Phys 123: Classical Mechanics — Reference Sheet
Illinois Institute of Technology

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1 Vectors

Defn 1 (Vector)**.** A *vector* is a way to show both magnitude of displacement and direction of displacement. Vectors are drawn as rays.

Remark 1.1. Vectors and Scalars may seem similar, but are different.

Defn 2 (Scalar)**.** A *scalar* is a way to show ***ONLY*** the magnitude of a displacement, without any direction information.

1.1 Vector Properties

(i) $\vec{A} + \vec{B} = \vec{C}$
(ii) $\vec{0} = \langle 0, 0, 0, \ldots, 0 \rangle$
(iii) $\vec{A} + \vec{0} = \vec{A}$
(iv) $\vec{A} + -\vec{A} = \vec{0}$
(v) $\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$
(vi) $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
(vii) Magnitude of vector: $\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

1.1.1 Getting Components

Getting the components of a vector involves solving the imaginary pythagorean triangle around the vector.

For a 2-dimensional vector, $\vec{V}$, you have the components $\langle V_x, V_y \rangle$. You find their values with this equation:

$$\begin{aligned} V_x &= V \cos\theta \\ V_y &= V \sin\theta \end{aligned} \tag{1.1}$$

1.1.2 3D Unit Vectors

3-dimensional vectors shouldn't be any too crazy by this point. They are just another variable that can be thrown around in the vector. However, the three 3D Unit Vectors are special. You can also use these to describe any lower-dimensional vector as well.

$$\begin{aligned} \hat{\imath} &= \langle 1, 0, 0 \rangle \\ \hat{\jmath} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle \end{aligned} \tag{1.2}$$

1.1.3 Addition

Vectors are additive, and are done from head-to-tail. This means that

$$\vec{A} + \vec{B} = \vec{C} \tag{1.3}$$

This means that in 3-dimensional vectors, they are added like this:

$$\begin{aligned} \vec{A} &= \langle A_x, A_y, A_z \rangle \\ \vec{B} &= \langle B_x, B_y, B_z \rangle \\ \vec{A} + \vec{B} &= \langle A_x + B_x, A_y + B_y, A_z + B_z \rangle \end{aligned} \tag{1.4}$$

1.1.4 Scalar Multiplication

When applying multiplication between a scalar and a vector, you perform Scalar Multiplication.

$$2 \times \vec{V} = 2\langle V_x, V_y \rangle = \langle 2V_x, 2V_y, 2V_z \rangle \tag{1.5}$$

This means that you do ***NOT*** modify the direction of the vector, you only change its magnitude.

1.1.5 Scalar (Dot) Product

The Scalar (Dot) Product is the first of two ways to multiply 2 vectors. The other is the Vector (Cross) Product. There are 2 ways to calculate the Scalar (Dot) Product.

The first involves using the magnitudes of each vector and multiplying those by the cosine of the angle between them.

$$\vec{A} \cdot \vec{B} = \|\vec{A}\|\|\vec{B}\| \cos(\theta) \tag{1.6}$$

The second is done by adding the product of each component of each vector.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.7}$$

Remark. This means that when you apply the Scalar (Dot) Product to 2 vectors, you return a Scalar.

1.1.5.1 Properties of Scalar (Dot) Product

(i) $(\vec{A})^2 = \vec{A} \cdot \vec{A}$

(ii) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

1.1.6 Vector (Cross) Product

The Vector (Cross) Product is the second of two ways to multiply 2 vectors. The other is the Scalar (Dot) Product. There are 2 ways to calculate the Vector (Cross) Product.

The first involves using the magnitudes of each vector and multiplying those by the sine of the angle between them.

$$\vec{A} \times \vec{B} = \|\vec{A}\|\|\vec{B}\| \sin\theta \tag{1.8}$$

The second is done by taking the determinant of a 2×2 or 3×3 matrix.

$$\begin{aligned}
\vec{A} \times \vec{B} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= (A_y B_z - A_z B_y)\,\hat{\imath} - (A_x B_z - A_z B_x)\,\hat{\jmath} + (A_x B_y - A_y B_x)\,\hat{k} \\
&= \left\langle A_y B_z - A_z B_y, -\left(A_x B_z + A_z B_x\right), A_x B_y - A_y B_x \right\rangle
\end{aligned} \tag{1.9}$$

Remark. This means that when you apply the Vector (Cross) Product to 2 vectors, you return a Vector.

1.1.6.1 Properties of Vector (Cross) Product

(i) $\vec{A} \times \vec{A} = \vec{0}$

(ii) $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$

(iii) $\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$

(iv) $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C}\left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)$

2 Kinematics

Defn 3 (Kinematics)**.** *Kinematics* is a way to describe macroscopic motion with equations. This includes anything moving, falling, thrown, shot, launched, etc. This forms the fundamental basis for all of classical mechanics.

2.1 1-D Kinematics

Defn 4 (1-D Displacement)**.** *One dimensional displacement* is calculated based on the change in position of the "thing."

$$s = x_2 - x_1 \tag{2.1}$$

Remark 4.1. *Displacement is different than path!* Displacement is the change in position of an object. Path is the length of the path takes between its starting and end point.

Defn 5 (1-D Velocity)**.** *One dimensional velocity* is calculated as the displacement per unit time. There is instantaneous velocity and average velocity. Average velocity is calculated with Equation (2.2).

$$v = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.2}$$

Instantaneous velocity is calculated by reducing the time interval Δt to 0. This can be summarized in Equation (2.3).

$$\begin{aligned} v &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \\ &= \frac{dx}{dt} \end{aligned} \tag{2.3}$$

Defn 6 (Acceleration)**.** *One dimensional acceleration* is the change in velocity over time. Again, there is average acceleration and instantaneous acceleration. Average acceleration is calculated with Equation (2.4)

$$a = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \tag{2.4}$$

Instantaneous acceleration is calculated by reducing the time interval Δt to 0. This can be summarized by Equation (2.5).

$$\begin{aligned} a &= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \\ &= \frac{dv}{dt} = \frac{d^2 x}{dt^2} \end{aligned} \tag{2.5}$$

2.2 Multi-Dimensional Kinematics

Because we can represent a two-dimensional and three-dimensional space in sets, and movement through this space as their respectively dimensioned vectors, we can construct multi-dimensional problems with multi-dimensional vectors! This is a massive simplification, because instead of solving for one equation with three variables, we can solve three equations for one variable each!!

For the following definitions, I have assumed that we are in a 3-dimensional space (x, y, z).

Defn 7 (Multi-Dimensional Position)**.** *Position* in multiple dimensions is done by referring to each of the constituent dimensions.

$$\vec{s} = \langle x(t), y(t), z(t) \rangle \tag{2.6}$$

Defn 8 (Multi-Dimensional Displacement)**.** *Displacement* in multiple dimensions can be broken down into several 1-D Displacements. Since 1-D Displacement is calculated as the difference between the start and end position, the same is true for the multi-dimensional case.

$$\begin{aligned} \vec{r} &= \Delta \vec{s} = \vec{s}_2 - \vec{s}_1 \\ &= \langle x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t) \rangle \\ &= \langle r_x(t), r_y(t), r_z(t) \rangle \end{aligned} \tag{2.7}$$

Defn 9 (Multi-Dimensional Velocity)**.** *Velocity* in multiple dimensions is described in much the same way as 1-D Velocity.

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \left\langle \frac{dr_x(t)}{dt}, \frac{dr_y(t)}{dt}, \frac{dr_z(t)}{dt} \right\rangle \\ &= \langle r'_x(t), r'_y(t), r'_z(t) \rangle \end{aligned} \tag{2.8}$$

Defn 10 (Multi-Dimensional Acceleration)**.** *Acceleration* in multiple dimensions is described in much the same way as Acceleration.

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt} \\ &= \left\langle \frac{dv_x(t)}{dt}, \frac{dv_y(t)}{dt}, \frac{dv_z(t)}{dt} \right\rangle = \left\langle \frac{d^2 r_x(t)}{dt}, \frac{d^2 r_y(t)}{dt}, \frac{d^2 r_z(t)}{dt} \right\rangle \\ &= \langle v'_x(t), v'_y(t), v'_z(t) \rangle = \langle r''_x(t), r''_y(t), r''_z(t) \rangle \end{aligned} \tag{2.9}$$

2.3 Projectile Motion

Defn 11 (Projectile)**.** A *projectile* is any body given an initial velocity that then follows a path determined by gravity and air resistance.

Remark 11.1. However, for most of our calculations, we will neglect air resistance. Air resistance can be a difficult thing to calculate for, especially in the variable cases that we will have.

There are a few things to keep in mind with projectiles in motion.

1. Origin is where the projectile starts from
2. The x-axis is the *distance* that the projectile travels. This is its displacement.
3. The y-axis is the *height* that the projectile travels.
4. The end point (landing point) is the only thing that may change on the x-axis.
5. The acceleration vector is as follows: $\langle 0, -g \rangle$.
6. *Trajectory* depends on $\vec{v}_0$ and $\vec{a}$ *ONLY*.
7. The two components of the projectile's initial velocity are *independent*, $\langle v_{0,x}, v_{0,y} \rangle$.

