EITF75: Systems and Signals - Reference Sheet

Karl Hallsby

Last Edited: October 13, 2019

Contents

1 Sinusoids

There are several ways to characterize Sinusoids. The first is by dimension:

1. Multidimensional/Multichannel Signals
2. Monodimensional/Monochannel Signals

You can also classify sinusoids by their independent variable (usually time) and the values they take.

1. Continuous-Time Signals or Analog Signals
2. Discrete-Time Signals
3. There is a third way to classify sinusoids and their signals: Digital Signals

Defn 1 (Continuous-Time Signals)**.** *Continuous-time signals* or *Analog signals* are defined for every value of time and they take on values in the continuous interval (a, b), where a can be $-\infty$ and b can be ∞. Mathematically, these signals can be described by functions of a continuous variable.

For example,

$$x_1(t) = \cos \pi t,\ x_2(t) = e^{-|t|},\ -\infty < t < \infty$$

Defn 2 (Discrete-Time Signals)**.** *Discrete-time signals* are defined only at certain specifed values of time. These time instants ***need not*** be equidistant, but in practice, they are usually taken at equally speced intervals for computation convenience and mathematical tractability.

For example,

$$x(t_n) = e^{-|t_n|},\ n = 0, \pm 1, \pm 2, \ldots$$

A Discrete-Time Signals can be represented mathematically by a sequence of real or complex numbers.

Remark 2.1. To emphasize the discrete-time nature of the signal, we shall denote the signal as $x(n)$, rather than $x(t)$.

Remark 2.2. If the time instants t_n are equally spaced (i.e., $t_n = nT$), the notation $x(nT)$ is also used.

1.1 Continuous-Time Signals

1.1.1 Frequency in Continuous-Time Signals

A simple harmonic oscillation is mathematically described by Equation (1.1).

$$x_a(t) = A \cos(\Omega t + \theta),\ -\infty < t < \infty \tag{1.1}$$

Remark. The subscript a is used with $x(t)$ to denote an analog signal.

This signal is completely characterized by three parameters:

1. A, the *amplitude* of the sinusoid
2. Ω, the *frequency* in radians per second (rad/s)
3. θ, the *phase* in radians.

Instead of Ω, the frequency F in cycles per second or hertz (Hz) is used.

$$\Omega = 2\pi F \tag{1.2}$$

Plugging (1.2) into (1.1), yields

$$x_a(t) = A \cos(2\pi F t + \theta),\ -\infty < t < \infty \tag{1.3}$$

1.1.2 Properties of Continuous-Time Sinusoidal Signals

The analog sinusoidal signal in equation (1.3) is characterized by the following properties:

(i) For every fixed value of the frequency F, $x_a(t)$ is periodic.

$$x_a(t + T_p) = x_a(t)$$

where $T_p = \frac{1}{F}$ is the fundamental period.

(ii) Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.

(iii) Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in the given time interval.

1.2 Discrete-Time Signals

These are usually found by sampling analog signals or Continuous-Time Signals. There are 2 ways to express this, both are shown in Equation (1.4).

$$
\begin{aligned}
x(n) &= x(t|t = nT_S) \\
x(n) &= x\left(t|t = \frac{n}{F_S}\right)
\end{aligned}
\tag{1.4}
$$

1.2.1 Frequency in Discrete-Time Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta)\,,\; n \in \mathbb{Z}\,,\; -\infty < n < \infty \tag{1.5}$$

The signal is characterized by these parameters:

1. n, the sample number. MUST be an integer.
2. A, the *amplitude* of the sinusoid
3. ω, the *angular frequency* in radians per sample
4. θ, is the *phase*, in radians.

Instead of ω, we use the frequency variable f defined by

$$\omega \equiv 2\pi f \tag{1.6}$$

Using (1.5) and (1.6) yields

$$x(n) = A\cos(2\pi f n + \theta)\,,\; n \in \mathbb{Z}\,,\; -\infty < n < \infty \tag{1.7}$$

1.2.2 Properties of Discrete-Time Sinusoidal Signals

(i) A discrete-time sinusoid is periodic ***ONLY*** if its frequency is a rational number.

(ii) Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical. This leads us to the idea of a Frequency Alias.

(iii) The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pm\pi$ or, equivalently, $f = \pm\frac{1}{2}$.

1.2.3 Frequency Aliases

The concept of a Frequency Alias is drawn from the idea that discrete-time sinusoids whose frequencies are separated by an integer mutliple of 2π are identical and that frequencies $|f| > \frac{1}{2}$ are identical. (Properties **(ii)** and **(iii)**)

Defn 3 (Frequency Alias)**.** A *frequency alias* is a sinusoid having a frequency $|\omega| > \pi$ or $|f| > \frac{1}{2}$. This is because this sinusoid is *indistinguishable* (*identical*) to one with frequency $|\omega| < \pi$ or $|f| < \frac{1}{2}$.

A *frequency alias* is a sequence resulting from the following assertion based on the sinusoid $\cos(\omega_0 n + \theta)$.
It follows that

$$\cos\left[(\omega_0 + 2\pi)\,n + \theta\right] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta$$

As a result, all sinusoidal sequences

$$x_k(n) = A\cos(\omega_k n + \theta),\; k = 0, 1, 2, \ldots$$

where

$$\omega k = \omega_0 + 2k\pi,\; -\pi \leq \omega_0 \leq \pi$$

are *indistinguishable* (i.e., *identical*).

Because of this, we regard frequencies in the range of $-\pi \leq \omega \leq \pi$ or $-\frac{1}{2} \leq f \leq \frac{1}{2}$ as unique, and all frequencies that fall outside of these ranges as aliases.

Remark 3.1. It should be noted that there is a difference between discrete-time sinusoids and continuous-time sinusoids here. Continuous-time sinusoids have distinct signals for Ω or F in the entire range $-\infty < \Omega < \infty$ or $-\infty < F < \infty$.

1.3 Sampling Rates and Sampling Frequency

Most signals of interest are analog. To process these signals, they must be collected and converted to a digital form, that is, to convert them to a sequence of numbers having finite precision. This is called *analog-to-digital (A/D) conversion*. Conceptually, we view this conversion as a 3-step process.

1. Sampling
2. Quantization
3. Coding

1.3.1 Nyquist Rate

1.3.2 Nyquist Frequency

1.4 Digital Signals

Defn 4 (Digital Signals)**.** *Digital signals* are a subset of Discrete-Time Signals. In this case, not only are the values being measured occuring at fixed points in time, the values themselves can only take certain, fixed values.

1.4.1 Quantization

Defn 5 (Quantization)**.** This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-value (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the Quantization Error.

1.4.1.1 Quantization Levels

1.4.1.2 Quantization Error

Defn 6 (Quantization Error)**.** The *quantization error* of something.

1.4.1.3 Bit Requirements

1.4.1.4 Bit Rate

2 Discrete-Time Systems

As discussed in Section 1.2, $x(n)$ is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is *not defined* at instants between the samples. Also, if n is not an integer, $x(n)$ is not defined.

Besides graphical representation of a discrete-time system, there are 3 ways to represent a discrete-time signal.

1. Functional Representation
2. Tabular Representation
3. Sequence Representation

2.1 Representing Discrete-Time Systems

2.1.1 Functional Representation

This representation of a discrete-time system is done as a mathematical function.

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1}$$

2.1.2 Tabular Representation

This representation of a discrete-time sysem is done as a table of corresponding values.

n	...	-2	-1	0	1	2	3	4	5	...
$x(n)$	...	0	0	0	1	4	1	0	0	...

