

EITF75: Systems and Signals - Reference Sheet

Karl Hallsby

Last Edited: October 12, 2019

Contents

1	Sinusoids	1
1.1	Continuous-Time Signals	1
1.1.1	Frequency in Continuous-Time Signals	1
1.1.2	Properties of Continuous-Time Sinusoidal Signals	1
1.2	Discrete-Time Signals	2
1.2.1	Frequency in Discrete-Time Signals	2
1.2.2	Properties of Discrete-Time Sinusoidal Signals	2
1.2.3	Frequency Aliases	2
1.3	Sampling Rates and Sampling Frequency	2
1.3.1	Nyquist Rate	3
1.3.2	Nyquist Frequency	3
1.4	Digital Signals	3
1.4.1	Quantization	3
1.4.1.1	Quantization Levels	3
1.4.1.2	Quantization Error	3
1.4.1.3	Bit Requirements	3
1.4.1.4	Bit Rate	3
2	Discrete-Time Systems	3
2.1	Representing Discrete-Time Systems	3
2.1.1	Functional Representation	3
2.1.2	Tabular Representation	3
2.1.3	Sequence Representation	3
2.2	Elementary Discrete-Time Signals	4
2.2.1	Unit Impulse Signal	4
2.2.2	Unit Step Signal	4
2.2.3	Unit Ramp Signal	4
2.2.4	Exponential Signal	4
2.3	Classification of Discrete-Time Signals	4
2.3.1	Energy Signal	5
2.3.2	Power Signal	5
2.3.3	Periodic and Aperiodic Signals	5
2.3.4	Symmetric and Antisymmetric Signals	5
2.4	Discrete-Time Signal Manipulations	5
2.4.1	Transformation of the Independent Variable (Time)	5
2.4.1.1	Shifting in Time	5
2.4.1.2	Folding	6
2.4.2	Addition, Multiplication, and Scaling	6
2.4.2.1	Addition	6
2.4.2.2	Multiplication	6
2.4.2.3	Amplitude Scaling	6
3	Convolutions	6

4	The \mathcal{Z}-Transform	6
4.1	The \mathcal{Z} -Transform	6
4.1.1	Region of Convergence	6
4.1.2	The One-Sided \mathcal{Z} -Transform	7
4.2	The Inverse \mathcal{Z} -Transform	7
4.2.1	The Inverse \mathcal{Z} -Transform by Contour Integration	8
4.2.2	The Inverse \mathcal{Z} -Transform by Power Series Expansion	8
4.2.3	The Inverse \mathcal{Z} -Transform by Partial-Fraction Expansion	8
4.3	Properties of the \mathcal{Z} -Transform	8
4.3.1	\mathcal{Z} -Transform Linearity	8
4.3.2	\mathcal{Z} -Transform Time Shifting	9
4.3.3	\mathcal{Z} -Domain Scaling	9
4.3.4	\mathcal{Z} -Transform Time Reversal	9
4.3.5	\mathcal{Z} -Domain Differentiation	10
4.3.6	\mathcal{Z} -Domain Convolutions	10
4.3.7	\mathcal{Z} -Transform 2 Sequence Correlation	11
4.3.8	\mathcal{Z} -Transform 2 Sequence Multiplication	11
4.3.9	Parsevals Relation for \mathcal{Z} -Transform	11
4.3.10	Initial Value Theorem for \mathcal{Z} -Transform	12
4.4	Properties of the One-Sided \mathcal{Z} -Transform	12
4.5	Rational \mathcal{Z} -Transforms	12
4.5.1	Poles and Zeros of a \mathcal{Z} -Transform	12
4.5.2	Decomposition of Rational \mathcal{Z} -Transforms	12
4.6	Analysis of LTI Systems in the \mathcal{Z} -Domain	12
4.7	Common \mathcal{Z} -Transforms	12
5	The Fourier Transform and Fourier Series	12
5.1	Fourier Transform Relations	14
5.1.1	Laplace Transform Fourier Transform Relation	14
5.1.2	\mathcal{Z} -Transform Discrete-Time Fourier Transform Relation	14
5.2	The Inverse Fourier Transform	15
5.3	Properties of the Discrete-Time Fourier Transform	15
5.3.1	Linearity	15
5.3.2	Time Shifting	16
5.3.3	Time Reversal	16
5.3.4	Convolution	16
5.3.5	Correlation	16
5.3.6	Wiener-Khintchine Theorem	16
5.3.7	Frequency Shifting	16
5.3.8	Modulation	16
5.3.9	Multiplication in Time Domain	17
5.3.10	Differentiation in Frequency Domain	17
5.3.11	Parseval's Theorem	17
6	The Fourier Transform and LTI Systems	17
6.1	Frequency Response	17
6.2	Magnitude Response	18
6.3	Phase Response	18
A	Trigonometry	19
A.1	Trigonometric Formulas	19
A.2	Euler Equivalents of Trigonometric Functions	19
A.3	Angle Sum and Difference Identities	19
A.4	Double-Angle Formulae	19
A.5	Half-Angle Formulae	19
A.6	Exponent Reduction Formulae	19
A.7	Product-to-Sum Identities	19
A.8	Sum-to-Product Identities	20
A.9	Pythagorean Theorem for Trig	20
A.10	Rectangular to Polar	20

A.11 Polar to Rectangular	20
B Calculus	21
B.1 Fundamental Theorems of Calculus	21
B.2 Rules of Calculus	21
B.2.1 Chain Rule	21
C Laplace Transform	22

1 Sinusoids

There are several ways to characterize Sinusoids. The first is by dimension:

1. Multidimensional/Multichannel Signals
2. Monodimensional/Monochannel Signals

You can also classify sinusoids by their independent variable (usually time) and the values they take.