2.3.1 Projectile Motion Equations

The following equations are used to solve for various questions that could be asked about projectile motion.

2.3.1.1 Initial Velocity Components

$$v_{0,x} = v_0 \cos(\theta) \quad v_{0,y} = v_0 \sin(\theta) \tag{2.10}$$

2.3.1.2 Velocity Components

$$v_x = v_{0,x} \cos(\theta) \quad v_y = v_{0,y} \sin(\theta) - gt \tag{2.11}$$

2.3.1.3 Projectile Position

$$x = v_0 t \cos(\theta) \quad y = v_0 t \sin(\theta) - \frac{1}{2} g t^2 \tag{2.12}$$

2.3.1.4 Projectile Time

$$t = \frac{x}{v_0 \cos(\theta)} \quad t = \frac{v_0 \sin(\theta)}{g} \tag{2.13}$$

2.3.1.5 Projectile Range

$$R = \frac{2v_0^2}{g} \cos(\theta) \sin(\theta) = \frac{v_0^2 \sin(2\theta)}{g} \tag{2.14}$$

2.3.1.6 Projectile Maximum Range

$$R_{\text{Max}} = \frac{v_0^2}{g} \tag{2.15}$$

This means that the θ in Equation (2.14) is 45°.

2.3.1.7 Projectile Height

$$h = \frac{v_0^2}{2g} \sin^2(\theta) \tag{2.16}$$

2.3.1.8 Projectile Maximum Height

$$h = \frac{v_0^2}{2g} \tag{2.17}$$

The lack of $\sin^2(\theta)$ from Equation (2.16) means that there is ***NO*** y-component to the velocity, meaning the projectile is at its instant of maximum height.

3 Uniform Circular Motion

Defn 12 (Uniform Circular Motion)**.** *Uniform circular motion* is when an object of moving in a perpetual circular motion. There is no outside source of acceleration changing the state of the system, meaning the Angular Velocity is 0.

Remark 12.1. This does *not* happen in real life. However, it is useful for modeling things under ideal conditions that do happen in real life.

3.1 Angular Part of Circular Motion

During Uniform Circular Motion, your terminology changes a little bit.

Defn 13 (Angular Position)**.** *Angular position* is determined with radians around a circle. It is denoted with

$$\vec{\theta}$$

Defn 14 (Angular Velocity)**.** *Angular velocity* is orthogonal to the flat 2-D plane that the object is traveling in. If you are looking at the thing moving in the circle from above, the angular velocity is either pointing directly at you, or directly away from you. It is the derivative of the Angular Position.

$$\begin{aligned} \vec{\omega} &= \frac{d\theta}{dt} \\ &= \langle 0, 0, \omega \rangle \end{aligned} \tag{3.1}$$

Defn 15 (Angular Acceleration)**.** *Angular acceleration* is the derivative of the Angular Velocity.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt} \tag{3.2}$$

Remark 15.1. Note that under Uniform Circular Motion, by its very definition, there cannot be any acceleration on the object. Therefore, when an object is in uniform circular motion, $\vec{\alpha} = 0$.

However, when the object is ***NOT*** in Uniform Circular Motion the object is undergoing Linear Acceleration.

3.2 Linear Part of Circular Motion

Defn 16 (Linear Position)**.** *Linear position* relates the position of an object from the Cartesian coordinate plane to the polar. This means that:

$$x = r\cos(\theta) \quad y = r\sin(\theta) \tag{3.3}$$

Defn 17 (Linear Velocity)**.** *Linear velocity* relates the velocity of an object in a line to its Angular Velocity.

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ \frac{d\vec{r}}{dt} &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, 0 \right\rangle = \left\langle \frac{d}{dt} r\cos(\theta), \frac{d}{dt} r\sin(\theta), 0 \right\rangle \\ &= \left\langle -r\sin(\theta)\omega, r\cos(\theta)\omega, 0 \right\rangle \\ &= \vec{\omega} \times \vec{r} \end{aligned} \tag{3.4}$$

Defn 18 (Linear Acceleration)**.** *Linear acceleration* is the derivative of Linear Velocity. It relates the acceleration of an object in a line relative to its Angular Acceleration

$$\begin{aligned} \frac{d\vec{a}}{dt} &= \frac{d\vec{v}}{dt} \\ &= \langle -r\omega^2\cos(\theta), -r\omega^2\sin(\theta), 0 \rangle = \omega^2 \langle -r\cos(\theta), -r\sin(\theta), 0 \rangle \\ &= -\omega^2\vec{r} \end{aligned} \tag{3.5}$$

3.3 Relation Between the Angular Part of Circular Motion and the Linear Part of Circular Motion

There are a few equations that relate both the Angular Part of Circular Motion and the Linear Part of Circular Motion.

3.3.0.1 Velocity and Angular Velocity

$$v = \omega r \tag{3.6}$$

3.3.0.2 Acceleration and Angular Acceleration

$$\begin{aligned} a &= \omega^2 r \\ &= \frac{v^2}{r} \end{aligned} \tag{3.7}$$

4 Reference Frames

Defn 19 (Reference Frames)**.** An *inertial reference frame* is a frame for the world. It is easiest to think of of an inertial reference frame with an example. For instance, when you're in a car going 40 mph and you see someone going 45, you can only tell they're going 5 mph faster than you.

Defn 20 (Galileo Transformation/Relativity Principle)**.** This is a transformation that happens when you're calculating in one Reference Frames and the event is happening in a different Reference Frames.

$$\begin{aligned} \frac{d\vec{R}}{dt} &= \frac{d\vec{r}}{dt} + \frac{d\vec{r'}}{dt} \\ \vec{V} &= \vec{v} + \vec{v'} \end{aligned} \tag{4.1}$$

5 Newton's Laws

There are 3 fundamental laws of classical mechanics.

1. An object in motion/at rest stays as such, unless acted upon by an outside force(s).
2. Force is equal to the change in momentum ($p = mv$) per change in time. For a constant mass, force equals mass times acceleration ($\vec{F} = m\vec{a}$).
3. For every force there is an equal and opposite force.

There are a few forces that are fundamental in our universe.

- Gravitational Force
- Magnetic Force
- Electric Force
- Frictional Force
- Normal Force
- Tension Force

Remark. ***REMEMBER*** that forces are vectors!!

To solve any force problem, you should construct a free-body diagram. This diagram should show a vector for every possible force in the system. This way, you can easily deconstruct the vectors, calculate their components, then use them to solve your problem.

5.1 Newton's Laws in 1-D

Example 5.1: Atwood Machine.

Find the acceleration that is the result of dropping one side of the machine? Neglect any friction that may occur because of the pulley.

$$\begin{aligned} m_2\vec{a}_2 &= m_2\vec{g} + \vec{T}_2 \\ m_1\vec{a}_1 &= m_1\vec{g} + \vec{T}_1 \end{aligned}$$

One thing to remember is that both masses will have the same acceleration, because the pulley has no friction. So because $\vec{a}_2 = \vec{a}_1$, we can solve for a single $\vec{a}$ variable.

$$
\begin{aligned}
m_2\vec{a} &= m_2\vec{g} + \vec{T}_2 \\
m_1\vec{a} &= m_1\vec{g} + \vec{T}_1 \\
\vec{a}\,(m_2 + m_1) &= \vec{g}\,(m_2 - m_1) \\
\vec{a} &= \frac{g\,(m_2 - m_1)}{m_2 + m_1}
\end{aligned}
$$

5.2 Newton's Laws in Multi-D

For a multi-dimensional situation, you break the free-body problem's vectors down into their components.

Example 5.2: Sliding Block.

If a block of mass m slides down a ramp of angle θ, what is the coefficient of static and kinetic friction?

5.3 Common Forces

This section will discuss various forces in greater detail.

5.3.1 Gravitational Force

Defn 21 (Gravitational Force)**.** *Gravitational force* is a force that arises from Gravity. Gravity will be covered more in Section 14, Gravitation. This force arises when one object is being pulled towards another due to Gravity. Gravitational Force is generally the acceleration vector due to Earth's gravity.

$$\vec{g} = 9.81\text{m/s}^2 \tag{5.1}$$

5.3.2 Magnetic Force

This force is discussed in much greater detail in the Physics 221: Electromagnetics and Optics course.

5.3.3 Electric Force

This force is discussed in much greater detail in the Physics 221: Electromagnetics and Optics course.

5.3.4 Frictional Force

Frictional force is the common culprit for the "Equal and Opposite Force" in classical mechanics.