2.1.3 Sequence Representation

There are 2 methods of representation for this. The first includes all values for $-\infty < n < \infty$. In all cases, $n = 0$ is marked in the sequence, somehow. I will do this with an underline.

$$x(n) = \{\ldots, 0, \underline{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.2}$$

The second only works if all $x(n)$ values for $n < 0$ are 0.

$$x(n) = \{\underline{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.3}$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underline{-2}, 5, 0, 4, -1\} \tag{2.4}$$

This is identified as a seven-point sequence.

A finite-duration sequence where $x(n) = 0$ for all $n < 0$ is represented as

$$x(n) = \{\underline{0}, 1, 4, 1\} \tag{2.5}$$

This is identified as a four-point sequence.

2.2 Elementary Discrete-Time Signals

The following signals are basic signals that appear often and play an important role in signal processing.

2.2.1 Unit Impulse Signal

Defn 7 (Unit Impulse Signal)**.** The *unit impulse signal* or *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv= \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.6}$$

This function is a signal that is zero everywhere, except at $n = 0$, where its value is 1.

Remark 7.1. This signal is different that the analog signal $\delta(t)$, which is also called a unit impulse, and is defined to be 0 everywhere except $t = 0$. The discrete unit impulse sequence is much less mathematically complicated.

2.2.2 Unit Step Signal

Defn 8 (Unit Step Signal)**.** The *unit step signal* is denoted as $u(n)$ or as $\mathcal{U}(n)$ and is defined as

$$\mathcal{U}(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.7}$$

2.2.3 Unit Ramp Signal

Defn 9 (Unit Ramp Signal)**.** The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.8}$$

2.2.4 Exponential Signal

Defn 10 (Exponential Signal)**.** The *exponential signal* is a sequence of the form

$$x(n) = a^n \text{ for all } n \tag{2.9}$$

If a is real, then $x(n)$ is a real signal. When a is complex valued ($a \equiv b \pm cj$), it can be expressed as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n (\cos\theta n + j \sin\theta n) \end{aligned} \tag{2.10}$$

This can be expressed by graphing the real and imaginary parts

$$\begin{aligned} x_R(n) &\equiv r^n \cos\theta n \\ x_I(n) &\equiv r^n j \sin\theta n \end{aligned} \tag{2.11}$$

or by graphing the amplitude function and phase function.

$$\begin{aligned} |x(n)| &= A(n) \equiv r^n \\ \angle x(n) &= \phi(n) \equiv \theta n \end{aligned} \tag{2.12}$$

2.3 Classification of Discrete-Time Signals

In order to apply some mathematical methods to discrete-time signals, we must characterize these signals.

2.3.1 Energy Signal

Defn 11 (Energy Signal)**.** The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.13}$$

The energy of a signal can be finite or infinite. If E is finite ($0 < E < \infty$), then $x(n)$ is called an *energy signal.*

2.3.2 Power Signal

Defn 12 (Power Signal)**.** The average power of a discrete time signal $x(n)$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \tag{2.14}$$

This means that there are 2 potential outcomes:

1. If E is finite, $P = 0$
2. If E is infinite, P may be either finite or infinite

If P is finite and nonzero, the signal is called a *power signal.*

2.3.3 Periodic and Aperiodic Signals

A signal $x(n)$ is periodic with period N ($N > 0$) if and only if

$$x(n+N) = x(n) \text{for all } n \tag{2.15}$$

The smallest value of N for which (2.15) holds is called the fundamental period. If there is no value of N that satisfies (2.15), the signal is called *nonperiodic* or *aperiodic.*

2.3.4 Symmetric and Antisymmetric Signals

A real-valued signal $x(n)$ is called *symmetric* or *even* if

$$x(n) = x(-n) \tag{2.16}$$

On the other hand, a signal $x(n)$ is called *antisymmetric* or *odd* if

$$x(n) = -x(-n) \tag{2.17}$$

2.4 Classification of Discrete-Time Systems

2.4.1 Static versus Dynamic Systems

Defn 13 (Static)**.** A discrete-time system is called *static* or *memoryless* if its output at any instant n depends only on the input sample at the same time, but not on past or future samples of the input.

Defn 14 (Dynamic)**.** A discrete-time system is called *dynamic* if its output at any instant n depends not only on the input sample at the same time, but **also** on past and/or future samples of the input.

If the output of s system at tine n is completely determined by the input samples in the interval from $n - N$ to $n (N \geq 0)$, the system is said to have a *memory* of duration N. If $N = 0$, then the system is Static, whereas if $N = \infty$, the system is said to have *infinite memory.*

2.4.2 Time-Invariant versus Time-Variant Systems

Defn 15 (Time-Invariant)**.** A *time-invariant* system in one whose output is affected only in time, if the input's time is changed. A relaxed system $\mathcal{T}$ is *time-invariant* or *shift invariant* is and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

implies that

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k)$$

for every input signal $x(n)$ and every time shift k.

To determine if any given system is Time-Invariant, we need to perform a test drawn from Definition 15.

1. Excite the system with an arbitrary input sequence $x(n)$, which produces an output $y(n)$.
2. Delay the input sequence by some amount k and recompute the output.
3. If $y(n,k) = y(n-k)$ for all possible values of k, the system is Time-Invariant.

2.4.3 Linear versus Non-Linear Systems

A linear system is one that satisfies the *superposition principle*.

Defn 16 (Linear)**.** A system is *linear* if and only if

$$\mathcal{T}\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 \mathcal{T}\left[x_1(n)\right] + a_2 \mathcal{T}\left[x_2(n)\right] \tag{2.18}$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

The Linearity property can be broken down into 2 parts:

1. Multiplicative Property
2. Additive Property

Remark 16.1. The Linearity property can be extended to any number of terms.

2.4.3.1 Multiplicative Property

Defn 17 (Multiplicative Property)**.** The *multiplicative* or *scaling property* is one requirement of a Linear system and is part of the definition of the superposition principle. If the input is scaled, the output is scaled by a proportional amount.

$$\begin{aligned} \mathcal{T}\left[a_1 x_1(n)\right] &= a_1 \mathcal{T}\left[x_1(n)\right] \\ &= a_1 y_1(n) \end{aligned} \tag{2.19}$$

2.4.3.2 Additive Property

Defn 18 (Additive Property)**.** The *additive property* is one requirement of a Linear system and is part of the definition of the superposition principle.

$$\begin{aligned} \mathcal{T}\left[x_1(n) + x_2(n)\right] &= \mathcal{T}\left[x_1(n)\right] + \mathcal{T}\left[x_2(n)\right] \\ &= y_1(n) + y_2(n) \end{aligned} \tag{2.20}$$

Defn 19 (Nonlinear)**.** If a relaxed system does no satisfy the superposition principle, or the definition of a Linear system, it is *nonlinear*.

2.4.4 Causal versus Noncausal Systems

Defn 20 (Causal)**.** A system is said to *causal* if the output of the system, $y(n)$, at any time n depends only on present and past inputs [i.e., $x(n), x(n-1), x(n-2), \ldots$], but does not depends on future inputs [i.e., $x(n+1), x(n+2), \ldots$].

Mathematically, the output of a causal system satisfies an equation of the form

$$y(n) = F\left[x(n), x(n-1), x(n-2), \ldots\right] \tag{2.21}$$

where $F[\cdot]$ is some arbitrary function.

Defn 21 (Noncausal)**.** If a system does not satisfy the definition of a Causal system, then it is *noncausal*. A noncausal system depends not just on present and past inputs, but also on future inputs.

