1 General Stuff

- Density - $\rho = \frac{\Delta m}{\Delta V}$
  - Uniform Density - $\rho = \frac{m}{V}$
- Pressure - $p = \frac{\Delta F}{\Delta A}$
  - Uniform Force on Flat Area - $\rho = \frac{F}{A}$
  - Conversions - $1 atm = 1.01 \times 10^5 Pa = 760 torr = 14.7 lb/in^2$
- Trig Relations are in Trigonometric Formulas, Section A.2.1.

2 Fluids

We must satisfy several parameters to make life easier, and to use most of these formulae.

1. Incompressible - Density of the fluid is constant
2. Non-turbulent Flow - Think of fluids swirling around an object
3. Isostatic Pressure - Pressure inside the fluid is the same in all directions

- Pressure at Some Depth - $p_2 = p_1 + \rho g (y_1 - y_2)$
  - Pressure at Depth $h \rightarrow p = p_0 + \rho g h$
- Pascal's Principle - 2 Parts
  1. $\vec{F}_o = \vec{F}_i \frac{A_o}{A_i}$
  2. $d_o = d_i \frac{A_i}{A_o}$
  - When 2 pressures should be equal, their forces are inversely proportional
  - Set each pressure equal to each other, then solve for the missing variable.
- Archimedes' Principle - $\vec{F}_{Up} = \vec{F}_{Down}$
  - Usually breaks down to $\vec{F}_{Bouyant} = \vec{F}_{g,Object}$
  - $\vec{F}_{Bouyant} = m_{Object} g$
  - $\vec{F}_{Bouyant} = \rho_{Object} V_{Object} g$
- Continuity of Fluids
  - $A_1 v_1 = A_2 v_2$
- Bernoulli's Equation - $p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$
  - Fluids at Rest - $p_2 = p_1 + \rho g (y_1 - y_2)$
  - Fluids not Changing Height - $p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$

3 Waves

Usually of form $y = y_m \sin (kx \pm \omega t)$ There are two types of waves:

1. Transverse Waves - Waves where displacement from equilibrium is orthogonal to direction of propagation
   - String Waves
   - Electromagnetic Waves
2. Longitudinal Waves - Waves where displacement from equilibrium is parallel to direction of propagation
   - Pressure Waves
   - Sound Waves (Which are a type of pressure wave)

- y_m - Amplitude, m
- k - Angular Wave Number, rad/m
  - $k = \frac{2\pi}{\lambda}$
  - λ is wavelength, m
- ω - Angular Frequency, rad/s
  - $\omega = 2\pi f$
  - f is frequency, Hz
  - Sign of this goes the opposite the direction the wave is going
    1. Wave going in positive direction $(+)$, then the sign should be negative $(-)$
    2. Wave going in negative direction $(-)$, then the sign should be positive $(+)$

- $v = \lambda f$, Wave Velocity, m/s
    - $v = \frac{\omega}{2\pi} * \frac{2\pi}{k} = \frac{\omega}{k}$
    - This can be proven with the angular portion of any wave (inside the parentheses of trig function)

$$kx - \omega t = \text{Constant}$$
$$\frac{d}{dt}\left[kx - \omega t\right] = \frac{d}{dt}\left[\text{Constant}\right]$$
$$k\frac{dx}{dt} - \omega\frac{dt}{dt} = 0$$
$$kv - \omega = 0$$
$$kv = \omega$$
$$v = \frac{\omega}{k}$$

Since $\omega = 2\pi f$, then $k = \frac{\lambda}{2\pi}$

Wave Interference

Waves are nice, and they just sum when they interfere. Let:

$$y_1\left(x, t\right) = y\sin\left(kx - \omega t\right) \tag{3.1}$$
$$y_2\left(x, t\right) = y\sin\left(kx + \omega t + \varphi\right) \tag{3.2}$$
$$Y\left(x, t\right) = y\left[\sin\left(kx - \omega t\right) + \sin\left(kx + \omega t + \varphi\right)\right] \tag{3.3}$$

You can usually use Equation A.1 to simplify Equation 3.3.

