\nu_g\right)$$
$$= \left(1 - \frac{m_{Gas}}{m_{Total}}\right)\nu_f + \frac{m_{Gas}}{m_{Total}}\nu_g$$
$$x = \frac{m_{Gas}}{m_{Total}}$$
$$\nu = (1-x)\nu_f + x\nu_g$$
$$= \nu_f + x(\nu_g - \nu_f)$$

The quantities ν_f and ν_g **are fixed** once pressure and temperature have been established.

Example 4.2: Change in Mixture Properties.

Suppose there is a pot of water with $m = 3\,\text{kg}$ of water at $P = 101\,\text{kPa}$, and temperature $T = 100\,^\circ\text{C}$. The pot is sitting at T to start with and continues to heat until all water is evaporated.

- What is the volume of the water at the start?
- What is the Quality of the Mixture at the start?
- If $x = \frac{1}{2}$, what is the volume of the resulting system?
- If $x = 1$, what is the volume of the resulting system?

- -

Using the equations we have already, we know

$$V = \nu_f m_{Total}$$

From Table A.4 in the textbook, we know $\nu_f = 0.001\,043\,\text{m}^3/\text{kg}$. Thus, we can just plug that value into the equation and solve for V.

$$V = \nu_f m_{Total}$$

At this pressure and temperature, the liquid water is not becoming a vapor due to boiling, so $m_{Total} = m_{Liquid}$

$$= 0.001\,043\,\text{m}^3/\text{kg}(3\,\text{kg})$$
$$= 0.003\,129\,\text{m}^3$$
$$= 3.129\,\text{L}$$

Now to solve for the "initial" Quality. Because the water has ***JUST*** reached its Saturated Liquid state, the water has **begun** boiling, but not yet started to vaporize. Thus,

$$x = 0$$

We are given the Quality of the system, $x = \frac{1}{2}$, meaning we have converted half the Saturated Liquid to vapor. We can easily find the mass of the vapor, then find its volume.

$$x = \frac{1}{2}$$

Remember, due to the Law of Conservation of Mass, **all** $3\,\text{kg}$ of the water remains **IN** the system!

$$= \frac{m_{Gas}}{m_{Total}} = \frac{m_{Gas}}{3\,\text{kg}}$$
$$\frac{1}{2} = \frac{m_{Gas}}{3\,\text{kg}}$$
$$m_{Gas} = 1.5\,\text{kg}$$

Now that we know the mass of the vaporized liquid, we can use Equation (4.5) to solve for the total volume.

$$\nu = \nu_f + x(\nu_g - \nu_f)$$

The value for ν_f remains constant throughout this entire Process, so we can reuse that value. The value for ν_g is found in Table A.4 from the textbook.

$$= 0.001\,043\,\mathrm{m^3/kg} + \frac{1}{2}(1.6720\,\mathrm{m^3/kg} - 0.001\,043\,\mathrm{m^3/kg})$$
$$= 0.8365\,\mathrm{m^3/kg}$$

Now that we have the Specific Volume of the system, we can find the total volume of the system.

$$V_{Total} = \nu m_{Total}$$
$$= 0.8365\,\mathrm{m^3/kg}(3\,\mathrm{kg})$$
$$= 2509\,\mathrm{L}$$

Remember that V_{Total} contains the volume for **both** the steam **and** the water. If we are curious about the separate values for each, then we can solve for the individual terms in Equation (4.6).

$$V_{Water} = m_{Water}\nu_f$$
$$= 1.5\,\mathrm{kg}(0.001\,043\,\mathrm{m^3/kg})$$
$$= 1.56\,\mathrm{L}$$
$$V_{Steam} = m_{Steam}\nu_g$$
$$= 1.5\,\mathrm{kg}(1.6720\,\mathrm{m^3/kg})$$
$$= 2508\,\mathrm{L}$$

Thus,

$$V_{Water} = 1.56\,\mathrm{L}$$
$$V_{Steam} = 2508\,\mathrm{L}$$

For this last part, $x = 1$, meaning that **all** the water has become steam. Thus, $m_{Water} = 0\,\mathrm{kg}$ and $m_{Steam} = 3\,\mathrm{kg}$. Because we also know $\nu_g = 1.6720\,\mathrm{m^3/kg}$, we can easily just solve for Equation (4.6) as the total volume.

$$V_{Total} = V_{Steam}$$
$$V_{Steam} = \nu_g m_{Steam}$$
$$= (1.6720\,\mathrm{m^3/kg})(3\,\mathrm{kg})$$
$$= 5016\,\mathrm{L}$$

So, after the water has completely become steam, the total volume of the system is $V_{Total} = 5016\,\mathrm{L}$.

Example 4.3: What's My State. Problem 4.23

Given certain intensive properties of the Pure Substance, water, fill in the table?

1. $T = 50\,°\mathrm{C}$ and a volume of $V = 4.16\,\mathrm{m^3/kg}$.
2. $P = 200\,\mathrm{kPa}$ and the water is in the Saturated Vapor phase.
3. $T = 250\,°\mathrm{C}$, $P = 400\,\mathrm{kPa}$, and $V = 0.595\,\mathrm{m^3/kg}$.
4. $T = 110\,°\mathrm{C}$, $P = 600\,\mathrm{kPa}$.

	Temperature °C	Pressure kPa	Volume m^3/kg	Phase
1	50	12.352	4.16	Saturated Mixture
2	120.21	200	0.88578	Saturated Vapor
3	250	400	0.595	Superheated Vapor/Steam
4	110	600	0.001052	Compressed Liquid

1. Phase 1:
   (a) We need to look at Table A.4 for phase 1's saturated pressure, and find it P_{sat}.
   (b) Now, looking at the values of ν_f and ν_g for water at $T = 50\,°\mathrm{C}$, we see the value we're given is somewhere

between those. Thus, this is a mixture of a Saturated Liquid and a Saturated Vapor, a Saturated Mixture.

2. Phase 2:

    (a) We were told a pressure, so we should use Table A.5.
    (b) Look in Table A.5 and find $P = 200$ kPa, and get the saturation temperature, T_{sat}.
    (c) They told us it was a Saturated Vapor, so we know the value is somewhere on the saturated vapor line, meaning the volume is the same as ν_g.

3. Phase 3:

    (a) Because we are given both pressure and temperature, either Table A.4 or Table A.5 is applicable.
    (b) However, neither table seems to make much sense, as on the temperature table, the pressure is too low, and on the pressure table, the temperature is too low.
    (c) On both tables, our volume is significantly greater than even the ν_g.
    (d) Because $V > \nu_g$, we go to Table A.6 for superheated water.
    (e) Using Table A.6, we see that ν_g in the table matches the volume given to us.
    (f) This means the water is now in its superheated vapor phase.

4. Phase 4:

    (a) You can start by using Table A.4 or Table A.5, but I will start with Table A.4.
    (b) Find 110 °C in the table.
    (c) Once there, you'll notice their P_{sat} is **much** lower than what we were given, so we move to the next table.
    (d) So, we move to Table A.5, and the temperature in the table is lower than what we were given, so we move onto the next table.
    (e) This means that the temperature is lower than is needed to start boiling the water at 600 kPa and the pressure is too high for the water to start boiling at 110 °C.
    (f) This means that it must be a Compressed Liquid.
    (g) But, we said to treat Compressed Liquids as Saturated Liquid at temperature T, so use Table A.4 instead.

Example 4.4: Changing State of R-134a. Problem 4.112

Given $m = 1$ kg of R-134a fills a rigid container of volume $V = 0.1450$ m³ at an initial temperature of $T_i = -40$ °C, at a final pressure of $P_f = 200$ kPa. If the container is heated, determine the inital pressure (P_i) and the final temperature (T_f)?

- -

Concepts:
The vessel has rigid walls, meaning the vessel's volume will remain constant. The Law of Conservation of Mass states that the gas's mass will remain constant throughout the Process.

$$\nu = \frac{V}{m}$$

Finding a state requires knowing 2 of 3 Intensive Property, (P, T, ν).
Explore:
Because both mass and volume are constant, the Specific Volume remains constant. Values for R-134a are present in Tables A.11 and A.12 in the textbook, which mirror Tables A.4 and A.5, respectively.
Plan:
Using P or T and ν, solve for the state.
Solve:
We use Table A.11 in the textbook, because we are given a temperature. So, we know:

T	P_{sat}	ν_f	ν_g
-40 °C	51.25 kPa	0.000 705 3 m³/kg	0.360 64 m³/kg

Now, we can also find ν:

$$\nu = \frac{V}{m}$$
$$= \frac{0.1450 \, \text{m}^3}{1 \, \text{kg}}$$
$$= 0.1450 \, \text{m}^3/\text{kg}$$

Now, we notice that ν is somewhere between ν_f and ν_g, meaning the R-134a is in its Saturated Mixture phase right now.

We use Table A.12 in the textbook, because we are given a pressure for the final state. So, we know:

P	T_{sat}	ν_f	ν_g
200 kPa	$-10.09\,°\text{C}$	$0.000\,738\,1\,\text{m}^3/\text{kg}$	$0.099\,951\,\text{m}^3/\text{kg}$

Because we know $\nu = 0.1450\,\text{m}^3/\text{kg}$, and we have the values for ν_f and ν_g, we can tell that the gas is not in a superheated vapor state. Because the gas is superheated, we need to use Table A.13 instead.

That table says that a gas with $\nu = 0.145\,04\,\text{m}^3/\text{kg}$ is at $T_{\text{sat}} = 90\,°\text{C}$.

Thus,

$$P_i = 51.25\,\text{kPa}$$
$$T_f = 90\,°\text{C}$$

Example 4.5: Changing State of Water English. Problem 4.113

Given $m = 1\,\text{lbm}$ of water at an initial temperature $T_i = 400°\text{F}$ fills a weighted piston cylinder device with movable lid of volume $V_i = 2.649\,\text{ft}^3$. If the vessel is cooled to $T_f = 100°\text{F}$, determine the final pressure (P_f) and the final volume of the container V_f?

- -

Concepts:
Weighted piston cylinder devices ensure that the pressure exerted by the fluid remains constant, although we don't know what it is. The mass of the water in this problem remained unchanged throughout the Process as well. We know the initial volume, initial temperature and the final temperature. Because we were given each temperature, we know the fluid in the vessel is being cooled.

Explore:
We want to get 2 of those 3 those Intensive Property (P, T, or ν). We can find the ν_i for the initial point in time, because we know the volume of the vessel initially and the mass is constant. We also know the initial temperature, so we have all the Intensive Propertys that we need to solve for the initial state.

Plan:
Use Table A.4E to find P. Because the pressure is constant, we can use P and the final temperature T_f to solve for the final state's V_f.

Solve:
Fome Table A.4E, we can find and know

T	P_{sat}	ν_f	ν_g
400°F	247 psia	$0.018\,64\,\text{ft}^3/\text{lbm}$	$1.864\,\text{ft}^3/\text{lbm}$

We also know that the Specific Volume of the water is:

$$\nu = \frac{V}{m}$$
$$= \frac{2.649\,\text{ft}^3}{1\,\text{lbm}}$$
$$= 2.649\,\text{ft}^3/\text{lbm}$$

Because $\nu > \nu_g$, we know that the fluid is a superheated vapor, steam.

Now, because the fluid is a superheated vapor, we need to use Table A.6E to find the pressure that the water/steam is under. Using the temperature and the Specific Volume of the water, we look those values up in Table A.6E, we see that $P = 180\,\text{psia}$. The saturation pressure $T_{\text{sat}} = 373°\text{F}$, which means that the final temperature T_f is likely a liquid. Because the water is likely a liquid at its final temperature, we use Table A.4E and look up the value of water at 100°F.

T	P_{sat}	ν_f	ν_g
100°F	$0.950\,52\,\text{psia}$	$0.016\,13\,\text{ft}^3/\text{lbm}$	$349.83\,\text{ft}^3/\text{lbm}$

If we use the pressure we got from Table A.6E $P = 180$ psia, and look that value up in Table A.5E, we see that $T_{\text{sat}} = 373.07°\text{F}$ to even boil the water. This likely means that the water is a Compressed Liquid, and because of our rule, $\nu_f = 0.016\,13\,\text{ft}^3/\text{lbm}$. Therefore, by multiplying ν_f by the mass of the water, we can find the volume it is consuming.

$$
\begin{aligned}
V_f &= \nu_f m \\
&= 0.016\,13\,\text{ft}^3/\text{lbm}(1\,\text{lbm}) \\
&= 0.016\,13\,\text{ft}^3
\end{aligned}
$$

4.3   Ideal Gas Equation

The Ideal Gas Equation is used when:

- We have dry air.
- Air is **not** good under high pressures, like in turbines.
- Not good on for liquid water.
- Can occasionally be used for steam, when the pressure is $P \leq 10\,\text{kPa}$.
- Not good for steam at high temperatures or pressures.
- Good for the noble gases, and hydrogen.

$$
P\nu = RT \tag{4.10}
$$

where

P: The **absolute** pressure.
ν: The Specific Volume.
R: Gas constant, specified in terms of gases and units.
T: **Absolute** temperature (K or R).

There is an alternative to Equation (4.10), Equation (4.11).

$$
PV = mRT \tag{4.11}
$$

Example 4.6: Ideal Gas in Tire. Problem 4.79

The air in a tire has its temperature and pressure measured, as the pressure inside the tire depends on the temperature of the air in the tire. Initially, at $T_i = 25\,°\text{C}$ the pressure is $P_{i,\text{Gage}} = 210\,\text{kPa}$. The volume of the tire is $V = 0.025\,\text{m}^3$. Determine the air pressure in the tire when the tmeperature is $T_f = 50\,°\text{C}$? Also determine the amount of air that must be let out to restore the original pressure at the new temperature, assuming $P_{atm} = 100\,\text{kPa}$?

- -

Concepts:

- The tire warms up when driven.
$$
T_i = 25\,°\text{C} \to T_f = 50\,°\text{C}
$$

- Air is an ideal gas.
- The pressure of the outside air is $P_{atm} = 100\,\text{kPa}$
- We are given an initial volume, but no final volume.
- We are given a gage pressure, which is the atmospheric pressure plus the pressure in the tire.
$$
P_{\text{gage}} = P_{abs} - P_{atm}
$$

Explore:
The Ideal Gas Equation needs both P_{abs} and T_{abs}. We should use Equation (4.11), because we need to know how much air to let out.
$$
PV = mRT
$$
The volume is either unchanged, or changes very little.
$$
\left(\frac{PV}{T}\right)_i = \left(\frac{PV}{T}\right)_f
$$

5 Energy Analysis of Closed Systems

Gases can perform work, meaning they can expend Energy.

For example, in a piston, there is a contained gas, which can press one side up and out. Remember the equation for Work, Equation (1.4). But, because we are dealing with gas pressures, we rewrite Equation (1.4).

$$W = PAd$$
$$\Delta W = PA\Delta d$$
$$= P\Delta V$$

We can express the change in work using differentials too, so we can integrate.

$$dW = PdV$$

This leads to Equation (5.1)

$$W = \int_{\text{State 1}}^{\text{State 2}} PdV \tag{5.1}$$

Equation (5.1) has two possibilities:

Volume is Constant $W = 0$
Pressure is Constant $W = P(V_2 - V_1)$

Now, we may be interested in the total energy required to change or create a system. This is Enthalpy. For example, Enthalpy of Vaporization is the amount of energy required to convert a system from a Saturated Liquid to a Saturated Vapor.

Defn 56 (Enthalpy). Another way to think of this, is *enthalpy* is the Energy required to create a new System. This can only be done if the current system performs work on its boundary, i.e. moves the boundary.

Enthalpy is typically the total Heat content of a System. It is equal to the Internal Energy of the system plus the product of Pressure and volume.

$$H = U + PV \ \text{J} \tag{5.2}$$

However, Enthalpy is an Extensive Property, but we would prefer to work with an Intensive Property. Thus, we more frequently use Specific Enthalpy.

Defn 57 (Specific Enthalpy). *Specific enthalpy* is also commonly referred to as just *enthalpy*, as context makes clear which one is being discussed. Specific enthalpy is the total Heat content of a System per unit mass of that system. Specific Enthalpy is defined in Equation (5.3).

$$h = u + P\nu \ \text{J/g} \tag{5.3}$$

5.1 Mechanisms of Energy Transfer

From our discussion of changes in energy in Section 3.3, we know that there are 3 main mechanisms of energy transfer:

1. Heat Transfer
2. Work
3. Mass Flow

In addition, according to the Law of Conservation of Energy, we know that the total amount of energy must be conserved throughout a process. This leads to 2 separate types of processes occurring.

Cyclical A cyclic process is one where the beginning and end states are identical. This also means that $Q_{In} = W_{Out}$ and $Q_{Out} = W_{In}$, making $\Delta E = 0$, and $E_{In} = E_{Out}$.
Regular A regular process is the general way to deal with processes, and cyclic processes are ones that are special cases of a regular process. In this case, there is a change in energy $\Delta E = (Q_{In} - Q_{Out}) + (W_{In} - W_{Out})$.

Now, we can generalize Equation (3.2) even more, to allow for the changing of systems. We start with an equation that represents the energy of the entire System, Equation (5.4).

$$U + \text{KE} + \text{PE} + \text{FE} = Q + W \tag{5.4}$$

Now, if we want to discuss the change in the total energy of the system, we can use Equation (5.5).

$$\Delta U + \Delta\text{KE} + \Delta\text{PE} + \Delta\text{FE} = \Delta Q + \Delta W$$
$$= (Q_{In} - Q_{Out}) + (W_{In} - W_{Out}) \tag{5.5}$$

The implications of Equation (5.5) are listed below.

- For stationary systems, $\Delta KE = 0$, and $\Delta PE = 0$.
- For stationary and Closed Systems (when there is **no** mass transfer) $\Delta KE = 0$, $\Delta PE = 0$, $\Delta FE = 0$. Therefore, $\Delta U = \Delta Q + \Delta W$.

Example 5.1: Enthalpy of Saturated Water. Textbook Problem 5.29

Saturated water vapor at $T = 200\,°C$ and is Isothermally condensed into a Saturated Liquid in a piston-cylinder device. Calculate the Heat Transfer and Work done during this process in kJ/kg?

- -

Concepts:
A piston-cylinder means that the pressure is constant throughout the duration of a Process.
An Isothermal process is one in which the temperature is constant.
The water is moving from a Saturated Vapor to a Saturated Liquid.
The total energy is conserved throughout the Process

Explore:
From our understanding of Saturated Mixtures, we know that we should use Table A.4 from the textbook's appendix. This is both a stationary system and a Closed System, so $\Delta KE = 0$, $\Delta PE = 0$, and $\Delta FE = 0$. Therefore, the only terms in Equation (5.5) we need to care about are

$$\Delta U = \Delta Q + \Delta W$$

To simplify this further, we can assume that at the starting state, there was no Heat going in and no Work being done. Therefore, $Q_{In} = 0$ and $W_{In} = 0$. So, the equation above simplifes down to

$$U_2 - U_1 = -Q_{Out} + -W_{Out}$$
$$= -Q_{Out} - P(V_2 - V_1)$$

The difference in Internal Energy is actually recorded in the textbook's Table A.4 as U_{fg}.

Plan:
Solve for the above equation, using Table A.4 in the textbook to find the value of the change in Internal Energy of the system.

Solve:
From Table A.4, we know that at $T = 200\,°C$, $P_{\text{sat}} = 1555\,kPa$. We also know $\nu_f = 0.001\,157\,m^3/kg$ and $\nu_g = 0.127\,21\,m^3/kg$. Lastly, we can also find $u_g = 2594\,kJ/kg$ and $u_f = 850.46\,kJ/kg$.

$$W_{Out} = 1555(0.001157 - 0.12621)$$
$$= -196\,kPa\,m^3/kg$$
$$= -196\,kJ/kg$$
$$u_f - u_g = 850 - 2594$$
$$= -1744\,kJ/kg$$
$$u_f - u_g = -Q_{Out} - W_{Out}$$
$$-1744\,kJ/kg = -Q_{Out} - -196\,kJ/kg$$
$$Q_{Out} = 1940\,kJ/kg$$

Thus, the vapor is cooling and condensing in the piston-cylinder, which makes sense in this case.

Validate:
We can validate this by performing the same calculations using the change in Specific Enthalpy instead.

$$\Delta h = -Q_{Out}$$
$$h_f - h_g = -Q_{Out}$$

The change in specific enthalpy can be found in Table A.4 again.

$$(852.3 - 2792)kJ/kg = -Q_{Out}$$
$$Q_{Out} = 1939.7\,kJ/kg$$

Now, taking significant figures.

$$Q_{Out} = 1940\,kJ/kg$$

Generalize:
Any time the system is changing states, we can use Enthalpy and/or Specific Enthalpy. This helps us avoid these longer calculations using all the other forms of Energy in the System.

Rigid $V = 1\,\text{ft}^3$ vessel that contains R-134a. The refrigerant is at $T_1 = -20°$F with Quality 27.7 %. It is heated until it reaches $T_2 = 100°$F. Calculate the Heat Transfer required to do this?