1. Continuous-Time Signals or Analog Signals
2. Discrete-Time Signals
3. There is a third way to classify sinusoids and their signals: Digital Signals

Defn 1 (Continuous-Time Signals). *Continuous-time signals* or *Analog signals* are defined for every value of time and they take on values in the continuous interval (a, b) , where a can be $-\infty$ and b can be ∞ . Mathematically, these signals can be described by functions of a continuous variable.

For example,

$$x_1(t) = \cos \pi t, x_2(t) = e^{-|t|}, -\infty < t < \infty$$

Defn 2 (Discrete-Time Signals). *Discrete-time signals* are defined only at certain specified values of time. These time instants **need not** be equidistant, but in practice, they are usually taken at equally spaced intervals for computation convenience and mathematical tractability.

For example,

$$x(t_n) = e^{-|t_n|}, n = 0, \pm 1, \pm 2, \dots$$

A Discrete-Time Signals can be represented mathematically by a sequence of real or complex numbers.

Remark 2.1. To emphasize the discrete-time nature of the signal, we shall denote the signal as $x(n)$, rather than $x(t)$.

Remark 2.2. If the time instants t_n are equally spaced (i.e., $t_n = nT$), the notation $x(nT)$ is also used.

1.1 Continuous-Time Signals

1.1.1 Frequency in Continuous-Time Signals

A simple harmonic oscillation is mathematically described by Equation (1.1).

$$x_a(t) = A \cos(\Omega t + \theta), -\infty < t < \infty \quad (1.1)$$

Remark. The subscript a is used with $x(t)$ to denote an analog signal.

This signal is completely characterized by three parameters:

1. A , the *amplitude* of the sinusoid
2. Ω , the *frequency* in radians per second (rad/s)
3. θ , the *phase* in radians.

Instead of Ω , the frequency F in cycles per second or hertz (Hz) is used.

$$\Omega = 2\pi F \quad (1.2)$$

Plugging (1.2) into (1.1), yields

$$x_a(t) = A \cos(2\pi F t + \theta), -\infty < t < \infty \quad (1.3)$$

1.1.2 Properties of Continuous-Time Sinusoidal Signals

The analog sinusoidal signal in equation (1.3) is characterized by the following properties:

- (i) For every fixed value of the frequency F , $x_a(t)$ is periodic.

$$x_a(t + T_p) = x_a(t)$$

where $T_p = \frac{1}{F}$ is the fundamental period.

- (ii) Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.
- (iii) Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in the given time interval.

1.2 Discrete-Time Signals

1.2.1 Frequency in Discrete-Time Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A \cos(\omega n + \theta), n \in \mathbb{Z}, -\infty < n < \infty \quad (1.4)$$

The signal is characterized by these parameters:

1. n , the sample number. MUST be an integer.
2. A , the *amplitude* of the sinusoid
3. ω , the *angular frequency* in radians per sample
4. θ , is the *phase*, in radians.

Instead of ω , we use the frequency variable f defined by

$$\omega \equiv 2\pi f \quad (1.5)$$

Using (1.4) and (1.5) yields

$$x(n) = A \cos(2\pi f n + \theta), n \in \mathbb{Z}, -\infty < n < \infty \quad (1.6)$$

1.2.2 Properties of Discrete-Time Sinusoidal Signals

- (i) A discrete-time sinusoid is periodic **ONLY** if its frequency is a rational number.
- (ii) Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical. This leads us to the idea of a Frequency Alias.
- (iii) The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pm\pi$ or, equivalently, $f = \pm\frac{1}{2}$.

1.2.3 Frequency Aliases

The concept of a Frequency Alias is drawn from the idea that discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical and that frequencies $|f| > \frac{1}{2}$ are identical. (Properties (ii) and (iii))

Defn 3 (Frequency Alias). A *frequency alias* is a sinusoid having a frequency $|\omega| > \pi$ or $|f| > \frac{1}{2}$. This is because this sinusoid is *indistinguishable* (*identical*) to one with frequency $|\omega| < \pi$ or $|f| < \frac{1}{2}$.

A *frequency alias* is a sequence resulting from the following assertion based on the sinusoid $\cos(\omega_0 n + \theta)$.

It follows that

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta)$$

As a result, all sinusoidal sequences

$$x_k(n) = A \cos(\omega_k n + \theta), k = 0, 1, 2, \dots$$

where

$$\omega_k = \omega_0 + 2k\pi, -\pi \leq \omega_0 \leq \pi$$

are *indistinguishable* (i.e., *identical*).

Because of this, we regard frequencies in the range of $-\pi \leq \omega \leq \pi$ or $-\frac{1}{2} \leq f \leq \frac{1}{2}$ as unique, and all frequencies that fall outside of these ranges as aliases.

Remark 3.1. It should be noted that there is a difference between discrete-time sinusoids and continuous-time sinusoids here. Continuous-time sinusoids have distinct signals for Ω or F in the entire range $-\infty < \Omega < \infty$ or $-\infty < F < \infty$.

1.3 Sampling Rates and Sampling Frequency

Most signals of interest are analog. To process these signals, they must be collected and converted to a digital form, that is, to convert them to a sequence of numbers having finite precision. This is called *analog-to-digital* (*A/D*) *conversion*. Conceptually, we view this conversion as a 3-step process.

1. Sampling
2. Quantization
3. Coding

1.3.1 Nyquist Rate

1.3.2 Nyquist Frequency

1.4 Digital Signals

Defn 4 (Digital Signals). *Digital signals* are a subset of Discrete-Time Signals. In this case, not only are the values being measured occurring at fixed points in time, the values themselves can only take certain, fixed values.

1.4.1 Quantization

Defn 5 (Quantization). This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-value (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the Quantization Error.

1.4.1.1 Quantization Levels

1.4.1.2 Quantization Error

Defn 6 (Quantization Error). The *quantization error* of something.