Constructive/Destructive Interference

$$\phi = 2\pi\frac{PathLengthDiff}{\lambda} = 2\pi\frac{\Delta\text{PLD}}{\lambda}$$
$$n = \frac{\phi}{2\pi} = 4\frac{PathLengthDiff}{\lambda}$$

- $PathLengthDiff$ - Is the Difference in path lengths that the waves must travel
- phi - Angular Location of points of Complete Constructive/Destructive Interference
- n - Number of locations where there is Complete Constructive/Destructive Interference

Standing Waves

This is actually the superposition of 2 waves, traveling in opposite directions, on a medium that is fixed at both ends, i.e. a taut string held by a wall.

Location of Nodes and Antinodes

- Nodes - $x = n\frac{\lambda}{2}$ for $n = 0, 1, 2, \ldots$
    - Always at closed ends of tubes
- Antinodes - $x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}$ for $n = 0, 1, 2, \ldots$
    - Always at open ends of tubes

Resonant Frequencies/Harmonics

These can also be called harmonics. There is a resonant frequency for every number of nodes/antinodes on tthe standing wave.
$f = \frac{v}{\lambda} = n\frac{v}{2L}$

- L is the length of the medium (The String).
- λ is the wavelength of the wave formed.

This can be extended to find the base resonant frequency, if you know how many node levels are between the two resonant frequencies given, i.e. they say that the **NEXT** frequency, means $n + 1$.
$f_{n+m} - f_n = (n + m)\frac{v}{2L} - n\frac{v}{2L} = m\frac{v}{2L}$

Reflecting Sound

- $D = (n + 1) d = vt$
  - n is the number of reflections that occurred
  - $n + 1$ is used when we want the distance the wave covers

Sound in Different Mediums

Frequency is a property of a wave, and **CANNOT BE ALTERED.** This means that:

$$v = \lambda f$$
$$v_{Sound,Material1} = \lambda_{Material1} f_{Unique,Material1}$$
$$v_{Sound,Material2} = \lambda_{Material2} f_{Unique,Material2}$$
$$f_{Unique,Material1} = f_{Unique,Material2}$$
$$\frac{v_{Sound,Material1}}{\lambda_{Material1}} = \frac{v_{Sound,Material2}}{\lambda_{Material2}}$$

Doppler Effect

$f' = f \frac{v \pm v_D}{v \pm v_S}$

- Moving ***TOWARDS*** each other: Frequency Increase
- Moving ***AWAY*** from each other: Frequency Decrease
- f - Initial Frequency, Hz
- v - Sound Speed, m/s
- v_D - Detector Speed, m/s
- v_S - Source Speed, m/s
- **For Numerator**:
  - If detector is moving towards the source, $+$
  - If detector is moving away from the source, $-$
- **For Denominator**:
  - If source is moving away from detector, $+$
  - If source is moving towards the detector, $-$

4 Thermodynamics

Defn 1 (Thermodynamics). *Thermodynamics* is the study of energy transfer between two macroscopic bodies driven by temperature differences.

Defn 2 (Temperature). *Temperature* is a direct measurement of internal energy of a system

Laws of Thermodynamics

Defn 3 (0th Law of Thermodynamics). If 2 bodies, A and B are in thermal equilibrium with a third body "T", then they are in thermal equilibrium with each other.

Defn 4 (1st Law of Thermodynamics).

$$dE_{Internal} = dQ - dT, \ dQ \text{ and } dT \text{ are inexact (path-dependent) differentials.}$$
$$\Delta E_{Internal} = Q - T$$

(4.1)

Note 4.1. **Special Cases for the 1st Law of Thermodynamics**:
1. Adiabatic Processes - $dE_{Int} = -dW$
   - No heat exchange
   - Insulating
   - Something happens too quickly for system to keep up
2. Isothermal Processes - $dT = 0 \rightarrow dE_{Int} = 0 \rightarrow dQ = dW$
3. Isobaric Processes - $dW = pdV$, p(pressure) is constant
4. Constant Volume - $W = 0$

5. Cyclical Processes - $dE_{Int} = 0 \rightarrow dQ = dW$
   - You end a cycle with the same internal energy when the cycle started

Defn 5 (2nd Law of Thermodynamics). If a *cyclical process occurs in a CLOSED system*, the entropy of the system increases for irreversible processes and remains constant for reversible processes. **IT NEVER DECREASES!!**

$$\Delta S \geq 0 \tag{4.2}$$

$$\Delta S = \int_a^b \frac{dQ\,(T)}{T} \tag{4.3}$$

Note 5.1. **ΔS is a state-function, meaning it is path-independent.**

Heat and Work

- $dW = \vec{F} \cdot \vec{ds}$
- $\vec{F} = p\,(V, T)\,dV$
- $W = \int p\,(V, T)\,dV$
- $W = \frac{dQ}{dt}$
- $W = \frac{\Delta Q}{\Delta t}$

Work done by thermal energy is path independent.