- -

Concepts:
Because we are told the Quality, we know this is a Saturated Mixture.
This is a rigid vessel, so the volume is constant, implying this is both a stationary and closed system.
Because Q is an Energy, not Specific Energy, we need to keep it as an Extensive Property. Using all of these facts, $\Delta U = \Delta Q + \Delta W$.
Explore:
We can start by looking at the various components of the equation above. There is no cooling going on in this system, so $Q_{Out} = 0$. There is no work being done on the refrigerant, so both $W_{In} = 0$ and $W_{Out} = 0$. Therefore, $\Delta U = Q_{In}$, which makes sense, as we were told we were only heating the refrigerant. Lastly, because the refrigerant starts as a Saturated Mixture, we need to use:

$$\nu_x = \nu_f + x(\nu_g - \nu_f)$$
$$u_x = U_f + x(U_g - U_g)$$

Plan:
Solve $\Delta u = Q_{In}$ by finding the Internal Energy values in the English appendix table for R-134a.
Solve:
When $T = -20°$F, $P_{\text{sat}} = 12.9\,\text{psia}$. In addition, the Quality is $x = 0.227$. Because this is in a rigid vessel, the volume is constant, which also means the Specific Volume is constant.

$$\nu_x = \nu_f + x(\nu_g - \nu_f)$$

Grabbing ν_f and ν_g from the Table in the textbook's appendix.

$$= 0.01156 + 0.277(3.4424 - 0.01156)$$
$$= 0.961\,843\,\text{ft}^3/\text{lbm}$$

Now we focus on finding the Internal Energy of the Saturated Mixture at the given Quality.

$$u_x = u_f + x(u_g - u_f)$$
$$= 6.014 + 0.277(91.90 - 6.014)$$
$$= 29.8044\,\text{btu/lbm}$$

Now, we focus on the end state. Keep in mind that the Specific Volume is constant. Looking in the Saturated Mixture tables for this temperature shows us the refrigerant is actually a Superheated Vapor now. When looking at Table A.13E, we notice that the Specific Volume does not have a single entry in the table. So, we must Interpolate instead.

$$y = y_0 + (x_1 - x)\frac{y_1 - y_0}{x_1 - x_0}$$

Interpolate the Pressure.

$$P = P_0 + (\nu - \nu_1)\frac{P_1 - P_0}{\nu_1 - \nu_0}$$
$$= 50\,\text{psia} + (0.961843 - .9072)\frac{60 - 50}{1.1043 - 0.9072}$$
$$= 52.7\,\text{psia}$$

Now, interpolating the Internal Energy.

$$u_2 = u_1 + (\nu - \nu_1)\frac{u_1 - u_0}{\nu_1 - \nu_0}$$
$$= 111.16\,\text{btu/lbm} + (0.961843 - 0.9072)\frac{111.17 - 111.56}{0.90762 - 1.1043}$$
$$= 111.3\,\text{btu/lbm}$$

Now that we have the beginning and end values of the pressure and internal energy, we can calculate the change in internal energy.

$$\Delta u = u_1 - u_x$$
$$= 111.3 - 29.8044$$
$$= 81.49 \, \text{btu/lbm}$$

However, because the change we actually got was the change in specific Internal Energy, we must multiply it by the mass of the refrigerant in the system. We know the Specific Volume and the volume of the vessel, so we can find the mass.

$$\nu = \frac{V}{m}$$
$$m = \frac{V}{\nu}$$
$$= \frac{1}{0.961843}$$
$$= 1.039\,67 \, \text{lbm}$$

Now, multiplying Δu with the mass, we get

$$U = mu$$
$$= 1.03967 \times 81.49$$
$$= 84.7228$$
$$= 84.73 \, \text{btu}$$

5.2 Heat and Specific Heat for Energy Transfer

Defn 58 (Specific Heat). *Specific heat* is the Energy required to raise the termperature of a unit mass of a material by one degree. Specific heat is denoted with c. This is a **linear** term between temperature and energy. Like any other Specific Property, specific heat is also an Intensive Property.

Remark 58.1 (Possible Materials to Apply to). This Intensive Property only works for the following materials, given they **do not undergo phase changes**:

- Noble Gases and Air
- Liquids
- Solids

Remark 58.2 (Other Notable Properties). Specific Heat is **not** always linear over large temperature changes or when under very high pressures or temperatures.

There are 2 cases of using Specific Heat:

1. Constant Volume
2. Constant Pressure

Both of these cases are **NOT** the same, meaning $c_V \neq c_P$. **However**, there **is** a relation between c_V and c_P, shown in Equation (5.6).

$$c_P = c_V + R \tag{5.6}$$

5.2.1 Constant Volume

If the material in question has a Specific Heat and undergoes a process at constant volume, then:

$$c_V \equiv \left[\frac{\partial U}{\partial T} \right]_V$$

The equation above is approximately linear for smaller changes in both temperature and volume, which yields Equation (5.7)

$$\Delta U = c_V m \Delta T \tag{5.7}$$

5.2.2 Constant Pressure

If the material in question has a Specific Heat and undergoes a process at constant pressure, then:

$$c_P \equiv \left[\frac{\partial H}{\partial T}\right]_P$$

The equation above is approximately linear for smaller changes in both temperature and volume, which yields Equation (5.8)

$$\Delta H = c_P m \Delta T \tag{5.8}$$

Example 5.3: Energy Required by Heater. Textbook Problem 5.79

A mass of $m = 15\,\text{kg}$ of air in a piston-cylinder device is heated from $T_1 = 25\,°\text{C}$ to $T_2 = 77\,°\text{C}$ by passing electric current through a resistance heater inside the cylinder. The Pressure inside the cylinder is held constant at $P = 300\,\text{kPa}$ throughout the Process. The Heat loss in the cylidner is $Q = 60\,\text{kJ}$. Determine the electric energy required by the electric heater in kW h?

- -

Concepts:
Because this is a piston-cylinder, the Pressure is constant.
This has no limitation on the volume of the air inside the piston-cylinder. Because the heating is greater than the heat loss, the air will expand.
This is a Closed System, so there is a heat transfer and it is possible to do work.
The electric output of the heater is measured in kW h, which is power multiplied by time. Thus, $1\,\text{kW h} = 3600\,\text{kJ}$.
The energy balance for this problem is summarized by $\Delta U = \Delta Q + \Delta W$.
Enthalpy exists in a system only when a new System is being created. There was work done by the air as it expanded on the piston, and it moved the air outside of the piston too, so we can use Enthalpy here.

Explore:
We start by looking at ΔQ. Because there is no heat going into the system from the surroundings, $Q_{In} = 0$. However, because the system is losing heat to the surroundings, $Q_{Out} = 60\,\text{kJ}$.
Now, looking at ΔW. The electric heater is doing work on the air, by heating it up, W_{In} is equal to the work done by the electric heater. Then, because the air is expanding, it is also performing work on the piston-cylinder to move the piston. Thus, we determine this is flow energy, where the pressure is constant, so $W_{Out} = PV$.
Lastly, because we are dealing with a system that *is* changing, we can use Enthalpy instead. In $\Delta H = -Q_{Out} + W_{In}$, the W_{Out} is actually implicitly added to the ΔH side, so it doesn't need to be recorded twice. This is because $H = U + PV$, from Equation (5.2).

$$\Delta U = -Q_{Out} + W_{In} - W_{Out}$$
$$= -Q_{Out} + W_{In} - P\Delta V$$
$$\Delta U + P\Delta V = -Q_{Out} + W_{In}$$
$$\Delta H = -Q_{Out} + W_{In}$$

Plan:
Solve $\Delta H = -Q_{Out} + W_{In}$.
Find the states from tables in the back of the textbook, namely Table A.21. We need to keep in mine that the tables will have the Specific Enthalpy, so we must change it into just Enthalpy with $H = mh$.

Solve:
From Table A.21 in the textbook: at $T_1 = 25\,°\text{C}$, $h_1 = 298.18\,\text{kJ/kg}$. Likewise, at $T_2 = 77\,°\text{C}$, $h_2 = 350.5\,\text{kJ/kg}$.
Now that we have every term in our equation to use for solving, we solve it.

$$\Delta H = -Q_{Out} + W_{In}$$
$$m\Delta h = -60 + W_{In}$$
$$m(h_2 - h_1) + 60 = W_{In}$$
$$W_{In} = 15(350.5 - 298.18) + 60$$
$$= 844.8\,\text{kJ}$$

Now, we have to divide by $1\,\mathrm{kW\,h} = 3600\,\mathrm{kJ}$.

$$W_{In} = 844.8\,\mathrm{kJ}$$
$$= \frac{844.8}{3600}$$
$$= 0.235\,\mathrm{kW\,h}$$

Validate:
We could do this with Specific Heat as well.

$$\Delta U = c_V m \Delta T$$
$$\Delta H = c_P m \Delta T$$

But, because this Process was done under constant Pressure, we need to use the second one. $c_P = 1.0065\,\mathrm{kJ\,K/kg}$ is taken from Table A.2, and was averaged over the c_P values for the range $0\,^\circ\mathrm{C}$ and $100\,^\circ\mathrm{C}$. Because of the way c_P is defined, we must convert our temperatures into Kelvin too, which is simple. Once done, we can solve this equation.

$$\Delta H = c_P m \Delta T$$
$$= 1.0065(15\,\mathrm{kg})(350 - 298)\mathrm{K}$$
$$= 785.07\,\mathrm{kJ}$$

Now, if we account for the $60\,\mathrm{kJ}$ of heat loss, then $785.07 + 60 = 845.07\,\mathrm{kJ}$, which is about the same.
Generalize:
We can use the state of the System and the energy balance equation or the Specific Heat to find the total Energy of the system.

5.3   Incompressible Substances

In general, because there is little that the substance *can* do, because it is incompressible, it leads to the Specific Heats being equal to each other.

$$c_P = c_V$$

Example 5.4: Using Specific Heat. Textbook Problem 5.66

A $4\,\mathrm{m} \times 5\,\mathrm{m} \times 6\,\mathrm{m}$ room is to be heated by a baseboard resistance heater. This heater must raise the temperature from $7\,^\circ\mathrm{C}$ to $23\,^\circ\mathrm{C}$ within 15 minutes. Assuming no heat losses in the room, and an absolute pressure of $P = 100\,\mathrm{kPa}$ determine the required power of the resistance heater assuming a constant Specific Heat at room temperature?

- -

Concepts:
Room has air at roughly $1\,\mathrm{atm}$.
This is ideally a Closed System, but if the room is leaky, then it is an Open System.
Air can be treated as an ideal gas.
The resistance heater is inside the room and no mention is made of temperature outside the room, meaning there is no heat gain nor is there any heat loss.
Explore:
We could use different Specific Heats depending on the system types. However, we can assume that the problem meant this system to be a Closed System. This implies a constant volume, so c_V should be used. We need Table A.2 in the textbook's appendix. Under the implication of constant volume, for air, $c_V = 0.718\,\mathrm{kJ\,K/kg}$. However, for air under constant pressure, an Open System, $c_P = 1.005\,\mathrm{kJ\,K/kg}$.
Now, as a discussion of the energy balance of this system, we use the equation below.

$$\Delta U = \Delta Q + \Delta W$$

We were tole that $\Delta Q = 0$ because there is no heat loss in the room, and likewise, there are no heat gains. Therefore,

$$\Delta U = W_{In} - W_{Out}$$

If the room were leaky, and this were an Open System, then $W_{Out} \neq 0$. We know that W_{In} is the work done on the room due to the heater.

In addition, we know that we can calculate the change in Internal Energy by using the Specific Heat.

$$\Delta U = c_V m \Delta T$$
$$= \dot{W} \Delta t$$

Plan:
Find all the mass in the system. Use c_V to solve for this.
Solve:
Start by finding the mass of the air in the room.

$$m = \frac{PV}{RT}$$
$$= 149\,\mathrm{kg}$$

Now, move onto finding the change in Internal Energy.

$$\Delta U = c_V m \Delta T$$
$$= 0.718(149)(23 - 7)$$
$$= 1711\,\mathrm{kJ}$$

Now plug that into our equation about the work done multiplied by the time.

$$\Delta U = \dot{W} \Delta t$$
$$1711\,\mathrm{kJ} = \dot{W}\left(15\,\mathrm{min}\frac{60\,\mathrm{s}}{1\,\mathrm{min}}\right)$$
$$\dot{W} = 1.9\,\mathrm{kW}$$

6 Mass and Energy Analysis of Control Volumes

6.1 Continuity Equation

We perform this analysis when we are interested in the matter flow of a System. We still need to have Law of Conservation of Energy. The important thing to remember here is that there is no change of mass of the control volume, meaning there is **no** storage of the mass in the system. This means we have *steady flow*. This also means we could have an unsteady flow, but we deal with that later in this section.

Now, to motivate the Continuity Equation, imagine an expandable tank. If there are n flows inwards and m flows outwards, we can construct an equation that looks like this.

$$(\dot{m_{In,1}} + \dot{m_{In,2}}) - \dot{m_{Out}} = \left[\frac{dm}{dt}\right]_{\text{Control Volume}}$$
$$\sum_{\text{Inlets}} \dot{m} - \sum_{\text{Outlets}} \dot{m} = \dot{m}_{\text{Control Volume}}$$

Defn 59 (Continuity Equation)**.** The *continuity equation* states the relationship between the mass flow into a point in a System and its output. Succinctly, it can be stated:

$$\sum_{\text{Inlets}} \dot{m} - \sum_{\text{Outlets}} \dot{m} = \dot{m}_{\text{Control Volume}}$$
$$\sum_{\text{Inlets}} \dot{V} - \sum_{\text{Outlets}} \dot{V} = \dot{V}_{\text{Control Volume}}$$

(6.1)

$\dot{m}_{\text{Control Volume}}$ is the storage possible in a vessel, in mass. Likewise, $\dot{V}_{\text{Control Volume}}$ is the possible storage of a vessel, in volume.

Remark 59.1 (Alternate Definitions of Mass Flowrate)*.* It will be useful to remember alternative versions of mass flowrate for this particular section. Namely,

$$\dot{m}_1 = \dot{m}_2$$
$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

There are 2 situations that can arise from the Continuity Equation.

1. Rigid control volumes, $\dot{m}_{\text{Control Volume}} = 0$.
2. Non-rigid control volumes, $\dot{m}_{\text{Control Volume}} \neq 0$.

Example 6.1: Use Continuity Equation.

A nozzle is attached to a 20 gal bucket to fill it. The inner diameter of the hose is 1 in and reduces to 0.5 in at the nozzle's exit. If the average velocity of the water in the hose is 8 ft/s determine the volume and mass flow rates, $\dot{V}$ and $\dot{m}$ of the water through the hose and how long it will take to fill the bucket? Also determine the average velocity at the tip of the nozzle, v_{Out}?

- -

Concepts:
A nozzle has 1 inlet and 1 outlet. The diameter of the nozzle at the inlet is $d_{In} = 1$ in, and the outlet $d_{Out} = 0.5$ in.
There is no temperature change in the hose, so the density of the water is constant, $\Delta\rho_{Water} = 0$.
Because the host does not have any storage, $\dot{m}_{\text{Control Volume}} = 0$. Therefore, $\dot{m}_{In} = \dot{m}_{Out}$ and $\dot{V}_{In} = \dot{V}_{Out}$.
Explore:
$\rho_{\text{Water}} = 62.3\,\text{lbm/ft}^3$, as 32°F.
The bucket has a capacity of 20 gal, which means $20 = \dot{V} \times$ Time to fill.
Plan:
Solve for $\dot{V}$, $\dot{m}$, V_2.
Solve:
Start with a definition of volume flowrate.

$$\dot{V}_{In} = A_{In} v_{In}$$

$$= \frac{\pi d_{In}^2}{4} 8\,\text{ft/s}$$

$$= 0.0436\,\text{ft}^3/\text{s}$$

Now, we can find the mass flowrate through the inlet of the hose.

$$\dot{m} = \rho_{Water}\dot{V}$$

$$= 62.3\,\text{lbm/ft}^3(0.0436\,\text{ft}^3/\text{s})$$

$$= 2.72\,\text{lbm/s}$$

Because the flowrates at the inlet and outlet of the hose are the same, we can reuse these values to solve for the time to fill the bucket.

$$20\,\text{gal}\frac{1\,\text{ft}^3}{7.48\,\text{gal}} = \dot{V}t$$

$$t = 61.3\,\text{s}$$

Lastly, we need to solve for the velocity of the flowing water at the tip of the nozzle. We can use the relation between mass flowrates, shown in Remark 59.1.

$$\rho_{Water} A_{In} v_{In} = \rho_{Water} A_{Out} v_{Out}$$

$$A_{In} v_{In} = A_{Out} v_{Out}$$

$$v_{Out} = \frac{A_{In} v_{In}}{A_{Out}}$$

$$= \frac{\frac{\pi d_{In}^2}{4} v_{In}}{\frac{\pi d_{Out}^2}{4}}$$

$$v_{Out} = 32\,\text{ft/s}$$

Validate:
We can validate that our answer for volume flowrate makes sense by multiplying the outlet volume flowrate by the outlet velocity and the outlet cross-sectional area.

$$\dot{V}_{Out} = v_{Out} \times A_{Out}$$

$$= 32\,\text{ft/s}\frac{\pi d_{Out}^2}{4}$$

If we look at this closely, we notice that when the diameter of the outlet was halved, the total area was quartered. Correspondingly, there was a four times increase in the velocity of the water at the outlet.
Generalize:
Constant mass flow and constant fluid density reuislts in a change at the outlet based on the change in area.

Example 6.2: Flow with Changing Mass Density. Textbook Problem 6.7

Air enters a nozzle at a steady rate of $\rho_{In} = 2.21\,\text{kg/m}^3$ at $v_{In} = 40\,\text{m/s}$ and leaves at $\rho_{Out} = 0.762\,\text{kg/m}^3$ and $v_{Out} = 180\,\text{m/s}$. If the inlet of the nozzle is $A_{In} = 90\,\text{cm}^2$, determine the mass flowrate, $\dot{m}$, through the nozzle and the outlet area of the nozzle, A_{Out}?

- -

Concepts:
Nozzles take a low v_{In} and increase it to a high v_{Out}.
This is a steady flow problem, meaning we can use the Continuity Equation freely. Also, the nozzle has no storage capacity for the air, so $\dot{m}_{In} = \dot{m}_{Out}$.
The mass density is changing in this problem. This is happening because air is an ideal gas, and/thus, it is compressible.
Explore:
Remember, because the nozzle does **not** store any air, $\dot{m}_{In} = \dot{m}_{Out}$. This occurs even though ρ is changing. If we expand this equation, we get:
$$(\rho v A)_{In} = (\rho v A)_{Out}$$

Plan:
Solve the expanded mass flowrate equation above.
Solve:

$$\dot{m}_{In} = \rho_{In} v_{In} A_{In}$$
$$= 2.21\,\text{kg/m}^3 (40\,\text{m/s})(90\,\text{m}^2)(\frac{1\,\text{m}}{100\,\text{cm}})^2$$
$$= 0.7956\,\text{kg/s}$$

Now, we can solve for the other side of the equation.

$$\dot{m}_{Out} = \rho_{Out} v_{Out} A_{Out}$$
$$0.7956\,\text{kg/s} = (0.762\,\text{kg/m}^3)(180\,\text{m/s}) A_{Out}$$
$$A_{Out} = 0.0058\,\text{m}^2$$
$$= 58\,\text{cm}^2$$

Validate:
Our equations have been consistent with our knowledge of how nozzles work.
We checked our units as we solved, so we are likely correct.
If we wanted to be thorough, we should check that $\dot{m}_{In} = \dot{m}_{Out}$ is actually a valid assumption.

6.2   Energy Flows

	Internal Energy	Kinetic Energy	Potential Energy	Flow Energy
Specific Energy, e	u	$\frac{v^2}{2}$	gd	$\frac{P}{\rho}$
Energy, E	U	$\frac{mv^2}{2}$	mgd	PV

Table 6.1: Types of Energy for Mass Flow

We can often connect the Internal Energy and the Flow Energy of a process and call that Enthalpy, h or H for a particular

control volume.

$$h = u + \frac{P}{\rho}$$

$$H = U + \text{FE}$$

And we know that power is the time-rate of change of energy.

$$P = \dot{E}$$

$$= \dot{m}e$$

Just like with mass flows, if a control volume has not storage capacity, then it also has no energy. Thus, if there is no storage, and the Law of Conservation of Energy is abided by, then:

$$\dot{m}_{In} - \dot{m}_{Out} = \dot{m}_{\text{Control Volume}}$$

$$\dot{E}_{In} - \dot{E}_{Out} = \dot{E}_{\text{Control Volume}}$$

$$\dot{m}_{In} = \dot{m}_{Out}$$

$$\dot{E}_{In} = \dot{E}_{Out}$$

Looking at the equations received, it is obvious that we can apply any kind of Energy that may be present in the system to the above equations and solve for any term we may be interested in. If we now expand our Continuity Equation to energies, we get Equation (6.2).

$$\dot{Q}_{In} + \dot{W}_{In} + \sum_{In} \dot{m}e = \dot{Q}_{Out} + \dot{W}_{Out} + \sum_{Out} \dot{m}e$$

$$\dot{Q}_{In} + \dot{W}_{In} + \sum_{In} \dot{m}(h + \frac{v^2}{2} + gd) = \dot{Q}_{Out} + \dot{W}_{Out} + \sum_{Out} \dot{m}(h + \frac{v^2}{2} + gd) \tag{6.2}$$

If we have a steady flow, then Equation (6.2) can be simplified a bit.

$$\Delta\dot{Q} - \Delta\dot{W} = \dot{m}\left(\Delta h + \Delta\text{KE} + \Delta\text{PE}\right)$$

$$= \dot{m}\left((h_2 - h_1) + \frac{v_2^2 - v_1^2}{2} + g(d_2 - d_1)\right) \tag{6.3}$$

The work in Equation (6.3) is shaft work. This could be a propeller turning the fluid, or the fluid doing work on the shaft to achieve something.