Remark 21.1. You can never have a noncausal system in real-time signal processing applications. However, if the signal has been recorded and will be processed offline, then a noncausal system can be constructed.

2.4.5 Stable versus Unstable Systems

Stability is incredibly important. Unstable stystems usually have erratic and extreme behavior.

Defn 22 (Stable)**.** An arbitrary relaxed system is said to be *Bounded Input-Bounded Output Stable (BIBO)* if and only if every bounded input produces a bounded output.

Mathematically, this means the input sequence $x(n)$ and the output sequence $y(n)$ are bounded, where there are some finite numbers M_x and M_y such that

$$|x(n)| \leq M_x < \infty \;\; |y(n)| \leq M_y < \infty \, \forall n \tag{2.22}$$

Defn 23 (Unstable)**.** If the some bounded input $x(n)$, the output is unbounded (infinite), the system is *unstable*.

2.4.6 Linear Time-Invariant Systems

Defn 24 (Linear Time-Invariant)**.** A *Linear Time-Invariant (LTI)* signal or system is one that is:

(i) Linear

(ii) Time-Invariant

2.5 Discrete-Time Signal Manipulations

2.5.1 Transformation of the Independent Variable (Time)

It is important to note that Shifting in Time and Folding are not commutative. For example,

$$\mathrm{TD_k}\{\mathrm{FD}\left[x(n)\right]\} = \mathrm{TD_k}\left[x(-n)\right] = x(-n+k) \tag{2.23}$$

whereas

$$\mathrm{FD}\{\mathrm{TD_k}\left[x(n)\right]\} = \mathrm{FD}\left[x(n-k)\right] = x(-n-k) \tag{2.24}$$

2.5.1.1 Shifting in Time

A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time (moves left). If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units of time (moves right).

This could be denoted by

$$\mathrm{TD_k}\left[x(n)\right] = x(n-k) \tag{2.25}$$

You cannot advance a signal that is being generated in real-time. Because that would involve signal samples that haven't been generated yet. So, you can only advance a signal that is stored on something. However, you can always introduce a delay to a signal.

2.5.1.2 Folding

Another useful modification of the time base is to replace n with $-n$. The result is a *folding* or *reflection* of the original signal around $n = 0$.

This could be denoted by

$$\mathrm{FD}\left[x(n)\right] = x(-n) \tag{2.26}$$

2.5.2 Addition, Multiplication, and Scaling

Amplitude modifications include Addition, Multiplication, and Amplitude Scaling.

2.5.2.1 Addition

The *sum* of 2 signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the sum of the values of these two signals at that instant.

$$y(n) = x_1(n) + x_2(n), \; -\infty < n < \infty \tag{2.27}$$

2.5.2.2 Multiplication

The *product* of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the product of the values of these two signals at that instant.

$$y(n) = x_1(n)x_2(n), \; -\infty < n < \infty \tag{2.28}$$

2.5.2.3 Amplitude Scaling

Amplitude scaling of a signal by a constant A is accomplished by multiplying every signal sample by A. Consequently, we obtain

$$y(n) = Ax(n), \; -\infty < n < \infty \tag{2.29}$$

2.6 Discrete-Time System Difference Equation

There exists an equation that describes any Linear Time-Invariant discrete-time system. This equation works for both Infinite Impulse Response and Finite Impulse Response filters.

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{l=0}^{L} b_l x(n-l) \tag{2.30}$$

Occasionally, Equation (2.30) will be written like below.

$$y(n) = \sum_{l=0}^{L} b_l x(n-l) - \sum_{k=1}^{N} a_k y(n-k)$$

Defn 25 (Infinite Impulse Response)**.** An *Infinite Impulse Response (IIR)* filter is one that has an impulse response which does not become exactly zero past a certain point, but continues indefinitely. This is opposite to a Finite Impulse Response Filter.

Defn 26 (Finite Impulse Response)**.** A *Finite Impulse Response (FIR)* filter is a filter whose impulse response (or response to any finite length input) is of finite duration, because it settles to zero in finite time. This is opposite to a Infinite Impulse Response Filter.

3 Convolutions

Defn 27 (Linear Convolution)**.** The *linear convolution* is more commonly called a *convolution.* It is a mathematical operation that involves infinite sums. It defines the relationship between 2 signals to produce an output.

$$y(t) = \sum_{k=-\infty}^{\infty} x(k) * h(n-k) \tag{3.1}$$

Because of associativity, commutativity, and distributivity, Equation (3.1) can be equivalently rewritten as

$$y(t) = \sum_{k=-\infty}^{\infty} h(k) * x(n-k) \tag{3.2}$$

The length of the resulting sequence from a linear convolution is

$$2L - 1 \tag{3.3}$$

where L is the length of the input sequences.

To compute the Linear Convolution of Equations (3.1) to (3.2):

1. Perform a Folding on one of the two signals
2. If necessary, pad a signal with 0s to ensure the 2 signals are the same length, L
3. "Run" both signals by each other.
   - This is illustrated in Example 3.1.
4. Perform this for all values of n.

Example 3.1: Linear Convolution. Problem 2.16b, Part 2

Compute the Linear Convolution $y(n) = h(n) * x(n)$ of the following signal. Check your result with this formula $\sum\limits_{y \in Y} = \sum\limits_{h \in H} \sum\limits_{x \in X}$.

$$x(n) = \{\underline{1}, 2, -1\}$$
$$h(n) = \{\underline{1}, 2, -1\}$$

Start by Folding a signal, I chose $h(n)$ to get

$$h(-n) = \{-1, 2, \underline{1}\}$$

Now we "run" each signal past each other. The $x(n)$ signal is the left operand and the $h(-n)$ signal is the right operand in the multiplications below.

$$\begin{aligned}
y(0) &= (0 \cdot -1) + (0 \cdot 2) + (1 \cdot 1) + (2 \cdot 0) + (-1 \cdot 0) = 1 \\
y(1) &= (0 \cdot -1) + (1 \cdot 2) + (2 \cdot 1) + (-1 \cdot 0) = 4 \\
y(2) &= (1 \cdot -1) + (2 \cdot 2) + (-1 \cdot 1) = 2 \\
y(3) &= (1 \cdot 0) + (2 \cdot -1) + (-1 \cdot 2) + (0 \cdot 1) = -4 \\
y(4) &= (1 \cdot 0) + (2 \cdot 0) + (-1 \cdot -1) + (0 \cdot 2) + (0 \cdot 1) = 1 \\
y(5) &= (1 \cdot 0) + (2 \cdot 0) + (-1 \cdot 0) + (0 \cdot -1) + (0 \cdot 2) + (0 \cdot 1) = 0
\end{aligned}$$

Thus, our output sequence is

$$y(n) = \{\underline{1}, 4, 2, -4, 1\}$$

We can verify the length of the output to be $2L - 1$. Since $2(3) - 1 = 5$, and the convolution is of length 5, we are correct here.
Now we check our solution

$$\sum_{y \in Y} = 4$$
$$\sum_{x \in X} = 2$$
$$\sum_{h \in H} = 2$$

so, according to the equation provided

$$4 = 2 \cdot 2$$

is true and correct.
So, our answer is: $y(n) = \{\underline{1}, 4, 2, -4, 1\}$.