Thermal Expansion

Occurs because the "springs" between each of the atoms in a lattice have energy applied by Heat (Temperature Change).

- $\frac{\Delta L}{L_0} = \alpha \Delta T$ (One-Dimensional Expansion)

- $\frac{dL}{L_0} = \alpha dT$ (One-Dimensional Expansion)
  - α is a material-specific constant

Specific Heat/Heat Capacity

- $C = \frac{dQ}{dT} \leftarrow$ Specific Heat

- $c = \frac{dQ}{mdT} \leftarrow$ Mass Specific Heat

Heat of Phase Transitions

This is a constant unique to the material and the phase transition it is going through.
- $Q = Lm$
- $Q = \int_{m_i}^{m_f} L_f dm$
  - $l_{Fusion} = $ Heat required to turn things from **SOLID TO LIQUID**
  - $l_{Vapor} = $ Heat required to turn things from **LIQUID TO GAS**

Conduction Heat Transfer

$$P_{Conduction} = \frac{(T_H - T_C)}{L} Ak \tag{4.4}$$

- $L = $ Length
- $A = $ Cross-Sectional Area
- $k = $ Material's Thermal Conductivity

For multiple materials between 2 thermal reservoirs:
- $P_1 = P_2 = \ldots = P_n$
- Heat will only flow as fast as the slowest thermal conductor

5 Kinetic Theory of Ideal Gases

Defn 6 (Ideal Gas). An *ideal gas* is a gas that obeys the ideal gas law.

$$pV = nRT, \ R \cong 8.31 \,\text{J/mol K} \tag{5.1}$$

$$E_{Internal} = K_{Translate} + K_{Rotate} \tag{5.2}$$

$$\Delta E_{Internal} = \Delta K_{Translate} + \Delta K_{Rotate} \tag{5.3}$$

$$v_{rms} = \sqrt{\frac{3RT}{M}} \ \text{M=molar mass of gas} \tag{5.4}$$

Mean Free Path

Defn 7 (Mean Free Path). The *mean free path*, λ is the average distance traversed by a molecule between collisions.

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 \frac{N}{V}} \tag{5.5}$$

Where:
- d is the diameter of the atoms, or distance between centers during collision (m)
- N is the number of molecules (mol)
- V is the volume of gas being handled (L)
- λ is the wavelength, the same as in $v = f\lambda$

Note 7.1. If all the particles, except 1 are stationary, then you can use:

$$\lambda = \frac{1}{\pi d^2 \frac{N}{V}} \tag{5.6}$$

Work Done by Ideal Gases

Isothermally

$$W = \int \vec{F}\vec{ds}$$

$$W = \int_{V_1}^{V_2} p\,(V,T)\,dV$$

$$W = \int_{V_1}^{V_2} \frac{nRT}{V}dV \tag{5.7}$$

$$W = nRT \int_{V_1}^{V_2} \frac{1}{V}dV$$

$$W = nRT \ln\left(\frac{V_2}{V_1}\right)$$

Constant Pressure

$$W = p\,(V_{final} - V_{init}) \tag{5.8}$$

Translational Kinetic Energy

Defn 8 (Degrees of Freedom). *Degrees of freedom* represent the number of variables that are needed to describe a system. Represented with d, occasionally.

Note 8.1. In an ideal gas, these are a means to store energy.

$$K_{Translate} = \frac{3}{2}nRT$$

$$\Delta K_{Translate} = \frac{3}{2}nR\Delta T \tag{5.9}$$

	Translational	Rotational	Total
Monatomic	3	0	3
Diatomic	3	2	5
Polyatomic	3	3	6

Table 1: Degrees of Freedom Table for Gases

- An ideal gas has **ONLY** kinetic energy
- Completely elastic collisions

$$E_{int} = \frac{DoF}{2} nRT, \text{ where } DoF = \text{Degrees of Freedom} \tag{5.10}$$