6.2.1 Nozzles and Diffusers

There are some general properties about nozzles and diffusers that are useful to remember.

- For nozzles, $v_2 \gg v_1$. The energy gained from the change in kinetic energy is changed from the Flow Energy, because $P_2 < P_1$.
- For diffusers, $v_2 \ll v_1$. The energy lost from the change in kinetic energy is given to the Flow Energy, because $P_2 > P_1$.
- In general, $\Delta\dot{Q} \approx 0$. There is very little Heat Transfer.
- In general, $\Delta\dot{W} \approx 0$. There are no propellers or outside shafts.
- In general, $\dot{\text{KE}} \neq 0$. The kinetic energy changes dramatically.
- In general, $h \neq 0$. There is a large volume change.

Example 6.3: Behavior of Nozzles. Textbook Problem 6.31

Given a nozzle that pressures air in an Adiabatic Process at an initial pressure of $P_1 = 300\,\text{kPa}$, initial temperature of $T_1 = 200\,°\text{C}$, an inlet velocity of $v_1 = 30\,\text{m/s}$, and an inlet cross-sectional area of $A_1 = 80\,\text{cm}^2$. The outlet pressure is $P_2 = 100\,\text{kPa}$ and a velocity of $v_2 = 180\,\text{m/s}$. Find the mass flow rate, $\dot{m}$, the temperature of the output air, T_2, and the cross-sectional area of the outlet, A_2?

- -

Concepts:
A nozzle has a much greater outlet velocity compared to its inlet velocity, $v_{Out} \gg v_{In}$, and $P_{In} > P_{Out}$.
Air is an ideal gas and is compressible. So, we might have to deal with different densities of the air at the inlet and outlet, ρ_1 may not necessarily be equal to ρ_2.

This system seems to be under steady flow conditions, so $\dot{m}_{In} = \dot{m}_{Out}$.
Due to the Law of Conservation of Energy, we can say $\dot{E}_{In} = \dot{E}_{Out}$. Namely, there is no energy storage in this system.
Explore:

$$\dot{m}_{In} = \dot{m}_{Out}$$
$$(\rho v A)_{In} = (\rho v A)_{Out}$$

Now, we can find the density of the inlet air by using the tables for it. But, to find the density of the air at the outlet, we can use Equation (4.10), and take the inverse of the Specific Volume.

$$\dot{m}_{In} = \rho_2 v_2 A_2$$
$$= \frac{1}{\nu_2} v_2 A_2$$

Find the Specific Volume using Equation (4.10).

$$P\nu = RT$$
$$\nu = \frac{RT}{P}$$
$$\dot{m}_{In} = \frac{P}{RT} v_2 A_2$$

We also have several other key pieces of information to solve this.

$$\Delta \dot{Q} = 0$$
$$\Delta \dot{W} = 0$$
$$\Delta PE = 0$$

Therefore, only kinetic energy and enthalpy change, ΔKE and Δh, respectively. We can calculate the change in Specific Enthalpy using a slightly modified version of Equation (5.8).

$$\Delta H = c_P m \Delta T$$
$$\Delta h = \frac{\Delta H}{m}$$
$$= \frac{c_P m \Delta T}{m}$$
$$= c_P \Delta T$$

We use c_P, Specific Heat under constant pressure, because once the air leaves the nozzle, it is in an environment with constant pressure.
Plan:
Solve for ν_2 using the Equation (4.10).
Solve for A_2 using the relation $\dot{m} = \frac{A_1 v_1}{v_2}$.
Use our energy conservation relation to solve for T_2 using ΔT.
Solve:

$$\nu_1 = \frac{RT_1}{P}$$
$$= 0.4525 \, \text{m}^3/\text{kg}$$

Now that we have the initial Specific Volume, we can find the mass flowrate.

$$\dot{m}_1 = \frac{A_1 v_1}{\nu_1}$$
$$= 0.5304 \, \text{kg/s}$$

Now, we can solve for our energy conservation equation, Equation (6.2) to find the final temperature, T_2.

$$\Delta\dot{Q} - \Delta\dot{W} = \dot{m}\left(\Delta h - \frac{\Delta\text{KE}}{m}\right)$$

$$0 - 0 = \dot{m}\left(c_P\Delta T + \frac{v_2^2 - v_1^2}{2}\right)$$

Refer to the textbook's Table A.2 for c_P. Some interpolation is needed.

$$0 = 0.5304\,\text{kg/s}\left(1.02\,\text{kJ/kg}(T_2 - (200 + 273.15)\text{K}) - \frac{180\,\text{m/s} - 30\,\text{m/s}}{2}\right)$$

Now, rearrange the equation such that you solve for T_2.

$$T_2 = 184.25\,°\text{C}$$

Now, we can solve for the final Specific Volume.

$$\nu_2 = \frac{RT_2}{P_2}$$
$$= 1.312\,\text{m}^3/\text{kg}$$

Plugging that into the mass flow equation we have, to find the outlet cross-sectional area, A_2.

$$\Delta m_1 = \Delta m_2$$
$$0.5304\,\text{kg/s} = \frac{A_2 v_2}{\nu_2}$$
$$A_2 = 0.003\,86\,\text{m}^2$$
$$= 38.6\,\text{cm}^2$$

Validate:
The area decreased, so it definitely is a nozzle. In particular, all the other factors help us confirm that. The pressure decreased, the enthalpy decreased, the velocity increased, and the cross-sectional area decreased. Thus, this is definitely a nozzle.

In	Out
Pressure High	Pressure Low
Velocity Low	Velocity High
Start Temperature	Temperature Decreased
Start Enthalpy	Enthalpy Decreased

Generalize:
In a nozzle, velocity and pressure change occurs at the expense of the Enthalpy.

Example 6.4: Steam Nozzle. Textbook Problem 6.35

A nozzle is handling steam that enters at $P_1 = 5\,\text{MPa}$ and $T_1 = 400\,°\text{C}$ steadily at $v_1 = 80\,\text{m/s}$. The steam leaves at $P_2 = 2\,\text{MPa}$ and $T_2 = 300\,°\text{C}$. The inlet of the nozzle is $A_1 = 50\,\text{cm}^2$. Heat is being lost at a rate of $Q_{Out} = 120\,\text{kJ/s}$. Find the mass flowrate of the steam $\dot{m}$, the exit velocity of the steam v_2, and the cross-sectional area of the nozzle's outlet A_2?

- -

Concepts:
The **nozzle** has steady flow, with *steam.*
A nozzle has no storage capacity, so $\dot{m}_{In} = \dot{m}_{Out}$.
Steam is *not* an ideal gas, so we have to use the Tables in the back of the textbook.
Explore:
Tables A.4, A.5, and A.6 will be of use. Initially, the steam is superheated, so we start by using Table A.6.

Expanding the definition of mass flowrate, we get the relation:

$$\dot{m} = \rho A v$$

It is also useful to remember

$$\rho = \frac{1}{\nu}$$

We assume that $\dot{Q}_{In} = 0$, because we are not told otherwise.
A nozzle is a rigid body, so it does not undergo any pressure changes. Thus, $\dot{W}_{In} = \dot{W}_{Out}$, meaning $\Delta \dot{W} = 0$.
Again, $\Delta PE = 0$, because there is no change in height.
Plan:
Find the Specific Volume for each state (ν_1 and ν_2), and their respective Enthalpy (h_1 and h_2).
Use these values, along with the ones given in the problem statement to solve Equation (6.3).
Solve:
From Table A.6 at $P_1 = 5\,\text{MPa}$ and $T_1 = 400\,\text{°C}$:

$$\nu_1 = 0.057\,84\,\text{m}^3/\text{kg}$$
$$h_1 = 3196.7\,\text{kJ/kg}$$

Likewise, for the output, at $P_2 = 2\,\text{MPa}$ and $T_2 = 300\,\text{°C}$:

$$\nu_2 = 0.125\,51\,\text{m}^3/\text{kg}$$
$$h_2 = 3024.2\,\text{kJ/kg}$$

First, we find the mass flowrate through the nozzle, $\dot{m}$.

$$\dot{m} = \frac{v_1 A_1}{\nu_1}$$
$$= 6.92\,\text{kg/s}$$

Now, solve Equation (6.3), to find the final velocity v_2:

$$\dot{Q}_{In} - \dot{Q}_{Out} + \Delta \dot{W} = \dot{m}\left(\Delta h + \frac{\Delta \text{KE}}{m}\right)$$

The Flow Energy is handled by the use of Specific Enthalpy in this problem.

$$0 - \dot{Q}_{Out} + 0 = \dot{m}\left(h_2 - h_1 + \frac{v_2^2 - v_1^2}{2}\right)$$
$$-\dot{Q}_{Out} = \dot{m}\left(h_2 - h_1 + \frac{v_2^2 - v_1^2}{2}\right)$$
$$\dot{m}(h_1 - h_2) - \dot{Q}_{Out} = \dot{m}\left(\frac{v_2^2}{2} - \frac{v_1^2}{2}\right)$$
$$\dot{m}\left(h_1 - h_2 + \frac{v_1^2}{2}\right) - \dot{Q}_{Out} = \dot{m}\frac{v_2^2}{2}$$
$$v_2^2 = 562\,\text{m/s}$$

Lastly, we find the outlet's cross-sectional area, A_2.

$$\dot{m} = \frac{v_2 A_2}{\nu_2}$$
$$A_2 = \dot{m}\left(\frac{\nu_2}{v_2}\right)$$
$$= 0.001\,54\,\text{m}^2$$

Validate:

In	Out
Pressure High	Pressure Low
Velocity Low	Velocity High
Start Temperature	Temperature Decreased
Start Enthalpy	Enthalpy Decreased

Generalize:

The reduction in Enthalpy allowed the steam to gain its velocity, increasing the steam's kinetic energy. The temperature also decreased, because the enthalpy decreased.

From the energy balance:

$$\dot{Q}_{Out} = \Delta h + \Delta \text{KE}$$
$$-120 \, \text{kJ/s} = -1194 \, \text{kJ/s} + 1073 \, \text{kJ/s}$$

6.2.2 Turbines and Compressors

Defn 60 (Turbine). A *turbine* takes some of the Energy in the fluid and transfers it to a shaft. The mass flowrate is constant. This means that there is a flowrate of Work out of the System.

A Turbine is the opposite of a Compressor.

Defn 61 (Compressor). A *compressor* increases the Energy of a fluid by performing Work with the shaft. The mass flowrate is constant. This means that there is a flowrate of Work into the System.

Turbines	Compressors
$\dot{W}_{In} = 0$	$\dot{W}_{In} \neq 0$
$\dot{W}_{Out} \neq 0$	$\dot{W}_{Out} = 0$
$\dot{Q}_{In} \approx 0$	$\dot{Q}_{In} \neq 0$
$\dot{Q}_{Out} \approx 0$	$\dot{Q}_{Out} \neq 0$
$\Delta \text{PE} = 0$	$\Delta \text{PE} = 0$
$\Delta \text{KE} \neq 0$	$\Delta \text{KE} \approx 0$
$\Delta H \neq 0$	$\Delta H \neq 0$

Table 6.2: Properties of Turbines and Compressors

Example 6.5: Adiabatic Turbines. Textbook Problem 6.52

Steam flows steadily through an Adiabatic Turbine. The inlet conditions of the steam are $P_{In} = 10 \, \text{MPa}$, $T_{In} = 450 \, °\text{C}$, and $v_{In} = 80 \, \text{m/s}$. The outlet conditions are $P_{Out} = 10 \, \text{kPa}$, $x = 0.92$, and $v_{Out} = 50 \, \text{m/s}$. The mass flowrate of the steam is $\dot{m} = 12 \, \text{kg/s}$. Find the change in kinetic energy ΔKE, the power output $\dot{W}_{Out}$, and the inlet cross-sectional area A_{In}?

- -

Concepts:

This is a Turbine with a steady flow. So, we use the properties in Table 6.2.

$\dot{W}_{Out}$ is at the shaft. This is gathered at the expense of the mass flow's energy, both kinetic and enthalpy. Thus, $\Delta \text{KE} \neq 0$ and $\Delta h \neq 0$.

The fluid at the outlet is a Saturated Mixture.

$\dot{m}_{In} = \dot{m}_{Out}$.

$\dot{E}_{In} = \dot{E}_{Out}$.

Explore:

For kinetic energy, we need $\dot{m}$ and the velocities at the inlet and outlet.

To solve for the output power, we need the Specific Enthalpy for the steam at the states at the inlet and the outlet. To find these, we can use Tables A.4, A.5, and A.6 from the textbook.

Due to the Law of Conservation of Energy, and the steady mass flow, we get Equation (6.3).

$$\Delta \dot{Q} + \Delta \dot{W} = \sum_{\text{Inlet}} \dot{m} e - \sum_{\text{Outlet}} \dot{m} e$$

$$(\dot{Q}_{In} - \dot{Q}_{Out}) + (\dot{W}_{In} - \dot{W}_{Out}) = \sum_{\text{Inlet}} \dot{m} e - \sum_{\text{Outlet}} \dot{m} e$$

$$(0 - 0) + (0 - \dot{W}_{Out}) = \sum_{\text{Inlet}} \dot{m} e - \sum_{\text{Outlet}} \dot{m} e$$

Lastly, at the outlet, we know that the steam is a Saturated Mixture, so we need to use the Quality to determine the Specific Enthalpy of the steam.

$$h = h_f + x(h_g - h_f)$$

Plan:
Solve for the change in kinetic energy ΔKE.
Solve the energy balance equation we have to find the power output, $\dot{W}_{Out}$.
Using the mass flowrate and the inlet velocity, find the inlet's cross-sectional area, A_{In}. This can be found using the equations in Remark 59.1.
Solve:
Find the inlet's information from Table A.6 in the textbook. At $P_{In} = 10 \, \text{MPa}$, $T_{In} = 450 \, ^\circ\text{C}$

$$\nu_{In} = 0.029\,78 \, \text{m}^3/\text{kg}$$
$$h_{In} = 3242 \, \text{kJ/kg}$$

Find the outlet's information from Table A.5 in the textbook. At $P_{Out} = 10 \, \text{kPa}$, $x = 0.92$

$$T_{\text{sat}} = 45.81 \, ^\circ\text{C}$$
$$h_f = 191.81 \, \text{kJ/kg}$$
$$h_g = 2583.9 \, \text{kJ/kg}$$
$$h = h_f + x(h_g - h_f)$$
$$= 2392 \, \text{kJ/kg}$$

Now that we have some background information, we will solve for the change in kinetic energy.

$$\Delta \text{KE} = \frac{\text{KE}_{Out} - \text{KE}_{In}}{m}$$
$$= \frac{v_{Out}^2 - v_{In}^2}{2}$$
$$= \frac{50^2 - 80^2}{2} \left(\frac{1 \, \text{kJ/kg}}{1000 \, \text{m}^2/\text{s}^2} \right)$$
$$= -1.95 \, \text{kJ/kg}$$

Now, we solve our energy balance equation for the power output.

$$-\dot{W}_{Out} = \dot{m}(\Delta h + \Delta \text{KE})$$
$$\dot{W}_{Out} = -\dot{m}\big((h_{Out} - h_{In}) + \Delta \text{KE}\big)$$
$$= \dot{m}\big((h_{In} - h_{Out}) - \Delta \text{KE}\big)$$
$$= 12 \, \text{kg/s}\big((3242 - 2392)\text{kJ/kg} - (-1.95 \, \text{kJ/kg})\big)$$
$$= 10.2 \, \text{MJ/s}$$
$$= 10.2 \, \text{MW}$$

Lastly, find the inlet's cross-sectional area:

$$\dot{m}\nu_{In} = A_{In}v_{In}$$
$$A_{In} = 44.67 \, \text{cm}^2$$

Validate:

We need to double check that we actually *have* an energy balance.

$$\dot{W}_{Out} = 10.22\,\text{MW}$$
$$= 10.2\,\text{MW} + 0.027\,\text{MW}$$

The values for properties of the states were from the textbook's tables.
For Turbine operation, Δh, ΔKE, ΔP, and ΔT.

Generalize:

Most of the work comes from Δh, very little comes from ΔKE.
If this was not Adiabatic, then ΔQ would be very small, like ΔKE.
The mass flowrate as a large effect on the power generated.

Example 6.6: Compressor. Textbook Problem 6.58

Helium is to be compressed from $P_{In} = 120\,\text{kPa}$ and $T_{In} = 310\,\text{K}$ to $P_{Out} = 700\,\text{kPa}$ and $T_{Out} = 430\,\text{K}$. A heat loss of $\dot{Q}_{Out} = 20\,\text{kJ/kg}$ occurs during the compression process. Neglecting kinetic energy changes, determine the power required to complete this $\dot{W}_{In}$ for $\dot{m} = 90\,\text{kg/min}$?

- -

Concepts:

Helium is an ideal gas, we can use Equation (4.10).
A compressor will increase the energy of a fluid by increasing its pressure and enthalpy.
We are told that the helium is flowing steadily because $\dot{m}$ is provided for us.
We can find the change in enthalpy by using the Specific Heat under constant pressure. We are told enough to say that our energy balance equation in this problem is Equation (6.3).

$$\Delta\dot{Q} + \Delta\dot{W} = \dot{m}(\Delta h + \Delta\text{KE} + \Delta\text{PE})$$

Explore:

We are told to neglect kinetic energy changes, so $\Delta\text{KE} = 0$.
We also know that there is no change in potential energy, because the height doesn't change, $\Delta\text{PE} = 0$.
We can find the change in enthalpy by using the Specific Heat under constant pressure, $\Delta h = c_P\Delta T$.
We are not told anything about any heat in or work out, so assume those are zero, $\dot{Q}_{In} = 0$, $\dot{W}_{Out} = 0$. Using all of this,

$$-\dot{Q}_{Out} + \dot{W}_{In} = \dot{m}(c_P\Delta T)$$

Plan:

Find the Specific Heat under constant pressure from Table A.3.
Solve for $\dot{W}_{In}$.

Solve:

From the textbook,

$$c_P = 5.1426\,\text{kJ K/kg}$$

$$-20\,\text{kJ/kg} + \dot{W}_{In} = 90\,\text{kg/min}\left(\frac{60\,\text{s}}{1\,\text{min}}\right)(5.1426(450-310))$$

$$\dot{W}_{In} = 965\,\text{kJ/s}$$
$$= 965\,\text{kW}$$

6.2.3 Throttling Valves

Defn 62 (Throttling Valve)**.** *Throttling valves* are flow restriction devices that produce large pressure and temperature drops. This is achieved with either an actual valve or a highly porous material.

Defn 63 (Isenthalpic)**.** A Process that is *isenthalpic*, just like a process that is Isothermal, is one where the Enthalpy (or Specific Enthalpy) is constant.

Because the Specific Enthalpy remains constant for a Throttling Valve, we can expand the definition of Specific Enthalpy to solve.

$$(u + PV)_{In} = (u + PV)_{Out} \tag{6.4}$$

$$\begin{array}{c} \hline \text{Throttling Valve} \\ \hline \Delta \dot{Q} = 0 \\ \Delta \dot{W} = 0 \\ \Delta \text{KE} = 0 \\ \Delta \text{PE} = 0 \\ h_{In} = h_{Out} \\ \hline \end{array}$$

Table 6.3: Properties of Throttling Valves

Example 6.7: Throttling Valve. Textbook Problem 6.68

An Adiabatic capilliary tube is used in a refrigeration system to drop the pressure from the condenser level to the evaporator level. R-134a enters the capilliary tube as a Saturated Liquid at $T_{In} = 50\,°\text{C}$ and leaves at $T_{Out} = -12\,°\text{C}$. Determine the Quality of the refrigerant at the inlet of the evaporator?