Defn 28 (Linear Time-Invariant System Convolution)**.** If there is a relaxed Linear Time-Invariant system to an input $x(n)$, then the output can be found by computing the Linear Convolution of the input with the sample response on the system. This results in the equation shown below.

$$y(n) = x(n) * h(n) \tag{3.4}$$

3.1 Properties of the Convolution

3.1.1 Identity Property

Defn 29 (Identity Property)**.** The Unit Impulse Signal is the identity element for the Linear Convolution.

$$y(n) = x(n) * \delta(n) = x(n) \tag{3.5}$$

Identity Property	$y(n) = x(n) * \delta(n) = x(n)$
Shifting Property	$x(n) * \delta(n-k) = y(n-k) = x(n-k)$
Commutative Law	$x(n) * h(n) = h(n) * x(n)$
Associative Law	$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
Distributive Law	$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

Table 3.1: Properties of the Convolution

3.1.2 Shifting Property

Defn 30 (Shifting Property)**.** Since the $\delta(n)$ function is the Identity function, if we shift $\delta(n)$ by k, the convolution sequence is also shifted by k.

$$x(n) * \delta(n-k) = y(n-k) = x(n-k) \tag{3.6}$$

3.1.3 Commutative Law

Defn 31 (Commutative Law for Convolutions)**.** The *commutative law for Linear Convolutions* is just like many other operations.

$$x(n) * h(n) = h(n) * x(n) \tag{3.7}$$

3.1.4 Associative Law

Defn 32 (Associative Law for Convolutions)**.** The *associative law for Linear Convolutions* is just like many other operations.

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)] \tag{3.8}$$

3.1.5 Distributive Law

Defn 33 (Distributive Law for Convolutions)**.** The *distributive law for Linear Convolutions* is just like many other operations.

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n) \tag{3.9}$$

3.2 Correlation

Defn 34 (Correlation)**.** *Correlation* measures the similarity between two signals. The greater the correlation, the more similar they are.

There are 2 types of Correlation.

1. Cross Correlation
2. Auto Correlation

3.2.1 Cross Correlation

Defn 35 (Cross Correlation)**.** *Cross correlation* measures the similarity between time shifted versions of ***different*** signals.

The defining equation is shown below:

$$r_{y,x}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) \tag{3.10}$$

However, there is a way to express Equation (3.10) in terms of a Linear Convolution.

$$r_{y,x}(k) = y(n) * x(-n) \tag{3.11}$$

3.2.2 Auto Correlation

Defn 36 (Auto Correlation)**.** *Auto correlation* measures the similarity between time shifted version of the ***same*** signal.

The defining equation is:

$$r_{x,x}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) \tag{3.12}$$

However, there is a way to express Equation (3.12) in terms of a Linear Convolution.

$$r_{x,x}(k) = x(n) * x(-n) \tag{3.13}$$

Remark 36.1. It is good to note that the Auto Correlation is technically a type of Cross Correlation where the second function is the same as the first.

4 The $\mathcal{Z}$-Transform

The $\mathcal{Z}$-Transform plays the same role in the analysis of Discrete-Time Signals and LTI systems as the Laplace Transform does in the analysis of Continuous-Time Signals and LTI systems.

4.1 The $\mathcal{Z}$-Transform

Defn 37 ($\mathcal{Z}$-Transform)**.** The *z-transform* is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.1}$$

Remark 37.1. For convenience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv \mathcal{Z}\{x(n)\} \tag{4.2}$$

and the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \overset{\mathrm{z}}{\longleftrightarrow} X(z) \tag{4.3}$$

4.1.1 Region of Convergence

Defn 38 (ROC)**.** The *ROC* or *region of convergence* is the region for which the infinite power series in the z-transform has a convergent solution.

Remark 38.1. ***Any time we cite a z-transform, we should also indicate its ROC***

Example 4.1: Simple Z-Transform.

Determine the z-transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n \mathcal{U}(n)$$

The z-transform is the infinite power series

$$\begin{aligned} X(z) &= 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^{-2} + \cdots + \left(\frac{1}{2}\right)^n z^{-n} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n \end{aligned}$$

Because this is an infinite geometric series, we can solve with with our equivalency:

$$1 + A + A^2 + \cdots + A^n + \cdots = \frac{1}{1-A} \text{ if } |A| < 1$$

Thus, $X(z)$ converges to

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC} : |z| > \frac{1}{2}$$

4.1.2 The One-Sided $\mathcal{Z}$-Transform

Defn 39 (One-Sided $\mathcal{Z}$-Transform)**.** The *one-sided z-transform* is the same as the $\mathcal{Z}$-Transform, but is only defined at n values greater than or equal to 0.

$$X(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n} \tag{4.4}$$

The One-Sided $\mathcal{Z}$-Transform is generally used when there are initial conditions on a causal signal. This captures the normal causal portion of the signal, while also showing the effect of the initial condition.

Signal	ROC
Finite-Duration Signals	
Causal	Entire z-plane except $z = 0$
Anticausal	Entire z-plane except $z = \infty$
Two-Sided	Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals	
Causal	$\lvert z\rvert > r_2$
Anticausal	$\lvert z\rvert < r_1$
Two-Sided	$r_2 < \lvert z\rvert < r_1$

Table 4.1: Characteristic Familes of Signals with Their Corresponding ROCs

4.2 The Inverse $\mathcal{Z}$-Transform

This is the formal definition of The Inverse $\mathcal{Z}$-Transform.

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz \tag{4.5}$$

where the integrals is a contour integral over a closed path C that encloses the origin and lies within the region of convergence of $X(z)$.

There are 3 methods that are often used for the evaluation of the inverse z-transform in practice:

1. Direct evaluation of (4.5).
2. Expansion into a series of terms, in the variable sz and z^{-1}.
3. Partial-fraction expansion and table lookup.

4.2.1 The Inverse $\mathcal{Z}$-Transform by Contour Integration

Defn 40 (Cauchy's Integral Theorem)**.** Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the derivative $\frac{\mathrm{d}f(z)}{\mathrm{d}z}$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0}\, \mathrm{d}z = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \tag{4.6}$$

More generally, if the $(k + 1)$-order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k}\, \mathrm{d}z = \begin{cases} \left.\frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1} f(z)}{\mathrm{d}z^{k-1}}\right|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \tag{4.7}$$

4.2.2 The Inverse $\mathcal{Z}$-Transform by Power Series Expansion

4.2.3 The Inverse $\mathcal{Z}$-Transform by Partial-Fraction Expansion

4.3 Properties of the $\mathcal{Z}$-Transform

4.3.1 $\mathcal{Z}$-Transform Linearity

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1 X_1(z) + a_2 X_2(z) \tag{4.8}$$

for any constants a_1 and a_2.

The linearity property can be generalized to an arbitrary number of signals.

Property	Time Domain	z-Domain	ROC
	$x(n)$	$X(z)$	ROC : $r_2 < \|z\| < r_1$
Notation	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
$\mathcal{Z}$-Transform Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
$\mathcal{Z}$-Transform Time Shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
$\mathcal{Z}$-Domain Scaling	$a^nx(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
$\mathcal{Z}$-Transform Time Reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real Part	$\text{Re}\{x(n)\}$	$\frac{1}{2}\left[X(z) + X^*(z^*)\right]$	Includes ROC
p Imaginary Part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j\left[X(z) - X^*(z^*)\right]$	Includes ROC
$\mathcal{Z}$-Domain Differentiation	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\|r_1$
$\mathcal{Z}$-Domain Convolutions	$x_1 * x_2$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
$\mathcal{Z}$-Transform 2 Sequence Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)x_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial Value Theorem for $\mathcal{Z}$-Transform	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
$\mathcal{Z}$-Transform 2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least, $r_{1l}r_{2l} < \|a\| < r_{1u}r_{2u}$
Parsevals Relation for $\mathcal{Z}$-Transform	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$= \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

Table 4.2: Properties of the $\mathcal{Z}$-Transform

Example 4.2: Simple Z-Transform Linearity Problem. Example 3.2.1

Deermine the z-transform and the ROC of the signal

$$x(n) = \left[3\left(2^n\right) - 4\left(3^n\right)\right]\mathcal{U}(n)$$

Solution on Page 158.