Molar Specific Heats of Ideal Gases

Molar Specific Heat @ Constant Volume

$$C_V = \frac{\Delta E}{n\Delta T}$$
$$C_V = \frac{dE}{dT} \tag{5.11}$$
$$C_V = \left(\frac{DoF}{2}\right) R$$

$$Q = nC_V\Delta T \tag{5.12}$$

Molar Specific Heat @ Constant Pressure

$$C_P = C_V + R \tag{5.13}$$

$$Q = nC_P\Delta T \tag{5.14}$$

Adiabatic Processes in Ideal Gases

Defn 9 (Adiabatic Process). An *adiabatic process* is one in which no heat exchange occurs, namely:

$$dE_{Internal} = dQ - dW$$
$$dQ = 0 \tag{5.15}$$
$$dE_{Internal} = -dW$$

This leads to:

$$pV^\gamma = \text{Constant}, \gamma = \frac{C_P}{C_V} \tag{5.16}$$

$$p_1V_1^\gamma = p_2V_2^\gamma \tag{5.17}$$

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \tag{5.18}$$

You get Equation (5.18) by plugging $p = \frac{nRT}{V}$ into Equation (5.17).

6 Entropy

There is a heavy relationship between this and the 2nd Law of Thermodynamics.

Defn 10 (Entropy). *Entropy* is a measure of the number of available states/configurations. The *units for entropy* are J/K.

$$S = k_B \ln(\Omega) \tag{6.1}$$

Note 10.1. **Entropy is NOT disorder.**

Defn 11 (Change in Entropy). *Change in entropy* is the change in the number of available states/configurations for energy in a system.

$$\Delta S = S_f - S_i$$
$$= \int_a^b \frac{dQ\,(T)}{T}$$
$$= nR \ln \left(\frac{V_f}{V_i} \right) \tag{6.2}$$
$$= mC \ln \left(\frac{T_f}{T_i} \right)$$

Note 11.1. Because ΔS is based off of heat,

$$\Delta S_{Total} = \sum_{states} \Delta S \tag{6.3}$$

6.1 Engines

Engines are inherently cyclical.

Defn 12 (Efficiency). *Efficiency* is a measure of how effect an engine is in turning the heat taken in and turning it into work out.

$$\text{Efficiency} = \frac{W_{Out}}{Q_{In}}$$
$$\text{Eff} = \frac{Q_{In} - Q_{Out}}{Q_{In}} \tag{6.4}$$
$$\text{Eff} = \frac{T_{Hot} - T_{Cold}}{T_{Hot}}$$

6.1.1 Stirling Engine

Based on the Ideal Gas. You can find Efficiency with Equation (6.4).

6.1.2 Carnot Engine

Uses Isothermal expansion and adiabatic expansion of gases to achieve work. The total work done is the area of the Carnot figure. You can find Efficiency with Equation (6.4).

Defn 13 (Coefficient of Performance). *Coefficient of performance* is a measurement of the amount of energy a Carnot engine moves in comparison to its cold reservoir.

$$K_C = \frac{T_L}{T_H - T_L}$$
$$K_C = \frac{|Q_L|}{|Q_H| - |Q_L|} \tag{6.5}$$
$$K = \frac{|Q_L|}{|W|}$$

7 Light

Defn 14 (Photon). A *photon* is an electric field (wave) propagating through another electric field.

The various θ used in Equations (7.1), (7.2) are measured from the surface normal.

Chromatic Dispersion is the breaking up of polychromatic light by spectra. Think of Pink Floyd's *Dark Side of the Moon* album cover. This happens because shorter wavelength, higher frequency light has a slightly higher index of refraction.

Reflection

$$\theta_{reflected} = \theta_{incident} \tag{7.1}$$

Refraction

Defn 15 (Snell's Law).
$$n_{refract} \sin\left(\theta_{refract}\right) = n_{incident} \sin\left(\theta_{incident}\right) \tag{7.2}$$

Defn 16 (Index of Refraction).
$$n_i = \frac{c}{v_i}$$
$$\lambda = \frac{\lambda_0}{n} \tag{7.3}$$

Total Internal Reflection

Defn 17 (Total Internal Reflection). Total Internal Reflection occurs when the *refracted light's angle is* $\frac{\pi}{2}$.
$$n_{refract} \sin\left(\theta_{refract}\right) = n_{incident} \sin\left(\theta_{incident}\right), \text{ where } \theta_{refract} = \frac{\pi}{2} \tag{7.4}$$

Interference and Diffraction

Defn 18 (Huygen's Principle). Any point on a plane wavefront can be treated as a source of outgoing spherical waves.