Defn 22 (Frictional Force)**.** Frictional force is defined as

$$\vec{F}_f = \mu\vec{N} \tag{5.2}$$

μ: The coefficient of the type of friction you are dealing with (Either the Coefficient of Static Friction or Coefficient of Kinetic Friction)

$\vec{N}$: The Normal Force

There are 2 types of frictional force:

1. Static Friction
2. Kinetic Friction

Defn 23 (Static Friction)**.** *Static friction* is friction that arises when an object is starting from a static position, and is being moved. This force tends to be stronger than Kinetic Friction because of electrostatic bonds between the object and it's supporting surface, along with other reasons. However, static friction is drawn from the Coefficient of Static Friction. This is a Scalar number that represents how much the object *does not* want to move.

$$\vec{F}_{f,s} = \mu_s\vec{N} \tag{5.3}$$

μ_s: The Coefficient of Static Friction
$\vec{N}$: The Normal Force

Defn 24 (Coefficient of Static Friction)**.** *Coefficient of static friction* is a Scalar number that represents how much the object does not want to **START** moving. This value tends to be greater than the Coefficient of Kinetic Friction. This is because there are additional bonds and forces at play that resist the start of an objects motion. The Coefficient of Kinetic Friction is denoted as such:

$$\mu_s \tag{5.4}$$

Defn 25 (Kinetic Friction)**.** *Kinetic friction* is a friction that arises when an object is in motion. This force is weaker than Static Friction. The value we may use for kinetic friction is drawn from the Coefficient of Kinetic Friction.

$$\vec{F}_{f,k} = -\mu_k \vec{N} \tag{5.5}$$

- μ_k - The Coefficient of Kinetic Friction
- $\vec{N}$ - The Normal Force that is going ***IN*** the direction of motion. You might need to break the normal force's vector down into its components.

Defn 26 (Coefficient of Kinetic Friction)**.** *Coefficient of kinetic friction* is Scalar number that represents how much the object does not want to **CONTINUE** moving. This value tends to be smaller than its Coefficient of Static Friction counterpart. The Coefficient of Kinetic Friction is denoted as such:

$$\mu_k \tag{5.6}$$

5.3.5 Normal Force

Defn 27 (Normal Force)**.** The *normal force* is, as the name implies, a normalizing force. This does not necessarily make it special, as both Kinetic Friction and Static Friction can be considered normalizing forces as well. However, the normal force is generally considered whenever something is being held against something. For example, a book on a table. The table is exerting a Normal Force on the book to prevent it from falling through the table. Likewise, the ground is exerting a Normal Force on the table to prevent it from falling through the ground.

This normal force is defined as

$$\vec{N} = -m\vec{g} \tag{5.7}$$

6 Dynamics of Circular Motion

There are three cases when the Dynamics of Circular Motion are easily visible. Each of these is illustrated with an example.

1. Turning on a flat curve, neglecting friction (Example 6.1)
2. Turning on an angled curve, neglecting friction (Example 6.2)
3. Turning on an angled curve, with friction (Example 6.3)

Example 6.1: Car Turning on a Flat Curve.

A car of mass m is turning on a flat curve with radius $r = 50\,\text{m}$. The Coefficient of Static Friction is $\mu_s = 0.7$. What is the car's maximum velocity, v_{Max} before it cannot turn anymore?

First, we start by finding the net force in the system, considering the Normal Force and Frictional Force to be positive.

All Forces Present:

$$\vec{F}_{Net} = \vec{N} - m\vec{g} + \vec{F}_f$$

Newton's Second Law:

$$\vec{F}_{Net} = m\vec{a}$$

Now that all forces have been identified, we can set both equations equal to each other and being substituting other expressions in.

$$\vec{N} - m\vec{g} + \vec{F}_f = m\vec{a}$$

Remember the formula for static friction:

$$\vec{F}_f = \mu_s \vec{N}$$

With those 2 relations gathered, we now need to consider the Normal Force. This *must* be 0, otherwise the car is starting to float or fall into the ground.

$$\vec{N} - m\vec{g} = 0$$
$$\vec{N} = m\vec{g}$$

Now, we can substitute everything back into our main equation.

$$\vec{N} - m\vec{g} + \vec{F}_f = m\vec{a}$$

Subsitute F_f:

$$\vec{N} - m\vec{g} + \mu_s\vec{N} = m\vec{a}$$

Substitute $\vec{N}$:

$$m\vec{g} - m\vec{g} + \mu_s m\vec{g} = m\vec{a}$$

Reduce:

$$\mu_s m\vec{g} = m\vec{a}$$

Substitute $\vec{a}$ with Equation (3.7):

$$\mu_s m\vec{g} = m\frac{v_{Max}^2}{r}$$

Simplify:

$$v_{Max} = \sqrt{\mu_s\vec{g}r}$$

With this expression, we can solve for the maximum velocity this vehicle can move. The expression $v_{Max} = \sqrt{\mu_s\vec{g}r}$ can be thought of as the speed where the Coefficient of Static Friction "breaks."

$$v_{Max} = \sqrt{\mu_s\vec{g}r} = \sqrt{0.7(9.81\,\mathrm{m/s})50\,\mathrm{m}} = 18.5\,\mathrm{m/s}$$

Example 6.2: Car Turning on Angled Curve with No Friction.

A car of mass m is turning on a curve r meters from the center, with an angle θ from the horizontal. If there is no friction, what is its maximum velocity, v_{Max}?

Here, because there is an angle involved, we need to decompose the vectors into their components. Starting with the vertical component, we find the normalizing force at an angle to the force of gravity.

$$\vec{N}\cos(\theta) - m\vec{g} = \vec{0}$$
$$\vec{N}\cos(\theta) = m\vec{g}$$

Now, we need to consider the horizontal components, which is the sliding that happens up or down the curve of the turn.

$$\vec{N}\sin(\theta) = ma_c$$

Substitute a_c with Equation (3.7):

$$\vec{N}\sin(\theta) = m\frac{v_{Max}^2}{r}$$

With both components of the final movement calculated, we single out $\vec{N}$ and set each relation equal to each other.

Vertical:

$$\vec{N} = \frac{m\vec{g}}{\cos(\theta)}$$

Horizontal:

$$\vec{N} = \frac{mv_{Max}^2}{r\sin(\theta)}$$

Substitute for $\vec{N}$:

$$\frac{m\vec{g}}{\cos(\theta)} = \frac{mv_{Max}^2}{r\sin(\theta)}$$

Simplify:

$$v_{Max}^2 = \frac{\vec{g}r\sin(\theta)}{\cos(\theta)}$$
$$v_{Max} = \sqrt{\vec{g}r\tan(\theta)}$$

Thus, our final expression for this problem is

$$v_{Max} = \sqrt{\vec{g}r\tan(\theta)}$$

Example 6.3: Car Turning on Angled Curve with Friction.

A car of mass m is turning on a curve r meters from the center, with an angle θ from the horizontal. There is friction, with the coefficient of static friction being μ_s and coefficient of kinetic friction being μ_k. What is its maximum velocity, v_{Max}?

First, we can eliminate the use of μ_k. This is because we are not concerned with the vehicle's sliding forward or backward in the direction of travel. Instead we want to know about the car's movement up or down the ramp due to its velocity.

Again, just like the previous example, the car is traveling at an angle, so the forces acting on the vehicle must be broken down into their components. Like Example 6.1:

$$\vec{F}_f = \mu_s\vec{N}$$

Starting with the vertical again:

$$\vec{N}\cos(\theta) - m\vec{g} + \vec{F}_f\sin(\theta) = 0$$

Substitute F_f:

$$\vec{N}\cos(\theta) - m\vec{g} + \mu_s\vec{N}\sin(\theta) = 0$$

Reduce and Simplify:

$$\vec{N}\cos(\theta) + \mu_s\vec{N}\sin(\theta) = m\vec{g}$$
$$\vec{N}\big(\cos(\theta) + \mu_s\sin(\theta)\big) = m\vec{g}$$
$$\vec{N} = \frac{m\vec{g}}{\cos(\theta) + \mu_s\sin(\theta)}$$

Now onto the horizontal:

$$\vec{N}\sin(\theta) - \vec{F}_f\cos(\theta) = ma_c$$

Substitute a_c with Equation (3.7):

$$\vec{N}\sin(\theta) - \vec{F}_f\cos(\theta) = m\frac{v^2}{r}$$

Substitute F_f:

$$\vec{N}\sin(\theta) - \mu_s\vec{N}\cos(\theta) = m\frac{v^2}{r}$$

Reduce and Simplify:

$$\vec{N}\big(\sin(\theta) - \mu_s \cos(\theta)\big) = m\frac{v^2}{r}$$
$$\vec{N} = \frac{mv^2}{r\big(\sin(\theta) - \mu_s \cos(\theta)\big)}$$

With both components simplified, we can subsitute one for the other.