Concepts:
Throttling Valves are Isenthalpic, their Enthalpy remains constant. We also know Equation (6.4).

$$(u + PV)_{In} = (u + PV)_{Out}$$

We assume that this operates under steady flow conditions, so $\dot{m}_{In} = \dot{m}_{Out}$.
We are told this is Adiabatic, so $\Delta Q = 0$.
There are no moving parts, so $\Delta W = 0$.
Explore:
We can find the Specific Enthalpy for each state from the tables in the back of the textbook.
It is implied that the refrigerant at the outlet is a Saturated Mixture, which has a quality.

$$h = h_f + x(h_g - h_f)$$

Plan:
Find Specific Enthalpy values from Table A.11 from the textbook.
Solve for h_{Out}.
Then solve for Quality, x.
Solve:
From Table A.11 at $T_{In} = 50\,°\text{C}$, we know that this fluid is a Saturated Liquid, so we can use h_f directly. However, we must Interpolate the results.

$$h_{In} = h_f = 123.45\,\text{kJ/kg}$$

From Table A.11 at $T_{In} = -12\,°\text{C}$.

$$h_f = 35.92\,\text{kJ/kg}$$
$$h_{fg} = 207.38\,\text{kJ/kg}$$

Now, we plug these values into our Isenthalpic equation.

$$h_{In} = h_{Out}$$
$$h_{Out} = h_f + x(h_{fg})$$
$$h_{In} = h_f + x(h_{fg})$$
$$123.45 = 35.92 + x(207.38)$$
$$x = 0.422$$

Validate:
We know at the inlet that

$$h = u + PV$$
$$123 = 122 + PV$$
$$PV \approx 1$$

Then, at the outlet, we see

$$h = u + PV$$

Remember that a Throttling Valve is Isenthalpic.

$$123 = 36 + PV$$
$$PV \approx 87$$

Because the Internal Energy decreased, the temperature of the fluid decreased as well.
Generalize:
A Throttling Valve is similar to a nozzle or a diffuser, because they re-apportion the Energy in the mass flow.

6.2.4 Mixing Chambers

Defn 64 (Mixing Chamber). *Mixing chamber*s are used whenever two different fluid streams together that have the same ρ. Each of the different streams comes in through a different inlet. There is a steady flow in from each of the inlets, so

$$\dot{m}_{Out} = \dot{m}_{In,1} + \dot{m}_{In,2}$$

In general, we assume that the fluids entering the chamber are the same, preventing possible chemical reactions. For example, we do not mix water and alcohol.

Mixing Chamber
$\Delta Q = 0$
$\Delta W = 0$
$\Delta \mathrm{KE} = 0$
$\Delta \mathrm{PE} = 0$
$\dot{E}_{In} = \dot{E}_{Out}$

Table 6.4: Mixing Chamber Properties

Due to these properties, the general energy balance equation for Mixing Chambers is shown in Equation (6.5).

$$\sum_{Inlets} h\dot{m} = \sum_{Outlets} h\dot{m} \tag{6.5}$$

Example 6.8: Mixing Chamber. Textbook Problem 6.78

A hot water stream of $T_{In,1} = 80\,°\mathrm{C}$ enters a Mixing Chamber with a mass flowrate of $\dot{m}_{In,1} = 0.5\,\mathrm{kg/s}$. It is mixed with a stream of cold water at $T_{In,2} = 20\,°\mathrm{C}$. If it is desired that the water leave the chamber at $T_{Out} = 42\,°\mathrm{C}$, determine the mass flowrate of the cold water stream, $\dot{m}_{In,2}$? Assume all streams are at a pressure of $P = 250\,\mathrm{kPa}$.

- -

Concepts:
This is a Mixing Chamber with steady flow.
Explore:
Our initial concepts about a Mixing Chamber yield the equations:

$$\dot{m}_{In,1} + \dot{m}_{In,2} = \dot{m}_{Out}$$
$$(h\dot{m})_{In,1} + (h\dot{m})_{In,2} = (h\dot{m})_{Out}$$

Use the tables in the back of the textbook to find the various h, by the given P and T for water. This water is under a lot of pressure, and is in a liquid state, so we should start by going to Table A.4 and A.5. The thing that affects the Specific Enthalpy more is the temperature of the water, so we use Table A.4 rather than A.5.
There may need to be some extrapolation to get the Specific Enthalpy.
Plan:
Use the tables in the textbook to find the Specific Enthalpy for the various streams.
Solve for the energy balance equation and the mass flowrate equation simultaneously.
Solve:

From Table A.4 at $T_{In,1} = 80\,°\text{C}$:
$$h_{In,1} = 335\,\text{kJ/kg}$$

From Table A.4 at $T_{In,2} = 20\,°\text{C}$:
$$h_{In,2} = 84\,\text{kJ/kg}$$

From Table A.4 at $T_{Out} = 42\,°\text{C}$, which requires us to Interpolate:

$$h = 175.9\,\text{kJ/kg}$$

Now, solve

$$\dot{m}_{In,1}h_{In,1} + \dot{m}_{In,2}h_{In,2}+ = \dot{m}_{Out}h_{Out}$$

According to our relationship between our mass flowrates, the inlets are equal to the outlet.

$$= h_{Out}(\dot{m}_{In,1} + \dot{m}_{In,2})$$

Rearranging to get the second inlet's mass flowrate by itself, we can solve.

$$\dot{m}_{In,2} = \frac{h_{In,1} - h_{Out}}{h_{Out} - h_{In,2}}\dot{m}_{In,1}$$
$$= 0.865\,\text{kg/s}$$

Validate:
Now that we know the mass flowrate out of the chamber, we can solve the energy balance equation directly. $\dot{m}_{Out} = 1.365\,\text{kg/s}$.

$$h_{In,1}\dot{m}_{In,1} + h_{In,2}\dot{m}_{In,2} = h_{Out}\dot{m}_{Out}$$
$$167 + 73 = 240$$

Generalize:
For a mass flow for a liquid under constant pressure, the Specific Enthalpy (h) dominates the outcome.

6.2.5 Heat Exchangers

Defn 65 (Heat Exchanger). *Heat exchanger*s allow two or more flows of fluid which are separated so that the flows do **not** mix, but exchange Heat.

Heat Exchanger
$\Delta\dot{Q} = \dot{m}\Delta h \neq 0$
$\Delta W = 0$
$\Delta\text{KE} = 0$
$\Delta\text{PE} = 0$
$\Delta H \neq 0$

Table 6.5: Properties of Heat Exchangers

To solve these kinds of problems, we treat them as if they were two separate problems. We have two control volumes that we interact with, CONTROL VOLUME 1 and CONTROL VOLUME 2. Each control volume exists under steady flow (state) conditions, so $\dot{m}_{In} = \dot{m}_{Out}$ for each fluid flow. They are related by an energy balance.

$$(\dot{m}_{In}h_{In})_1 + (\dot{m}_{In}h_{In})_2 = (\dot{m}_{Out}h_{Out})_1 + (\dot{m}_{Out}h_{Out})_2 \tag{6.6}$$

Example 6.9: Heat Exchanger. Textbook Problem 6.82

R-134a at $P_1 = 700\,\text{kPa}$ and $T_{1,In} = 70\,°\text{C}$ and moving at $\dot{m}_1 = 8\,\text{kg/min}$ is cooled by water in a condenser until it exists as a Saturated Liquid at the same pressure. The cooling water enters the condenser at $P_2 = 300\,\text{kPa}$ and

$T_{2,In} = 15\,°\text{C}$ and leaves at $T_{2,Out} = 25\,°\text{C}$. Determine the mass flowrate $\dot{m}_2$ of the water required to cool the refrigerant?

- -

Concepts:

Heat Exchanger that is under steady state conditions. Water is cooling the R-134a, so the water is warmed.

Treat this as a mixing chamber problem combined with a nozzle problem.

Explore:

By exploring the R-134a pipe in more detail, we see:

$$\Delta\dot{W} = 0$$
$$\dot{Q}_{In} = 0$$
$$\dot{Q}_{Out} \neq 0$$
$$\Delta\text{KE} = 0$$
$$\Delta\text{PE} = 0$$

Using the tables for R-134a, we can find the Specific Enthalpy at the inlet and outlet (1 and 2, respectively). This yields an energy balance equation of:

$$-\dot{Q}_{1,Out} = \dot{m}_1(h_{1,Out} - h_{1,In})$$

For the water block, we assume that there is no heat entering or leaving the heat exchanger, so $\Delta\dot{Q}_2 = 0$. Now, we apply Equation (6.6):

$$\dot{E}_{In} = \dot{E}_{Out}$$
$$(\dot{m}h)_{1,In} + (\dot{m}h)_{2,In} = (\dot{m}h)_{1,Out} + (\dot{m}h)_{2,Out}$$
$$(\dot{m}h)_{\text{R-134a},In} + (\dot{m}h)_{\text{Water},In} = (\dot{m}h)_{\text{R-134a},Out} + (\dot{m}h)_{\text{Water},Out}$$

Plan:

Use the tables in the back of the textbook to find the various values of h.

Solve:

First, we know that we will eventually have to get the mass flowrate of the R-134a into seconds, so

$$\dot{m}_1 = 0.133\,\text{kg/s}$$

Next up, the various Specific Enthalpy values from the tables. From Table A.12 at $P_1 = 700\,\text{kPa}$ we find that $T_{1,Sat} = 27\,°\text{C}$. This means that the R-134a is a Superheated Vapor right now. We move to Table A.13.

$$h_{1,In} = 308\,\text{kJ/kg}$$

For the R-134a outlet's Specific Enthalpy, we are told the refrigerant is a Saturated Liquid, so $h_{1,Out} = h_f$. From Table A.12 at $P = 700\,\text{kPa}$, we know

$$T_{1,Out} = T_{1,Sat}$$
$$= 27\,°\text{C}$$
$$h_{1,Out} = 89\,\text{kJ/kg}$$

From Table A.4 at $P = 300\,\text{kPa}$, we see that $T_{2,In} < T_{Sat}$ and $T_{2,Out} < T_{Sat}$, so the water remains a liquid. For $T_{2,In} = 15\,°\text{C}$,

$$h_{2,In} = 63\,\text{kJ/kg}$$

For $T_{2,Out} = 25\,°\text{C}$,

$$h_{2,Out} = 105\,\text{kJ/kg}$$

Now, we can solve for the R-134a's energy balance equation.

$$-\dot{Q}_{1,Out} = \dot{m}_1(h_{1,Out} - h_{1,In})$$
$$= 0.133\,\text{kg/s}(89 - 365)\text{kJ/kg}$$
$$\dot{Q}_{1,Out} = 29\,\text{kJ/s}$$
$$= 29\,\text{kW}$$

If we rearrange the terms in the energy balance equation, for the larger system, we can get:

$$\dot{m}_2 = \dot{m}_1 \left(\frac{h_{1,In} - h_{1,Out}}{h_{2,In} - h_{2,Out}} \right)$$

$$= 8\frac{89 - 308}{67 - 105}$$

$$= 42 \, \text{kg/min}$$

Validate:

Check that we have energy balance. Namely, directly solve

$$(\dot{m}h)_{\text{R-134a},In} + (\dot{m}h)_{\text{Water},In} = (\dot{m}h)_{\text{R-134a},Out} + (\dot{m}h)_{\text{Water},Out}$$

6.2.6 Pipe/Duct Flow

This is a subset of problems from the Heat Exchanger.

Pipe/Duct Flow
$\Delta\dot{Q} = \dot{m}\Delta h \neq 0$
$\Delta\dot{W} = 0$
$\Delta\text{KE} = 0$
$\Delta\text{PE} = 0$
$\Delta H \neq 0$

Table 6.6: Pipe/Duct Flow Properteis

Something to note is that in Table 6.6, we say $\Delta\dot{W} = 0$. This only holds true for the purest case of this. If a propeller were inserted to the piping, then the work done *would* change.

Likewise, if the gas that is moving through the pipe is an ideal gas, we can use an ideal gas equation to determine $\Delta h = c_P \Delta T$.

Example 6.10: Pipe Flow. Textbook Problem 6.98

Saturated Liquid water is heated in a steady-flow steam boiler at a constant pressure of $P = 5\,\text{MPa}$ at a rate of $\dot{m} = 10\,\text{kg/s}$ to an outlet temperature of $T_{Out} = 350\,°\text{C}$. Determine the rate of Heat Transfer in the boiler?

- -

Concepts:

Same set of concepts of a Heat Exchanger. Namely, this is a sub-class of problems.

Explore:

The only change in heat comes from the surrounding boiler, with $\dot{Q}_{In}$.

$$\dot{Q}_{In} = \dot{m}\Delta h$$

To find the Specific Enthalpy, we can use Table A.4 or A.7 (if Superheated Vapor).

Plan:

Use Table A.4 and/or Table A.7 from the back of the textbook to find the Specific Enthalpy, h. Solve the heat energy-balance equation.

Solve:

From Table A.5 at $P = 5\,\text{MPa}$, we get:

$$T_{Sat} = 263\,°\text{C}$$

$$SpecificEnthalpy_f = 1154\,\text{kJ/kg}$$

From the previous table, we notice that the final temperature, is greater than the saturation temperature, $T_{Out} > T_{Sat}$. This means the fluid becomes a Superheated Vapor.

$$h_{Out} = 3069\,\text{kJ/kg}$$

Now, solving our energy balance equation:

$$\dot{Q}_{In} = \dot{m}\Delta h$$
$$= \dot{m}(h_{Out} - h_{In})$$
$$= 10(3069 - 1154)$$
$$= 19\,150\,\text{kJ/kg}$$

7 The Second Law of Thermodynamics

Defn 66 (2nd Law of Thermodynamics)**.** The *2nd law of thermodynamics* states that the total Entropy of an Isolated System can **never** decrease over time, and is constant if and only if all Processes are reversible. Isolated systems spontaneously evolve towards thermodynamic equilibrium, the state with maximum entropy.

In terms of total Energy in a system, this law states that the total energy of the system is minimized at all times during a Process.

Remark 66.1. We do not discuss Entropy in this section. Rather, we discuss it in great detail in Section 8.

In more common terms, the 2nd Law of Thermodynamics explains why some Processes only move in one direction. For example, water going over a waterfall is *technically* symmetric. Meaning, that there is a chance the water will actually go back **up** the cliff. However, the 2nd Law of Thermodynamics explains that this reversal is so thermodynamically unfavorable that it will never happen.

Another example is a container of water that is greater than that of its surroundings. For instance, the water is $T = 90°\text{F}$ and the surroundings are at $T = 20°\text{F}$. The 2nd Law of Thermodynamics explains why the heat always flows **out** of the water and to the surroundings, rather than the other way around.

This means that **energy is dispersed**. This spread (dispersion) of energy is Entropy.

> Energy naturally flows from being concentrated in one place to another such that the dispersal of energy is maximized. This limits the Work that can be performed.

Defn 67 (Kelvin-Planck 2nd Law of Thermodynamics)**.** There are a variety of definitions of the 2nd law of thermodynamics. The one in Definition 66 is just one of them. The *Kelvin-Planck 2nd law of thermodynamics* states that it is impossible for any device that operates on a Cycle to receive heat from a single Thermal Reservoir and to use to produce Work.

Remark 67.1 (Thermal Reservoir). A thermal source with a constant temperature. For example, the Earth, a large boiler, the air, a lake, or river are considered thermal reservoirs. More concretely, this means that if Heat is removed, there is no temperature change.

Defn 68 (Cycle)**.** A *cycle* is a Process whose ending state is the same as its starting state, allowing the process to continue again.

Remark 68.1 (Cyclic). A *cyclic* Process is one that behaves as a Cycle.

7.1 Thermodynamic Cycles

Cycles absorb work, limiting the amount of energy that can be extracted. You cannot perfectly convert Heat into Work. This is because the cycle itself requires some amount of work to even occur.

Example 7.1: Thermodynamic Cycles and Efficiency. Textbook Problem 7.22

Given a steam power plant, with a boiler, working shaft, condenser, and a source of water outputs $P = 150\,\text{MW}$ consumes coal at $\dot{m} = 60\,\text{t/h}$. If the overall power available from coal is $P_{\text{Coal}} = 30\,000\,\text{kJ/kg}$, determine the overall efficiency of this plant?

- -

The mass flowrate can be converted to kilograms, $\dot{m} = 60\,000\,\text{kg/h}$.
Concepts:
The efficiency can be defined as $\eta = \frac{\dot{W}_{\text{Net Out}}}{\dot{Q}_{In}}$.
Explore:
There was no input work provided, so $\dot{W}_{In} = 0$.

Solve:
So, we can find the heat in.

$$\dot{Q}_{In} = \dot{m}P_{\text{Coal}}$$
$$= 1.8 \times 10^9 \text{kJ/h}$$
$$= 500\,\text{MW}$$

Now, to solve for the efficiency, η.

$$\eta = \frac{\dot{W}_{\text{Net Out}}}{\dot{Q}_{In}}$$
$$= \frac{150\,\text{MW}}{500\,\text{MW}}$$
$$= 0.3$$

Validate:
Because $\eta = 1 - \frac{Q_{Out}}{Q_{In}}$, we can say $\dot{W}_{Net} = \dot{Q}_{In} - \dot{Q}_{Out}$. Plugging our known values in for $\dot{Q}_{Out}$, we fine that $\dot{Q}_{Out} = 350\,\text{MW}$. This is the same as the energy lost by the system, which means we are correct.
Generalize:
You must waste Heat in order to have a Cycle.

7.2 Heat Engines

One common system we will see is the Heat Engine.
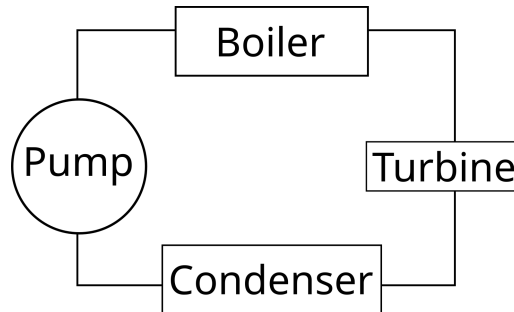
Figure 7.1: Heat Engine

The boundary of the system is such that the entire system is contained. This means that each element of the system is a different part of the thermodynamic energy balance.

- The boiler takes the Heat in. Because this is typically the hot side of the System, we call this $\dot{Q}_H$.
- The turbine is for the Work out.
- The condenser is for the Heat out. Because this is typically the cold side of the System, we call this $\dot{Q}_L$.
- The pump is for the Work in.

Thus, we can define the efficiency as shown in Equation (7.1).

$$\eta = \frac{\dot{W}_{\text{Net Out}}}{\dot{Q}_H}$$
$$= 1 - \frac{\dot{Q}_C}{\dot{Q}_H} \tag{7.1}$$

If we examine Equation (7.1), we see that:

- $\dot{Q}_H$ sits on the boiler, which is at a higher temperature.
- $\dot{Q}_C$ sits on the condenser, which is at a lower temperature.

The greater the temperature difference between the system and its surroundings, the greater the efficiency of the System.

If there is no Work out, but work in is present, then Energy is being forced out of the lower temperature reservoir ($\dot{Q}_C$) and put into the higher temperature reservoir.

Equation (7.1) is actually completely general. It works for car engines, steam-based electricity generating power plants, Refrigerators, Heat Pumps, etc.

7.2.1 The Second Law and Heat Engines

The 2nd Law of Thermodynamics states that $\dot{Q}_C = 0$ is impossible, for Cycles. If we think in terms of Thermal Reservoirs, the second law states we cannot turn all the Heat provided by one reservoir into usable work.

However, $\dot{Q}_C = 0$ **if the Process is NOT Cyclic**.

Defn 69 (Coefficient of Performance). The *coefficient of performance* is a scalar value, like efficiency. It shows the relationship between a certain amount of Energy out given a certain amount of energy in. Or vice versa, it shows the potential output of the system given a certain input.

$$\text{CoP} = \frac{\text{Desired Output}}{\text{Required Input}} \tag{7.2}$$

Now, applying Coefficient of Performance to a heat engine, we get Equation (7.3).

$$\begin{aligned}\text{CoP} &= \frac{\dot{Q}_C}{\dot{W}_{Out}} \\ &= \frac{\dot{Q}_C}{\dot{Q}_H - \dot{Q}_C} \\ &= \frac{1}{\frac{\dot{Q}_C}{\dot{Q}_H} - 1}\end{aligned} \tag{7.3}$$

7.3 Heat Pumps

Defn 70 (Heat Pump). *Heat pump*s move Heat from a colder Thermal Reservoir to a warmer one. A *heat pump* is just another term for a Refrigerator.

Because Heat Pumps function identically to Refrigerators and air conditioners, these equations are common between them.

$$\eta = \frac{\dot{Q}_H}{\dot{W}_{In}} \tag{7.4}$$

$$\text{CoP} = \frac{\dot{Q}_H}{\dot{W}_{In}} \tag{7.5}$$

Example 7.2: Heat Pumps. Textbook Problem 7.41

A commercial Heat Pump removes $\dot{Q}_C = 10\,000\,\text{btu/h}$ from the source and rejects $\dot{Q}_H = 15\,090\,\text{btu/h}$ to the sink, and has a $\dot{W}_{In} = 2\,\text{hp}$ of power. What is the Coefficient of Performance?

- -

Concepts:
A Heat Pump is quite similar to a Refrigerator.
The energy flow for the problem looks identical to a refrigerator's.
A refrigerator has a Coefficient of Performance as seen in Equation (7.6).
Solve:
First, change the Work into a more useful set of units.

$$2\,\text{hp} \left(\frac{2544\,\text{btu/h}}{1\,\text{hp}} \right) = 5088\,\text{btu/h}$$

Now, we can solve Equation (7.6).

$$\text{CoP} = \frac{\dot{Q}_H}{\dot{W}_{In}}$$
$$= \frac{15\,090\,\text{btu/h}}{5088\,\text{btu/h}}$$
$$= 2.97$$

Example 7.3: Air Conditioner. Textbook Problem 7.48

When a man returns to his well-sealed house on a summer day, he finds the house to be at $T_1 = 32\,°\text{C}$. He turns on the air conditioner which cools the entire house to $T_2 = 20\,°\text{C}$ in $t = 15\,\text{min}$. If the Coefficient of Performance of the air conditioning unit is CoP = 2.5, determine the power drawn by the air conditioner? Assume the entire mass of the house is $m = 800\,\text{kg}$ of air for which $c_V = 0.72\,\text{kJ/(kg\,°C)}$ and $c_P = 1.0\,\text{kJ/(kg\,°C)}$.