1.4.1.3 Bit Requirements

1.4.1.4 Bit Rate

2 Discrete-Time Systems

As discussed in Section 1.2, $x(n)$ is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is *not defined* at instants between the samples. Also, if n is not an integer, $x(n)$ is not defined.

Besides graphical representation of a discrete-time system, there are 3 ways to represent a discrete-time signal.

1. Functional Representation
2. Tabular Representation
3. Sequence Representation

2.1 Representing Discrete-Time Systems

2.1.1 Functional Representation

This representation of a discrete-time system is done as a mathematical function.

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \quad (2.1)$$

2.1.2 Tabular Representation

This representation of a discrete-time system is done as a table of corresponding values.

n		...	-2	-1	0	1	2	3	4	5	...
$x(n)$...	0	0	0	1	4	1	0	0	...

2.1.3 Sequence Representation

There are 2 methods of representation for this. The first includes all values for $-\infty < n < \infty$. In all cases, $n = 0$ is marked in the sequence, somehow. I will do this with an underline.

$$x(n) = \{\dots, 0, \underline{0}, 1, 4, 1, 0, 0, \dots\} \quad (2.2)$$

The second only works if all $x(n)$ values for $n < 0$ are 0.

$$x(n) = \{\underline{0}, 1, 4, 1, 0, 0, \dots\} \quad (2.3)$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underline{-2}, 5, 0, 4, -1\} \quad (2.4)$$

This is identified as a seven-point sequence.

A finite-duration sequence where $x(n) = 0$ for all $n < 0$ is represented as

$$x(n) = \{0, 1, 4, 1\} \quad (2.5)$$

This is identified as a four-point sequence.

2.2 Elementary Discrete-Time Signals

The following signals are basic signals that appear often and play an important role in signal processing.

2.2.1 Unit Impulse Signal

Defn 7 (Unit Impulse Signal). The *unit impulse signal* or *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \quad (2.6)$$

This function is a signal that is zero everywhere, except at $n = 0$, where its value is 1.

Remark 7.1. This signal is different that the analog signal $\delta(t)$, which is also called a unit impulse, and is defined to be 0 everywhere except $t = 0$. The discrete unit impulse sequence is much less mathematically complicated.

2.2.2 Unit Step Signal

Defn 8 (Unit Step Signal). The *unit step signal* is denoted as $u(n)$ or as $\mathcal{U}(n)$ and is defined as

$$\mathcal{U}(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \quad (2.7)$$

2.2.3 Unit Ramp Signal

Defn 9 (Unit Ramp Signal). The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \quad (2.8)$$

2.2.4 Exponential Signal

Defn 10 (Exponential Signal). The *exponential signal* is a sequence of the form

$$x(n) = a^n \text{ for all } n \quad (2.9)$$

If a is real, then $x(n)$ is a real signal. When a is complex valued ($a \equiv b \pm cj$), it can be expressed as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n (\cos \theta n + j \sin \theta n) \end{aligned} \quad (2.10)$$

This can be expressed by graphing the real and imaginary parts

$$\begin{aligned} x_R(n) &\equiv r^n \cos \theta n \\ x_I(n) &\equiv r^n j \sin \theta n \end{aligned} \quad (2.11)$$

or by graphing the amplitude function and phase function.

$$\begin{aligned} |x(n)| &= A(n) \equiv r^n \\ \angle x(n) &= \phi(n) \equiv \theta n \end{aligned} \quad (2.12)$$

2.3 Classification of Discrete-Time Signals

In order to apply some mathematical methods to discrete-time signals, we must characterize these signals.

2.3.1 Energy Signal

Defn 11 (Energy Signal). The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad (2.13)$$

The energy of a signal can be finite or infinite. If E is finite ($0 < E < \infty$), then $x(n)$ is called an *energy signal*.

2.3.2 Power Signal

Defn 12 (Power Signal). The average power of a discrete time signal $x(n)$ is defined as

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2 \quad (2.14)$$

This means that there are 2 potential outcomes:

1. If E is finite, $P = 0$
2. If E is infinite, P may be either finite or infinite

If P is finite and nonzero, the signal is called a *power signal*.

2.3.3 Periodic and Aperiodic Signals

A signal $x(n)$ is periodic with period N ($N > 0$) if and only if

$$x(n+N) = x(n) \text{ for all } n \quad (2.15)$$

The smallest value of N for which (2.15) holds is called the fundamental period. If there is no value of N that satisfies (2.15), the signal is called *nonperiodic* or *aperiodic*.

2.3.4 Symmetric and Antisymmetric Signals

A real-valued signal $x(n)$ is called *symmetric* or *even* if

$$x(n) = x(-n) \quad (2.16)$$

On the other hand, a signal $x(n)$ is called *antisymmetric* or *odd* if

$$x(n) = -x(-n) \quad (2.17)$$

2.4 Discrete-Time Signal Manipulations

2.4.1 Transformation of the Independent Variable (Time)

It is important to note that Shifting in Time and Folding are not commutative. For example,

$$\text{TD}_k\{\text{FD}[x(n)]\} = \text{TD}_k[x(-n)] = x(-n+k) \quad (2.18)$$

whereas

$$\text{FD}\{\text{TD}_k[x(n)]\} = \text{FD}[x(n-k)] = x(-n-k) \quad (2.19)$$

2.4.1.1 Shifting in Time A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time (moves left). If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units of time (moves right).

This could be denoted by

$$\text{TD}_k[x(n)] = x(n-k) \quad (2.20)$$

You cannot advance a signal that is being generated in real-time. Because that would involve signal samples that haven't been generated yet. So, you can only advance a signal that is stored on something. However, you can always introduce a delay to a signal.

2.4.1.2 Folding Another useful modification of the time base is to replace n with $-n$. The result is a *folding* or *reflection* of the original signal around $n = 0$.

