1 Convolution

$$r_{y,x}(k) = y(n) * x(-n) \tag{1.1}$$

2 $\mathcal{Z}$-Transform

Signal, $x(n)$	z-Transform, $X(z)$	ROC
$\delta(n)$	1	All z
$\mathcal{U}(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$a^n\,\mathcal{U}(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$na^n\,\mathcal{U}(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-a^n\,\mathcal{U}(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$-na^n\,\mathcal{U}(-n-1)$	$\frac{az^{-z}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$(\cos\omega_0 n)\,\mathcal{U}(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
$(\sin\omega_0 n)\,\mathcal{U}(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
$(a^n\cos\omega_0 n)\,\mathcal{U}(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
$(a^n\sin\omega_0 n)\,\mathcal{U}(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Table 2.1: Common $\mathcal{Z}$-Transforms

2.1 Properties of the $\mathcal{Z}$-Transform

Property	Time Domain	z-Domain	ROC
	$x(n)$	$X(z)$	ROC : $r_2 < \|z\| < r_1$
Notation	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time Shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time Reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real Part	$\text{Re}\{x(n)\}$	$\frac{1}{2}\left[X(z) + X^*(z^*)\right]$	Includes ROC
Imaginary Part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j\left[X(z) - X^*(z^*)\right]$	Includes ROC
Differentiation	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\|r_1$
Convolution	$x_1 * x_2$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
2 Sequence Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)x_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial Value Theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least, $r_{1l}r_{2l} < \|a\| < r_{1u}r_{2u}$
Parsevals Relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$= \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

Table 2.2: Z-Transform Properties

2.2 One-Sided $\mathcal{Z}$-Transform

The *one-sided z-transform* is the same as the $\mathcal{Z}$-Transform, but is only defined at n values greater than or equal to 0.

$$X(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n} \tag{2.1}$$

2.2.1 Time Delay

If

$$x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$$

then

$$x(n-k) \overset{z^+}{\longleftrightarrow} z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n) z^n\right], \quad k > 0 \tag{2.2}$$

3 DTFT

$$\begin{aligned} z &= e^{j2\pi f} \\ z &= e^{j\omega} \end{aligned} \tag{3.1}$$

$$\begin{aligned} X(f) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \\ X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \end{aligned} \tag{3.2}$$

4 DFT

The N-point DFT is shown as:

$$X_{DFT}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n} \text{ for } k = 0, 1, 2, \ldots, N-1 \tag{4.1}$$

If N is specified, then replace all occurrences of N in Equation (4.1) with that value.

Remark 0.1. If the length, N of the DFT is not specified, it is assumed that $N =$ length of the signal. If the length of the DFT N is greater than the length of the signal, you are sampling the DTFT of the signal.

$$\begin{aligned} x(n) &= A\cos\left(2\pi \frac{k_0}{N} n\right), \quad 0 < k_0 \leq N-1 \\ &= \frac{A}{2}\left(e^{j\frac{2\pi k_0}{N} n} + e^{-j\frac{2\pi k_0}{N} n}\right) \\ X(k) &= \frac{AN}{2}\Big[(\delta(k-k_0) \bmod N) + (\delta(k+k_0) \bmod N)\Big] \end{aligned} \tag{4.2}$$

$$x_{IDFT}(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi \frac{k}{N} n} \text{ for } n = 0, 1, \ldots, N-1 \tag{4.3}$$

$$x_1(n) \circledast x_2(n) = \sum_{k=0}^{N-1} x_1(k) x_2(n-k \bmod N) \tag{4.4}$$

It is important to remember that the modulus (mod) operator yields 0 when the input is a multiple of the divisor.

Defn 1 (Decimation)**.** *Decimation* takes an input signal and compresses it. Decimation uses the symbol $D \in \mathbb{Z}^+$.

$$y(m) = x(mD)$$

If decimation occurs later in the system, then if just the input and output are compared, $y(m)$ appears it was sampled at

$$f = \frac{F_S}{D} \tag{4.5}$$

Property	Time Domain $x(n)$	DFT Domain $X(k)$
Notation	$x(n)$, $y(n)$	$X(k)$, $Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time Reversal	$x(N-n)$	$X(N-k)$
Circular Time Shifting	$x(n - n_0 \bmod N)$	$X(k)e^{-j2\pi \frac{k}{N} n_0}$
Circular Frequency Shift	$x(n)e^{j2\pi ln/N}$	$X(k - l \bmod N)$
Complex Conjugate	$X^*(n)$	$X^*(N-k)$
Circular Convolution	$x(n) \circledast y(n)$	$X(k)Y(k)$
Circular Correlation	$x(n) \circledast y^*(-n)$	$X(k)Y^*(k)$
2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \circledast X_2(k)$
Parsevals Theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Table 4.1: Properties of the DFT

Thus, when we perform sampling on the input signal, then there is folding at

$$f = \frac{F_S}{2D} \tag{4.6}$$