- -

Concepts:
An air conditioner behaves the same as a Refrigerator.
Continuing with this analogy, the air inside the house behaves as the interior of the refrigerator and is to be cooled. The air outside the house is analogous to the air outside of the refrigerator (the condenser is outside the house).
Need to find $\dot{Q}_C$.

Explore:
We can find the Work in by negating the change in Heat of the air inside the house. We can do this because the house is well-sealed and that is the only thing that can change the Internal Energy of the house.

$$-\dot{W}_{In} = \frac{\dot{Q}_C}{\text{CoP}}$$

Because the house is well-sealed, if the air is cooled, the pressure will change. If the house were leaky, then air will flow somewhat freely across the system boundary. In this case, we will use c_P.
More thoroughly, we will use both c_V and c_P to bracket our solution.

$$\Delta Q = cm\Delta T$$

We can use this to find the Heat removed from the house Q_C and then determine its flowrate by dividing by the time taken, $\dot{Q}_C$.

Plan:
Solve for the change in energy of the house using the equation for changes in heat given the fluid in question is an ideal gas.

Solve:
Starting with c_V for a leaky house.

$$Q_{C,V} = c_V m\Delta T$$
$$= 6912\,\text{kJ}$$

Now, using c_P for a well-sealed house.

$$Q_{C,P} = c_P m\Delta T$$
$$= 9600\,\text{kJ}$$

Now, finding the flowrates.

$$\dot{Q}_{C,V} = 7.68\,\text{kJ/s} \qquad\qquad Q_{C,P} = 10.6\,\text{kJ/s}$$
$$\dot{Q}_{C,V} = 7.68\,\text{kW} \qquad\qquad Q_{C,P} = 10.6\,\text{kW}$$

Now, solving for the net Work in:

$$\dot{W}_{In,V} = 3.077\,\text{kW} \qquad\qquad \dot{W}_{In,P} = 4.264\,\text{kW}$$

Validate:

The sealed room doesn't take as much energy to cool, because we don't have to constantly cool newly warmed air from the outside, which makes sense.

We can find $\dot{Q}_C$ using Enthalpy, from Table A.21.

$$\dot{Q} = m\Delta h$$
$$= 9000 \, \text{kW}$$

Using the Enthalpy we can see that the best designed system is one that is between both the c_V and c_P.

7.4 Refrigerators

In the case of a refrigerator (which is just a Heat Engine run in reverse), we get Equation (7.6).

Defn 71 (Refrigerator). A *refrigerator* takes Work in to remove Heat from its interior and moves it to the exterior.

$$
\begin{aligned}
\text{CoP} &= \frac{\dot{Q}_L}{\dot{W}_{In}} \\
&= \frac{1}{1 - \frac{\dot{Q}_L}{\dot{Q}_H}}
\end{aligned}
\tag{7.6}
$$

Example 7.4: Refrigerators. Textbook Problem 7.40

A household Refrigerator with a Coefficient of Performance of CoP $= 1.2$ removes heat from the refrigerated space at a rate of $\dot{Q}_{Out} = 60 \, \text{kJ/min}$. Determine the electric power consumed by the refrigerator ($\dot{W}_{In}$) and the rate of Heat Transfer to the kitchen air ($\dot{Q}_H$)?

- -

Concepts:

A refrigerator has a Coefficient of Performance as seen in Equation (7.6).

We are told $\dot{Q}_C = 60 \, \text{kJ/min}$, which is $\dot{Q}_C = 1 \, \text{kJ/s}$.

The $\dot{Q}_C$ is the inside of the fridge and its air because of the evaporator. $\dot{Q}_H$ is the warm air inside the kitchen and the air released by the condenser.

Explore:

From our simplified depiction of a refrigerator, we can figure out an energy balance for this problem, $\dot{Q}_H = \dot{Q}_L + \dot{W}$.

Plan:

Solve Equation (7.6) to find $\dot{Q}_H$.

Solve the energy balance to find $\dot{W}_{In}$.

Solve:

Using Equation (7.6), solve for $\dot{W}_{In}$.

$$
\begin{aligned}
\dot{W}_{In} &= \frac{\dot{W}_C}{\text{CoP}} \\
&= \frac{1 \, \text{kJ/s}}{1.2} \\
&= 50 \, \text{kJ/min} \\
&= 0.83 \, \text{kW}
\end{aligned}
$$

Now, we can solve the energy balance.

$$
\begin{aligned}
\dot{Q}_H &= \dot{Q}_C + \dot{W}_{In} \\
&= 60 \, \text{kJ/min} + 50 \, \text{kJ/min} \\
&= 110 \, \text{kJ/min}
\end{aligned}
$$

Validate:

We cal validate this by double-checking against the general equation for Coefficient of Performance for a refrigerator,

Equation (7.6).

$$\text{CoP} = \frac{1}{\frac{\dot{Q}_C}{\dot{Q}_H} - 1}$$
$$= 1.2$$

Generalize:
For a refrigerator, the higher the Coefficient of Performance, the more Heat removed from the inside.

7.5 Reversible and Irreversible Processes

Defn 72 (Reversible Process)**.** A *reversible process* is a thermodynamic Process that can reversed without affecting the system surroundings. This means the system is always in equilibrium.

Defn 73 (Irreversible Process)**.** An *irreversible process* is a thermodynamic Process where the state of the System or surroundings has been altered in a non-reversible way. This means that whatever the process is, some part of it is lost and cannot be recovered.

Some things that make a Process irreversible are:

- Friction
- Heat Transfer

Irreversible Processes are what we encounter in the real world. Reversible Processes are idealized situations that we use to simplify System.

Since most everyday Processes are **not** Reversible Processes, then the ideal process is the Reversible Process and reflects the **mot** efficient process possible. This means that $\eta = 1 - \frac{Q_{Out}}{Q_{In}} < 1$ for all realistic Processes.

7.6 Carnot Cycle

The Process path of the Carnot Cycle is shown in Figure 7.2.

Defn 74 (Carnot Cycle)**.** The *Carnot cycle* is a thermodynamically ideal Cycle. The temperatures T_H and T_L are isotherms. The vertical cases are times when the System is perfectly Adiabatic, meaning only Work, W, is done. The interior of the cycle's process path is the net work done by the System.

If we look at the efficiency of the Carnot Cycle, we see something interesting.

$$\eta_C = 1 - \frac{Q_{Out}}{Q_{In}}$$
$$= 1 - \frac{Q_{Out} \propto T_L}{Q_{In} \propto T_H}$$
$$= 1 - \frac{T_L}{T_H}$$

$$\text{CoP}_{C,\text{Heat Pump}} = \frac{1}{1 - \frac{Q_L}{Q_H}}$$
$$\text{CoP}_{C,\text{HP}} = \frac{1}{1 - \frac{T_L}{T_H}} \tag{7.7}$$

Because the Carnot Cycle is idealized, we can also run it backwards, making it a refrigerator. This is seen in Figure 7.3. The Coefficient of Performance for the reversed Carnot Cycle is:

$$\text{CoP}_{C,R} = \frac{1}{\frac{Q_H}{Q_L} - 1}$$
$$= \frac{1}{\frac{T_H}{T_L} - 1} \tag{7.8}$$

Remark. For both Equation (7.7) and Equation (7.8), the temperatures **MUST** be in Kelvin or Rankine (K, R).
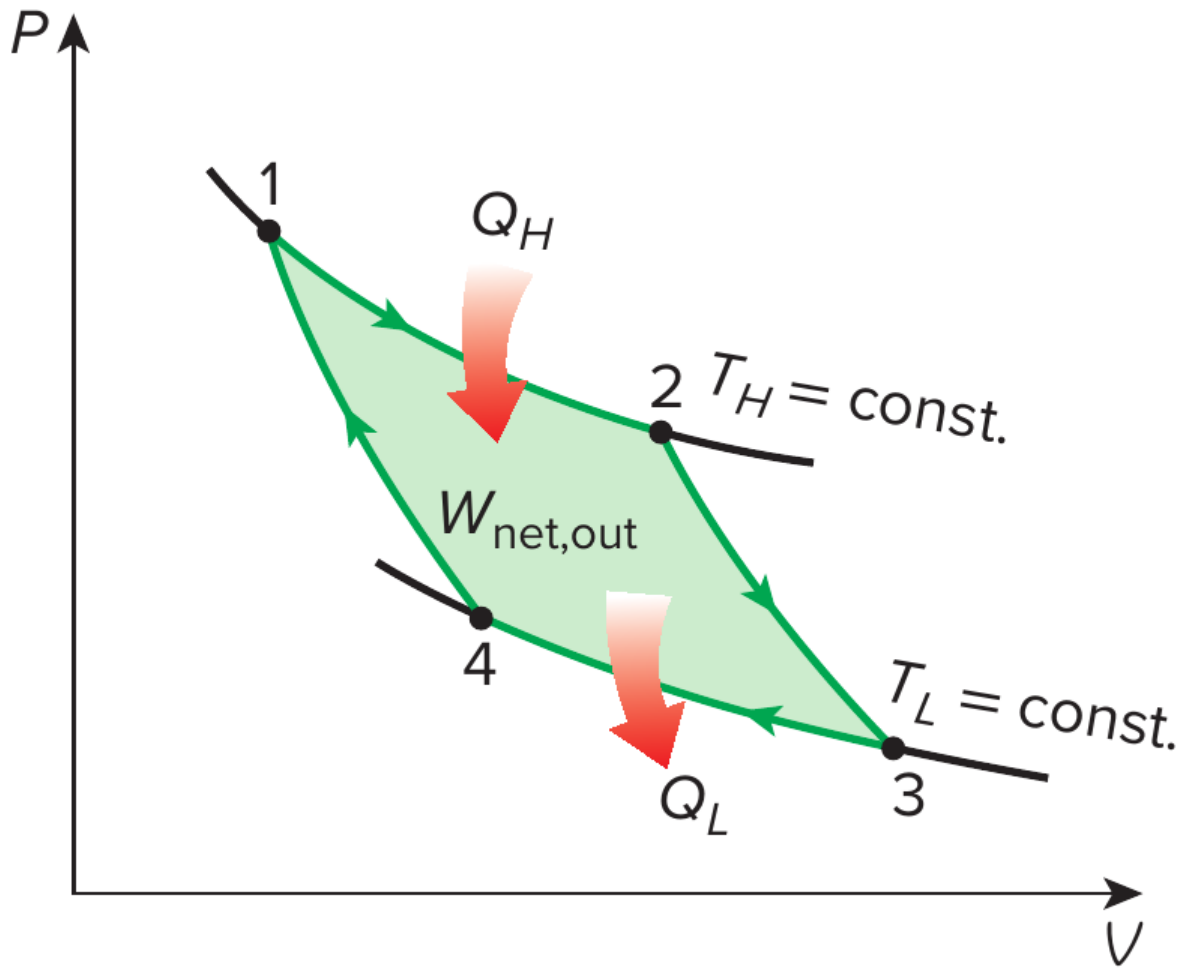
Figure 7.2: Carnot Cycle (Çengel, Cimbala, and Turner 2017, pg. 252)
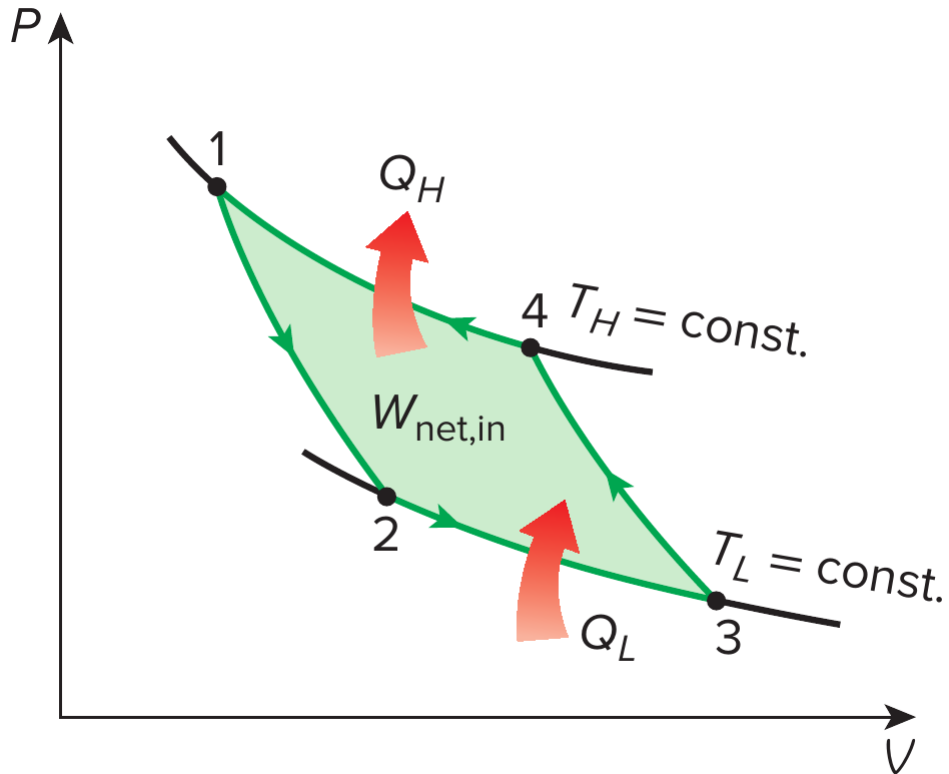
Figure 7.3: Carnot Cycle, in Reverse (Çengel, Cimbala, and Turner 2017, pg. 253)

Example 7.5: Carnot Cycles on Power Plants. Textbook Problem 7.88

A geothermal power plant extracts geothermal water at $T_H = 160\,°C$ at a rate of $\dot{m} = 440\,kg/s$ as the heat source and produces $\dot{W} = 22\,MW$ of power. If the environment temperature is $T_L = 25\,°C$ determine the actual thermal efficiency, the maximum possible thermal efficiency, and the actual rate of heat rejection from the power plant?

- -

Concepts:

This is a power plant, so there must be a Turbine. We can refer back to Section 6 to help solve this. Namely,

- $\dot{W}_{Out}$ is large.
- $\dot{W}_{In} \approx 0$
- $\dot{Q} \approx 0$, meaning the turbine is adiabatic.
- $\Delta KE = 0$
- $\Delta H \neq 0$

We can also call this a Carnot Cycle, where $\dot{Q}_H = \dot{Q}_L + \dot{W}_{Out}$. In this case, the $\dot{Q}_H$ is the water coming from the ground, the $\dot{Q}_L$ is the atmosphere, and the $\dot{W}_{Out}$ was provided.

$$\eta_C = \frac{\dot{W}_{Out}}{\dot{Q}_H}$$

$$= 1 - \frac{T_L}{T_H}$$

Explore:

Because we weren't given any additional information about the input fluid, we assume that the fluid remains in a single state throughout the entire Process.

For the Turbine, we have a mass flow that changes the energy balance. The energy balance is

$$\dot{Q}_{Net} - \dot{W}_{Out} = \dot{m}(h_{Out} - h_{In})$$

$\dot{Q}_{Net}$ means there is heat added and/or removed. If it is positive, then heat is being added. If it is negative, then heat is being removed.

We can assume this fluid does not change state. We will also assume that hsi fluid is water, rather than steam. We can refer to Tables A.4 to find the specific enthalpies, because we have no pressures.

Plan:

Solve the Turbine energy balance.

Solve the Carnot efficiency problem.

The actual efficiency is $\eta_C = \frac{\dot{W}_{Out}}{\dot{Q}_H}$.

Solve:

Solve for the net Heat for the Turbine.

$$\dot{Q}_{Net} - \dot{W}_{Out} = \dot{m}(h_{Out} - h_{In})$$

Finding the enthalpies from Table A.4.

$$\dot{Q}_{Net} - 22\,MW = 440\,kg/s(104.8 - 675.5)kJ/kg$$

$$= -251\,000\,kJ/s$$

$$= -251\,MW$$

$$\dot{Q}_{Net} = -229\,MW$$

The flowrate we found for the net heat from the turbine is negative, meaning the heat is flowing **out** of the turbine. Now, we find the heat the system rejects, $\dot{Q}_L$.

$$\dot{Q}_L = \dot{Q}_H - \dot{W}_{Out}$$

$$= 229\,MW - 22\,MW$$

$$= 251\,MW$$

We can now find the efficiency of the system.

$$\eta_{Out} = \frac{\dot{W}_{Out}}{\dot{Q}_{In}}$$

$$= \frac{22}{251}$$

$$= 0.088$$

We can now find the best case efficiency for the system using the efficiency of the Carnot Cycle.

$$\eta_C = 1 - \frac{T_L}{T_H}$$
$$= 1 - 25\,160$$
$$= 0.312$$

Validate:
Double check our actual efficiency by solving

$$\eta_{Out} = 1 - \frac{\dot{Q}_L}{\dot{Q}_H}$$

Generalize:
We had to consider two separate boundaries for the system. First was the Turbine's boundary, the second was the Carnot Cycle boundary.

Example 7.6: Carnot Refrigerators. Textbook Problem 7.98

A Carnot Refrigerator removes Heat from a cool space at a rate of $\dot{Q}_L = 300\,\text{kJ/min}$ to maintain a temperature of $T_L = -8\,°\text{C}$. If the air surrounding the refrigerator is at $T_H = 25\,°\text{C}$, determine the minimum power input, $\dot{W}_{In}$, required by this refrigerator?

- -

Concepts:
A Refrigerator rejects heat from the interior (which it keeps cool) into the surroundings, in this case, the room. This heats up the room while keeping the inside of the fridge cool.
If we model this as a Carnot Cycle, we can find the Coefficient of Performance and the macroscopic energy flow.

$$\dot{Q}_H = \dot{Q}_L + \dot{W}_{In}$$

The Coefficient of Performance tells us how efficient the fridge is. If we relate it using the fact this refrigerator is a Carnot refrigerator, then we have this equation.

$$\text{CoP}_{C,R} = \frac{1}{\frac{T_H}{T_L} - 1}$$

Lastly, the CoP equation above is related to the work out by Equation (7.6).

$$\text{CoP}_R = \frac{\dot{Q}_L}{\dot{W}_{In}}$$

Solve:
First, we find $\text{CoP}_{C,R}$, remembering that the temperatures must be in absolute terms.

$$\text{CoP}_{C,R} = \frac{1}{\frac{T_H}{T_L} - 1}$$
$$= \frac{1}{\frac{25 + 273.15\text{K}}{-8 + 273.15\text{K}} - 1}$$
$$= 8.03$$

Now, rearranging CoP_R, we can solve for the Work in required.

$$\dot{W}_{In} = \frac{\dot{Q}_L}{\text{CoP}}$$
$$= \frac{300\,\text{kJ/min}}{8.03}$$
$$= 32.63\,\text{kJ/min}$$
$$= 0.623\,\text{kJ/s}$$
$$= 0.623\,\text{kW}$$

Example 7.7: Carnot Heat Pump. Textbook Problem 7.107

The structure of a house is such that it loses heat at a rate $\dot{Q} = 5400\,kJ/(h\,°C)$ between the indoors and outdoors. A Heat Pump that requires an input of $\dot{W}_{In} = 6\,kW$ is used to maintain the house at $T_L = 29\,°C$. Determine the lowest outdoor temperature in which the heat pump can meet the heating requirements of the house?

Concepts:
The flowrate of heat looks a bit strange at first glance, but after looking at it for a little while, it starts to look familiar.

$$\dot{Q} = \frac{\dot{Q}_{Out}}{\Delta T}$$

In addition, because the denominator is the change in temperature, in celsius, we can keep the number the same and treat it as the change in temperature in kelvin.

$$\left(5400\,kJ/(kg\,K)\right)(\Delta T)K$$

We were given very few details, so we have to take what we were given in a macroscopic view. Looking at the Carnot energy balance, we can see:

$$\dot{Q}_H = \dot{Q}_L + \dot{W}_{In}$$

We can view this Heat Pump in two ways. The first is an idealized, Carnot depiction:

$$\text{CoP}_{C,HP} = \frac{1}{1 - \frac{T_L}{T_H}}$$

The second is the realistic depiction:

$$\text{CoP}_{HP} = \frac{\dot{Q}_H}{\dot{W}_{In}}$$

Explore:
For $\dot{Q}_L$, the Thermal Reservoir we will use is the air surrounding the hosue, rather than the earth.
The value we were given was power per change in termperature.

$$\frac{5400\,kg}{h\Delta T} = \frac{1.5\,kW}{\Delta T}$$

Let's keep everything in absolute temperatures until the very end.

$$= \frac{1.5\,kW}{294.15 - T_L}$$

Now, we need to look at the Coefficient of Performance.

$$\text{CoP}_{HP} = \frac{\dot{Q}_H}{\dot{W}_{In}}$$

$$= \frac{1}{1 - \frac{T_L}{294.15}}$$

Plan:
Solve the Coefficient of Performance quadratic equation above.