This could be denoted by

$$\text{FD}[x(n)] = x(-n) \quad (2.21)$$

2.4.2 Addition, Multiplication, and Scaling

Amplitude modifications include Addition, Multiplication, and Amplitude Scaling.

2.4.2.1 Addition The *sum* of 2 signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the sum of the values of these two signals at that instant.

$$y(n) = x_1(n) + x_2(n), \quad -\infty < n < \infty \quad (2.22)$$

2.4.2.2 Multiplication The *product* of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the product of the values of these two signals at that instant.

$$y(n) = x_1(n)x_2(n), \quad -\infty < n < \infty \quad (2.23)$$

2.4.2.3 Amplitude Scaling *Amplitude scaling* of a signal by a constant A is accomplished by multiplying every sample by A . Consequently, we obtain

$$y(n) = Ax(n), \quad -\infty < n < \infty \quad (2.24)$$

3 Convolutions

Defn 13 (Convolution). The *convolution* operator.

$$y(t) = \sum_{k=-\infty}^{\infty} x(k) * h(n-k) \quad (3.1)$$

4 The \mathcal{Z} -Transform

The \mathcal{Z} -Transform plays the same role in the analysis of Discrete-Time Signals and LTI systems as the Laplace Transform does in the analysis of Continuous-Time Signals and LTI systems.

4.1 The \mathcal{Z} -Transform

Defn 14 (\mathcal{Z} -Transform). The *z-transform* is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Remark 14.1. For convenience, the z -transform of a signal $x(n)$ is denoted by

$$X(z) \equiv \mathcal{Z}\{x(n)\} \quad (4.2)$$

and the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z) \quad (4.3)$$

4.1.1 Region of Convergence

Defn 15 (ROC). The *ROC* or *region of convergence* is the region for which the infinite power series in the z -transform has a convergent solution.

Remark 15.1. Any time we cite a z -transform, we should also indicate its ROC

Example 4.1: Simple \mathcal{Z} -Transform.

Determine the z -transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n \mathcal{U}(n)$$

The z -transform is the infinite power series

$$\begin{aligned} X(z) &= 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^{-2} + \cdots + \left(\frac{1}{2}\right)^n z^{-n} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n \end{aligned}$$

Because this is an infinite geometric series, we can solve with with our equivalency:

$$1 + A + A^2 + \cdots + A^n + \cdots = \frac{1}{1 - A} \text{ if } |A| < 1$$

Thus, $X(z)$ converges to

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC} : |z| > \frac{1}{2}$$

Signal	ROC
Finite-Duration Signals	
Causal	Entire z -plane except $z = 0$
Anticausal	Entire z -plane except $z = \infty$
Two-Sided	Entire z -plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals	
Causal	$ z > r_2$
Anticausal	$ z < r_1$
Two-Sided	$r_2 < z < r_1$

Table 4.1: Characteristic Families of Signals with Their Corresponding ROCs

4.1.2 The One-Sided \mathcal{Z} -Transform

Defn 16 (One-Sided \mathcal{Z} -Transform). The *one-sided z -transform* is the same as the \mathcal{Z} -Transform, but is only defined at n values greater than or equal to 0.

$$X(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n} \quad (4.4)$$

The One-Sided \mathcal{Z} -Transform is generally used when there are initial conditions on a causal signal. This captures the normal causal portion of the signal, while also showing the effect of the initial condition.

4.2 The Inverse \mathcal{Z} -Transform

This is the formal definition of The Inverse \mathcal{Z} -Transform.

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz \quad (4.5)$$

where the integrals is a contour integral over a closed path C that encloses the origin and lies within the region of convergence of $X(z)$.

There are 3 methods that are often used for the evaluation of the inverse z -transform in practice:

1. Direct evaluation of (4.5).
2. Expansion into a series of terms, in the variable sz and z^{-1} .
3. Partial-fraction expansion and table lookup.

4.2.1 The Inverse \mathcal{Z} -Transform by Contour Integration

Defn 17 (Cauchy's Integral Theorem). Let $f(z)$ be a function of the complex variable z and C be a closed path in the z -plane. If the derivative $\frac{df(z)}{dz}$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \quad (4.6)$$

More generally, if the $(k+1)$ -order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k} dz = \begin{cases} \left. \frac{1}{(k-1)!} \frac{d^{k-1}f(z)}{dz^{k-1}} \right|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \quad (4.7)$$

4.2.2 The Inverse \mathcal{Z} -Transform by Power Series Expansion

4.2.3 The Inverse \mathcal{Z} -Transform by Partial-Fraction Expansion

4.3 Properties of the \mathcal{Z} -Transform

Property	Time Domain	z -Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC : $r_2 < z < r_1$ ROC ₁ ROC ₂
\mathcal{Z} -Transform Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC ₁ and ROC ₂
\mathcal{Z} -Transform Time Shifting	$x(n - k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
\mathcal{Z} -Domain Scaling	$a^n x(n)$	$X(a^{-1}z)$	$ a r_2 < z < a r_1$
\mathcal{Z} -Transform Time Reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < z < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real Part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
p Imaginary Part	$\text{Im}\{x(n)\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Includes ROC
\mathcal{Z} -Domain Differentiation	$nx(n)$	$-z \frac{dX(z)}{dz}$	$r_2 < z r_1$
\mathcal{Z} -Domain Convolutions	$x_1 * x_2$	$X_1(z)X_2(z)$	At least, the intersection of ROC ₁ and ROC ₂
\mathcal{Z} -Transform 2 Sequence Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial Value Theorem for \mathcal{Z} -Transform	If $x(n)$ causal	$x(0) = \lim_{z \rightarrow \infty} X(z)$	
\mathcal{Z} -Transform 2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j} \oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least, $r_{1l}r_{2l} < a < r_{1u}r_{2u}$
Parsevals Relation for \mathcal{Z} -Transform	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$= \frac{1}{2\pi j} \oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

Table 4.2: Properties of the \mathcal{Z} -Transform

4.3.1 \mathcal{Z} -Transform Linearity

If

$$\begin{aligned} x_1(n) &\xrightarrow{z} X_1(z) \\ x_2(n) &\xrightarrow{z} X_2(z) \end{aligned}$$

then

$$x(n) = a_1x_1(n) + a_2x_2(n) \xrightarrow{z} X(z) = a_1X_1(z) + a_2X_2(z) \quad (4.8)$$

for any constants a_1 and a_2 .