Vertical:

$$\vec{N} = \frac{m\vec{g}}{\cos(\theta) + \mu_s \sin(\theta)}$$

Horizontal:

$$\vec{N} = \frac{mv^2}{r\big(\sin(\theta) - \mu_s \cos(\theta)\big)}$$

Substitute:

$$\frac{m\vec{g}}{\cos(\theta) + \mu_s \sin(\theta)} = \frac{mv^2}{r\big(\sin(\theta) - \mu_s \cos(\theta)\big)}$$

Reduce and Simplify for v_{Max}:

$$v_{Max}^2 = \frac{\vec{g}r\big(\sin(\theta) - \mu_s \cos(\theta)\big)}{\cos(\theta) + \mu_s \sin(\theta)}$$
$$v_{Max} = \sqrt{\vec{g}r\left(\frac{\sin(\theta) - \mu_s \cos(\theta)}{\cos(\theta) + \mu_s \sin(\theta)}\right)}$$

Thus, our final expression for the maximum velocity of a car on a curved turn with friction is:

$$v_{Max} = \sqrt{\vec{g}r\left(\frac{\sin(\theta) - \mu_s \cos(\theta)}{\cos(\theta) + \mu_s \sin(\theta)}\right)}$$

However, one thing to note is the way I chose to interpret F_f. I chose to point it out of the curve, which is what happens when the car is moving slowly enough. If you were to point the F_f vector towards the inside of the curve, you would end up with this instead.

$$v_{Max} = \sqrt{\vec{g}r\left(\frac{\sin(\theta) + \mu_s \cos(\theta)}{\cos(\theta) - \mu_s \sin(\theta)}\right)}$$

Another thing to note is that if we consider $\mu_s = 0$, i.e. there is no friction, we get the same result back as in Example 6.2.

7 Springs

Springs are useful things. We can model many things as a spring for Statics

7.1 Hooke's Law

Defn 28 (Hooke's Law)**.** *Hooke's law* relates the distance a spring is pulled from equilibrium to the force it will exert as it returns to equilibrium. Because of Newton's Laws, this will also tell us the force required to move a spring from its equilibrium position.

$$\vec{F} = -k\Delta x \tag{7.1}$$

- k - The spring constant. A unique value for the "springyness" of any spring
- Δx - The distance displaced from the equilibrium position

Remark 28.1. Hooke's Law only works for distances where Δx are relatively small. If Δx were to become too large, your spring would cease to be a spring and become a straight piece of metal.

8 Energy

There is a way to rewrite Newton's Laws such that we get the Work-Kinetic Energy Theorem.

Defn 29 (Work)**.** *Work* is defined as the amount of force done over a distance. This is summarized with its equation.

$$\mathrm{W} = \vec{F} \cdot \vec{s} \tag{8.1}$$

- $\vec{F}$ - The force applied
- $\vec{s}$ - The distance the thing travelled while under influence of the force

Remark 29.1. One outcome of Equation (8.1) is that if a force is acting orthogonally to the direction of motion, the force is doing ***NO*** Work.

Defn 30 (Power)**.** *Power* is defined as the amount of Work done per unit time.

$$\mathrm{P} = \frac{d\mathrm{W}}{dt} \tag{8.2}$$

Defn 31 (Work-Kinetic Energy Theorem)**.** The *work-energy theorem* rewrites Newton's Second law.

$$\vec{F}_{\text{Net}} = m\frac{d\vec{v}}{dt}$$

This differential equation can be solved for.

$$\begin{aligned} m\left(\vec{v}\right) d\vec{v} &= \vec{F}_{\text{Net}}\left(\vec{v}\, dt\right) \\ \int m\vec{v}\, d\vec{v} &= \int \vec{F}_{\text{Net}} \cdot d\vec{r} \\ \frac{1}{2}m\vec{v}^2 &= \mathrm{W} \end{aligned}$$

8.1 Kinetic Energy

Defn 32 (Kinetic Energy)**.** Kinetic Energy is the energy an object has due to its velocity. Using the Work-Kinetic Energy Theorem, we can define the *Kinetic Energy* of an object as:

$$\mathrm{K} = \frac{1}{2}m\vec{v}^2 \tag{8.3}$$

- m - Mass of the object
- $\vec{v}$ - Velocity of the object
- K - The total kinetic energy of the object

Example 8.1: Kinetic Energy of Rolling Downhill.

If something is moving downhill with a force of friction $F_f = 71\text{N}$, mass of $m = 58\text{kg}$ an initial velocity of $v_0 = 3.6\text{m/s}$ over a distance of $s = 57\text{m}$, what is its final velocity?

Solution on Work/Kinetic Energy note page.

8.2 Potential Energy

Defn 33 (Potential Energy)**.** *Potential energy* is the energy an object has because of a change in height. It is derived as shown.

$$\begin{aligned} \vec{F}_h &= -m\vec{g} \\ \mathrm{W} &= \int_{h_1}^{h_2} \vec{F}_h\, dh \\ \mathrm{W} &= \int_{h_1}^{h_2} -m\vec{g}\, dh \\ \mathrm{W} &= -m\vec{g}\left(h_2 - h_1\right) \end{aligned}$$

This means that the Potential Energy of something is defined as

$$\mathrm{U} = m\vec{g}h \tag{8.4}$$

- m - Mass of the object
- $\vec{g}$ - Gravity of Earth
- h - Height of the object above some reference height that we call "0"
- U - The Potential Energy of the object

Example 8.2: Potential Energy of Skiier.

If a skiier is on super slick ice such that there is no friction , starting from a height of $h_0 = 200\mathrm{m}$, is starting from rest, $v_0 = 0$, what is their final velocity, v?

To start this off, lets use the Law of Conservation of Energy

$$\mathrm{K}_0 + \mathrm{U}_0 = \mathrm{K} + \mathrm{U}$$
$$\frac{1}{2}mv_0^2 + m\vec{g}h_0 = \frac{1}{2}mv^2 + m\vec{g}h$$

To simplify this, we can say that $h = 0$, meaning that the final height is our reference height. This means that our final potential energy will be 0, $\mathrm{U} = 0$. Also, since the skiier starts at rest, their initial kinetic energy, $\mathrm{K}_0 = 0$.

$$\begin{aligned} 0 + m\vec{g}h_0 &= \frac{1}{2}mv^2 + 0 \\ \vec{g}h_0 &= \frac{1}{2}v^2 \\ v^2 &= 2\vec{g}h_0 \\ v &= \sqrt{2\vec{g}h_0} \\ v &= \sqrt{2(9.81)(200)} \\ v &= 62.94\mathrm{m/s} \end{aligned}$$

So, the final velocity of the skiier after going down the hill is 62.94 m/s.

8.2.1 Hooke's Law and Potential Energy

Hooke's Law can be applied to the concept of potential energy as well.

$$\begin{aligned} F_{\mathrm{H}} &= -k\Delta x \\ \mathrm{U} &= -\int F_{\mathrm{H}}\,dx \\ \mathrm{U} &= -\int -k\Delta x\,dx \\ \mathrm{U} &= \frac{1}{2}k\left(\Delta x\right)^2 \end{aligned}$$

$$\mathrm{U} = \frac{1}{2}k\left(\Delta x\right)^2 \tag{8.5}$$

8.3 Conservation of Energy

Defn 34 (Law of Conservation of Energy)**.** The *law of conservation of energy* states that energy can never be created, nor destroyed. Energy can only change forms.

$$\sum \mathrm{E} = \mathrm{Constant} \tag{8.6}$$

In this classical mechanical context, it means that kinetic energy and potential energy are always going to have to add up and be equal between the start and end of an experiment.

$$\begin{aligned} \mathrm{K} + \mathrm{U} &= \text{Constant} \\ \mathrm{K}_0 + \mathrm{U}_0 &= \mathrm{K} + \mathrm{U} \end{aligned} \tag{8.7}$$

Remark 34.1. This law always hold true. However, if you do not make your experiment a closed system, it might seem like energy was created or destroyed. When really is was provided by or lost to the environment. Types of systems are discussed much more in Physics 224: Modern Physics.

Example 8.3: Conservation of Energy.

If you drop a pendulum, where its initial conditions were:

- $\ell = 2\text{m}$ - Length of the string the mass is attached to
- $\theta = 30°$ - Angle from equilibrium pendulum was pulled to

What is the maximum velocity that the pendulum achieves v_{Max}?

Let's start by using Equation (8.7).

$$\begin{aligned} \mathrm{K}_0 + \mathrm{U}_0 &= \mathrm{K} + \mathrm{U} \\ \frac{1}{2}mv_0^2 + m\vec{g}h_0 &= \frac{1}{2}mv^2 + m\vec{g}h \end{aligned}$$

We can start by assuming that the velocity when it is dropped is $v_0 = 0$. This makes $\mathrm{K}_0 = 0$. The height that the pendulum is dropped from will be $h_0 = \ell - (\ell\cos(\theta))$. The point where the pendulum has the greatest velocity is right when it reaches the vertex of its swing, at the very bottom. This means that we want to know about $h = 0$.

$$\begin{aligned} 0 + m\vec{g}h_0 &= \frac{1}{2}mv_{\text{Max}}^2 + m\vec{g}h \\ 0 + m\vec{g}\left(\ell - \ell\cos(\theta)\right) &= \frac{1}{2}mv_{\text{Max}}^2 + 0 \\ \vec{g}\ell\left(1 - \cos(\theta)\right) &= \frac{1}{2}v_{\text{Max}}^2 \\ v_{\text{Max}}^2 &= 2\vec{g}\ell\left(1 - \cos(\theta)\right) \\ v_{\text{Max}} &= \sqrt{2\vec{g}\ell\left(1 - \cos(\theta)\right)} \\ v_{\text{Max}} &= \sqrt{2(9.81)(2)\left(1 - \cos(30°)\right)} \\ v_{\text{Max}} &= 2.4\text{m/s} \end{aligned}$$

So, the maximum velocity of the pendulum is 2.4 m/s.

