1 Introduction

1.1 Definitions and Terminology

Defn 1 (Differential Equation)**.** A *differential equation (DE)* is an equation with 1 or more derivatives.

Remark 1.1. The highest differential determines the order of the differential equation. This means that the differential equation below is of order 2.

$$y'' + y = 0$$
$$\frac{d^2y}{dx^2} + y = 0$$

Defn 2 (Initial Value Problem)**.** A differential equation with one or more initial conditions is called an *initial value problem (IVP).*

Remark 2.1. To solve an initial value problem, you must have the same number of initial conditions as the order of the differential equation.

Remark 2.2 (Existence of Unique Solution). R is a rectangular region on the xy-plane $a \leq x \leq b$, $c \leq y \leq d$ that contains (x_0, y_0) interior. If $f(x, y)$ and $\frac{df}{dy}$ are continuous on R, then an interval exists I_0 such that $(x_0 - h, x_0 + h)$ where $h > 0$, on the interval $[a, b]$, and a unique function $y(x)$, defined on I_0 that is a solution of the initial value problem.

1.2 Separable Differential Equation

Defn 3 (Separable)**.** A *separable* differential equation allows you to move various elements around to solve the equation. For example,

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P}dP = kdt$$
$$\ln(P) = kt + C$$
$$P = Ce^{kt}$$

Remark 3.1. These are used extensively in modelling phenomena with differential equations. These include: Population Growth, Radioactive Decay, Newton's Law of Cooling/Heating, and Spread of Disease.

1.3 Modeling with Differential Equations

1.3.1 Population Growth

Defn 4 (Population Growth)**.** *Population growth* can be modelled with a separable differential equation. Namely,

$$\frac{dP}{dt} = kP \tag{1}$$

Remark 4.1 (Population Growth Parameters). The parameters for the Population Growth equation are given below.

- $k > 0$
- $P > 0$

1.3.2 Radioactive Decay

Defn 5 (Radioactive Decay)**.** *Radioactive decay* is the process that some particularly heave atoms undergo.

Defn 6 (Half-Life)**.** The *half-life* is the usual reported metric, and is defined as the amount of time required for an element to half its mass through Radioactive Decay.

$$\frac{1}{2}A_0 = A_0e^{kt} \tag{2}$$

Remark 6.1 (Radioactive Decay Parameters). The parameters for the Radioactive Decay equation are given below.

- $k < 0$
- $A > 0$

1.3.3 Newton's Law of Cooling/Heating

Defn 7 (Newton's Law of Cooling/Heating)**.** *Newton's Law of Cooling/Heating* is the same equation, but some of the parameters change. This equation is defined as:

$$\frac{dT}{dt} = k\left(T - T_m\right) \tag{3}$$

Remark 7.1. The parameters for the Newton's Law of Cooling/Heating equation are given below.

- $\frac{dT}{dt}$; The rate of change of temperature in the object per unit time.
- $k < 0$; The cooling constant and is unique to every object.
- T; The starting temperature.
- T_m; The temperature of the surrounding medium.

1.3.4 Spread of Disease

Defn 8 (Spread of Disease)**.** This is used to model the spread of something throughout a society or group of people.

$$\frac{dx}{dt} = kxy \tag{4}$$

Remark 8.1. The parameters for the Spread of Disease equation are given below.

- $\frac{dx}{dt}$; Change in the number of infected per unit time.
- $k < 0$; Transmission Constant
- x; Number of Infected
- y; Number of non-infected, y is really a function of x
  - $y = n + 1 - x$

A Reference Material

A.1 Trigonometry

A.1.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.1.2 Euler Equivalents of Trigonometric Functions

$$\sin(x) = \frac{e^{\imath x} + e^{-\imath x}}{2} \tag{A.3}$$

$$\cos(x) = \frac{e^{\imath x} - e^{-\imath x}}{2\imath} \tag{A.4}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.5}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.6}$$

A.2 Calculus

A.2.1 Fundamental Theorems of Calculus

Defn 9 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{A.7}$$

Defn 10 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{A.8}$$