The linearity property can be generalized to an arbitrary number of signals.

Example 4.2: Simple Z-Transform Linearity Problem. Example 3.2.1

Determine the z -transform and the ROC of the signal

$$x(n) = [3(2^n) - 4(3^n)]\mathcal{U}(n)$$

Solution on Page 158.

Example 4.3: Z-Transform Linearity on Trig Functions. Example 3.2.2

Determine the z -transform of the signals

(a) $x(n) = (\cos \omega_0 n)\mathcal{U}(n)$

(b) $x(n) = (\sin \omega_0 n)\mathcal{U}(n)$

Solution on Pages 158-159.

4.3.2 Z-Transform Time Shifting

If

$$x(n) \xleftrightarrow{z} X(z)$$

then

$$x(n-k) \xleftrightarrow{z} z^{-k}X(z) \quad (4.9)$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z=0$ if $k>0$ and $z=\infty$ if $k<0$.

Example 4.4: Z-Transform Time Shifting. Example 3.2.3

By applying the time-shifting property, determine the z -transform of the signals

$$x_1(n) = \{1, 2, \underline{5}, 7, 0, 1\}$$

$$x_2(n) = \{\underline{0}, 0, 1, 2, 5, 7, 0, 1\}$$

from the z -transform of

$$x_0(n) = \{1, 2, 5, 7, 0, 1\}$$

$$X_0(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}, \text{ROC : entire } z\text{-plane except } z=0$$

Solution on Page 160.

4.3.3 Z-Domain Scaling

If

$$x(n) \xleftrightarrow{z} X(z), \text{ROC : } r_1 < |z| < r_2$$

then

$$a^n x(n) \xleftrightarrow{z} X(a^{-1}z), \text{ROC : } |a|r_1 < |z| < |a|r_2 \quad (4.10)$$

4.3.4 Z-Transform Time Reversal

If

$$x(n) \xleftrightarrow{z} X(z), \text{ROC : } r_1 < |z| < r_2$$

then

$$x(-n) \xleftrightarrow{z} X(z^{-1}), \text{ROC : } \frac{1}{r_2} < |z| < \frac{1}{r_1} \quad (4.11)$$

Example 4.5: Z-Transform Time Reversal. Example 3.2.6

Determine the z -transform of the signal

$$x(n) = \mathcal{U}(-n)$$

The transform for $\mathcal{U}(n)$ is given in Table 4.3.

$$\mathcal{U}(n) \xleftrightarrow{z} \frac{1}{1 - x^{-1}}, \text{ ROC : } |z| > 1$$

By using (4.11), we obtain

$$\mathcal{U}(-n) \xleftrightarrow{z} \frac{1}{1 - z}, \text{ ROC : } |z| < 1$$

4.3.5 Z-Domain Differentiation

If

$$x(n) \xleftrightarrow{z} X(z)$$

then

$$nx(n) \xleftrightarrow{z} -z \frac{dX(z)}{dz} \quad (4.12)$$

Example 4.6: Z-Domain Differentiation. Example 3.2.7

Determine the z -transform of the signal

$$x(n) = na^n \mathcal{U}(n)$$

The signal $x(n)$ can be expressed as $nx_1(n)$, where $x_1(n) = a^n \mathcal{U}(n)$. By passing this through the z -transform, we have

$$x_1(n) = a^n \mathcal{U}(n) \xleftrightarrow{z} X_1(z) = \frac{1}{1 - az^{-1}}, \text{ ROC : } |z| > |a|$$

Then by using (4.12), we obtain

$$na^n \mathcal{U}(n) \xleftrightarrow{z} X(z) = -z \frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1 - az^{-1})^2}$$

4.3.6 Z-Domain Convolutions

If

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X(z) = X_1(z)X_2(z) \quad (4.13)$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

Example 4.7: Z-Domain Convolutions. Example 3.2.9

Compute the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \leq n \leq 6 \\ 0, & \text{elsewhere} \end{cases}$$

When

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

According to (4.13) we carry out the multiplication of $X_1(z)$ and $X_2(z)$. Thus

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underline{1}, -1, 0, 0, 0, 0, -1, 1\}$$

4.3.7 Z-Transform 2 Sequence Correlation

If

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \xleftrightarrow{z} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) \quad (4.14)$$

Example 4.8: Z-Transform 2 Sequence Correlation. Example 3.2.10

Determine the autocorrelation of the signal

$$x(n) = a^n \mathcal{U}(n), \quad -1 < a < 1$$

Solution on Page 166.

4.3.8 Z-Transform 2 Sequence Multiplication

If

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

then

$$x(n) = x_1(n)x_2(n) \xleftrightarrow{z} X_z = \frac{1}{2\pi j} \oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv \quad (4.15)$$

where C is a closed contour that encloses the origin and lies within the region of convergence common to both $X_1(v)$ and $X_2(\frac{1}{v})$.

4.3.9 Parsevals Relation for Z-Transform

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv \quad (4.16)$$

4.3.10 Initial Value Theorem for \mathcal{Z} -Transform

If $x(n)$ is *causal* [i.e., $x(n) = 0$ for $n < 0$], then

$$x(0) = \lim_{z \rightarrow \infty} X(z) \quad (4.17)$$

4.4 Properties of the One-Sided \mathcal{Z} -Transform

(i)

4.5 Rational \mathcal{Z} -Transforms

An important family of z -transforms are those for which $X(z)$ is a rational function, a ratio of two polynomials in z^{-1} (or z).