Solve:

$$\frac{\dot{Q}_H}{\dot{W}_{In}} = \frac{1}{1 - \frac{T_L}{294.15}}$$

$$T_L = 259.7\,\text{K}$$

$$= -13.3\,\text{°C}$$

Validate:
To validate, we double-check by solving the Coefficient of Performance equation in the two different ways.

$$\text{CoP} = \frac{\frac{1.5\,\text{kW}}{294.15 - 259.7}}{6\,\text{kW}}$$

$$= 8.57$$

$$\text{CoP} = \frac{1}{1 - \frac{259.7}{294.15}}$$

$$= 8.57$$

Generalize:
An air Heat Pump is limited in how low the exterior temperature can get before they become ineffective/inefficient.

Example 7.8: Carnot Air Conditioner. Textbook Problem 7.113

An air conditioner with the refrigerant R-134a as a working fluid is used to keep a room at $T_L = 26\,\text{°C}$ by rejecting the heat to the exterior at $T_H = 34\,\text{°C}$. The room gains heat at a rate of $Q_{In} = 50\,\text{kJ/min}$ while the room has $W_{In} = 900\,\text{W}$ of heat-generating components. The refrigerant enters the compressor at $P_{In} = 500\,\text{kPa}$ as a Saturated Vapor at a rate of $\dot{V} = 100\,\text{L/min}$ and leaves at $P_{Out} = 1200\,\text{kPa}$ and $T_{Out} = 50\,\text{°C}$. Find the actual and the theoretical Coefficient of Performance, the minimum volume flowrate at the compressor?

- -

Concepts:
We are given the air conditioner with "microscopic" energy values/variables. However, we will need the macroscopic values to be able to calculate what we need to.
Microscopic Variables:

- State
- Specific Enthalpy
- Actual $\dot{W}_{In}$

Macroscopic Variables:

- Coefficient of Performance modeled as a Carnot engine
- Coefficient of Performance modeled realistically

Explore:
Because the compressor doesn't change anything other than the Specific Enthalpy of a fluid, the energy balance is:

$$\dot{W}_{In} = \dot{m}(h_{Out} - h_{In})$$

To find the mass flowrate,

$$\dot{V} = \dot{m}\nu$$

Because the mass flowrate is constant, the Specific Volume is constant, so $\nu_{In} = \nu_{Out} = \nu$.
To find the specific enthalpies, we use Tables A.11, A.12, and A.13.
Plan:
Solve for $\dot{m}$.
Solve for $\dot{W}_{In}$.
Solve the Coefficient of Performance equation, both for the actual and the Carnot, using $\dot{W}_{In} + \dot{Q}_L = \dot{Q}_H$.
Solve:
Starting with the inlet state, we are told it is a Saturated Vapor, so:

$$x = 1$$

$$P_{In} = 500\,\text{kPa}$$

So we use Table A.13.

$$T_{Sat} = 15.7\,°\text{C}$$
$$h_g = 259.3\,\text{kJ/kg}$$
$$\nu_g = 0.041\,118\,\text{m}^3/\text{kg}$$

Moving onto the outlet's state at $P_{Out} = 1.2\,\text{MPa}$ and $T_{Out} = 50\,°\text{C}$, we see that the R-134a is a Superheated Vapor. So, we use Table A.12.

$$h = 178.3\,\text{kJ/kg}$$
$$\nu = 0.017\,301\,\text{m}^3/\text{kg}$$

Now, we can find the mass flowrate entering/leaving the compressor.

$$\dot{V} = \dot{m}\nu$$
$$\dot{V} = 100\,\text{L/min} = 1.66 \times 10^{-3}\text{m}^3/\text{s}$$
$$\dot{m} = \frac{\dot{V}}{\nu}$$
$$= 0.040\,53\,\text{kg/s}$$

Now, we can find the Work required to compress the R-134a.

$$\dot{W}_{In} = \dot{m}(h_{Out} - h_{In})$$
$$= 0.040\,53\,\text{kg/s}(278.3 - 259.3)\text{kJ/kg}$$
$$= 0.7686\,\text{kW}$$

We now find out the heat that is being added to the room.

$$\dot{Q}_L = 250\,\text{kJ/min} + 900\,\text{W}$$
$$= 5.067\,\text{kW}$$

Now, we can find the Coefficient of Performance, starting with the actual value.

$$\text{CoP}_{AC} = \frac{\dot{Q}_L}{\dot{W}_{In}}$$
$$= \frac{5.067\,\text{kW}}{0.7686\,\text{kW}}$$
$$= 6.59$$

Now, for the idealized, maximum, Coefficient of Performance.

$$\text{CoP}_{C,AC} = \frac{1}{\frac{T_H}{T_L} - 1}$$
$$= 37.4$$

Because the Carnot version of this cycle is the most efficient, that means it will require the least power, and the least flowrate.

$$P_{Min} = \dot{W}_{In}$$
$$= \frac{5.067\,\text{kW}}{37.4}$$
$$= 0.1356\,\text{kW}$$

The actual minimum power required is shown below.

$$= 0.7686\,\text{kW}$$

Now for the minimum flowrate required.

$$\dot{m} = \frac{\dot{W}_{In}}{\Delta h}$$
$$= 0.007\,149\,\text{kg/s}$$

Lastly, the volume flowrate required.

$$\dot{V}_{Min} = \dot{m}\nu$$
$$= 17.64\,\text{L/min}$$

The mass flowrate out of the compressor is:

$$\dot{m}_{Out} = \frac{\dot{W}_{In}}{\Delta h}$$
$$= 0.04\,\text{kg/s}$$

8   Entropy

8.1   Clausius Inequality

We have 2 equations for the efficiency of a Carnot Cycle.

$$\eta_{\text{Carnot}} = \frac{T_H - T_L}{T_H} = 1 - \frac{T_L}{T_H}$$
$$= \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H}$$

8.1.1   Reversible Carnot Cycles

If we equate these (set them equal to each other) we get:

$$1 - \frac{T_L}{T_H} = 1 - \frac{Q_L}{Q_H}$$
$$\frac{T_L}{T_H} = \frac{Q_L}{Q_H}$$
$$\frac{Q_L}{T_L} = \frac{Q_H}{T_H}$$
$$\frac{Q_L}{T_L} - \frac{Q_H}{T_H} = 0$$

This derivation's result is valid for any Reversible Process which is a Carnot Cycle.

8.1.2   Irreversible Carnot Cycles

Looking at the energy balance present for an Irreversible Process

$$Q_H - Q_L = W_{Out} + W_{\text{Friction}}$$

Because of the presence of W_{Friction}, Q_L gets smaller (in comparison to a reversible Carnot cycle). That means that the heat/temperature ratio equation has changed.

$$\frac{Q_L}{T_L} - \frac{Q_H}{T_H} < 0$$

Here, we start getting into the topic of Entropy a little bit more.

$$\sum_i \frac{dQ_i}{T_i} \leq 0 \tag{8.1}$$

There are some things to say about Equation (8.1).

- Assume that you know the directions for the heats.
- The $\frac{dQ_i}{T_i}$ is called Entropy, or S.
- If we divide the Entropy by the mass, we have the Specific Entropy, $\frac{\frac{dQ_i}{T_i}}{m}$.
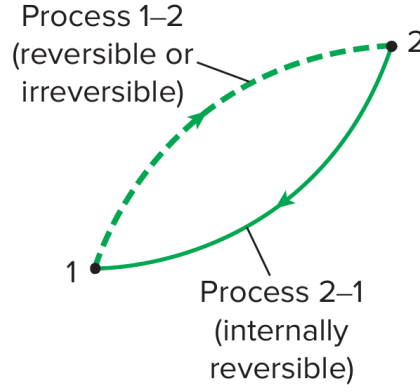
Figure 8.1: Heat Engines and the Clausius Inequality (Çengel, Cimbala, and Turner 2017, pg. 280)

8.1.3 Heat Engines and the Clausius Inequality

If we imagine a heat engine running through a set of cycles, as shown in Figure 8.1, then we can say some things about it. Namely, the thing we can say is:

$$\int_1^2 \frac{dQ}{T} + \int_2^1 \frac{dQ}{T} \leq 0$$

$$\int_1^2 S + \int_2^1 S \leq 0$$

$$(S_1 - S_2) + \int_2^1 S \leq 0$$

$$S_2 - S_1 \geq \int_2^1 \frac{dQ}{T}$$

This last point is actually quite important and is restated here.

$$S_2 - S_1 \geq \int_2^1 \frac{dQ}{T} \tag{8.2}$$

Equation (8.2) states that for an Irreversible Process, Work is lost **forever**.

- For Irreversible Processes, S **always** increases.
- For Reversible Processes, $\Delta S = 0$.

8.1.4 Entropy and its Definitions

Defn 75 (Entropy). *Entropy* is an thermodynamically Extensive Property. It is defined from the Clausius Inequality, in Equation (8.1).

$$S = \frac{Q_i}{T_i} \tag{8.3}$$

Another equation to describe entropy is based on the concept of the number of energy states.

$$S = k \ln(W) \tag{8.4}$$

where:

k is Boltzmann's Constant.
W is the number of energy states in the system. This value is usually proportional to temperature or some energy.

Defn 76 (Specific Entropy). *Specific Entropy*, like all the other specific properties of a state, is a thermodynamically Intensive Property. Similarly to Entropy, specific entropy is defined from the Clausius Inequality, in Equation (8.1), but is also divided by the mass.

$$s = \frac{\frac{Q_i}{T_i}}{m} \tag{8.5}$$

There also exist Cycles where the Entropy remains constant (which only happens if the Process is a Reversible Process), which are called Isentropic. These can only occur if the Process is Adiabatic.

Defn 77 (Isentropic). An *isentropic* Process/Cycle has constant Entropy.

Example 8.1: Isentropic Ideal Gases. Textbook Problem 8.38

$m = 1\,\mathrm{lbm}$ R-134a is expanded Isentropicly in a Closed System that goes from $P_1 = 100\,\mathrm{psia}$ and $T_1 = 100°\mathrm{F}$ to $P_2 = 10\,\mathrm{psia}$. Determine the total heat transfer and the work required for this Process?

- -

Concepts:
This is a Closed System, so mass is constant, there is **no** mass flow. However, there can still be a heat and work flow. We are told this is an Isentropic process, so there is no change in temperature.

$$T_1 = T_2$$
$$\Delta S = 0$$
$$\frac{\Delta Q}{T} = 0$$
$$\Delta Q = 0$$

There is a change in pressure, meaning the vessel itself expands.
Explore:
Pressure is **not** constant.
The vessel is expanding, meaning there needs to be a change in energy. The energy is coming from the internal energy, U, of the gas because the R-134a is not "pushing" against anything.
We discovered that $\Delta Q = 0$.
We are not told about any work in, so assume $W_{In} = 0$.
The energy balance is now:

$$-W_{Out} = m(u_2 - u_1)$$

We can use Tables A.11E through A.13E to find values for R-134a.
Plan:
Using the pressures and temperatures, use the Tables to find the entropies and the internal energies. Remember that $s_1 = s_2$, so we can find the different internal energies.
Solve:
Start by using Table A.13E at $P_1 = 100\,\mathrm{psia}$ and $T_1 = 100°\mathrm{F}$ to find the initial properties of the R-134a.

$$s_1 = 0.229\,02\,\mathrm{btu/(lbm\,R)}$$
$$u_1 = 109.46\,\mathrm{btu/lbm}$$

Now, using the facts that $s_1 = s_2$, and $P_2 = 10\,\mathrm{psia}$, we use Table A.13E again, looking for a s_2 that is as similar to s_1 as possible.

$$s_2 = 0.229\,49\,\mathrm{btu/(lbm\,R)}$$
$$T_2 = -29.52°\mathrm{F}$$
$$u_2 = 113.02\,\mathrm{btu/lbm}$$

Validate:

Generalize:

8.2 Isentropic Efficiencies

Consider a Heat Engine, like the one shown in Figure 7.1.

- If each sub-Process is Isentropic, then on a $T - S$ graph, then the process is perfectly vertical.
- If any part is even slightly irreversible, then the ending entropy is further to the right compared ot any previous part of the Cycle.

If we are talking in a non-ideal case, then we can say:

$$\eta = \frac{W_{Out}}{Q_{In}}$$

8.2.1 Turbine

If we deal with the actual work out and ideal heat in, we say the Isentropic efficiency is:

$$\eta_T = \frac{W_{Out,\text{Actual}}}{Q_{In,\text{Ideal}}}$$
$$= \frac{\dot{m}(h_{In} - h_{Out,\text{Actual}})}{\dot{m}(h_{In} - h_{Out,\text{Ideal}})}$$
$$= \frac{h_{In} - h_{Out,\text{Actual}}}{h_{In} - h_{Out,\text{Ideal}}}$$

For a Turbine,

$$\eta_T = \frac{h_{In} - h_{Out,\text{Actual}}}{h_{In} - h_{Out,\text{Ideal}}} \tag{8.6}$$

Example 8.2: Turbine Entropy.

Given an Adiabatic Turbine that uses steam where the inlet steam is at $P_{In} = 1000\,\text{psia}$ and $T_{In} = 800\text{°F}$. The outlet is $P_{Out} = 400\,\text{psia}$. Find $\frac{\dot{W}_{Out}}{\dot{m}}$?

- -

Concepts:
This is a Turbine, so it has constant flow. Meaning $\dot{m}_{In} = \dot{m}_{Out}$.
This is an Adiabatic Turbine, which means the Process is a Reversible Process, and is an Isentropic process. Meaning $s_{In} = s_{Out}$.
Lastly, energy is conserved, so $\dot{E}_{In} = \dot{E}_{Out}$.
Explore:
The change in energy is due to the Enthalpy changes in the steam.
The Specific Entropy remains unchanged throughout the process, so we can use these values from Tables.
Tables A.4E — A.7E will help us solve this.
Plan:
Find the steam's initial state properties, h_{In} and s_{In}.
Use $s_{In} = s_{Out}$ to help find h_{Out}.
Use the energy balance equation for a Turbine $\dot{W}_{Out} = \dot{m}(h_{Out} - h_{In})$ and solve for $\frac{\dot{W}_{Out}}{\dot{m}}$.
Solve:
The steam enters the Turbine as a Superheated Vapor, so we use Table A.6E. From Table A.6E at $P_{In} = 1000\,\text{psia}$ and $T_{In} = 800\text{°F}$, we see:

$h_{In} = 1389.0\,\text{btu/lbm}$
$s_{In} = 1.5670\,\text{btu/(lbm R)}$

Now, for the outlet state. Start by looking at Table A.5E at $P_{Out} = 400\,\text{psia}$. If we look at the Specific Entropy entries, we see that the highest value is when the steam is a Saturated Vapor, with $s_g = 1.4852\,\text{btu/(lbm R)}$. However, because $s_{In} = s_{Out}$, we see that this is not enough. So, that means the steam is still a Superheated Vapor at the outlet.
From Table A.6E, at $P_{Out} = 400\,\text{psia}$, we look for $s_{Out} = 1.5670\,\text{btu/(lbm R)}$. The closest value to that s is at $T_{Out} = 550\text{°F}$, with $h_{Out} = 1277.3\,\text{btu/lbm}$. (In reality, we should Interpolate our values, but this is close enough for this example.)

Now, solving the energy balance equation for $\frac{\dot{W}_{Out}}{\dot{m}}$.

$$-\dot{W}_{Out} = \dot{m}(h_{Out} - h_{In})$$

$$\frac{-\dot{W}_{Out}}{\dot{m}} = h_{Out} - h_{In}$$

$$\frac{\dot{W}_{Out}}{\dot{m}} = h_{In} - h_{Out}$$

$$= 1389.0 \,\text{btu/lbm} - 1277.3 \,\text{btu/lbm}$$

$$= 111.7 \,\text{btu/lbm}$$

Generalize:
We can use Specific Entropy to find the state of the working fluid, just like the Specific Enthalpy, temperature, or pressure.

Example 8.3: Isentropic Efficiency of a Steam Turbine. Textbook Example 8.14

Steam enters an Adiabatic Turbine steadily at $P_{In} = 3\,\text{MPa}$ and $T_{In} = 400\,°\text{C}$ and leaves at $P_{Out} = 50\,\text{kPa}$ and $T_{Out} = 100\,°\text{C}$. If the power output of the turbine is $\dot{W}_{Out} = 2\,\text{MW}$, determine the isentropic efficiency of the turbine and the mass flowrate of the steam flowing through the turbine?

- -

Concepts:
We are given the information for both the inlet and the outlet. These are the actual values of the turbine. If we want the idealized values, we need to interpret the inlet data a little bit.
For a turbine be Isentropic, it needs to be Adiabatic, which we are told it is. This means that $s_{In} = s_{Out}$.
The isentropic efficiency of a turbine is seen in Equation (8.6).
Explore:
Because this is steam, Tables A.4 through A.7 will be of use.
The idealized values for the ideal case can only be found by treating this system as Adiabatic and ensuring that $s_{In} = s_{Out}$.
Plan:
Use the tables to find the actual inlet and outlet specific enthalpies.
Use the idea that $s_{In} = s_{Out}$ to find the ideal outlet specific enthalpy.
Solve Equation (8.6) for η_T.

Example 8.4: Power Output of Isentropically Efficienct Steam Turbine. Textbook Problem 8.128

Steam at $P_{In} = 3\,\text{MPa}$ and $T_{In} = 400\,°\text{C}$ is expanded to $P_{Out} = 30\,\text{kPa}$ in an Adiabatic Turbine with an Isentropic efficiency of $\eta_T = 0.92$. Determine the power produced by this turbine ($\dot{W}_{Out}$) in kW when the mass flowrate is $\dot{m} = 2\,\text{kg/s}$?

- -

Concepts:
This is a Turbine, with Adiabatic steady flow. A turbine gets all its energy from the change in Enthalpy, Δh.
The equation for Isentropic efficiency is:

$$\eta_T = \frac{\text{Actual Work Out}}{\text{Isentropic Work Out}}$$

Explore:
To solve for the work out, we need to know the maximum work out that we could gather. This only occurs if the Turbine is perfectly Isentropic, meaning $S_{In} = S_{Out}$.
If we expand the isentropic efficiency equation, then we get:

$$\eta_T = \frac{h_{In} - h_{Out,\text{Actual}}}{h_{In} - h_{Out,\text{Ideal}}}$$

The energy balance for this turbine is:

$$\dot{W}_{Out,\text{Actual}} = \dot{m}(h_{In} - h_{Out})$$

This is a steam turbine, so Tables A.4 through A.7 will be of use.

Plan:

Find properties of the initial state, h_{In}, s_{In}.

Use our assumption of isentropic to find $s_{Out,\text{Ideal}}$.

Solve:

We see that from the inlet fluid's pressure and temperature, it is a Superheated Vapor, so we use Table A.6. From Table A.6 at $P_{In} = 3\,\text{MPa}$ and $T_{In} = 400\,°\text{C}$, we have:

$$h_{In} = 3231\,\text{kJ/kg}$$
$$s_{In} = 6.923\,\text{kJ/(kg K)}$$

If we assume that the Turbine is perfectly Isentropic, then we notice that the steam has become a Saturated Mixture, so we use Table A.5 to find its properties.

$$\begin{aligned}
x &= \frac{s_{In} - s_f}{s_g - s_f} \\
&= \frac{6.923 - 0.9441}{7.7675 - 0.9441} \\
x &= 0.8763
\end{aligned}$$

Now, we find the ideal $h_{Out,\text{Ideal}}$.

$$\begin{aligned}
h_{Out,\text{Ideal}} &= h_f + x(h_g - h_f) \\
&= 289.27 + 0.8763(2624.6 - 289.27) \\
&= 2335.72\,\text{kJ/kg}
\end{aligned}$$

So, the Isentropic work out is:

$$\begin{aligned}
\dot{W}_{Out,\text{Ideal}} &= \dot{m}(h_{In} - h_{Out,\text{Ideal}}) \\
&= 2(3231 - 2335.72) \\
&= 1790.56\,\text{kJ/s}
\end{aligned}$$

Now, we plug that into our isentropic efficiency.

$$\eta_T = \frac{\dot{W}_{Out,\text{Actual}}}{\dot{W}_{Out,\text{Ideal}}}$$

Rearranging the equation.

$$\begin{aligned}
\dot{W}_{Out,\text{Actual}} &= \eta_T \dot{W}_{Out,\text{Ideal}} \\
&= 0.92(1790.56) \\
&= 1647.32\,\text{kJ/s}
\end{aligned}$$

Validate:

Generalize:

8.2.2 Compressor

For a Compressor, we have:

$$\begin{aligned}
\eta_T &= \frac{\text{Isentropic Work In}}{\text{Actual Work In}} \\
&= \frac{h_{Out,\text{Ideal}} - h_{In}}{h_{Out,\text{Actual}} - h_{In}}
\end{aligned} \tag{8.7}$$

Example 8.5: Isentropic Compressor. Textbook Problem 8.24

Air is compressed by a $\dot{W}_{In} = 12\,\text{kW}$ Compressor. The air temperature is maintained at $T = 25\,°\text{C}$ during the process. Then, as a result of Heat Transfer, the surrounding air, which is at $T = 10\,°\text{C}$. Determine the rate of entropy change

of the air?