Example 4.3: Z-Transform Linearity on Trig Functions. Example 3.2.2

Determine the z-transform of the signals

(a) $x(n) = (\cos\omega_0 n)\mathcal{U}(n)$

(b) $x(n) = (\sin\omega_0 n)\mathcal{U}(n)$

Solution on Pages 158-159.

4.3.2 $\mathcal{Z}$-Transform Time Shifting

If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z) \tag{4.9}$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

Example 4.4: $\mathcal{Z}$-Transform Time Shifting. Example 3.2.3

By applying the time-shifting property, determine the z-transform of the signals

$$x_1(n) = \{1, 2, \underline{5}, 7, 0, 1\}$$
$$x_2(n) = \{\underline{0}, 0, 1, 2, 5, 7, 0, 1\}$$

from the z-transform of

$$x_0(n) = \{\underline{1}, 2, 5, 7, 0, 1\}$$
$$X_0(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}, \text{ROC : entire } z\text{-plane except } z = 0$$

Solution on Page 160.

4.3.3 $\mathcal{Z}$-Domain Scaling

If

$$x(n) \overset{\text{z}}{\longleftrightarrow} X(z),\ \text{ROC} : r_1 < |z| < r_2$$

then

$$a^n x(n) \overset{\text{z}}{\longleftrightarrow} X\left(a^{-1}z\right),\ \text{ROC} : |a|r_1 < |z| < |a|r_2 \tag{4.10}$$

4.3.4 $\mathcal{Z}$-Transform Time Reversal

If

$$x(n) \overset{\text{z}}{\longleftrightarrow} X(z),\ \text{ROC} : r_1 < |z| < r_2$$

then

$$x(-n) \overset{\text{z}}{\longleftrightarrow} X(z^{-1}), \text{ROC} : \frac{1}{r_2} < |z| < \frac{1}{r_1} \tag{4.11}$$

Example 4.5: $\mathcal{Z}$-Transform Time Reversal. Example 3.2.6

Determine the z-transform of the signal

$$x(n) = \mathcal{U}(-n)$$

The transform for $\mathcal{U}(n)$ is given in Table 4.3.

$$\mathcal{U}(n) \overset{\text{z}}{\longleftrightarrow} \frac{1}{1 - x^{-1}},\ \text{ROC} : |z| > 1$$

By using (4.11), we obtain

$$\mathcal{U}(-n) \overset{\text{z}}{\longleftrightarrow} \frac{1}{1 - z}, \text{ROC} : |z| < 1$$

4.3.5 $\mathcal{Z}$-Domain Differentiation

If

$$x(n) \overset{\text{z}}{\longleftrightarrow} X(z)$$

then

$$nx(n) \overset{\text{z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} \tag{4.12}$$

Example 4.6: $\mathcal{Z}$-Domain Differentiation. Example 3.2.7

Determine the z-transform of the signal

$$x(n) = na^n\,\mathcal{U}(n)$$

The signal $x(n)$ can be expressed as $nx_1(n)$, where $x_1(n) = a^n\,\mathcal{U}(n)$. By passing this through the z-transform, we have

$$x_1(n) = a^n\,\mathcal{U}(n) \overset{\text{z}}{\longleftrightarrow} X_1(z) = \frac{1}{1 - az^{-1}},\ \text{ROC} : |z| > |a|$$

Then by using (4.12), we obtain

$$na^n\,\mathcal{U}(n) \overset{\text{z}}{\longleftrightarrow} X(z) = -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{\left(1 - az^{-1}\right)^2}$$

4.3.6 $\mathcal{Z}$-Domain Convolutions

If

$$x_1(n) \overset{\text{z}}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{\text{z}}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \overset{\text{z}}{\longleftrightarrow} X(z) = X_1(z)X_2(z) \tag{4.13}$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

Example 4.7: $\mathcal{Z}$-Domain Convolutions. Example 3.2.9

Compute the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$
$$x_2(n) = \begin{cases} 1, & 0 \le n \le 6 \\ 0, & \text{elsewhere} \end{cases}$$

When

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

According to (4.13) we carry out the multiplication of $X_1(z)$ and $X_2(z)$. Thus

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underline{1}, -1, 0, 0, 0, 0, -1, 1\}$$

4.3.7 $\mathcal{Z}$-Transform 2 Sequence Correlation

If

$$x_1(n) \overset{\text{z}}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{\text{z}}{\longleftrightarrow} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \overset{\text{z}}{\longleftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) \tag{4.14}$$

Example 4.8: $\mathcal{Z}$-Transform 2 Sequence Correlation. Example 3.2.10
Determine the autocorrelation of the signal $$x(n) = a^n\,\mathcal{U}(n),\ -1 < a < 1$$
Solution on Page 166.

4.3.8 $\mathcal{Z}$-Transform 2 Sequence Multiplication

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n)x_2(n) \overset{z}{\longleftrightarrow} X_z = \frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv \tag{4.15}$$

where C is a closed contour that encloses the origin and lies within th eregion of convergence common to both $X_1(v)$ and $X_2(\frac{1}{v})$.

4.3.9 Parsevals Relation for $\mathcal{Z}$-Transform

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv \tag{4.16}$$

4.3.10 Initial Value Theorem for $\mathcal{Z}$-Transform

If $x(n)$ is *causal* [i.e., $x(n) = 0$ for $n < 0$], then

$$x(0) = \lim_{z\to\infty} X(z) \tag{4.17}$$

4.4 Properties of the One-Sided $\mathcal{Z}$-Transform

(i)

4.5 Rational $\mathcal{Z}$-Transforms

An important family of z-transforms are those for which $X(z)$ is a rational function, a ratio of two polynomials in z^{-1} (or z).

4.5.1 Poles and Zeros of a $\mathcal{Z}$-Transform

Defn 41 (Zeros)**.** The *zeros* of a z-transform $X(z)$ are the values of z for which $X(z) = 0$.
This is analogous to "setting the numerator equal to zero."

Defn 42 (Poles)**.** The *poles* of a z transform $X(z)$ are the values of z for which $X(z) = \infty$.
This is analogous to "setting the denominator equal to zero."

If $X(z)$ is a rational function, then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1z^{-1} + \cdots + b_Mz^{-M}}{a_0 + a_1z^{-1} + \cdots + a_Nz^{-N}} = \frac{\sum_{k=0}^{M} b_kz^{-k}}{\sum_{k=0}^{N} a_kz^{-k}}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid negative powers of z by factoring out the terms z^{-M} and z^{-N}.

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M}}{z^{-N}}\frac{b_0z^M + b_1z^{M-1} + \cdots + b_M}{a_0z^N + a_1z^{N-1} + \cdots + a_N}$$

Since $B(z)$ and $A(z)$ are polynomials in z, they can be expressed in factored form as

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M}}{z^{-N}}\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)} \tag{4.18}$$

Thus, $X(z)$ has M finite Zeros at $z = z_1, z_2, \ldots, z_M$ (the roots of the numerator polynomial), N finite Poles at $z = p_1, p_2, \ldots, p_N$ (the roots of the denominator polynomial, and $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$. Poles and zeroes may occur at $z = \infty$. A zero exists at $z = \infty$ if $X(\infty) = 0$ and a pole exists at $z = \infty$ if $X(\infty) = \infty$.