4.5.1 Poles and Zeros of a \mathcal{Z} -Transform

Defn 18 (Zeros). The *zeros* of a z -transform $X(z)$ are the values of z for which $X(z) = 0$.

This is analogous to “setting the numerator equal to zero.”

Defn 19 (Poles). The *poles* of a z transform $X(z)$ are the values of z for which $X(z) = \infty$.

This is analogous to “setting the denominator equal to zero.”

If $X(z)$ is a rational function, then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid negative powers of z by factoring out the terms z^{-M} and z^{-N} .

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M} b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^{-N} a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

Since $B(z)$ and $A(z)$ are polynomials in z , they can be expressed in factored form as

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M} (z - z_1)(z - z_2) \cdots (z - z_M)}{z^{-N} (z - p_1)(z - p_2) \cdots (z - p_N)} \quad (4.18)$$

Thus, $X(z)$ has M finite Zeros at $z = z_1, z_2, \dots, z_M$ (the roots of the numerator polynomial), N finite Poles at $z = p_1, p_2, \dots, p_N$ (the roots of the denominator polynomial), and $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$. Poles and zeroes may occur at $z = \infty$. A zero exists at $z = \infty$ if $X(\infty) = 0$ and a pole exists at $z = \infty$ if $X(\infty) = \infty$.

Defn 20 (Pole-Zero Plot). Poles and Zeros of a z -transform can be shown graphically by a *pole-zero plot* in the complex plane, which shows the location of poles by crosses (\times) and the location of zeros by circles. Multiplicity is shown by a number close to the corresponding cross or circle. The ROC of a z -transform should not contain any poles, by definition.

4.5.2 Decomposition of Rational \mathcal{Z} -Transforms

4.6 Analysis of LTI Systems in the \mathcal{Z} -Domain

4.7 Common \mathcal{Z} -Transforms

5 The Fourier Transform and Fourier Series

When a signal is decomposed with either the Fourier Transform or the Fourier Series, you receive either sinusoids or complex-valued exponentials. This decomposition is said to be represented in the *frequency domain*.

Defn 21 (Fourier Transform). When decomposing the class of signals with finite energy, you perform a *Fourier transform*. This is generally shown as the function

$$c_k = F\{x(t)\}$$

There are 2 possible equations for the Fourier Transform, depending of the function is continuous-time or discrete-time.

1. Continuous-Time: Equation (5.1)
2. Discrete-Time: Equation (5.2)

Signal, $x(n)$	z -Transform, $X(z)$	ROC
$\delta(n)$	1	All z
$\mathcal{U}(n)$	$\frac{1}{1-z^{-1}}$	$ z > 1$
$a^n \mathcal{U}(n)$	$\frac{1}{1-az^{-1}}$	$ z > a $
$na^n \mathcal{U}(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z > a $
$-a^n \mathcal{U}(-n-1)$	$\frac{1}{1-az^{-1}}$	$ z < a $
$-na^n \mathcal{U}(-n-1)$	$\frac{az^{-z}}{(1-az^{-1})^2}$	$ z < a $
$(\cos \omega_0 n) \mathcal{U}(n)$	$\frac{1-z^{-1} \cos \omega_0}{1-2z^{-1} \cos \omega_0 + z^{-2}}$	$ z > 1$
$(\sin \omega_0 n) \mathcal{U}(n)$	$\frac{z^{-1} \sin \omega_0}{1-2z^{-1} \cos \omega_0 + z^{-2}}$	$ z > 1$
$(a^n \cos \omega_0 n) \mathcal{U}(n)$	$\frac{1-az^{-1} \cos \omega_0}{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}}$	$ z > a $
$(a^n \sin \omega_0 n) \mathcal{U}(n)$	$\frac{az^{-1} \sin \omega_0}{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}}$	$ z > a $

Table 4.3: Common \mathcal{Z} -Transforms

The Fourier Transform is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt \quad (5.1)$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad (5.2)$$

Remark 21.1. Sometimes $X(F)$ and $X(f)$ will be denoted with Ω and ω ($X(\Omega)$ and $X(\omega)$) respectively. In both cases, Ω and ω mean something similar.

$$\begin{aligned} \Omega &= 2\pi F \\ \omega &= 2\pi f \end{aligned}$$

This means that we can rewrite Equations (5.1) to (5.2) as

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \quad (5.3)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad (5.4)$$

Remark 21.2. Generally, when people say the Fourier Transform, they are referring to the transform on Continuous-Time Signals. There is a distinction that occurs with the *DTFT* or *Discrete-Time Fourier Transform*.

This document explains them side-by-side, but will primarily focus on the Discrete-Time Fourier Transform.

Defn 22 (Fourier Series). When decomposing the class of periodic signals, you are returned a *Fourier series*. This is generally shown as the function

$$X(F) = F\{x(t)\}$$

Defn 23 (Discrete-Time Fourier Transform). The *Discrete-Time Fourier Transform*, *DTFT* is a special case of the Fourier Transform that occurs when the input function $x(n)$ is a case of Discrete-Time Signals.

The transformation (analysis) equations are:

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad (5.5a)$$

$$\omega = 2\pi f$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad (5.5b)$$

The reverse (synthesis) equations are:

$$x(n) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f n} df \quad (5.6a)$$

$$x(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega n} d\omega \quad (5.6b)$$

These equations are expanded more upon in Section 5.2, The Inverse Fourier Transform.