9 Systems of Particles

Up until now we have only been considering the objects that we work with to be single, equally distributed masses. However, in the real world, we have strangely shaped things and objects that have multiple materials inside of them for balance and strength.

To account for oddities in these objects, we calculate something called the Center of Mass.

9.1 Center of Mass

Defn 35 (Center of Mass)**.** The *center of mass* of an object is a weighted average of all particles in a system. Center of mass takes the mass of a point and the distance from the center the point is into account, then normalizes by mass.

$$\vec{R} = \frac{\sum\limits_{i=1}^{n} m_i\vec{r}_i}{\sum\limits_{i=1}^{n} m_i} \tag{9.1}$$

Example 9.1: Center of Mass of Atomic Bond.

Take a carbon monoxide molecule, CO. Carbon has a mass of 12 u and Oxygen has a mass of 16 u. The bond between them is a long. What is the center of mass of this system?

We need to start by having a reference point, which we can define to be the center of the carbon atom. So, $r_{\mathrm{C}} = 0$ and $r_{\mathrm{O}} = a$.

$$\vec{R} = \frac{m_{\mathrm{C}}(0) + m_{\mathrm{O}}(a)}{m_{\mathrm{C}} + m_{\mathrm{O}}}$$
$$\vec{R} = \frac{12(0) + 16(a)}{12 + 16}$$
$$\vec{R} = \frac{16}{28}a$$
$$\vec{R} = \frac{4}{7}a$$

So, the center of mass of this carbon monoxide molecule is $\frac{4}{7}$ of the way towards the Oxygen atom.

9.1.1 Center of Mass in Multiple Dimensions

Center of Mass can be applied to systems in multiple dimensions. All that must be done is that the distances be calculated for each dimension separately.

Example 9.2: Center of Mass of Atomic Bond in 2-D.

Given a water atom, H_20, what is its center of mass?

- Mass of Hydrogen is 1 u
- Mass of Oxygen is 16 u
- Distance between hydrogen bonds is a

We can start by placing the hydrogen atoms on the x-axis of an xy-plane. The oxygen atom will sit on the y-axis. The variable h stands for the distance between the oxygen atom and the hydrogen bond location.

The center of mass, $\vec{R}$ will be broken down into its components. The x portion of the center of mass will be given the variable $\vec{X}$. The y portion of the center of mass will be given the variable $\vec{Y}$.

$$\vec{X} = \frac{m_{\mathrm{H}}\left(\frac{-a}{2}\right) + m_{\mathrm{H}}\left(\frac{-a}{2}\right) + m_{\mathrm{O}}(0)}{m_{\mathrm{H}} + m_{\mathrm{H}} + m_{\mathrm{O}}} = \frac{0}{2m_{\mathrm{H}} + m_{\mathrm{O}}}$$
$$\vec{X} = 0$$

Since the oxygen atom is at $x = 0$, it does not contribute to the center of mass in the x direction.

$$\vec{Y} = \frac{m_{\mathrm{O}}(h) + m_{\mathrm{H}}(0) + m_{\mathrm{H}}(0)}{m_{\mathrm{O}} + m_{\mathrm{H}} + m_{\mathrm{H}}}$$
$$\vec{Y} = \frac{m_{\mathrm{O}}}{m_{\mathrm{O}} + 2m_{\mathrm{H}}}h$$
$$\vec{Y} = \frac{16}{16 + 2(1)}h = \frac{16}{18}h$$
$$\vec{Y} = \frac{8}{9}h$$

Since the hydrogen atoms are on $y = 0$, they do not contribute to the center of mass in the y direction.
Combining both the x and y solutions, we come up with

$$\vec{R} = \left\langle 0, \frac{8}{9}h \right\rangle$$

10 Momentum

Along with Energy, another key takeaway from Newton's Laws is Momentum.

Defn 36 (Momentum)**.** *Momentum* is the product of mass and velocity. It can be thought of as the amount of moving "power" something has. Keep in mind this "power" is ***NOT*** the same as Power.

$$\vec{p} = \sum_{i=1}^{n} m_i \vec{v}_i \tag{10.1}$$

Defn 37 (Conservation of Momentum)**.** Momentum is always *conserved*, when it is a closed system.

$$\begin{aligned} \vec{p} &= \text{Constant} \\ \vec{p}_0 &= \vec{p} \end{aligned} \tag{10.2}$$

Example 10.1: Conservation of Momentum.

If a cannon, initially at rest, fires a cannonball from rest, what is the recoil velocity of the cannon, V? The cannonball is fired at $v = 150\text{m/s}$. The cannon is $M = 500\text{kg}$ and $m = 10\text{kg}$.

Let's start with Equation (10.2).

$$\vec{p}_0 = \vec{p}$$

Since everything is at rest when we start, $\vec{p}_0 = 0$. Since things are moving after the cannon was fired, we need to find $\vec{p}$.

$$\begin{aligned} \vec{p} &= MV + mv = 0 \\ \vec{p} &= 500V + 10(150) = 0 \\ 500V &= 0 - 10(150) \\ V &= -3\text{m/s} \end{aligned}$$

So, the cannon will move away from the starting position at 3 m/s.

11 Collisions

Collisions are a fundamental part of classical physics. For our uses, we will assume that these are closed systems.

In practice, there are 2 types of collisions.

1. Elastic Collisions
2. Inelastic Collisions

11.1 Elastic Collision

Defn 38 (Elastic Collision)**.** An *elastic collision* is a collision where the objects colliding:

- "Bounce" apart
- The objects' internal structure is unchanged

There are some conclusions that we can draw about these collisions based on the criteria above, assuming it is a closed system.

1. Mechanical energy is conserved. No kinetic energy is lost, it is converted to potential energies.

2. Linear Momentum is conserved.

The set of equations you will need to solve to solve an Elastic Collision problem are:

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 &= \sum_{i=1}^{n} \frac{1}{2} m_i {v'_i}^2 \\ \sum_{i=1}^{n} m_i v_i &= \sum_{i=1}^{n} m_i v'_i \end{aligned} \tag{11.1}$$

- The v_i' are ***NOT*** derivatives, but the velocities after the collision
- Each equation is just the sum of each object's Momentum or Kinetic Energy before and after the collision

Remark 38.1. Remember to keep your signs $(+, -)$ in order!! The initial velocity and final velocity might be going in different directions, keep that in mind!!

Example 11.1: Elastic Collision with Stationary Object.

Say we have 2 cars, with springs attached to them. The first car is moving at $v_1 = 5$m/s and has a mass of $m_1 = 2$kg. The second car is not moving $v_2 = 0$m/s and has a mass of $m_2 = 1$kg. What is the final velocities of the cars?

Start by referencing Equation (11.1).

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1{v_1'}^2 + \frac{1}{2}m_2{v_2'}^2$$
$$m_1v_1 + m_2v_2 = m_1{v_1'}^2 + m_2{v_2'}^2$$

Now we start plugging our given values in. Starting with the Kinetic Energy portion of the above equations.

$$\frac{1}{2}(2)5^2 + \frac{1}{2}(1)0^2 = \frac{1}{2}(2){v_1'}^2 + \frac{1}{2}(1){v_2'}^2$$
$$25 + 0 = {v_1'}^2 + \frac{1}{2}{v_2'}^2$$
$$25 = {v_1'}^2 + \frac{1}{2}{v_2'}^2$$

Now, moving to the Momentum portion of the equation.

$$2(5) + 1(0) = 2{v_1'}^2 + 1{v_2'}^2$$
$$10 = 2{v_1'}^2 + 1{v_2'}^2$$

Now we have to solve this system of equations.

$$25 = {v_1'}^2 + \frac{1}{2}{v_2'}^2$$
$$10 = 2{v_1'}^2 + 1{v_2'}^2$$

When solved, you end up with

$$v_1' = \frac{5}{3}\text{m/s}$$
$$v_2' = \frac{20}{3}\text{m/s}$$

After the collision, both cars are going the same direction. The originally moving one is moving at $\frac{5}{3}$ m/s. The originally stationary one is moving at $\frac{20}{3}$ m/s.

Example 11.2: Elastic Collision with Non-Stationary Objects.

Say we have 2 cars, with springs attached to them. The first car is moving at $v_1 = 5$m/s and has a mass of $m_1 = 2$kg. The second car is moving towards the first $v_2 = 10$m/s and has a mass of $m_2 = 1$kg. What is the final velocities of the cars?

Start by referencing Equation (11.1).