Defn 43 (Pole-Zero Plot)**.** Poles and Zeros of a z-transform can be shown graphically by a *pole-zero plot* in the complex plane, which shows the location of poles by crosses ($\times$) and the location of zeros by circles ($\circ$). Multiplicity is shown by a number close to the corresponding cross or circle. The ROC of a z-transform should not contain any poles, by definition.

4.5.2 Decomposition of Rational $\mathcal{Z}$-Transforms

The short end of this story is that you should group complex-conjugate pairs together.

4.6 Analysis of LTI Systems in the $\mathcal{Z}$-Domain

There are a few steps to move from the time-domain to the $\mathcal{Z}$-domain to perform analysis.

1. Convert all your time-based to terms to the $\mathcal{Z}$-domain.
   - $y(n-k) \rightarrow z^{-k}Y(z)$
   - $x(n-k) \rightarrow z^{-k}X(z)$
2. Express $H(z)$ as $\frac{Y(z)}{X(z)}$, the System Function.
3. Find the roots of the numerator and the denominator.
   - When solving for the roots, you should solve in terms of z^k, not z^{-k}. Factor our z^{-k} to achieve this.
   - If the degree of the numerator is greater than or equal to the degree of the numerators, you have to reduce the degree of the numerator.
     (a) Use Long Polynomial Division
     (b) Use Partial-Fraction Expansion on the Remainder

   (a) The roots of the numerator is/are Zeros. This is where the $\mathcal{Z}$-plane becomes 0.
   (b) The roots of the denominator is/are Poles. This is where the $\mathcal{Z}$-plane tends towards ∞.

Defn 44 (System Function)**.** The *system function* or *system equation* is the $\mathcal{Z}$-transform of the filter response.

$$H(z) = \mathcal{Z}\{h(n)\} \tag{4.19}$$

Because of Equation (4.13), and the relation shown in Equation (3.4), we can write the system equation like so

$$\begin{aligned} Y(z) &= X(z)H(z) \\ H(z) &= \frac{Y(z)}{X(z)} \end{aligned} \tag{4.20}$$

This forms the basis for Rational $\mathcal{Z}$-Transforms and Analysis of LTI Systems in the $\mathcal{Z}$-Domain

4.7 Common $\mathcal{Z}$-Transforms

The most common $\mathcal{Z}$-transforms are shown in Table 4.3.

5 The Fourier Transform and Fourier Series

When a signal is decomposed with either the Fourier Transform or the Fourier Series, you receive either sinusoids or complex-valued exponentials. This decomposition is said to be represented in the *frequency domain*.

Defn 45 (Fourier Transform)**.** When decomposing the class of signals with finite energy, you perform a *Fourier transform*. This is generally shown as the function

$$c_k = \mathrm{F}\{x(t)\}$$

There are 2 possible equations for the Fourier Transform, depending of the function is continuous-time or discrete-time.

1. Continuous-Time: Equation (5.1)
2. Discrete-Time: Equation (5.2)

Signal, $x(n)$	z-Transform, $X(z)$	ROC
$\delta(n)$	1	All z
$\mathcal{U}(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n\,\mathcal{U}(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n\,\mathcal{U}(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n\,\mathcal{U}(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$-na^n\,\mathcal{U}(-n-1)$	$\frac{az^{-z}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$(\cos\omega_0 n)\,\mathcal{U}(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(\sin\omega_0 n)\,\mathcal{U}(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(a^n\cos\omega_0 n)\,\mathcal{U}(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$(a^n\sin\omega_0 n)\,\mathcal{U}(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 4.3: Common $\mathcal{Z}$-Transforms

The Fourier Transform is defined as

$$X(F) = \int\limits_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt \tag{5.1}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \tag{5.2}$$

Remark 45.1. Sometimes $X(F)$ and $X(f)$ will be denoted with Ω and ω ($X(\Omega)$ and $X(\omega)$) respectively. In both cases, Ω and ω mean something similar.

$$\Omega = 2\pi F$$
$$\omega = 2\pi f$$

This means that we can rewrite Equations (5.1) to (5.2) as

$$X(\Omega) = \int\limits_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{5.3}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.4}$$

Remark 45.2. Generally, when people say the Fourier Transform, they are referring to the transform on Continuous-Time Signals. There is a distinction that occurs with the *DTFT* or *Discrete-Time Fourier Transform.*

This document explains them side-by-side, but will primarily focus on the Discrete-Time Fourier Transform.

Defn 46 (Fourier Series)**.** When decomposing the class of periodic signals, you are returned a *Fourier series.* This is generally shown as the function

$$X(F) = \mathrm{F}\{x(t)\}$$

Defn 47 (Discrete-Time Fourier Transform)**.** The *Discrete-Time Fourier Transform, DTFT* is a special case of the Fourier Transform that occurs when the input function $x(n)$ is a case of Discrete-Time Signals.

The transformation (analysis) equations are:

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \tag{5.5a}$$

$$\omega = 2\pi f$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.5b}$$

The reverse (synthesis) equations are:

$$x(n) = \int_{-\infty}^{\infty} X(f)e^{j2\pi fn}df \tag{5.6a}$$

$$x(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega n}d\omega \tag{5.6b}$$

These equations are expanded more upon in Section 5.2, The Inverse Fourier Transform.

5.1 Fourier Transform Relations

Each of these relations is just a side-note, the only relation of real importance is Equation (5.7). The Fourier Transform is just a special case in each of these scenarios. The Fourier Transform is evaluated around the unit circle on the real-imaginary plane.

5.1.1 Laplace Transform Fourier Transform Relation

There is a correlation between the Laplace Transform and the Fourier Transform. The Fourier Transform is a more specific case of the Laplace Transform, when

$$e^{-st} = e^{-j2\pi ft}$$

5.1.2 $\mathcal{Z}$-Transform Discrete-Time Fourier Transform Relation

There is a relationship between the $\mathcal{Z}$-Transform and the Discrete-Time Fourier Transform.

$$\begin{aligned} z &= e^{j2\pi f} \\ z &= e^{j2\pi n} \end{aligned} \tag{5.7}$$

The Discrete-Time Fourier Transform can be viewed as the $\mathcal{Z}$-transform of the sequence evaluated at the unit circle. If $X(z)$ does not converge in the region $|z| = 1$ (i.e., if the unit circle is not contained within the ROC of $X(z)$), the Fourier Transform $X(f)$ does not exist.

The existence of the $\mathcal{Z}$-transform requires that the sequence $\{x(n)r^{-n}\}$ be absolutely summable for some value of r, that is,

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \tag{5.8}$$

Therefore, if Equation (5.8) converges only for values of $r < r_0 < 1$, the ***$\mathcal{Z}$-transform exists***, but the ***Discrete-Time Fourier Transform DOES NOT EXIST***. This is the case for causal sequences of the form $x(n) = a^n\,\mathcal{U}(n)$, where $|a| > 1$.

There are sequences that do not satisfy Equation (5.8), for example

$$x(n) = \frac{\sin \omega_c n}{\pi n},\ -\infty < n < \infty$$

This sequences does not have a $\mathcal{Z}$-transform. However, since it is a finite Energy Signal, it has a Discrete-Time Fourier Transform that converges to

$$X(f) = \begin{cases} 1, & |f| < f_c \\ 0, & f_c < |f| \leq \frac{1}{2} \end{cases}$$

The existece of the $\mathcal{Z}$-transform requires that Equation (5.8) be satisfied for some region in the z-plane. If this region contains the unit circle, the Discrete-Time Fourier Transform, $X(f)$ exists. However, the existence of the Discrete-Time Fourier Transform, which is defined for finite Energy Signals, does not necessarily ensure the existence of the $\mathcal{Z}$-transform.