5.1 Fourier Transform Relations

Each of these relations is just a side-note, the only relation of real importance is Equation (5.7). The Fourier Transform is just a special case in each of these scenarios. The Fourier Transform is evaluated around the unit circle on the real-imaginary plane.

5.1.1 Laplace Transform Fourier Transform Relation

There is a correlation between the Laplace Transform and the Fourier Transform. The Fourier Transform is a more specific case of the Laplace Transform, when

$$e^{-st} = e^{-j2\pi ft}$$

5.1.2 Z-Transform Discrete-Time Fourier Transform Relation

There is a relationship between the Z-Transform and the Discrete-Time Fourier Transform.

$$\begin{aligned} z &= e^{j2\pi f} \\ z &= e^{j2\pi n} \end{aligned} \quad (5.7)$$

The Discrete-Time Fourier Transform can be viewed as the Z-transform of the sequence evaluated at the unit circle. If $X(z)$ does not converge in the region $|z| = 1$ (i.e., if the unit circle is not contained within the ROC of $X(z)$), the Fourier Transform $X(f)$ does not exist.

The existence of the Z-transform requires that the sequence $\{x(n)r^{-n}\}$ be absolutely summable for some value of r , that is,

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \quad (5.8)$$

Therefore, if Equation (5.8) converges only for values of $r < r_0 < 1$, the **Z-transform exists**, but the **Discrete-Time Fourier Transform DOES NOT EXIST**. This is the case for causal sequences of the form $x(n) = a^n \mathcal{U}(n)$, where $|a| > 1$.

There are sequences that do not satisfy Equation (5.8), for example

$$x(n) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

This sequences does not have a Z-transform. However, since it is a finite Energy Signal, it has a Discrete-Time Fourier Transform that converges to

$$X(f) = \begin{cases} 1, & |f| < f_c \\ 0, & f_c < |f| \leq \frac{1}{2} \end{cases}$$

The existence of the Z-transform requires that Equation (5.8) be satisfied for some region in the z -plane. If this region contains the unit circle, the Discrete-Time Fourier Transform, $X(f)$ exists. However, the existence of the Discrete-Time Fourier Transform, which is defined for finite Energy Signals, does not necessarily ensure the existence of the Z-transform.

5.2 The Inverse Fourier Transform

Defn 24 (Inverse Fourier Transform). Since the Fourier Transform is a “lossless” function (the definition of a transformation), the *inverse fourier transform* is just the opposite setup of Equations (5.1) to (5.2).

In both cases, a Continuous-Time signal and a Discrete-Time signal, you use the below synthesis equations (Equations (5.9) to (5.10)).

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF \\ x(n) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi fn} df \end{aligned} \tag{5.9}$$

If you’re calculating with Ω or ω instead of F or f , then use these synthesis equations.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \\ x(n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega n} d\omega \end{aligned} \tag{5.10}$$

5.3 Properties of the Discrete-Time Fourier Transform

One thing to keep in mind with all of these properties is that $\omega = 2\pi f$.

Property	Time Domain $x(n)$	Frequency Domain $X(f)$ or $X(\omega)$
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(\omega)$ $X_1(\omega)$ $X_2(\omega)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
Time Shifting	$x(n - k)$	$e^{-j\omega k} X(\omega)$
Time Reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega) X_2(\omega)$
Correlation	$r_{x_1, x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1, x_2}(\omega) = X_1(\omega) X_2(\omega)$ $= X_1(\omega) X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener-Khintchine Theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency Shifting	$e^{j\omega_0 n} x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n) \cos(\omega_0 n)$	$\frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$
Multiplication in Time Domain	$x_1(n) x_2(n)$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$
Differentiation in Frequency Domain	$n x(n)$	$j \frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval’s Theorem	$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n)$	$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega) d\omega$

Table 5.1: Properties of the Fourier Transform for Discrete-Time Signals

5.3.1 Linearity

If

$$\begin{aligned} x_1(n) &\xleftrightarrow{F} X_1(f) \\ x_2(n) &\xleftrightarrow{F} X_2(f) \end{aligned}$$

then

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{F} a_1 X_1(f) + a_2 X_2(f) \tag{5.11}$$

5.3.2 Time Shifting

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$x(n-k) \xleftrightarrow{F} e^{-j\omega k} X(f) \quad (5.12)$$

5.3.3 Time Reversal

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$x(-n) \xleftrightarrow{F} X(-f) \quad (5.13)$$

5.3.4 Convolution

If

$$x_1(n) \xleftrightarrow{F} X_1(f)$$

$$x_2(n) \xleftrightarrow{F} X_2(f)$$

then

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{F} X(f) = X_1(f)X_2(f) \quad (5.14)$$

Remark. There is one thing to note here. Both $x_1(n)$ and $x_2(n)$ must be reasonably well-behaved and have be BIBO-stable for this relation to hold.

5.3.5 Correlation

If

$$x_1(n) \xleftrightarrow{F} X_1(f)$$

$$x_2(n) \xleftrightarrow{F} X_2(f)$$

then

$$r_{x_1x_2}(m) \xleftrightarrow{F} S_{x_1x_2}(f) = X_1(f)X_2(-f) \quad (5.15)$$

5.3.6 Wiener-Khintchine Theorem

Let $x(n)$ be a real signal. Then

$$r_{xx}(l) \xleftrightarrow{F} S_{xx}(f) \quad (5.16)$$

That is, the energy spectral density of an energy signal is the Fourier Transform of its autocorrelation sequence. This is a special case of Equation (5.15).

5.3.7 Frequency Shifting

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$e^{-i2\pi f_0 n} x(n) \xleftrightarrow{F} X(f - f_0) \quad (5.17)$$

5.3.8 Modulation

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$x(n) \cos(2\pi f_0 n) \xleftrightarrow{F} \frac{1}{2} [X(f + f_0) + X(f - f_0)] \quad (5.18)$$

5.3.9 Multiplication in Time Domain

This is also called the *Windowing Theorem*.