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1{v_1'}^2 + \frac{1}{2}m_2{v_2'}^2$$
$$m_1v_1 + m_2v_2 = m_1{v_1'}^2 + m_2{v_2'}^2$$

Now we start plugging our given values in. Starting with the Kinetic Energy portion of the above equations. Keep in mind that the velocity of one car will have to be defined as negative, since the two cars are moving towards each other.

$$\begin{aligned}\frac{1}{2}(2)5^2 + \frac{1}{2}(1)(-10)^2 &= \frac{1}{2}(2){v_1'}^2 + \frac{1}{2}(1){v_2'}^2 \\ 25 + 50 &= {v_1'}^2 + \frac{1}{2}{v_2'}^2 \\ 75 &= {v_1'}^2 + \frac{1}{2}{v_2'}^2\end{aligned}$$

Now, moving to the Momentum portion of the equation.

$$\begin{aligned}2(5) + 1(-10) &= 2{v_1'}^2 + 1{v_2'}^2 \\ 10 + -10 &= 2{v_1'}^2 + {v_2'}^2 \\ 0 &= 2{v_1'}^2 + {v_2'}^2\end{aligned}$$

Now we have to solve this system of equations.

$$\begin{aligned}{v_1'}^2 + \frac{1}{2}{v_2'}^2 &= 75 \\ 2{v_1'}^2 + {v_2'}^2 &= 0\end{aligned}$$

When solved, you end up with

$$\begin{aligned}v_1' &= -5\text{m/s} \\ v_2' &= 10\text{m/s}\end{aligned}$$

After the collision, the cars are going away from each other. The first car is going backward at 5 m/s. The second car is moving forward (what we defined as forward) at 10 m/s.

11.2 Inelastic Collisions

Defn 39 (Inelastic Collision)**.** An *inelastic collision* is a collision where the objects colliding:

- "Stick" together
- The objects' internal structure is changed

There are some conclusions that we can draw about these collisions based on the criteria above, assuming it is a closed system.

1. Mechanical energy is ***NOT*** conserved.
2. Linear Momentum is conserved.

The equation that you will need to solve an Inelastic Collision problem is:

$$\sum_{i=1}^{n} m_i v_i = \left(\sum_{i=1}^{n} m_i\right) v' \tag{11.2}$$

- v' is the final velocity of the objects after they have collided. It is the same for all objects because they stuck together.

Remark 39.1. Remember to keep your signs (+, −) in order!! The initial velocity and final velocity might be going in different directions, keep that in mind!!

11.3 Collisions in Multiple Dimensions

To solve a collision problem that is in multiple dimensions, you break it down into its component vectors, and solve each separately.

Example 11.3: Multiple Dimension Collision.

Say 2 objects are travelling in an xy-plane. When they have an *inelastic* collision, they veer off at different angles. Object A has an angle of α from the x-axis and is travelling at $\vec{v}_{A,f}$. Object A is initally travelling on the x-axis, with no y-component. Object B has an angle of β from the x-axis and is travelling at $v_{B,f}$. Object B initially starts at rest, $\vec{v}_{B,i} = \langle 0, 0 \rangle$.
What are the initial velocity of object A?

Since this is an Inelastic Collision, we want to use Equation (11.2).

$$m_A \, \vec{v}_{A,i} + m_B \, \vec{v}_{B,i} = m_A \, \vec{v}_{A,f} + m_B \, \vec{v}_{B,f}$$

Now start plugging in information that we know.

$$\begin{aligned} m_A \, \vec{v}_{A,i} + m_B \left(\vec{0}\right) &= m_A \, \vec{v}_{A,f} + m_B \, \vec{v}_{B,f} \\ m_A \, \vec{v}_{A,i} + 0 &= m_A \, \vec{v}_{A,f} + m_B \, \vec{v}_{B,f} \\ m_A \, \vec{v}_{A,i} &= m_A \, \vec{v}_{A,f} + m_B \, \vec{v}_{B,f} \end{aligned}$$

Since this is a multi-dimensional collision problem, we need need to break the velocity vectors down into their component vectors.

$$\begin{aligned} &\mathbf{X} \rightarrow m_A \, \vec{v}_{A,i} \cos(0) = m_A v_{A,f} \cos(\alpha) + m_B \, v_{B,f} \cos(\beta) \\ &\mathbf{Y} \rightarrow m_A \, \vec{v}_{A,i} \sin(0) = m_A v_{A,f} \sin(\alpha) - m_B \, v_{B,f} \sin(\beta) \end{aligned}$$

Object B is moving the the negative y-direction, so it has a negative in front of its term. We know that Object A was not moving in the y-direction at all, so the above equations simplify down to:

$$\begin{aligned} &\mathbf{X} \rightarrow m_A \, \vec{v}_{A,i} \cos(0) = m_A v_{A,f} \cos(\alpha) + m_B \, v_{B,f} \cos(\beta) \\ &\mathbf{Y} \rightarrow 0 = m_A v_{A,f} \sin(\alpha) - m_B \, v_{B,f} \sin(\beta) \end{aligned}$$

Since we know the final velocity of Object A, the y-component equation can give us the one unknown we have, $v_{B,f}$.

$$\begin{aligned} 0 &= m_A \, v_{A,f} \sin(\alpha) - m_B \, v_{B,f} \sin(\beta) \\ m_A \, v_{A,f} \sin(\alpha) &= m_B \, v_{B,f} \sin(\beta) \\ v_{B,f} &= \frac{m_A v_{A,f} \sin(\alpha)}{m_B \sin(\beta)} \end{aligned}$$

Now, plugging in that value to the x-component equation, we can solve for $v_{A,i}$.

$$\begin{aligned} m_A \, v_{A,i} &= m_A \, v_{A,f} \cos(\alpha) + m_B \left(\frac{m_A v_{A,f} \sin(\alpha)}{m_B \sin(\beta)} \right) \cos(\beta) \\ v_{A,i} &= v_{A,f} \cos(\alpha) + \left(\frac{v_{A,f} \sin(\alpha)}{\sin(\beta)} \right) \cos(\beta) \end{aligned}$$

So, the answer for the inital velocity of Object A is:

$$\vec{v}_{A,i} = \left\langle v_{A,f} \cos(\alpha) + \frac{v_{A,f} \sin(\alpha) \cos(\beta)}{\sin(\beta)}, 0 \right\rangle \text{m/s}$$

12 Rotational Kinematics

Defn 40 (Rotational Kinematics)**.** *Rotational kinematics* is the application of Kinematics to rotating objects. There are some stipulations that need to be made for rotational kinematics to make sense.

1. The thing ***MUST*** be macroscopic. If the thing gets too small, rotation doesn't make sense.
2. The thing must be an absolutely Rigid Body

If an object is undergoing rotation, then there are is both Linear Kinematics and Rotational Kinematics.

Defn 41 (Rigid Body)**.** A *rigid body* is one that that experiences no or negligible deformation during the experiment. Another way to define this is that any 2 arbitrary points do not change during rotation.

12.1 Linear Kinematics and Rotational Kinematics Correlation

Rotational Kinematics are closely related to Linear Kinematics. Many of the definitions that were presented in Uniform Circular Motion can be translated between the two.

	Linear	Rotational
Coordinates	x	θ
Velocity	$\vec{v} = \frac{dx}{dt}$	$\vec{\omega} = \frac{d\theta}{dt}$
Acceleration	$\vec{a} = \frac{d\vec{v}}{dt}$	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Table 12.1: Linear Kinematics and Rotational Kinematics Correlation

12.2 Linear Dynamics and Rotational Dynamics Correlation

	Linear	Rotational
Kinetic Energy	$\text{K} = \frac{1}{2}mv^2$	$\text{K} = \frac{1}{2}I\omega^2$
Center of Mass/Moment of Inertia	$\vec{R} = \sum_{i=1}^{n} \frac{m_i r_i}{m_i}$	$I = \sum_{i=1}^{n} m_i r_i^2$
Force	$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$
Work	$\text{W} = \int \vec{F} \cdot d\vec{r}$	$\text{W} = \int \vec{\tau} \cdot d\theta$
Power	$\text{P} = \frac{d\text{W}}{dt}$	$\text{P} = \vec{\tau} \cdot \vec{\omega}$
Momentum	$\vec{p} = m\vec{v}$	$\vec{\ell} = I\vec{\omega}$

Table 12.2: Linear Dynamics and Rotational Dynamics Correlation

13 Rotational Dynamics

Rotational dynamics applies the concepts from Newton's Laws, Energy, Systems of Particles, and Momentum to rotating objects.

13.1 Moment of Inertia

Defn 42 (Moment of Inertia)**.** A *moment of inertia* or *moment* is quite similar to Center of Mass. It is the point on the object where the mass and distance from this point are equal throughout the object. It can be thought of as the "balancing" point for rotation.

$$I = \sum_{i=1}^{n} m_i r_i^2 \tag{13.1}$$

Remark 42.1. Because we have made the constraint that all the objects we will be dealing with are Rigid Body objects, that means our angular velocity, ω is constant everywhere.

Remark 42.2. Moments of Inertia are additive.

