1 Introduction

Example 1.1: Confirm Differential Solution.

Test example.

1.1 Definitions and Terminology

Defn 1 (Differential Equation)**.** A *differential equation (DE)* is an equation with 1 or more derivatives.

Remark 1.1. The highest differential determines the order of the differential equation. This means that the differential equation below is of order 2.

$$y'' + y = 0$$
$$\frac{d^2y}{dx^2} + y = 0$$

Defn 2. Initial Value Problem A differential equation with one or more initial conditions is called an *initial value problem (IVP).*

Remark 2.1. To solve an initial value problem, you must have the same number of initial conditions as the order of the differential equation.

Remark 2.2 (Existence of Unique Solution). R is a rectangular region on the xy-plane $a \leq x \leq b$, $c \leq y \leq d$ that contains (x_0, y_0) interior. If $f(x, y)$ and $\frac{df}{dy}$ are continuous on R, then an interval exists I_0 such that $(x_0 - h, x_0 + h)$ where $h > 0$, on the interval $[a, b]$, and a unique function $y(x)$, defined on I_0 that is a solution of the initial value problem.

1.2 Confirm If Differential Equation

You can confirm if the solution $y(x)$ found for a differential equation $y(x)'$ is the solution by differentiating the solution and putting that in the solved differential equation and verfiying that the equation holds true. This is shown in Example 1.2

Example 1.2: .

Given the differential equation, $2y' + y = 0$, is $y = e^{\frac{-x}{2}}$ a solution?

$$y'$$

1.3 Separable Differential Equation

Defn 3 (Separable)**.** A *separable* differential equation allows you to move various elements around to solve the equation. For example,

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P}dP = kdt$$
$$\ln(P) = kt + C$$
$$P = Ce^{kt}$$

Remark 3.1. These are used extensively in modelling phenomena with differential equations. These include: Population Growth, Radioactive Decay, Newton's Law of Cooling/Heating, and Spread of Disease.

1.4 Modeling with Differential Equations

1.4.1 Population Growth

Defn 4 (Population Growth)**.** *Population growth* can be modelled with a separable differential equation. Namely,

$$\frac{dP}{dt} = kP \tag{1.1}$$

Remark 4.1 (Population Growth Parameters). The parameters for the Population Growth equation are given below.

- $k > 0$
- $P > 0$

1.4.2 Radioactive Decay

Defn 5 (Radioactive Decay). *Radioactive decay* is the process that some particularly heave atoms undergo.

Defn 6 (Half-Life). The *half-life* is the usual reported metric, and is defined as the amount of time required for an element to half its mass through Radioactive Decay.

$$\frac{1}{2}A_0 = A_0 e^{kt} \tag{1.2}$$

Remark 6.1 (Radioactive Decay Parameters). The parameters for the Radioactive Decay equation are given below.

- $k < 0$
- $A > 0$

1.4.3 Newton's Law of Cooling/Heating

Defn 7 (Newton's Law of Cooling/Heating). *Newton's Law of Cooling/Heating* is the same equation, but some of the parameters change. This equation is defined as:

$$\frac{dT}{dt} = k\left(T - T_m\right) \tag{1.3}$$

Remark 7.1. The parameters for the Newton's Law of Cooling/Heating equation are given below.

- $\frac{dT}{dt}$; The rate of change of temperature in the object per unit time.
- $k < 0$; The cooling constant and is unique to every object.
- T; The starting temperature.
- T_m; The temperature of the surrounding medium.

1.4.4 Spread of Disease

Defn 8 (Spread of Disease). This is used to model the spread of something throughout a society or group of people.

$$\frac{dx}{dt} = kxy \tag{1.4}$$

Remark 8.1. The parameters for the Spread of Disease equation are given below.

- $\frac{dx}{dt}$; Change in the number of infected per unit time.
- $k < 0$; Transmission Constant
- x; Number of Infected
- y; Number of non-infected, y is really a function of x
  - $y = n + 1 - x$

1.4.5 Chemical Reactions

1.4.6 Tank Mixture

1.4.7 Torricelli's Law

1.4.8 LRC Circuits

Defn 9 (LRC Circuits). An *LRC Circuit* is analyzed in terms of the energy moving through the circuit. There is a unique relationship for the energy in each element:

$$E\left(t\right) = \frac{q}{C} \tag{1.5}$$

$$E\left(t\right) = RI = R\frac{dq}{dt} \tag{1.6}$$

$$E\left(t\right) = L\frac{dI}{dt} = L\frac{d^2q}{dt^2} \tag{1.7}$$

Remark 9.1. Depending on the circuit given, you might use a combination of these, but you ***must*** have at least one capacitor or inductor, otherwise it is not a differential equation.

Remark 9.2. These equations *add* together when the entire circuit is in series, i.e. the elements are put together back-to-back.

1.5 Linear and Non-Linear Differential Equations

Defn 10 (Linear Differential Equation)**.** A *linear differential equation* is one that satisfies one of the following equations below.

$$\begin{aligned} a_1(x)\frac{dy}{dx} + a_0(x) &= g(x) \\ a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x) &= g(x) \end{aligned} \tag{1.8}$$

Remark 10.1. The equations in Equation (1.8) can be generalized to the nth order as shown below.

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x) = g(x) \tag{1.9}$$

Defn 11 (Non-Linear)**.** A *non-linear* differential equation is one that does not satisfy the definition of a Linear Differential Equation. It does not obey Equation (1.9).

A Reference Material

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm \imath\alpha} = \cos(\alpha) \pm \imath\sin(\alpha) \tag{B.3}$$

$$\sin(x) = \frac{e^{\imath x} + e^{-\imath x}}{2} \tag{B.4}$$

$$\cos(x) = \frac{e^{\imath x} - e^{-\imath x}}{2\imath} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\alpha) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

B.10 Rectangular to Polar

$$a + \imath b = \sqrt{a^2 + b^2} e^{\imath\theta} = re^{\imath\theta} \tag{B.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.25}$$

B.11 Polar to Rectangular

$$re^{\imath\theta} = r\cos(\theta) + \imath r\sin(\theta) \tag{B.26}$$

C Calculus

C.1 Fundamental Theorems of Calculus

Defn 12 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{C.1}$$

Defn 13 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.2}$$

Defn 14 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$