If

$$\begin{aligned} x_1(n) &\xleftrightarrow{F} X_1(f) \\ x_2(n) &\xleftrightarrow{F} X_2(f) \end{aligned}$$

then

$$x_3(n) \equiv x_1(n)x_2(n) \xleftrightarrow{F} X_3(f) = \int_{\frac{1}{2}}^{\frac{1}{2}} X_1(\lambda)X_2(f-\lambda)d\lambda \quad (5.19)$$

5.3.10 Differentiation in Frequency Domain

If

$$x(n) \xleftrightarrow{F} X(f)$$

then

$$nx(n) \xleftrightarrow{F} j \frac{dX(f)}{df} \quad (5.20)$$

5.3.11 Parseval's Theorem

If

$$\begin{aligned} x_1(n) &\xleftrightarrow{F} X_1(f) \\ x_2(n) &\xleftrightarrow{F} X_2(f) \end{aligned}$$

then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \int_{-0.5}^{0.5} X_1(f)X_2^*(f)df \quad (5.21)$$

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega \quad (5.22)$$

Both Equations (5.21) to (5.22) can be expressed in another format.

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \int_{-0.5}^{0.5} |X_1(f)|^2 df \quad (5.23)$$

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X_1(\omega)|^2 d\omega \quad (5.24)$$

6 The Fourier Transform and LTI Systems

6.1 Frequency Response

The Frequency Response of a function is defined in 2 parts.

1. Magnitude Response
2. Phase Response

$$\begin{aligned} H(\omega) &= \|H(\omega)\| \Theta(\omega) \\ H(f) &= \|H(f)\| \Theta(f) \end{aligned} \quad (6.1)$$

Remark. This is similar to the Rectangular to Polar conversion shown on Page 20.

6.2 Magnitude Response

Defn 25 (Magnitude Response). The *magnitude response* of a Fourier Transform is commonly denoted as $|H(\omega)|$ or $|H(f)|$. However, in this text, it is denoted as $\|H(\omega)\|$ or $\|H(f)\|$.

The equation that defines a Fourier Transform's Magnitude Response is

$$\begin{aligned}\|H(\omega)\| &= \sqrt{\left(\operatorname{Re}\{H(\omega)\}\right)^2 + \left(\operatorname{Im}\{H(\omega)\}\right)^2} \\ \|H(f)\| &= \sqrt{\left(\operatorname{Re}\{H(f)\}\right)^2 + \left(\operatorname{Im}\{H(f)\}\right)^2}\end{aligned}\tag{6.2}$$

Remark 25.1. It is important to note that the numerator will **NOT** have any imaginary terms (i or j) in it!

Remark. It is important to note that in this reference guide, the magnitude is denoted with double bars. For example, if a complex function $a(n)$ exists, then I will denote its magnitude as $\|a(n)\|$. This helps distinguish between magnitude of a function and its absolute value. These sometimes have different values, so it is useful to differentiate between the two.

6.3 Phase Response

Defn 26 (Phase Response). The *phase response* of a Fourier Transform is commonly denoted as $\angle H(\omega)$ or $\angle H(f)$. This is a function that defines a Fourier Transform's Phase Response is defined in the equation below.

$$\begin{aligned}\angle H(\omega) = \Theta(\omega) &= \tan^{-1}\left(\frac{\operatorname{Im}\{H(\omega)\}}{\operatorname{Re}\{H(\omega)\}}\right) \\ \angle H(f) = \Theta(f) &= \tan^{-1}\left(\frac{\operatorname{Im}\{H(f)\}}{\operatorname{Re}\{H(f)\}}\right)\end{aligned}\tag{6.3}$$

Remark 26.1. It is important to note that the numerator will **NOT** have any imaginary terms (i or j) in it!

Remark 26.2 (Complex Exponential to Unit Circle). **REMEMBER:**

$$\begin{aligned}e^{\pm j\omega} &= \cos(\omega) \pm j \sin(\omega) \\ e^{\pm j2\pi f} &= \cos(2\pi f) \pm j \sin(2\pi f)\end{aligned}\tag{6.4}$$

This is also defined in Equation (A.3) on Page 19.

A Trigonometry

A.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{A.2})$$

A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j \sin(\alpha) \quad (\text{A.3})$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \quad (\text{A.4})$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \quad (\text{A.5})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{A.6})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{A.7})$$

A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \quad (\text{A.8})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \quad (\text{A.9})$$

A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad (\text{A.10})$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (\text{A.11})$$

A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \quad (\text{A.12})$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \quad (\text{A.13})$$

A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2} \quad (\text{A.14})$$

$$\cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2} \quad (\text{A.15})$$

A.7 Product-to-Sum Identities

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad (\text{A.16})$$

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad (\text{A.17})$$

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (\text{A.18})$$

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (\text{A.19})$$

A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{A.20})$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.21})$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.22})$$

A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \quad (\text{A.23})$$

A.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2} e^{j\theta} = r e^{j\theta} \quad (\text{A.24})$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \quad (\text{A.25})$$

A.11 Polar to Rectangular

$$r e^{j\theta} = r \cos(\theta) + j r \sin(\theta) \quad (\text{A.26})$$

B Calculus

B.1 Fundamental Theorems of Calculus

Defn B.1.1 (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{B.1})$$

Defn B.1.2 (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I , and states that if F is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{B.2})$$

Defn B.1.3 (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname{argmax}_x$$

B.2 Rules of Calculus

B.2.1 Chain Rule

Defn B.2.1 (Chain Rule). The *chain rule* is a way to differentiate a function that has 2 functions multiplied together. If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \quad (\text{B.3})$$

C Laplace Transform

Defn C.0.1 (Laplace Transform). The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{C.1}$$