Note 18.1. This is a mathematical construct/model.

Defn 19 (Phase Difference). Waves from the same source, but measured in such a way that there is a Path Length Difference (PLD) between them will have a *phase difference*.
$$\varphi = \frac{2\pi \, \text{PLD}}{\lambda} \tag{7.5}$$

All Interference and Diffraction equations used from here on out are based on Equation (7.5).

Defn 20 (Single Slit Diffraction). When waves propagate through a single slit, there is a Probability Distribution Function that describes where intensity of light is greatest and smallest. The location of the ***minima*** are given in Equation (7.6)
$$m\lambda = a \sin\left(\theta_m\right) \tag{7.6}$$

Note 20.1. m is the number of minima from the central major distribution of intensity.
1. First minima means $m = 1$
2. Second minima means $m = 2$
3. etc.

Thus, the equation for the Single Slit Diffraction is given in Equation (7.7).
$$\varphi_{\text{Single Slit}} = \frac{2\pi}{\lambda} a \sin\left(\theta\right) \tag{7.7}$$

Defn 21 (Double Slit Diffraction). This is really interference. The ***maxima*** of the interference is given in Equation.
$$n\lambda = d \sin\left(\theta_n\right) \tag{7.8}$$

Note 21.1. n is the number of maxima from the central major distribution of intensity.
1. First maxima means $n = 1$
2. Second maxima means $n = 2$
3. etc.

Thus, the equation for the Double Slit Diffraction is given in Equation (7.9).
$$\varphi_{\text{Double Slit}} = \frac{2\pi}{\lambda} d \sin\left(\theta\right) \tag{7.9}$$

8 Quantum Mechanics

Quantum mechanics may seem strange or weird, but it really isn't.

Defn 22 (Hamiltonian). The *hamiltonian* is a function that contains all knowable information about a particle or system in classical mechanics.
$$\text{Hamiltonian} = K + U \tag{8.1}$$

- K is the Kinetic Energy of the particle of system of particles.
  - $K = \frac{1}{2} m v^2$
  - This contains the velocity of the system, which includes direction.
- U is the Potential Energy of the particle or system of particles.
  - $U = mg\left(y_2 - y_1\right)$
  - Contains the force of the system, its location, etc.

8.1 Schrödinger's Equation

Defn 23 (de Broglie Wavelength)**.** The *de Broglie wavelength, λ*, is the wavelength that is *associated* with a particle. In other words, a particle does **not** have a wavelength, but the equations describing the particle's actions have a wavelength.

Note 23.1. As a direction consequence of the de Broglie Wavelength,

$$p = \frac{h}{\lambda} \tag{8.2}$$

- h is Planck's Constant
- λ is the de Broglie Wavelength

Defn 24 (Schrödinger's Equation)**.** *Schrödinger's equation* is a 3-dimensional Cartesian, time-independent way to express the equivalent of a Hamiltonian for quantum effects, namely in terms of waves (Denoted with Ψ or ψ). There are 2 equivalent ways to express Schrödinger's Equation. The first way, Equation (8.3), is the most general form. The second way, Equation (8.4), has expanded the nabla operator, ∇.

$$\frac{-\hbar^2}{2m} \cdot \nabla^2 \Psi + V\Psi = E\Psi \tag{8.3}$$

$$\frac{-\hbar}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + V\Psi = E\Psi \tag{8.4}$$

- E is the **total** energy of the system
- V is the **potential energy** of the system. It may be:
    1. Constant
    2. A function of time and/or space
- Ψ is a solution to Schrödinger's Equation, and contains all that can be known about the system
- $\frac{-\hbar^2}{2m} \cdot \nabla \Psi$ is the **kinetic energy** operator
    - $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is in the Cartesian coordinate system (x, y, z), and is called "del squared", or "2nd-degree nabla."
- For each observable system, there exists a corresponding mathematical operator.