Concepts:

Compressors typically increase the air's temperature. Because there is no temperature change of the air, that means the whole compressor is being cooled, thus a $\dot{Q}_{Out} \neq 0$.

We find the $\dot{S}$, but we aren't given a mass flowrate, $\dot{m}$.

We should also treat air as an ideal gas.

Explore:

We can ignore PE and KE. The energy balance for a compressor is:

$$\dot{W}_{In} - \dot{Q}_{Out} = \dot{m}(h_{Out} - h_{In})$$

Inside the Compressor, it is Isothermal. Therefore, the Process **INSIDE** the compressor is a Reversible Process, so it is Isentropic.

Using the energy balance $\dot{E}_{In} = \dot{E}_{Out}$, we can say:

$$\dot{E}_{In} = \dot{E}_{Out}$$

$$\dot{W}_{In} + \dot{m}h_{In} = \dot{Q}_{Out} + \dot{m}h_{Out}$$

Because $h \propto T$, substitute for temperatures. We can do this because of the specific enthalpy-specific heat equations for ideal gases.

$$\dot{W}_{In} + \dot{m}T_{In} = \dot{Q}_{Out} + \dot{m}T_{In}$$

And, the temperatures are the same, so they cancel out.

$$\dot{W}_{In} = \dot{Q}_{Out}$$

Lastly, we can find the Entropy flowrate.

$$S = \frac{Q}{T}$$

$$\dot{S} = \frac{\dot{Q}}{T}$$

Because the heat is flowing outwards, there is actually a negative.

$$\dot{S} = \frac{-\dot{Q}}{T}$$

Plan:

Solve for the entropy flowrate, $\dot{S}$.

Solve:

$$\dot{S} = \frac{-\dot{Q}}{T}$$

$$= \frac{-12\,\text{kW}}{(25 + 273.15)\text{K}}$$

$$= -0.0403\,\text{kW/K}$$

Validate:

Even though the Entropy inside the system decreased, the overall entropy of the entire system (compressor and surroundings) increased by a similar amount.

$$\dot{S} = \frac{\dot{Q}}{T}$$

$$= \frac{12\,\text{kW}}{(10 + 273.15)\text{K}}$$

$$= 0.0424\,\text{kW/K}$$

Generalize:

If we add the entropy gained by the surroundings and the entropy lost by the Compressor together, we get a non-zero result. This aligns with what we understand about Entropy and Irreversible Processes and the fact that we assumed this process was actually a Reversible Process.

Example 8.6: Isentropic Efficiency of Compressor. Textbook Problem 8.130

An Adiabatic steady-flow device compresses argon at $P_{In} = 200\,\text{kPa}$ and $T_{In} = 27\,°\text{C}$ to $P_{Out} = 2\,\text{MPa}$. If the argon leaves this Compressor at $T_{Out} = 550\,°\text{C}$ what is the isentropic efficiency, η_T of the compressor?

- -

Concepts:

Because the working fluid is argon, it is an ideal gas. The equations we found for ideal gases are really just approximations. We have c_P, c_V, and k.

Since we need to assume that the system is Isentropic for some of the calculations, we have some equations that we can use for an ideal gas (Section 8.4).

Explore:

The efficiency of a compressor is:

$$\eta_T = \frac{\dot{W}_{In,\text{Ideal}}}{\dot{W}_{Out,\text{Actual}}}$$

The energy balance for a compressor is:

$$\dot{W}_{In} = \dot{m}(h_{Out} - h_{In})$$

If we substitute that into the isentropic efficiency equation:

$$\eta_T = \frac{\dot{W}_{In,\text{Ideal}}}{\dot{W}_{Out,\text{Actual}}}$$

$$= \frac{\dot{m}(h_{Out,\text{Ideal}} - h_{In})}{\dot{m}(h_{Out,\text{Actual}} - h_{In})}$$

We can cancel out like terms.

$$= \frac{\Delta h_{\text{Ideal}}}{\Delta h_{\text{Actual}}}$$

Because this is an ideal gas, we can substitute in Equation (5.8).

$$= \frac{c_P \Delta T_{\text{Ideal}}}{c_P \Delta T_{\text{Actual}}}$$

$$\approx \frac{\Delta T_{\text{Ideal}}}{\Delta T_{\text{Actual}}}$$

$$= \frac{T_{Out,\text{Ideal}} - T_{In}}{T_{Out,\text{Actual}} - T_{In}}$$

To find the Isentropic output fluid temperature, we have Equations (8.15a) to (8.15c).

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}}$$

$$T_{Out,\text{Ideal}} = T_{In}\left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}}$$

where $k = \frac{c_P}{c_V}$.

Table A.2 will have these values.

Plan:

Solve for the Isentropic output fluid temperature.

Solve:

From Table A.2:

$$c_P = 0.5203\,\text{kJ/(kg K)}$$
$$c_V = 0.3122\,\text{kJ/(kg K)}$$
$$k = 1.667$$

Solving for the ideal outlet fluid temperature:

$$T_{Out,\text{Ideal}} = T_{In} \left(\frac{P_2}{P_1} \right)^{\frac{k-1}{k}}$$

$$= (27 + 273.15) \left(\frac{2000\,\text{kPa}}{200\,\text{kPa}} \right)^{\frac{1.667-1}{1.667}}$$

$$T_{Out,\text{Ideal}} = 754.151\,\text{K}$$

$$= 481.001\,^{\circ}\text{C}$$

Now for the isentropic efficiency:

$$\eta_T = \frac{T_{Out,\text{Ideal}} - T_{In}}{T_{Out,\text{Actual}} - T_{In}}$$

$$= \frac{481.001 - 27}{550 - 27}$$

$$\eta_T = 0.868071$$

Validate:

Generalize:

8.2.3   Pump

Liquids are much more dense than gases. This means that the previously-used energy balance equation is not quite the same. Before, we had:

$$\dot{E} = \dot{m}e$$

$$= \dot{m}(\text{FE} + \text{KE} + \text{PE})$$

$$= \dot{m} \left(\frac{\Delta P}{\rho} + \frac{\Delta v^2}{2} + g\Delta d \right)$$

$$\Delta \dot{Q} - \Delta \dot{W} = \dot{m} \left((h_{Out} - h_{In}) + \frac{v_{Out}^2 - v_{In}^2}{2} + g(d_{Out} - d_{In}) \right)$$

If we remember the Clausius Inequality for a Reversible Process.

$$\Delta S = 0$$

$$\int_1^2 T dS = \int_1^2 dQ = \Delta Q$$

$$T \Delta S = \Delta Q$$

Now, we connect it to both Closed Systems and Open Systems.

$$Q - W = U$$

Now, looking at the individual definitions for each expression.

$$Q = \int T dS$$

$$W = \int P dV$$

$$H = U + PV$$

$$dH = dU + P dV + V dP$$

Take a differential of each term in the equation.

$$dQ - dW = dU$$

$$T dS - P dV = dU$$

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Substitute in for the internal energy.

$$TdS = dH - VdP$$

$$\int TdS = \int dH - \int VdP$$

In fact, we have created an important equation, called Gibbs Equation, shown in Equation (8.8).

$$TdS = dH - VdP \tag{8.8}$$

Using these results, we can substitute back into the original energy balance for heat.

$$\Delta Q - \Delta W = m(h_{Out} - h_{In}) + \left(\frac{v_{Out}^2 - v_{In}^2}{2}\right) + g(d_{Out} - d_{In})$$

Drop the mass.

$$\Delta q - \Delta w = (h_{Out} - h_{In}) + \left(\frac{v_{Out}^2 - v_{In}^2}{2}\right) + g(d_{Out} - d_{In})$$

Remember,

$$Q = (h_{Out} - h_{In}) - \int VdP$$

So,

$$W = -\int VdP - \Delta \text{KE} - \Delta \text{PE}$$

For a pump, we tend to ignore the kinetic and potential energy. Because it is difficult for liquids to change in pressure, we tend to treat them as if it is constant.

$$W = -V(P_{Out} - P_{In})$$

$$w = -\nu(P_{Out} - P_{In})$$

Using all of this information, we can now find the isentropic efficiency of a pump.

$$\eta_T = \frac{\nu(P_{Out,\text{Ideal}} - P_{In})}{h_{Out,\text{Actual}} - h_{In}}$$

$$= \frac{h_{Out,\text{Ideal}} - h_{In}}{h_{Out,\text{Actual}} - h_{In}} \tag{8.9}$$

8.2.4 Nozzle

Because the major energy present in a nozzle is the change in kinetic energy. Thus, Equation (8.10) is natural.

$$\eta_T = \frac{\Delta \text{KE}_{\text{Actual}}}{\Delta \text{KE}_{\text{Ideal}}}$$

$$= \frac{v_{Out,\text{Actual}}^2 - v_{In}^2}{v_{Out,\text{Ideal}}^2 - v_{In}^2} \tag{8.10}$$

8.3 Entropy Changes for Solids/Liquids

Gibbs Equation (Equation (8.8)) can be used to solve for these incompressible substances.

$$TdS = dU + PdV$$

$$dS = \frac{dU}{T} + \frac{PdV}{T}$$

Because the substance is incompressible, the volume is constant.

$$= \frac{dU}{T} + \frac{P(0)}{T}$$

$$\int dS = \int \frac{dU}{T}$$

Because we are under constant volume conditions, we can expand the internal energy factor out using the equation for Specific Heat (Equation (5.7)).

$$= \int \frac{c_V \, dT}{T}$$

$$S_2 - S_1 = c_V \ln\left(\frac{T_2}{T_1}\right)$$

Because of the nature of incompressible substances, $c_V = c_P = c$.

$$S_2 - S_1 = c \ln\left(\frac{T_2}{T_1}\right) \tag{8.11}$$

Example 8.7: Water Entropy Change. Textbook Problem 8.36

$m = 2\,\text{lbm}$ of water is inside a weighted piston-cylinder at $P_1 = 300\,\text{psia}$, which is heated. The volume of the cylinder is $V = 2.5\,\text{ft}^3$. The water is heated at a constant pressure until it reaches $T_2 = 500°\text{F}$. Determine the water's final volume, V_2 and the total change in Entropy ΔS?

- -

Concepts:
Piston-cylinders keep the Process under constant pressure.
Because we are heating water, the volume of the water increases. How the pressure, temperature, and specific volume are related is by the
Because the temperature is **not** constant, it is an Irreversible Process.
Explore:
Because both the mass remains constant throughout this process, we can at least solve for the initial Specific Volume.

$$\nu = \frac{V}{m}$$

$$\nu_1 = \frac{V_1}{m}$$

$$= \frac{2.5\,\text{ft}^3}{2\,\text{lbm}}$$

$$\nu_1 = 1.25\,\text{ft}^3/\text{lbm}$$

Using the initial pressure, temperature, and specific volume, we can find the state from Table A.5E. This will also provide us with the entropy for the given state.
We can repeat this same process for the final state too.
Plan:
Start with Table A.5E, and determine the state of the water.
Find S_1 in the appropriate table.
Solve:
The initial state has $P_1 = 300\,\text{psia}$ and $\nu_1 = 1.25\,\text{ft}^3/\text{lbm}$, so we use Table A.5E. From that table, we see that the water starts as a Saturated Mixture, so we need to find the Quality.

$$x_1 = \frac{\nu_1 - \nu_f}{\nu_g - \nu_f}$$

$$= \frac{1.25 - 0.01890}{1.5435 - 0.01890}$$

$$x_1 = 0.80749$$

Now, using the initial quality, we can find the initial Entropy.

$$s_1 = s_f + x(s_g - s_f)$$

$$= 0.58818 + 0.80749(1.5111 - 0.58818)$$

$$s_1 = 1.333\,43\,\text{btu}/(\text{lbm R})$$

Now, moving onto the final state. Because the pressure remains constant, $P_2 = 300\,\text{psia}$ and $T_2 = 500°\text{F}$. Looking in Table A.4E we see that $P_{\text{sat}} > P_2$ and in Table A.5E we see $T_{\text{sat}} < T_2$, which means that the steam is now a Superheated Vapor. Using Table A.6E at $P_2 = 300\,\text{psia}$ and $T_2 = 500°\text{F}$, we find:

$$s_2 = 1.5706\,\text{btu}/(\text{lbm R})$$

Therefore, we can find the total change in Entropy like so:

$$\Delta S = m\Delta s$$
$$= 2\,\text{lbm}(1.5706 - 1.33343)\text{btu}/(\text{lbm R})$$
$$\Delta S = 0.4744\,\text{btu}/\text{R}$$

Validate:
The easiest way to validate this is to plot the entropy against the pressure and look at the graph.
Generalize:

8.4 Entropy Changes for Ideal Gases

Gibbs Equation (Equation (8.8)) can again be used to solve for ideal gases.

$$TdS = dU + PdV$$
$$dS = \frac{dU}{T} + \frac{PdV}{T}$$

Using the ideal gas equation Equation (4.10) and the definition of change in internal energy for ideal gases Equation (5.7), we can "simplify" this.

$$= \frac{c_V dT}{T} + \frac{RdV}{\nu}$$
$$\int dS = \int \frac{c_V dT}{T} + \int \frac{RdV}{\nu}$$

Evaluating the integrals.

$$S_2 - S_1 = c_V \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\nu_2}{\nu_1}\right)$$

Thus, we can find the change in Entropy of an ideal gas if we know its Specific Volumes using Equation (8.12).

$$S_2 - S_1 = c_V \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\nu_2}{\nu_1}\right) \tag{8.12}$$

We can also expand using a slightly different formula, yielding a very similar result.

$$TdS = dH - \nu dP$$
$$dS = \frac{dH}{T} - \frac{\nu dP}{T}$$

Using the ideal gas equation Equation (4.10) and the definition of change in enthalpy for ideal gases Equation (5.8), we can "simplify" this.

$$= \frac{c_P dT}{T} - \frac{RdP}{P}$$
$$\int dS = \int \frac{c_P dT}{T} - \int \frac{RdP}{P}$$

Evaluating the integrals.

$$S_2 - S_1 = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$

Thus, we can find the change in Entropy of an ideal gas if we know its pressures using Equation (8.13).

$$S_2 - S_1 = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \tag{8.13}$$

Equations (8.12) and (8.13) both have the limitation that they will only work if the change in temperature is small. If we have to deal with large temperature differences, then we use Equation (8.14).

Example 8.8: Change in Entropy of Ideal Gas. Textbook Problem 8.83

There is an amount of air at an initial pressure of $P_1 = 200\,\text{psia}$ and a temperature of $T_1 = 500°\text{F}$. The air is then moved to a new vessel with a final pressure of $P_2 = 100\,\text{psia}$ and $T_2 = 50°\text{F}$. If the Specific Heat of the air remains constant, what is the change in Entropy, ΔS?

Concepts:
The reduction in both temperature and pressure of the gas implies that the gas expands in volume.
Air is an ideal gas.
Constant Specific Heats is a little questionable, because of the large temperature change in this problem. Because of this, we can approximate the solution to give us an idea, or we can use the most accurate method.

Explore:
There are 2 equations we can use to find the change in Entropy of an ideal gas.

$$\Delta S = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$

$$\Delta S = S_2^\circ - S_1^\circ - R\ln\left(\frac{P_2}{P_1}\right)$$

We can find values for S_1° and S_2° in Table A.21E. We also need to be careful that the pressures and temperatures are in absolute units.

Plan:
Solve both ΔS equations.
Use Table A.21E to solve the equation for large temperature differences.

Solve:
Starting with the equation we expect to be slightly wrong.

$$\Delta S = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$

We need to find an appropriate c_P value for the air. We can use Table A.2E for this. We also need to convert the given relative temperatures to absolute temperatures.
Starting with the c_P at 500°F, we can pull the value directly from Table A.2E(b).

$$c_P = 0.248\,\text{btu}/(\text{lbm R})$$

Now for the the c_P at 50°F, we do the same. I chose to approximate to 40°F for 50°F.

$$c_P = 0.240\,\text{btu}/(\text{lbm R})$$

Now, we take the average of the two c_P.

$$c_P = \frac{0.248 + 0.240}{2}$$
$$c_P = 0.244\,\text{btu}/(\text{lbm R})$$

Now, we can solve for the approximating equation.

$$\Delta s = c_P \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$
$$= 0.244\ln\left(\frac{509.67\,\text{R}}{959.67\,\text{R}}\right) - 0.06855\ln\left(\frac{100}{200}\right)$$
$$\Delta s = -0.106\,894\,\text{btu}/(\text{lbm R})$$

We can now also solve for the equation we expect to be more exact. We start by finding the s° values from Table A.21E.

$$T_1 = 959.67\,\text{R} \rightarrow s_1^\circ \approx 0.740\,30\,\text{btu}/(\text{lbm R})$$

I approximated the temperature to 960 R.

$$T_2 = 509.67\,\text{R} \rightarrow s_2^\circ \approx 0.587\,\text{btu}/(\text{lbm R})$$

I approximated the temperature to $510\,\text{R}$.

Using these values, we can now solve for the change in Entropy.

$$\Delta s = s_2^\circ - s_1^\circ - R\ln\left(\frac{P_2}{P_1}\right)$$

$$= 0.587 - 0.74030 - 0.06855\ln\left(\frac{100}{200}\right)$$

$$\Delta s = -0.105\,785\,\text{btu/(lbm R)}$$

Validate:

By solving for both Δs, we actually ended up validating our answer as we went. Both numbers are quite similar to one another, so it is likely that the approximation was correct.

The Entropy being negative is perfectly valid, because of two reasons:

1. We didn't consider the entire System. We only considered inside the boundary of the system. If we included the surroundings, then we would see the net change in entropy is positive.
2. The temperature of the gas is decreasing, reducing the overall number of possible energy states, which means the entropy decreases as well.

Generalize:

The approximation here was not significantly different than the exact calculation. This happened because the Specific Heat did not significantly differ over the temperature range. If the temperature change were larger, then the approximation would start becoming less accurate compared to the exact equation.

Now, for large temperature differences, we use an equation that has already been provided for us, Equation (8.14).

$$S_2 - S_1 = S_2^\circ - S_1^\circ - R\ln\left(\frac{P_2}{P_1}\right) \tag{8.14}$$

S_2° and S_1° come from Table A.21 in the textbook.

Example 8.9: Use Entropy Change of Ideal Gas. Textbook Problem 8.97

Air enters an Adiabatic nozzle at $P_{In} = 60\,\text{psia}$, $T_{In} = 540°\text{F}$, and $v_{In} = 200\,\text{ft/s}$ and exits at $P_{Out} = 12\,\text{psia}$. Assuming air to be an ideal gas with variable Specific Heats and disregarding any irreversibilities, determine the exit velocity, v_{Out} of the air?

- -

Concepts:

Because this is a nozzle, we know that $v_{Out} > v_{In}$.

All the energy of the fluid is in the Enthalpy and the kinetic energy.

We are told that the fluid has a variable Specific Heat.

Because we are told the the nozzle is Adiabatic and are told to ignore any irreversibilities, we can assume the nozzle is Isentropic meaning $S_{In} = S_{Out}$.

Explore:

Variable specific heats for an ideal gas under Isentropic conditions is:

$$S_{Out}^\circ = S_{In}^\circ + R\ln\left(\frac{P_{Out}}{P_{In}}\right)$$

We can also use the equation for relative pressures:

$$\frac{P_{Out}}{P_{In}} = \frac{P_{Out,r}}{P_{In,r}}$$

where the relative pressures come from Table A.21E and are functions of temperature.

The energy balance for a nozzle is:

$$\dot{m}_{In}\left(h_{In} + \frac{v_{In}^2}{2}\right) = \dot{m}_{Out}\left(h_{Out} + \frac{v_{Out}^2}{2}\right)$$

Plan:

For illustration purposes, we will use the relative pressure equation. We used the Entropy equation in Example 8.8.

Solve:

Using the relative pressure equation, we have:

$$\frac{P_{Out}}{P_{In}} = \frac{P_{Out,r}}{P_{In,r}}$$

Rearrange the terms.

$$P_{Out,r} = \frac{P_{Out}}{P_{In}} P_{In,r}$$

The relative pressure is a function of temperature and can be found in Table A.21E. The absolute temperature of the air at the inlet is:

$$T_{In} = 540°\text{F} + 459.67$$
$$= 999.67\,\text{R}$$

From Table A.21E, at 1000 R, we find a relative pressure.

$$P_{In,r} = 12.30$$
$$h_{In} = 240.98\,\text{btu/lbm}$$

Now, we can solve for the outlet's relative pressure.

$$P_{Out,r} = \frac{P_{Out}}{P_{In}} P_{In,r}$$
$$= \left(\frac{12\,\text{psia}}{60\,\text{psia}}\right) 12.30$$
$$= 2.46$$

Now, looking through Table A.21E, we look for an entry that is as close to the outlet's relative pressure as possible, and get the Specific Enthalpy at that temperature.