5.2 The Inverse Fourier Transform

Defn 48 (Inverse Fourier Transform)**.** Since the Fourier Transform is a "lossless" function (the definition of a transformation), the *inverse fourier transform* is just the opposite setup of Equations (5.1) to (5.2).

In both cases, a Continuous-Time signal and a Discrete-Time signal, you use the below synthesis equations (Equations (5.9) to (5.10)).

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}dF \\ x(n) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi fn}df \end{aligned} \tag{5.9}$$

If you're calculating with Ω or ω instead of F or f, then use these synthesis equations.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \\ x(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega n}d\omega \end{aligned} \tag{5.10}$$

5.3 Properties of the Discrete-Time Fourier Transform

One thing to keep in mind with all of these properties is that $\omega = 2\pi f$.

Property	Time Domain $x(n)$	Frequency Domain $X(f)$ or $X(\omega)$
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(\omega)$ $X_1(\omega)$ $X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time Shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time Reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1,x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1,x_2}(\omega) = X_1(\omega)X_2(\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener-Khintchine Theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency Shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos(\omega_0 n)$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication in Time Domain	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in Frequency Domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

Table 5.1: Properties of the Fourier Transform for Discrete-Time Signals

5.3.1 Linearity

If

$$\begin{aligned} x_1(n) &\overset{\text{F}}{\longleftrightarrow} X_1(f) \\ x_2(n) &\overset{\text{F}}{\longleftrightarrow} X_2(f) \end{aligned}$$

then

$$a_1x_1(n) + a_2x_2(n) \overset{\text{F}}{\longleftrightarrow} a_1X_1(f) + a_2X_2(f) \tag{5.11}$$

5.3.2 Time Shifting

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$x(n-k) \overset{\mathrm{F}}{\longleftrightarrow} e^{-j\omega k} X(f) \tag{5.12}$$

5.3.3 Time Reversal

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$x(-n) \overset{\mathrm{F}}{\longleftrightarrow} X(-f) \tag{5.13}$$

5.3.4 Convolution

If

$$x_1(n) \overset{\mathrm{F}}{\longleftrightarrow} X_1(f)$$
$$x_2(n) \overset{\mathrm{F}}{\longleftrightarrow} X_2(f)$$

then

$$x(n) = x_1(n) * x_2(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f) = X_1(f)X_2(f) \tag{5.14}$$

Remark. There is one thing to note here. Both $x_1(n)$ and $x_2(n)$ must be reasonably well-behaved and have be BIBO-stable for this relation to hold.

5.3.5 Correlation

If

$$x_1(n) \overset{\mathrm{F}}{\longleftrightarrow} X_1(f)$$
$$x_2(n) \overset{\mathrm{F}}{\longleftrightarrow} X_2(f)$$

then

$$r_{x_1x_2}(m) \overset{\mathrm{F}}{\longleftrightarrow} S_{x_1x_2}(f) = X_1(f)X_2(-f) \tag{5.15}$$

5.3.6 Wiener-Khintchine Theorem

Let $x(n)$ be a real signal. Then

$$r_{xx}(l) \overset{\mathrm{F}}{\longleftrightarrow} S_{xx}(f) \tag{5.16}$$

That is, the energy spectral density of an energy signal is the Fourier Transform of its autocorrelation sequence. This is a special case of Equation (5.15).

5.3.7 Frequency Shifting

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$e^{-i2\pi f_0 n} x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f-f_0) \tag{5.17}$$

5.3.8 Modulation

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$x(n)\cos(2\pi f_0 n) \overset{\mathrm{F}}{\longleftrightarrow} \frac{1}{2}\left[X(f+f_0) + X(f-f_0)\right] \tag{5.18}$$

5.3.9 Multiplication in Time Domain

This is also called the *Windowing Theorem*.

If

$$x_1(n) \xleftrightarrow{\text{F}} X_1(f)$$
$$x_2(n) \xleftrightarrow{\text{F}} X_2(f)$$

then

$$x_3(n) \equiv x_1(n)x_2(n) \xleftrightarrow{\text{F}} X_3(f) = \int_{\frac{1}{2}}^{\frac{1}{2}} X_1(\lambda)X_2(f-\lambda)d\lambda \tag{5.19}$$

5.3.10 Differentiation in Frequency Domain

If

$$x(n) \xleftrightarrow{\text{F}} X(f)$$

then

$$nx(n) \xleftrightarrow{\text{F}} j\frac{dX(f)}{df} \tag{5.20}$$

5.3.11 Parseval's Theorem

If

$$x_1(n) \xleftrightarrow{\text{F}} X_1(f)$$
$$x_2(n) \xleftrightarrow{\text{F}} X_2(f)$$

then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \int\limits_{-0.5}^{0.5} X_1(f)X_2^*(f)df \tag{5.21}$$

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega \tag{5.22}$$

Both Equations (5.21) to (5.22) can be expressed in another format.

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \int\limits_{-0.5}^{0.5} |X_1(f)|^2 df \tag{5.23}$$

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |X_1(\omega)|^2 d\omega \tag{5.24}$$

6 The Fourier Transform and LTI Systems

6.1 Frequency Response

The Frequency Response of a function is defined in 2 parts.

1. Magnitude Response
2. Phase Response

$$\begin{gathered} H(\omega) = \|H(\omega)\|\,\Theta(\omega) \\ H(f) = \|H(f)\|\,\Theta(f) \end{gathered} \tag{6.1}$$

Remark. This is similar to the Rectangular to Polar conversion shown on Page 27.

6.2 Magnitude Response

Defn 49 (Magnitude Reponse)**.** The *magnitude response* of a Fourier Transform is commonly denoted as $|H(\omega)|$ or $|H(f)|$. However, in this text, it is denoted as $\|H(\omega)\|$ or $\|H(f)\|$.

The equation that defines a Fourier Transform's Magnitude Reponse is

$$\begin{aligned} \|H(\omega)\| &= \sqrt{\Big(\mathrm{Re}\{H(\omega)\}\Big)^2 + \Big(\mathrm{Im}\{H(\omega)\}\Big)^2} \\ \|H(f)\| &= \sqrt{\Big(\mathrm{Re}\{H(f)\}\Big)^2 + \Big(\mathrm{Im}\{H(f)\}\Big)^2} \end{aligned} \tag{6.2}$$

Remark 49.1. It is important to note that the numerator will ***NOT*** have any imaginary terms (i or j) in it!

Remark. It is important to note that in this reference guide, the magnitude is denoted with double bars. For example, if a complex function $a(n)$ exists, then I will denote its magnitude as $\|a(n)\|$. This helps distinguish between magnitude of a function and its absolute value. These sometimes have different values, so it is useful to differentiate between the two.

6.3 Phase Response

Defn 50 (Phase Response)**.** The *phase response* of a Fourier Transform is commonly denoted as $\measuredangle H(\omega)$ or $\measuredangle H(f)$. This is a function that defines a Fourier Transform's Phase Response is defined in the equation below.

$$\begin{aligned} \measuredangle H(\omega) &= \Theta(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(\omega)\}}{\mathrm{Re}\{H(\omega)\}}\right) \\ \measuredangle H(f) &= \Theta(f) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(f)\}}{\mathrm{Re}\{H(f)\}}\right) \end{aligned} \tag{6.3}$$

Remark 50.1. It is important to note that the numerator will ***NOT*** have any imaginary terms (i or j) in it!