Note 24.1. The probability distribution function of the position of a particle in the system can be found according to Equation (8.5). This is the multiplication of Ψ with its complex conjugate, Ψ^*.

$$\text{PDF} = \Psi \cdot \Psi^* \tag{8.5}$$

8.1.1 Properties of Schrödinger's Equation

Solutions, Ψ, to Schrödinger's Equation have the following characteristics:
1. They are single-valued
2. They are piecewise continuous values
3. $\frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial z}$ are all continuous
4. $\iiint \Psi\Psi^* \partial x \partial y \partial z$ is square integrable, or $\iiint \Psi\Psi^* \partial x \partial y \partial z \neq \pm\infty$

Rough Justification of Schrödinger's Equation. If Ψ is to be a wave function, it must satisfy the following differential equation:

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$$

Since we are looking for Schrödinger's Equation, in which all the dynamical observables are contained within Ψ, we will set $v = c$, the upper limit of information transfer. We will first consider (without loss of generality) one dimension, the x-dimension. Then,

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$$

As a test solution, we will propose

$$\Psi = Ae^{-i(kx - \omega t)}$$

With respect to x:

$$\frac{\partial \Psi}{\partial x} = A\left(-ik\right) e^{-i(kx - \omega t)}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = A\left(-ik\right)^2 e^{-i(kx - \omega t)} = \left(-ik\right)^2 \Psi = k^2 \Psi$$

Therefore, Ψ is a satisfactory solution in x. With respect to t:

$$\frac{\partial \Psi}{\partial t} = A\left(i\omega\right)e^{-i(kx-\omega t)}$$

$$\frac{\partial^2 \Psi}{\partial t^2} = A\left(i\omega\right)^2 e^{-i(kx-\omega t)}$$

Therefore, Ψ is a satisfactory solution in t. Thus, Ψ is a solution to $\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2}$. (Also, $\frac{1}{c^2} = \frac{k^2}{\omega^2}$).
In our solution, $\Psi = Ae^{-i(kx-\omega t)}$, k has its usual meaning:

$$k = \frac{2\pi}{\lambda}$$

Where the wavelength, λ, is to be the de Broglie Wavelength, i.e. the wavelength that is associated with a particle. In otherwords, a particle does not have a wavelength, but the equations describing the particle's action have a wavelength.

$$\frac{\partial \Psi}{\partial x} = A\left(-ik\right)e^{-i(kx-\omega t)}$$

$$= A\left(-ik\right)e^{-i\left(\frac{2\pi}{\lambda}x - 2\pi ft\right)}$$

$$= A\left(-ik\right)e^{-i\frac{h}{h}\left(\frac{2\pi}{\lambda}x - 2\pi ft\right)}$$

$$= A\left(-ik\right)e^{-i\frac{2\pi}{h}\left(\frac{h}{\lambda}x - hft\right)}$$

According to the de Broglie Wavelength, $p = \frac{h}{\lambda}$; according to Einstein, $E = hf$; substituting:

$$\frac{\partial \Psi}{\partial x} = -i\left(\frac{2\pi}{h}\right)\left(\frac{h}{\lambda}\right)Ae^{-i\frac{2\pi}{h}\left(\frac{h}{\lambda}x - hft\right)}$$

$$\frac{\partial \Psi}{\partial x} = -i\left(\frac{2\pi}{h}\right)(p)\,Ae^{-i\frac{2\pi}{h}(px - Et)}$$

Define ($\hbar$ is pronounced "h-bar"):

$$\hbar = \frac{h}{2\pi}$$

This leads us to:

$$\frac{\partial \Psi}{\partial x} = \frac{i}{\hbar}(p)\,\Psi$$

$$\frac{-\hbar}{i}\frac{\partial \Psi}{\partial x} = p\Psi$$

Here, it is clear that p does not equal the classical mechanical $p = mv$. Rather, p is an *operator* such that:

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

It is possible to construct other operators that correspond to observable dynamical variables in a similar way. (Operators are expressed with a "hat" above the variable name).
In classical mechanics, the kinetic energy of a particle can be written as:

$$K = \frac{\vec{p}\cdot\vec{p}}{2m}$$

Since there exists a quantum mechanical operator that corresponds to any observable, we can write the kinetic energy operator for a one-dimensional system as:

$$\widehat{K} = \frac{\hat{p}\cdot\hat{p}}{2m} = \frac{\frac{-\hbar}{i}\frac{\partial}{\partial x}\cdot\frac{-\hbar}{i}\frac{\partial}{\partial x}}{2m} = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