Looking at the table, the outlet air is somewhere between 620 R and 640 R, so we must Interpolate.

$$h_{Out} = h_{620} + (2.46 - P_{r,620})\left(\frac{h_{640} - h_{620}}{P_{r,640} - P_{r,620}}\right)$$
$$= 148.28 + (2.46 - 2.249)\left(\frac{153.09 - 148.28}{2.514 - 2.249}\right)$$
$$h_{Out} = 152.11\,\text{btu/lbm}$$

Now, solving the energy balance equation, we can find the exit velocity of the air.

$$\dot{m}_{In}\left(h_{In} + \frac{v_{In}^2}{2}\right) = \dot{m}_{Out}\left(h_{Out} + \frac{v_{Out}^2}{2}\right)$$

Nozzles are steady-flow devices ($\dot{m}_{In} = \dot{m}_{Out}$), so we can cancel out the mass flowrates.

$$h_{In} + \frac{v_{In}^2}{2} = h_{Out} + \frac{v_{Out}^2}{2}$$
$$v_{Out}^2 = 2h_{In} + v_{In}^2 - 2h_{Out}$$
$$v_{Out} = \sqrt{2h_{In} + v_{In}^2 - 2h_{Out}}$$

Remember that we need to perform a unit conversion on the enthalpies.

$$v_{Out} = \sqrt{2(240.98\,\text{btu/lbm})\left(\frac{25\,037\,\text{ft}^2/\text{s}^2}{1\,\text{btu/lbm}}\right) + (200\,\text{ft/s})^2 - 2(152.11\,\text{btu/lbm})\left(\frac{25\,037\,\text{ft}^2/\text{s}^2}{1\,\text{btu/lbm}}\right)}$$
$$v_{Out} = 2118.98\,\text{ft/s}$$

If a Process involving an ideal gas is Isentropic, then it behaves according to a set of interesting equations (Equations (8.15a) to (8.15c)).

$$\frac{P_1}{P_2} = \left(\frac{\nu_1}{\nu_2}\right)^k \tag{8.15a}$$

$$\frac{T_2}{T_1} = \left(\frac{\nu_1}{\nu_2}\right)^{k-1} \tag{8.15b}$$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} \tag{8.15c}$$

Equations (8.15a) to (8.15c) all have a k value defined as:

$$k = \frac{c_P}{c_V}$$

8.4.1 Variable Specific Heats

$$S_2^\circ = S_1^\circ + R\ln\left(\frac{P_2}{P_1}\right) \tag{8.16}$$

A Complex Numbers

Defn A.0.1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted $\mathbb{C}$.

A complex number is written, in Cartesian form, as shown in Equation (A.1) below.

$$z = a \pm ib \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers)*.* Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

A.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn A.1.1 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\mathrm{Re}\{z\} = x \tag{A.3}$$

Remark A.1.1.1 (Alternative Notation)*.* The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\mathfrak{Re}$$

Defn A.1.2 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\mathrm{Im}\{z\} = y \tag{A.4}$$

Remark A.1.2.1 (Alternative Notation)*.* The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\mathfrak{Im}$$

A.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$z = x_1 + iy_1$$
$$w = x_2 + iy_2$$

A.2.1 Addition

The addition operation, still denoted with the $+$ symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w := (x_1 + x_2) + i(y_1 + y_2) \tag{A.5}$$

A.2.2   Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$
\begin{aligned}
zw &:= (x_1 + iy_1)(x_2 + iy_2) \\
&= (x_1 x_2) + (iy_1 x_2) + (ix_1 y_2) + (i^2 y_1 y_2) \\
&= (x_1 x_2) + i(y_1 x_2 + x_1 y_2) + (-1 y_1 y_2) \\
&= (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)
\end{aligned}
\tag{A.6}
$$

A.3   Complex Conjugates

Defn A.3.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$
z = a \pm bi
$$

then, the conjugate is denoted and calculated as shown below.

$$
\overline{z} = a \mp bi
\tag{A.7}
$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$
z^* = \overline{z}
\tag{A.8}
$$

A.3.1   Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$

2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$
\begin{aligned}
z\overline{z} &= (x + iy)(x - iy) \\
&= x^2 - xyi + xyi - i^2 y^2 \\
&= x^2 - (-1)y^2 \\
&= x^2 + y^2
\end{aligned}
$$

Thus,

$$
z\overline{z} = x^2 + y^2
\tag{A.9}
$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$
\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}
$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$
z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}
$$

Using our solution from Equation (A.9):

$$
\begin{aligned}
&= \frac{(x + iy)^2}{x^2 + y^2} \\
&= \frac{x^2 + 2xyi + i^2 y^2}{x^2 + y^2}
\end{aligned}
$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$= \frac{(x^2 - y^2) + 2xyi}{x^2 + y^2}$$
$$= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2}$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{A.10}$$

A.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.
First, start by defining some expressions so that we can prove some of these properties:

$$z = x + iy$$
$$\overline{z} = x - iy$$

(i) The conjugation operation is commutative.
(ii) The conjugation operation can be distributed over addition and multiplication.

$$\overline{z + w} = \overline{z} + \overline{w}$$
$$\overline{zw} = \overline{z}\,\overline{w}$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers $(z, w \in \mathbb{C})$ where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.
We start by simplifying the left-hand side of the equation $(\overline{z + w})$.

$$\overline{z + w} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$$
$$= \overline{(x_1 + x_2) + i(y_1 + y_2)}$$
$$= (x_1 + x_2) - i(y_1 + y_2)$$

Now, we simplify the other side $(\overline{z} + \overline{w})$.

$$\overline{z} + \overline{w} = \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)}$$
$$= (x_1 - iy_1) + (x_2 - iy_2)$$
$$= (x_1 + x_2) - i(y_1 + y_2)$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

∎

A.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn A.4.1 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in \mathbb{C}$. This is taken from Descarte's definition of a completely real plane.
The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

A.4.1 Modulus of a Complex Number

Defn A.4.2 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$|z|^2 = x^2 + y^2 = z\bar{z}$$
$$|z| = \sqrt{x^2 + y^2}$$

(A.11)

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in \mathbb{C}$). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (A.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\bar{z}\bar{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$= (z\bar{z})(w\bar{w})$$
$$= |z|^2|w|^2$$

Note how we never needed to define z or w, so this is as general a result as possible. ∎

A.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$zw = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$
$$|zw|^2 = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2$$
$$= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right)$$
$$= |z|^2|w|^2$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

A.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

A.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (A.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r$$

(A.12)

Example A.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$|5iz + 10| = 7$$
$$|5i(z + \frac{10}{5i})| = 7$$
$$|5i(z + \frac{2}{i})| = 7$$
$$|5i(z + \frac{2}{i}\frac{-i}{-i})| = 7$$
$$|5i(z - 2i)| = 7$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$|5i||z - 2i| = 7$$
$$5|z - 2i| = 7$$
$$|z - 2i| = \frac{7}{5}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

A.5.1 Annulus

Defn A.5.1 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (A.13).

$$r_1 \leq |z - a| \leq r_2 \tag{A.13}$$

In Equation (A.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

A.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (A.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{A.14}$$

Remark. Note that in the definition of polar form (Equation (A.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$r = |z|$$
$$\theta = \arg(z)$$

Example A.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

- -

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\text{Re}\{z\}^2 + \text{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3 + 1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\cos(\theta) = \frac{-\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{6}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$r = 2$$

$$\theta = \frac{5\pi}{6}$$

$$\arg(z) = \frac{5\pi}{6} + 2\pi k$$

A.6.1 Converting Between Cartesian and Polar Forms

Using Equation (A.14) and Equation (A.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{A.15}$$

A.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

A.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta + \phi) + i\sin(\theta + \phi)\big) \tag{A.16}$$

A.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta - \phi) + i\sin(\theta - \phi)\big) \tag{A.17}$$

A.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph A.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn A.6.1 (de Moiver's Law). Given a complex number z, $z \in \mathbb{C}$ and a rational number n, $n \in \mathbb{Q}$, the exponentiation of z^n is defined as Equation (A.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{A.18}$$

A.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{A.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (A.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{A.20}$$

A w that satisfies Equation (A.20) is an nth root of z.

Example A.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

- -

From Example A.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (A.19), we can find the general form of the roots.

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

$$z^{\frac{1}{3}} = \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right)$$

$$= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right)$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \dots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

---

k = 2

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(2)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(2)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)$$

k = 3

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(3)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(3)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + \frac{36\pi}{18}\right) + i\sin\left(\frac{\pi}{18} + \frac{36\pi}{18}\right)\right)$$

$$= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + 2\pi\right) + i\sin\left(\frac{\pi}{18} + 2\pi\right)\right)$$

$$= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

Thus, the 3 cube roots of z are:

$$z_1^{\frac{1}{3}} = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

$$z_2^{\frac{1}{3}} = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

$$z_3^{\frac{1}{3}} = \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)$$

A.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn A.8.1 (Argument). The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example A.4: Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Argument**?

- -

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$z = -1 - i$$

$$\arg(z) = \tan(\theta) = \frac{-i}{-1}$$

$$= \frac{\pi}{4}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z) = \frac{5\pi}{4} + 2\pi k$$

Thus, the argument of $z = -1 - i$ is $\arg(z) = \frac{5\pi}{4} + 2\pi k$.

Defn A.8.2 (Principal Argument). The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi < \text{Arg}(z) \leq \pi$.

Example A.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

- -

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$z = -1 - i$$

$$\arg(z) = \tan(\theta) = \frac{-i}{-1}$$

$$= \frac{\pi}{4}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\text{Arg}(z) = \frac{5\pi}{4}$.

A.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (A.21).

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big) \tag{A.21}$$

If instead of e as the base, we have some value a, then we have Equation (A.22).

$$a^z = e^{z\ln(a)}$$

$$= e^{\text{Re}\{z\ln(a)\}}\Big(\cos\big(\text{Im}\{z\ln(a)\}\big) + i\sin\big(\text{Im}\{z\ln(a)\}\big)\Big) \tag{A.22}$$

In the case of Equation (A.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example A.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

- -

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$x = -1$$
$$y = \sqrt{3}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\left|e^{-1+i\sqrt{3}}\right| = \left|e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right|$$

$$= e^{-1}$$

$$\arg\left(e^{-1+i\sqrt{3}}\right) = \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right)$$

$$= \sqrt{3} + 2\pi k$$

$$\text{Arg}\left(e^{-1+i\sqrt{3}}\right) = \text{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right)$$

$$= \sqrt{3}$$

Example A.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$e^{-i} = e^{0-i}$$
$$= e^0\big(\cos(-1) + i\sin(-1)\big)$$
$$= 1\big(\cos(-1) + i\sin(-1)\big)$$

Now, with that exponent simplified, we can solve the main question.

$$e^{-e^{-i}} = e^{-1\big(\cos(-1)+i\sin(-1)\big)}$$
$$= e^{-1\big(\cos(1)-i\sin(1)\big)}$$
$$= e^{-\cos(1)+i\sin(1)}$$

If we refer back to Equation (A.21), then it becomes obvious what x and y are.

$$x = -\cos(1)$$
$$y = \sin(1)$$
$$e^{-e^{-i}} = e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\left|e^{-e^{-i}}\right| = \left|e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right|$$
$$= e^{-\cos(1)}$$
$$\arg\left(e^{-e^{-i}}\right) = \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right)$$
$$= \sin(1) + 2\pi k$$
$$\mathrm{Arg}\left(e^{-e^{-i}}\right) = \mathrm{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right)$$
$$= \sin(1)$$

Example A.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^i$?

Use Equation (A.22) to simplify this to a base of e, where we can use the usual Equation (A.21) to solve this.

$$a^z = e^{z\ln(a)}$$
$$1^i = e^{i\ln(1)}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\ln(1) = \log_e|1| + i\arg(1)$$
$$= \log_e(1) + i(0 + 2\pi k)$$
$$= 0 + 2\pi ki$$
$$= 2\pi ki$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$e^{i(2\pi ki)} = e^{2\pi ki^2}$$
$$= e^{2\pi k(-1)}$$
$$z = e^{-2\pi k}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

A.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.23}$$

A.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn A.10.1 (Complex Logarithm). The *complex logarithm* is defined in Equation (A.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$
\begin{aligned}
e^z &= w \\
z &= \ln(w) \\
&= \log_e |w| + i \arg(w)
\end{aligned}
\tag{A.24}
$$

Remark A.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example A.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

- -

We can apply the definition of a Complex Logarithm (Equation (A.24)) directly.

$$
\begin{aligned}
\ln(z) &= \log_e |z| + i \arg(z) \\
&= \log_e |-1| + i \arg(-1) \\
&= \log_e(1) + i(\pi + 2\pi k) \\
&= 0 + i(\pi + 2\pi k) \\
&= i(\pi + 2\pi k)
\end{aligned}
$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \dots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example A.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

- -

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (A.24), to simplify this value.

$$
\begin{aligned}
\ln(w) &= \log_e |w| + i \arg(w) \\
\ln(1) &= \log_e |1| + i \arg(1) \\
&= \log_e 1 + i(0 + 2\pi k) \\
&= 0 + 2\pi k i \\
&= 2\pi k i
\end{aligned}
$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$
\begin{aligned}
\ln(z) &= \ln(2\pi k i) \\
&= \log_e |2\pi k i| + i \arg(2\pi k i) \\
&= \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases} \right)
\end{aligned}
$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i\begin{cases}\frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0\end{cases}\right)$.

A.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\bar{z}) \tag{A.25}$$

A.11 Complex Trigonometry

For the equations below, $z \in \mathbb{C}$. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{A.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{A.27}$$

Example A.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

- -

We start by using the defintion of complex cosine Equation (A.26).

$$\cos(z) = 5$$
$$\frac{e^{iz} + e^{-iz}}{2} = 5$$
$$e^{iz} + e^{-iz} = 10$$
$$e^{iz}\left(e^{iz} + e^{-iz}\right) = e^{iz}(10)$$
$$e^{iz^2} + 1 = 10e^{iz}$$
$$e^{iz^2} - 10e^{iz} + 1 = 0$$

Solve this quadratic equation by using the Quadratic Equation.

$$e^{iz} = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{10 \pm \sqrt{100 - 4}}{2}$$
$$= \frac{10 \pm \sqrt{96}}{2}$$
$$= \frac{10 \pm 4\sqrt{6}}{2}$$
$$= 5 \pm 2\sqrt{6}$$

Use the definition of complex logarithms to simplify the exponential.

$$iz = \ln(5 \pm 2\sqrt{6})$$
$$= \log_e\left|5 \pm 2\sqrt{6}\right| + i\arg(5 \pm 2\sqrt{6})$$
$$= \log_e\left|5 \pm 2\sqrt{6}\right| + i(0 + 2\pi k)$$
$$= \log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi ki$$
$$z = \frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi ki\right)$$
$$= \frac{-i}{-i}\frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right|\right) + 2\pi k$$
$$= 2\pi k - i\log_e\left|5 \pm 2\sqrt{6}\right|$$

Thus, $z = 2\pi k - i \log_e \left| 5 \pm 2\sqrt{6} \right|$.

A.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x \pm iy) = \big(\cos(x)\cosh(y)\big) \mp i\big(\sin(x)\sinh(y)\big) \tag{A.28}$$

$$\sin(x \pm iy) = \big(\sin(x)\cosh(y)\big) \pm i\big(\cos(x)\sinh(y)\big) \tag{A.29}$$

Example A.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

- -

We start by simplifying the cos using Equation (A.28).

$$\cos(x + iy) = \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big)$$
$$\cos(2 + 3i) = \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (A.21), we can solve this.

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big)$$
$$x = \cos(2)\cosh(3)$$
$$y = -\sin(2)\sinh(3)$$
$$e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} = e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$|z| = \left| e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big) \right|$$
$$= e^{\cos(2)\cosh(3)}$$
$$\arg(z) = \arg\!\big(e^{\cos(2)\cosh(3)}\big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\big)\big)$$
$$= -\sin(2)\sinh(3) + 2\pi k$$
$$\mathrm{Arg}(z) = \mathrm{Arg}\!\big(e^{\cos(2)\cosh(3)}\big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\big)\big)$$
$$= -\sin(2)\sinh(3)$$

A.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\overline{\cos(x)} = \cos(x)$$
$$= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \tag{A.30}$$

$$\overline{\sin(x)} = \sin(x)$$
$$= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{A.31}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin\left(\alpha\right) \pm \sin\left(\beta\right) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos\left(\alpha\right) + \cos\left(\beta\right) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos\left(\alpha\right) - \cos\left(\beta\right) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{B.24}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.26}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos\left(\theta\right) + jr\sin\left(\theta\right) \tag{B.27}$$

C Calculus

C.1 L'Hôpital's Rule

L'Hôpital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcilable results.

Lemma C.0.1 (L'Hôpital's Rule). *If the equation*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) holds.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$
$$F'(x) = f(x) \tag{C.3}$$

Defn C.2.3 (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule). The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.
If

$$f(x) = g(x) \cdot h(x)$$

then,

$$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$$
$$\frac{df(x)}{dx} = \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \tag{C.4}$$

C.4 Useful Integrals

$$\int \cos(x)\, dx = \sin(x) \tag{C.5}$$

$$\int \sin(x)\, dx = -\cos(x) \tag{C.6}$$

$$\int x \cos(x)\, dx = \cos(x) + x \sin(x) \tag{C.7}$$

Equation (C.7) simplified with Integration by Parts.

$$\int x \sin(x) \, dx = \sin(x) - x \cos(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2 \cos(x) \, dx = 2x \cos(x) + (x^2 - 2) \sin(x) \tag{C.9}$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2 \sin(x) \, dx = 2x \sin(x) - (x^2 - 2) \cos(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x} \cos(\beta x) \, dx = \frac{e^{\alpha x} \big(\alpha \cos(\beta x) + \beta \sin(\beta x)\big)}{\alpha^2 + \beta^2} + C \tag{C.11}$$

$$\int e^{\alpha x} \sin(\beta x) \, dx = \frac{e^{\alpha x} \big(\alpha \sin(\beta x) - \beta \cos(\beta x)\big)}{\alpha^2 + \beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x} \, dx = \frac{e^{\alpha x}}{\alpha} \tag{C.13}$$

$$\int x e^{\alpha x} \, dx = e^{\alpha x} \left(\frac{x}{\alpha} - \frac{1}{\alpha^2} \right) \tag{C.14}$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha + \beta x} = \int \frac{1}{\alpha + \beta x} \, dx = \frac{1}{\beta} \ln(\alpha + \beta x) \tag{C.15}$$

$$\int \frac{dx}{\alpha^2 + \beta^2 x^2} = \int \frac{1}{\alpha^2 + \beta^2 x^2} \, dx = \frac{1}{\alpha \beta} \arctan\left(\frac{\beta x}{\alpha} \right) \tag{C.16}$$

$$\int \alpha^x \, dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.17}$$

$$\frac{d}{dx} \alpha^x = \frac{d\alpha^x}{dx} = \alpha^x \ln(x) \tag{C.18}$$

C.5   Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule). *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x, t) \, dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt} g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} \, dx + f\big[b(t), t\big] \frac{db(t)}{dt} - f\big[a(t), t\big] \frac{da(t)}{dt} \tag{C.19}$$

C.6   Laplace's Equation

Laplace's Equation is used to define a harmonic equation. These functions are twice continuously differentiable $f : U \to R$, where U is an open subset of $\mathbb{R}^n$, that satisfies Equation (C.20).

$$\frac{\partial^2 f}{\partial x_1{}^2} + \frac{\partial^2 f}{\partial x_2{}^2} + \cdots + \frac{\partial^2 f}{\partial x_n{}^2} = 0 \tag{C.20}$$

This is usually simplified down to

$$\nabla^2 f = 0 \tag{C.21}$$

D   Laplace Transform

D.1   Laplace Transform

Defn D.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$

D.2   Inverse Laplace Transform

Defn D.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\,ds \tag{D.2}$$

D.3   Properties of the Laplace Transform

D.3.1   Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (D.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{D.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{D.3b}$$

D.3.2   Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (D.4).

$$\mathcal{L}\{x(\alpha t)\} = \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \tag{D.4}$$

D.3.3   Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (D.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\{x(t-a)\} = X(s)e^{-as} \tag{D.5}$$

D.3.4   Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\{X(s-a)\} = x(t)e^{at} \tag{D.6}$$

D.3.5   Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\,d\lambda\right\} = \frac{1}{s}X(s) \tag{D.7}$$

D.3.6   Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\{x(t) * v(t)\} = X(s)V(s) \tag{D.8}$$

D.3.7   Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (D.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \tag{D.9}$$

D.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem D.1 (Intial Value Theorem). *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

Theorem D.2 (Final Value Theorem). *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

D.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t - T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n \mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t - T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at} \mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at} \mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt) \mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt) \mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at} \cos(bt) \mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at} \sin(bt) \mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$

$$re^{-at}\cos(bt + \theta)\mathcal{U}(t) \quad \begin{cases} a: & \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: & \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb} + \frac{re^{-j\theta}}{s+a+jb}\right) \\ c: & \frac{As+B}{s^2+2as+c} \begin{cases} r &= \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta &= \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$$

$$e^{-at}\left(A\cos(\sqrt{c-a^2}t) + \frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t) \qquad \frac{As+B}{s^2+2as+c}$$

D.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0) - X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0) + X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n \frac{d^n}{ds^n} X(s) \; n \in \mathbb{N}$
$\frac{d^n}{dt^n} x(t)$	$s^n X(s) - \sum_0^{n-1} s^{n-1-i} \frac{d^i}{dt^i} x(t)\vert_{t=0^-} \; n \in \mathbb{N}$

References

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