Theorem 13.1 (Parallel Axis Theorem)**.** *Suppose a body of mass m is made to rotate about an axis z passing throught the body's Center of Mass. The body has a Moment of Inertia I_{CoM} with respect to this axis. The Parallel Axis Theorem states that if the body is made to rotate about a new axis z' which is parallel to the first axis and displaced from it by a distance d, then the Moment of Inertia I with respect to the new axis is related to I_{CoM} by*

$$I = I_{CoM} + md^2 \tag{13.2}$$

13.1.1 Common Moments of Inertia

Object Type	Moment of Inertia Formula
Slender Rod, Axis Through Center (Not Length-wise Axis)	$I = \frac{1}{12}mh^2$
Slender Rod, Axis at End	$I = \frac{1}{3}mh^2$
Rectangular Plate, Axis Through Center	$I = \frac{1}{12}m\left(a^2 + b^2\right)$
Rectangular Plate, Axis along Edge	$I = \frac{1}{3}\left(\perp \text{Side}\right)^2$
Hollow Cylinder	$I = \frac{1}{2}m\left(r_{\text{Outer}}^2 - r_{\text{Inner}}^2\right)$
Solid Cylinder	$I = mr^2$
Thin Walled Cylinder (Soda can thin)	$I = mr^2$
Solid Sphere	$\frac{2}{5}mr^2$
Thin Walled Sphere	$I = \frac{2}{3}mr^2$

13.2 Torque

Defn 43 (Torque)**.** *Torque* is the force that is applied when making an object rotate. There are two portions to it. The force that is being applied, $\vec{F}$ at some radius r from the point of rotation.

There are 2 equations that can be applied for Torque.

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{13.3}$$

$$\vec{\tau} = \|\vec{r}\|\|\vec{F}\| \sin(\theta) \tag{13.4}$$

13.3 Angular Momentum

Defn 44 (Angular Momentum)**.** *Angular momentum* is similar to Linear Momentum, but we define it in terms of rotation components.

$$\vec{\ell} = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i \tag{13.5}$$

Defn 45 (Conservation of Angular Momentum)**.** The *conservation of angular momentum* states that if $\vec{\tau}_{\text{Net}} = \vec{0}$, then $\vec{\ell} = \text{Constant}$.

14 Gravitation

Defn 46 (Gravity)**.** *Gravity* is one of The 4 Fundamental Interactions in the Universe. It is also one of the least understood. However, it it the thing that is keeping us "glued" to this plant.

There is an equation that describes the force caused by gravity on multiple objects.

$$\vec{F_g} = G\frac{m_1 m_2}{r^2} \cdot \frac{\vec{r}}{\hat{r}} \tag{14.1}$$

- G - The universal gravitational constant. The actual value is in Appendix A.
- m_1 - The mass of one of the objects.
- m_2 - The mass of the other object.

- r^2 - Distance between the objects.
- $\vec{r}$ - Vector describing the direction of the distance.
- $\hat{r}$ - Normalizing unit vector

Remark 46.1. Equation (14.1) only applies if:
- Relativistic effects are small
- Relatively low density, compared to a black hole for instance
- Relatively low velocity, compared to light speed

Defn 47 (Kepler's Laws)**.** Johannes Kepler developed these three laws for gravitation and planetary motion.
1. Planets move in ellipses with one focus at the sun.
2. Planets move fastest when close to the sun, and slowest when furthest from the sun, in the same time. The area of sectors in the ellipse are the same.
3. The sun's side is perihelion, the far side is the aphelion. Period^2 $\propto$ Semi-Major^3.

$$\frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3} \tag{14.2}$$

Using the information from Definition 46, we can construct an updated set of Kepler's Laws.
1. All trajectories are conic sections.
2. Angular momentum, $\vec{\ell}$ is conserved.
3. $\frac{T^2}{a^3} = \frac{4\pi^2}{G(m+M)}$.

15 Statics

Defn 48 (Statics)**.** *Statics* is the case when an object is at rest. It is when the sum of all forces in x, y, and z axes is equal to 0.
- Acceleration is 0, which means $\vec{F}_{\text{Net}} = \vec{0}$. This means that there is no linear motion.
- Angular Acceleration is 0, which means $\vec{\tau}_{\text{Net}} = \vec{0}$. This means that there is no rotational motion.
- The Center of Gravity is the Center of Mass.

To solve any Statics problem, you simply have to solve for Equation (15.1). The net force and net Torque both must sum to 0.

$$\begin{gathered}\sum_{i=1}^{n} \vec{F}_i = \vec{0} \\ \sum_{i=1}^{n} \vec{\tau}_i = \vec{0}\end{gathered} \tag{15.1}$$

16 The 4 Fundamental Interactions in the Universe

There are 4 fundamental interactions present in our universe, as we know it right now.
1. Gravity
2. Electromagnetic
3. "Weak" (Neutrinos and the like)
4. "Strong" (Holds atomic nucleus together)

A Physical Constants

Constant Name	**Variable Letter**	**Value**
Boltzmann Constant	R	$8.314\mathrm{J/mol\,K}$
Universal Gravitational	G	$6.67408 \times 10^{-11}\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Planck's Constant	h	$6.62607004 \times 10^{-34}\mathrm{mkg/s} = 4.163 \times 10^{-15}\mathrm{eV\,s}$
Speed of Light	c	$299792458\mathrm{m/s} = 2.998 \times 10^{8}\ \mathrm{m/s}$
Charge of Electron	e	$1.602 \times 10^{-19}\mathrm{C}$
Mass of Electron	m_{e^-}	$9.11 \times 10^{-31}\mathrm{kg}$
Mass of Neutron	m_{n^0}	$1.67 \times 10^{-31}\mathrm{kg}$
Mass of Earth	m_{Earth}	$5.972 \times 10^{24}\mathrm{kg}$
Diameter of Earth	d_{Earth}	$12742\mathrm{km}$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{B.24}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.26}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.27}$$

C Calculus

C.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcialable results.

Lemma C.0.1 (L'Hopital's Rule)**.** *If the equation*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) *holds.*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.3}$$

Defn C.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.
If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.4}$$

C.4 Useful Integrals

$$\int \cos(x)\, dx = \sin(x) \tag{C.5}$$

$$\int \sin(x)\, dx = -\cos(x) \tag{C.6}$$

$$\int x \cos(x)\, dx = \cos(x) + x \sin(x) \tag{C.7}$$

Equation (C.7) simplified with Integration by Parts.

$$\int x\sin(x)\;dx = \sin(x) - x\cos(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2\cos(x)\;dx = 2x\cos(x) + (x^2-2)\sin(x) \tag{C.9}$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2\sin(x)\;dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\;dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.11}$$

$$\int e^{\alpha x}\sin(\beta x)\;dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x}\;dx = \frac{e^{\alpha x}}{\alpha} \tag{C.13}$$

$$\int xe^{\alpha x}\;dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{C.14}$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha+\beta x} = \int \frac{1}{\alpha+\beta x}\;dx = \frac{1}{\beta}\ln(\alpha+\beta x) \tag{C.15}$$

$$\int \frac{dx}{\alpha^2+\beta^2x^2} = \int \frac{1}{\alpha^2+\beta^2x^2}\;dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{C.16}$$

$$\int \alpha^x\;dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.17}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{C.18}$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\big[b(t),t\big]\,\frac{db(t)}{dt} - f\big[a(t),t\big]\,\frac{da(t)}{dt} \tag{C.19}$$

D Complex Numbers

Defn D.0.1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted mathbbC.

A complex number is written, in Cartesian form, as shown in Equation (D.1) below.

$$z = a \pm ib \tag{D.1}$$

where

$$i = \sqrt{-1} \tag{D.2}$$

Remark (i vs. j for Imaginary Numbers)*.* Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

D.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn D.1.1 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\mathrm{Re}\{z\} = x \tag{D.3}$$

Remark D.1.1.1 (Alternative Notation)*.* The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re\mathfrak{e}$$

Defn D.1.2 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\mathrm{Im}\{z\} = y \tag{D.4}$$

Remark D.1.2.1 (Alternative Notation)*.* The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im\mathfrak{m}$$

D.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$z = x_1 + iy_1$$
$$w = x_2 + iy_2$$

D.2.1 Addition

The addition operation, still denoted with the + symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w \coloneqq (x_1 + x_2) + i(y_1 + y_2) \tag{D.5}$$

D.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$\begin{aligned} zw &\coloneqq (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\ &= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned} \tag{D.6}$$

D.3 Complex Conjugates

Defn D.3.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{D.7}$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{D.8}$$

D.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$
2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - xyi + xyi - i^2y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2 \end{aligned}$$

Thus,

$$z\overline{z} = x^2 + y^2 \tag{D.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}$$

Using our solution from Equation (D.9):

$$\begin{aligned} &= \frac{(x + iy)^2}{x^2 + y^2} \\ &= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2} \end{aligned}$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$\begin{aligned} &= \frac{\left(x^2 - y^2\right) + 2xyi}{x^2 + y^2} \\ &= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \end{aligned}$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{D.10}$$

D.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.
First, start by defining some expressions so that we can prove some of these properties:

$$\begin{aligned} z &= x + iy \\ \overline{z} &= x - iy \end{aligned}$$

(i) The conjugation operation is commutative.
(ii) The conjugation operation can be distributed over addition and multiplication.

$$\begin{aligned} \overline{z + w} &= \overline{z} + \overline{w} \\ \overline{zw} &= \overline{zw} \end{aligned}$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers ($z, w \in$ mathbbC) where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.
We start by simplifying the left-hand side of the equation ($\overline{z + w}$).