Remark 50.2 (Complex Exponential to Unit Circle)**.** ***REMEMBER:***

$$\begin{aligned} e^{\pm j\omega} &= \cos(\omega) \pm j\sin(\omega) \\ e^{\pm j2\pi f} &= \cos(2\pi f) \pm j\sin(2\pi f) \end{aligned} \tag{6.4}$$

This is also defined in Equation (A.3) on Page 26.

7 Sampling and Reconstruction

8 Discrete Fourier Transform

Defn 51 (Discrete Fourier Transform)**.** The *Discrete Fourier Transform* or *DFT* can be the Discrete-Time Fourier Transform sampled at certain values. This is only true if $N >$ Length of Signal DTFT.

Mathematically, this is shown as

$$X_{DFT}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{k}{N}n} \text{ for } k = 0, 1, 2, \ldots, N-1 \tag{8.1}$$

Remark 51.1. We now have several possible representations of the same signal with slight variations in the function description. We will list them out to ensure clarity.

- $x(t)$, a Continuous-Time Signals
- $x(n)$, a Discrete-Time Signals
- $X(z)$, the The $\mathcal{Z}$-Transform of $x(n)$
- $X(F)$, the Fourier Transform of $x(t)$
- $X(f)$, the Discrete-Time Fourier Transform of $x(n)$
- $X(k)$, the Discrete Fourier Transform of $x(n)$

Remark 51.2 (Time Complexity to Compute). If we are interested in the time complexity (Big-O) $O(n)$ of the Discrete Fourier Transform, it is $O(n^2)$.

1. N values of $X_{DFT}(k)$ to be computed

2. Each value of $X_{DFT}(k)$ requires N multiplications of $x(n)e^{-j2\pi \frac{k}{N} n}$
3. This means the time complexity is $O(n^2)$.

The Discrete Fourier Transform is used for many reasons, some of which are listed below:

1. Computers have limited memory, and the Discrete-Time Fourier Transform of a Discrete-Time Signals is a Continuous-Time Signals. Thus, the Discrete-Time Fourier Transform cannot be stored in memory.
2. The Fast Fourier Transform can be used to compute Circular Convolutions relatively quickly.

Defn 52 (Fast Fourier Transform)**.** This is a special case of the Discrete Fourier Transform that is only useful for computers. If $N = 2^k$, where k is an integer, then the Discrete Fourier Transform can be performed in $O(n \log_2(n))$ time. This can be used to calculate Linear Convolutions relatively quickly, especially when the number of terms in the sequence is quite large.

These 2 pieces of MATLAB/GNU Octave source produce the same output, but through different methods.

```
x = [1 2 3 4]
h = [2 2 1 1]
y = conv(x, h)
y = 2 6 11 17 13 7 4
```

```
x = [1 2 3 4 0 0 0 0]
h = [2 2 1 1 0 0 0 0]
Y_k = ifft(fft(x) .* fft(h))
Y_k = 2.0000 6.0000 11.0000 17.0000 13.0000 7.0000 4.0000 -0.0000
```

Remark 52.1 (Zero-Padding)*.* Note that the padding with zeros to a length greater than the output length of the Linear Convolution is required. Also, to take advantage of the Fast Fourier Transform's quick calculation property, the length of the inputs **MUST** be a power of 2, 2^k.

So, the below code produces something different because it calculates the Circular Convolution directly.

```
x = [1 2 3 4]
h = [2 2 1 1]
Y_k_bad = ifft(fft(x) .* fft(h))
Y_k_bad = 15 13 15 17
```

8.1 Inverse Discrete Fourier Transform

Defn 53 (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the inverse of the Discrete Fourier Transform.

$$x_{IDFT}(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi \frac{k}{N} n} \text{ for } n = 0, 1, \ldots, N-1 \tag{8.2}$$

8.2 Properties of the Discrete Fourier Transform

With the Discrete Fourier Transform, the usual properties do not apply.

- $x(n) * y(n) \neq X(k)Y(k)$
- $x(n - n_0) \not\leftrightarrow X(k)e^{j2\pi \frac{k}{N} n_0}$

These have to be modified instead.

Property	Time Domain $x(n)$	DFT Domain $X(k)$
Circular Convolution	$x(n) \circledast y(n)$	$X(k)Y(k)$
Time Shifting	$x(n - n_0 \bmod N)$	$X(k)e^{-j2\pi \frac{k}{N} n_0}$

Table 8.1: Properties of the Discrete Fourier Transform

8.2.1 Circular Convolution

Defn 54 (Circular Convolution)**.** The *circular convolution* is similar to the Linear Convolution. The key difference is that the circular convolution repeats its sequence indefinitely.

Mathematically, this type of convolution is represented as

$$x_N(n) \circledast h(n) = \sum_{m=-\infty}^{\infty} h(m) \sum_{k=-\infty}^{\infty} x(n-m-kN) \tag{8.3}$$

Computing a Circular Convolution is shown in Example 8.1.

The length of the resulting sequence from a Circular Convolution is

$$L \tag{8.4}$$

where L is the length of the input sequences.

Remark 54.1 (Linear vs. Circular Convolution Length)*.* The length of a Linear Convolution is $2L-1$, whereas the length of a Circular Convolution is L; given that the input signal lengths are L.

Example 8.1: Circular Convolution.

Perform a Circular Convolution on these signals.

$$x(n) = \{\underline{1}, 2, 3, 4\}$$
$$h(n) = \{\underline{2}, 2, 1, 1\}$$

There are 2 realistic ways to solve this by hand.

1. Perform a convolution where one of the signals is repeated periodically
2. Perform a normal convolution, but pad with 0s to get input signals that are longer than $2L-1$ of the original signals. Then, add the first term where 0s were padded to the first where they weren't.
   - This is the basis of performing a Linear Convolution with the Fast Fourier Transform and Circular Convolutions

For both methods, I will perform a Folding on $h(n)$ to get $h(-n)$.

Method 1

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$					
$x(k)$	2	3	4	$\underline{1}$	2	3	4	1	2	3
$y(k)$				$\underline{15}$	13	15	17			

Method 2

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$									
$x(k)$	0	0	0	$\underline{1}$	2	3	4	0	0	0	0	1	2	3
$y(k)$				$\underline{2}$	6	11	17	13	7	4	0			

Then, $\underline{2} + 13 = 15$, $6 + 7 = 13$, $11 + 4 = 15$, $17 + 0 = 17$.

Both methods yield $y(k) = \{\underline{15}, 13, 15, 17\}$.

Remark. Personally, I use Method 2 shown in Example 8.1 to calculate Circular Convolutions by hand.

8.2.2 Time Shifting

9 Implementation of Filters

A Trigonometry

A.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{A.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{A.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{A.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.7}$$

A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{A.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{A.9}$$

A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{A.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{A.11}$$

A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{A.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{A.13}$$

A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{A.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{A.15}$$

A.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{A.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{A.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{A.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{A.19}$$

A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{A.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{A.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{A.22}$$

A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{A.23}$$

A.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{A.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{A.25}$$

A.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{A.26}$$

B Calculus

B.1 Fundamental Theorems of Calculus

Defn B.1.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{B.1}$$

Defn B.1.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{B.2}$$

Defn B.1.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

B.2 Rules of Calculus

B.2.1 Chain Rule

Defn B.2.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{B.3}$$

C Laplace Transform

Defn C.0.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{C.1}$$