Extension to Three Dimensions

Recall that the "del-squared" can be written as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So, the kinetic energy part of the 3-dimensional Hamiltonian for a wave can be written as:

$$\widehat{K} = \frac{-\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

Define $\Psi_{x,y,z}\left(x, y, z\right)$:

$$\Psi_{x,y,z}\left(x, y, z\right) = \left\langle \Psi_x\left(x\right), \Psi_y\left(y\right), \Psi_z\left(z\right)\right\rangle$$

Therefore,

$$\begin{aligned}
\widehat{K} &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi_{x,y,z}\left(x, y, z\right)\\
&= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left\langle \Psi_x\left(x\right), \Psi_y\left(y\right), \Psi_z\left(z\right)\right\rangle\\
&= -\frac{\hbar^2}{2m}\left[\left(\frac{\partial^2\Psi_x\left(x\right)}{\partial x^2}\right) + \left(\frac{\partial^2\Psi_y\left(y\right)}{\partial y^2}\right) + \left(\frac{\partial^2\Psi_z\left(z\right)}{\partial z^2}\right)\right]
\end{aligned}$$

■

This means that the 3-Dimensional problem is broken down into 3 1-Dimensional problems! Each dimension can be approached exactly the same way as above.

8.2 Operators

Defn 25 (Operator). In quantum mechanics, for each observable dynamical variable, there exists a corresponding *operator*, which, if applied to the solution of the Schrödinger's Equation, will provide a value for that dynamical variable. Operator do something to the function immediately after the operator. An operator is nothing other than shorthand for instruction to do something to a function or number. For example, some common operators are:

1. $+$
2. $-$
3. $\times$
4. $*$
5. $\frac{\partial^2}{\partial t^2}$
6. $\int$
7. $\sum$
8. ∇^2

Operators have some vector-like properties, but they are not vectors. Further, they do not always always *commute*. Part of the main confusion arising from the public press about quantum mechanics is as statement like "$\dots A \times B$ is equal to $B \times A$." This is simply sensationalizing the fact that some operators have commutative properties. One example is:

$$y \times \frac{\partial f\left(y\right)}{\partial y} \neq \frac{\partial\left[x \times f\left(y\right)\right]}{\partial y}$$

For each classical observable, there exists a corresponding quantum mechanical operator. Some examples of simple quantum mechanical operators are:

- Kinetic Energy: $\frac{-\hbar^2}{2m}\nabla^2$
- Momentum: $\frac{-\hbar}{i}\frac{\partial}{\partial x}$
- Potential Energy: V is simply multiplication by V
- Angular Momentum: $L_z = \mathbf{r} \times \frac{-\hbar}{i}\nabla^2$

Further, some, but not all, operators are linear. Fortunately, the quantum mechanics kinetic energy operators, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}$ is a linear operator. This allows us to *reduce a 3-dimensional problem into 3 1-dimensional problems*.

8.3   Eigenvalues

Defn 26 (Eigenvalue). Let $\widehat{\mathbf{O}}$ be some quantum mechanical operator. If O and Ψ are *eigenvalues* and eigenfunctions of $\widehat{\mathbf{O}}\Psi = o\Psi$, the result o will always be a precise number. *Eigenvalues* represent the value of precise, exact values of the dynamical variables corresponding to the eigenfunction of the operator.

The **easy** way to find the value of a dynamical variable o, given the corresponding operator, $\widehat{\mathbf{O}}$ is as follows.

Solve for Eigenvalue. Let Ψ be a normalized function, i.e.

$$\iiint \Psi\Psi^* dxdydz = 1$$

Then,

$$\widehat{\mathbf{O}}\Psi = o\Psi$$

Multiply both sides by Ψ^*, the complex conjugate of Ψ.

$$\Psi^*\widehat{\mathbf{O}}\Psi = \Psi^* o\Psi$$

Integrate both sides over the space Ω.

$$\int \Psi^*\widehat{\mathbf{O}}\Psi d\Omega = \int \Psi^* o\Psi d\Omega = o\int Psi^*\Psi d\Omega = o, \text{ since } int Psi^*\Psi d\Omega = 1$$

∎

8.4   Expectation Values

Defn 27 (Expectation Values). If o and Ψ are **not** eigenvalues and eigenfunctions of $\widehat{\mathbf{O}}$, e.g. $\widehat{\mathbf{O}}\Psi = z\phi$, where z is some number, and ϕ is some function not equal to Ψ, the result z will be the average of many possible observations.
This is usually written as $\langle z \rangle$ and is called an *expectation value*. The expectation value is the expected average value of a number of measurements for a dynamical variable described by the operator $\widehat{\mathbf{O}}$.