$$\begin{aligned} \overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

Now, we simplify the other side ($\overline{z} + \overline{w}$).

$$\begin{aligned} \overline{z} + \overline{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

■

D.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn D.4.1 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in$ mathbbC. This is taken from Descarte's definition of a completely real plane.
The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

D.4.1 Modulus of a Complex Number

Defn D.4.2 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\overline{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{D.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in$ mathbbC). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (D.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\overline{z}\overline{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \end{aligned}$$

Note how we never needed to define z or w, so this is as general a result as possible. ∎

D.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right) \\ &= |z|^2|w|^2 \end{aligned}$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

D.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

D.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (D.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{D.12}$$

Example D.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$\begin{aligned}
|5iz + 10| &= 7 \\
|5i(z + \frac{10}{5i})| &= 7 \\
|5i(z + \frac{2}{i})| &= 7 \\
|5i(z + \frac{2}{i}\frac{-i}{-i})| &= 7 \\
|5i(z - 2i)| &= 7
\end{aligned}$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$\begin{aligned}
|5i||z - 2i| &= 7 \\
5|z - 2i| &= 7 \\
|z - 2i| &= \frac{7}{5}
\end{aligned}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

D.5.1 Annulus

Defn D.5.1 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (D.13).

$$r_1 \leq |z - a| \leq r_2 \tag{D.13}$$

In Equation (D.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

D.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (D.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{D.14}$$

Remark. Note that in the definition of polar form (Equation (D.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$\begin{aligned}
r &= |z| \\
\theta &= \arg(z)
\end{aligned}$$

Example D.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned} \cos(\theta) &= \frac{-\sqrt{3}}{2} \\ \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned} r &= 2 \\ \theta &= \frac{5\pi}{6} \\ \arg(z) &= \frac{5\pi}{6} + 2\pi k \end{aligned}$$

D.6.1 Converting Between Cartesian and Polar Forms

Using Equation (D.14) and Equation (D.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{D.15}$$

D.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

D.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \tag{D.16}$$

D.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big) \tag{D.17}$$

D.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph D.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn D.6.1 (de Moiver's Law)**.** Given a complex number z, $z \in$ mathbbC and a rational number n, $n \in$ mathbbQ, the exponentiation of z^n is defined as Equation (D.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{D.18}$$

D.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{D.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (D.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{D.20}$$

A w that satisfies Equation (D.20) is an nth root of z.

Example D.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example D.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (D.19), we can find the general form of the roots.

$$\begin{aligned} z &= 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right) \\ z^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right) \end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \ldots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

k = 2

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(2)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(2)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)$$

k = 3

$$\begin{aligned}\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(3)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(3)}{18}\right)\right) &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + \frac{36\pi}{18}\right) + i\sin\left(\frac{\pi}{18} + \frac{36\pi}{18}\right)\right)\\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + 2\pi\right) + i\sin\left(\frac{\pi}{18} + 2\pi\right)\right)\\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)\end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned}z_1^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)\\ z_2^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)\\ z_3^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)\end{aligned}$$

D.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn D.8.1 (Argument)**.** The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example D.4: Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i\\ \arg(z) = \tan(\theta) &= \frac{-i}{-1}\\ &= \frac{\pi}{4}\end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z) = \frac{5\pi}{4} + 2\pi k$$

Thus, the argument of $z = -1 - i$ is $\arg(z) = \frac{5\pi}{4} + 2\pi k$.

Defn D.8.2 (Principal Argument)**.** The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi < \mathrm{Arg}(z) \leq \pi$.

Example D.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) = \tan(\theta) &= \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\operatorname{Arg}(z) = \frac{5\pi}{4}$.

D.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (D.21).

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big) \tag{D.21}$$

If instead of e as the base, we have some value a, then we have Equation (D.22).

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ &= e^{\operatorname{Re}\{z\ln(a)\}}\Big(\cos\big(\operatorname{Im}\{z\ln(a)\}\big) + i\sin\big(\operatorname{Im}\{z\ln(a)\}\big)\Big) \end{aligned} \tag{D.22}$$

In the case of Equation (D.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example D.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$\begin{aligned} x &= -1 \\ y &= \sqrt{3} \end{aligned}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\begin{aligned} |e^{-1+i\sqrt{3}}| &= |e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)| \\ &= e^{-1} \\ \arg\left(e^{-1+i\sqrt{3}}\right) &= \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} + 2\pi k \\ \operatorname{Arg}\left(e^{-1+i\sqrt{3}}\right) &= \operatorname{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} \end{aligned}$$

Example D.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^{0}\big(\cos(-1) + i\sin(-1)\big) \\ &= 1\big(\cos(-1) + i\sin(-1)\big) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1\big(\cos(-1)+i\sin(-1)\big)} \\ &= e^{-1\big(\cos(1)-i\sin(1)\big)} \\ &= e^{-\cos(1)+i\sin(1)} \end{aligned}$$

If we refer back to Equation (D.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big) \end{aligned}$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} |e^{-e^{-i}}| &= |e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)| \\ &= e^{-\cos(1)} \\ \arg\left(e^{-e^{-i}}\right) &= \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) + 2\pi k \\ \operatorname{Arg}\left(e^{-e^{-i}}\right) &= \operatorname{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) \end{aligned}$$

Example D.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^i$?

Use Equation (D.22) to simplify this to a base of e, where we can use the usual Equation (D.21) to solve this.

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ 1^i &= e^{i\ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^2} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

D.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{D.23}$$

D.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn D.10.1 (Complex Logarithm)**.** The *complex logarithm* is defined in Equation (D.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e|w| + i\arg(w) \end{aligned} \tag{D.24}$$

Remark D.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example D.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (D.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e|z| + i\arg(z) \\ &= \log_e|-1| + i\arg(-1) \\ &= \log_e(1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \ldots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example D.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (D.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e|w| + i\arg(w) \\ \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e|2\pi k i| + i\arg(2\pi k i) \\ &= \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases} \right) \end{aligned}$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i\begin{cases}\frac{\pi}{2} + 2\pi m & k > 0\\ \frac{-\pi}{2} + 2\pi m & k < 0\end{cases}\right)$.

D.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\overline{z}) \tag{D.25}$$

D.11 Complex Trigonometry

For the equations below, $z \in$ mathbbC. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{D.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{D.27}$$

Example D.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

We start by using the defintion of complex cosine Equation (D.26).

$$\begin{aligned}
\cos(z) &= 5\\
\frac{e^{iz} + e^{-iz}}{2} &= 5\\
e^{iz} + e^{-iz} &= 10\\
e^{iz}\left(e^{iz} + e^{-iz}\right) &= e^{iz}(10)\\
{e^{iz}}^2 + 1 &= 10e^{iz}\\
{e^{iz}}^2 - 10e^{iz} + 1 = 0
\end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned}
e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)}\\
&= \frac{10 \pm \sqrt{100 - 4}}{2}\\
&= \frac{10 \pm \sqrt{96}}{2}\\
&= \frac{10 \pm 4\sqrt{6}}{2}\\
&= 5 \pm 2\sqrt{6}
\end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned}
iz &= \ln(5 \pm 2\sqrt{6})\\
&= \log_e|5 \pm 2\sqrt{6}| + i\arg(5 \pm 2\sqrt{6})\\
&= \log_e|5 \pm 2\sqrt{6}| + i(0 + 2\pi k)\\
&= \log_e|5 \pm 2\sqrt{6}| + 2\pi k i\\
z &= \frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}| + 2\pi k i\right)\\
&= \frac{-i}{-i}\frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}|\right) + 2\pi k\\
&= 2\pi k - i\log_e|5 \pm 2\sqrt{6}|
\end{aligned}$$

Thus, $z = 2\pi k - i\log_e|5 \pm 2\sqrt{6}|$.

D.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x+iy) = \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big) \tag{D.28}$$

$$\sin(x+iy) = \big(\sin(x)\cosh(y)\big) + i\big(\cos(x)\sinh(y)\big) \tag{D.29}$$

Example D.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (D.28).

$$\begin{aligned}
\cos(x+iy) &= \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big) \\
\cos(2+3i) &= \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)
\end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (D.21), we can solve this.

$$\begin{aligned}
e^z = e^{x+iy} &= e^x\big(\cos(y) + i\sin(y)\big) \\
x &= \cos(2)\cosh(3) \\
y &= -\sin(2)\sinh(3) \\
e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} &= e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)
\end{aligned}$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned}
|z| &= |e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)| \\
&= e^{\cos(2)\cosh(3)} \\
\arg(z) &= \arg(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)) \\
&= -\sin(2)\sinh(3) + 2\pi k \\
\operatorname{Arg}(z) &= \operatorname{Arg}(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)) \\
&= -\sin(2)\sinh(3)
\end{aligned}$$

D.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned}
\overline{\cos(x)} &= \cos(x) \\
&= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)
\end{aligned} \tag{D.30}$$

$$\begin{aligned}
\overline{\sin(x)} &= \sin(x) \\
&= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)
\end{aligned} \tag{D.31}$$