8.5   Determine Dynamical Variables in Quantum Mechanics

This section has an example where we solve for dynamical variables. Finding Ψ, the wave function that satisfies the Schrödinger's Equation for the one-dimensional infinite square well potential (particle in a box) is presented in the physics textbooks. Now we wish to find values for dynamical observables. Starting with the energy E.

Recall the Properties of Schrödinger's Equation, Section 8.1.1. The solution for the Schrödinger's Equation in a box of length L is

$$\Psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \text{ where } n \text{ is the quantum number of the wave function}$$

First, we operate on Ψ_n with the kinetic energy operator $\widehat{K}$:

$$\widehat{K}\Psi_n = E\Psi_n = \varepsilon_n\Psi_n$$

ε_n is the kinetic energy since E; the total energy is all kinetic with quantized values $varepsilon_n$.
Rewrite $\widehat{K}\Psi_n$.

$$\widehat{K}\Psi_n = -\frac{\hbar^2}{2m}\frac{d^2\Psi_n}{dx^2} = \varepsilon_n\Psi_n$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left[\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right] = \varepsilon_n\Psi_n$$

Multiply both sides by Ψ_n^*, the complex conjugate of Ψ_n (which in this case $\Psi_n^* = \Psi_n$ since Ψ_n is real) **before the operator** (since operators may not commute nor be distributive):

$$\Psi_n^*\widehat{K}\Psi_n = \Psi_n^*\left(-\frac{\hbar^2}{2m}\cdot\frac{d^2\Psi_n}{dx^2}\right) = \Psi_n^*\varepsilon_n\Psi_n$$

$$= -\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\cdot\left(-\frac{\hbar^2}{2m}\right)\frac{d^2}{dx^2}\left[\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= \varepsilon_n\sqrt{\frac{2}{L}}\left(\sin\left(\frac{n\pi x}{L}\right)\right)\frac{n^2\pi^2}{L^2}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Now integrate over both sides:

$$\int \Psi_n \left(-\frac{\hbar^2}{2m} \frac{d^2 \Psi_n}{dx^2} \right) dx = \int \Psi_n^* \varepsilon_n \Psi_n dx$$

$$= \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \cdot \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right] \right) dx$$

Take the second derivative of Ψ_n with respect to x:

$$\frac{d^2}{dx^2} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right] = \frac{n^2 \pi^2}{L^2} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Substitute into the integral

$$\int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \left(-\frac{\hbar^2}{2m} \right) \left(-\frac{n^2 \pi^2}{L^2} \right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx = \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \varepsilon_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

Pull out the constants

$$\left(-\frac{\hbar^2}{2m} \right) \left(-\frac{n^2 \pi^2}{L^2} \right) \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx = \varepsilon_n \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

Recall that

$$\int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx = 1$$

So, $\left(\frac{\hbar^2}{2m} \right) \left(\frac{n^2 \pi^2}{L^2} \right) = \varepsilon_n$ is the kinetic energy of a particle with n as its quantum number.

A Reference Material

A.1 Physical Constants

Constant Name	Variable Letter	Value
Boltzmann Constant	R	$8.314 \mathrm{J/mol\,K}$
Universal Gravitational	G	$6.67408 \times 10^{-11} \mathrm{m^3\,kg^{-1}\,s^{-2}}$
Planck's Constant	h	$6.62607004 \times 10^{-34} \mathrm{mkg/s}$
Speed of Light	c	$299792458 \mathrm{m/s}$
Mass of Earth	m_{Earth}	$5.972 \times 10^{24} \mathrm{kg}$
Diameter of Earth	d_{Earth}	$12742 \mathrm{km}$

A.2 Trigonometry

A.2.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.2.2 Euler Equivalents of Trigonometric Functions

$$\sin(x) = \frac{e^{\imath x} + e^{-\imath x}}{2} \tag{A.3}$$

$$\cos(x) = \frac{e^{\imath x} - e^{-\imath x}}{2\imath} \tag{A.4}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.5}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.6}$$