

Phys 224: Modern Physics — Reference Sheet

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1 General Stuff

- Density: $\rho = \frac{\Delta m}{\Delta V}$
 - Uniform Density: $\rho = \frac{m}{V}$
- Pressure: $p = \frac{\Delta F}{\Delta A}$
 - Uniform Force on Flat Area: $\rho = \frac{F}{A}$
 - Conversions: $1atm = 1.01 \times 10^5 Pa = 760torr = 14.7lb/in^2$
- Trig Relations are in Trigonometric Formulas, Section B.1.

2 Fluids

We must satisfy several parameters to make life easier, and to use most of these formulae.

1. Incompressible - Density of the fluid is constant
 2. Non-turbulent Flow - Think of fluids swirling around an object
 3. Isostatic Pressure - Pressure inside the fluid is the same in all directions
- Pressure at Some Depth - $p_2 = p_1 + \rho g (y_1 - y_2)$
 - Pressure at Depth $h \rightarrow p = p_0 + \rho gh$
 - Pascal's Principle - 2 Parts
 1. $\vec{F}_o = \vec{F}_i \frac{A_o}{A_i}$
 2. $d_o = d_i \frac{A_i}{A_o}$
 - When 2 pressures should be equal, their forces are inversely proportional
 - Set each pressure equal to each other, then solve for the missing variable.
 - Archimedes' Principle - $\vec{F}_{Up} = \vec{F}_{Down}$
 - Usually breaks down to $\vec{F}_{Bouyant} = \vec{F}_{g, Object}$
 - $\vec{F}_{Bouyant} = m_{Object}g$
 - $\vec{F}_{Bouyant} = \rho_{Object}V_{Object}g$
 - Continuity of Fluids
 - $A_1v_1 = A_2v_2$
 - Bernoulli's Equation - $p_1 + \frac{1}{2}\rho v_1^2 + \rho gy_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho gy_2$
 - Fluids at Rest - $p_2 = p_1 + \rho g (y_1 - y_2)$
 - Fluids not Changing Height - $p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$

3 Waves

Usually of form $y = y_m \sin(kx \pm \omega t)$ There are two types of waves:

1. Transverse Waves - Waves where displacement from equilibrium is orthogonal to direction of propagation
 - String Waves
 - Electromagnetic Waves
 2. Longitudinal Waves - Waves where displacement from equilibrium is parallel to direction of propagation
 - Pressure Waves
 - Sound Waves (Which are a type of pressure wave)
- y_m - Amplitude, m
 - k - Angular Wave Number, rad/m
 - $k = \frac{2\pi}{\lambda}$
 - λ is wavelength, m
 - ω - Angular Frequency, rad/s
 - $\omega = 2\pi f$
 - f is frequency, Hz
 - Sign of this goes the opposite the direction the wave is going
 1. Wave going in positive direction (+), then the sign should be negative (-)
 2. Wave going in negative direction (-), then the sign should be positive (+)

- $v = \lambda f$, Wave Velocity, m/s
 - $v = \frac{\omega}{2\pi} * \frac{2\pi}{k} = \frac{\omega}{k}$
 - This can be proven with the angular portion of any wave (inside the parentheses of trig function)

$$\begin{aligned}
 kx - \omega t &= \text{Constant} \\
 \frac{d}{dt} [kx - \omega t] &= \frac{d}{dt} [\text{Constant}] \\
 k \frac{dx}{dt} - \omega \frac{dt}{dt} &= 0 \\
 kv - \omega &= 0 \\
 kv &= \omega \\
 v &= \frac{\omega}{k}
 \end{aligned}$$

Since $\omega = 2\pi f$, then $k = \frac{\lambda}{2\pi}$

3.1 Wave Interference

Waves are nice, and they just sum when they interfere. Let:

$$y_1(x, t) = y \sin(kx - \omega t) \quad (3.1)$$

$$y_2(x, t) = y \sin(kx + \omega t + \varphi) \quad (3.2)$$

$$Y(x, t) = y [\sin(kx - \omega t) + \sin(kx + \omega t + \varphi)] \quad (3.3)$$

You can usually use Equation B.1 to simplify Equation 3.3.

3.1.1 Constructive/Destructive Interference

$$\begin{aligned}
 \phi &= 2\pi \frac{\text{PathLengthDiff}}{\lambda} = 2\pi \frac{\Delta \text{PLD}}{\lambda} \\
 n &= \frac{\phi}{2\pi} = 4 \frac{\text{PathLengthDiff}}{\lambda}
 \end{aligned}$$

- *PathLengthDiff* - Is the Difference in path lengths that the waves must travel
- ϕ - Angular Location of points of Complete Constructive/Destructive Interference
- n - Number of locations where there is Complete Constructive/Destructive Interference

3.2 Standing Waves

This is actually the superposition of 2 waves, traveling in opposite directions, on a medium that is fixed at both ends, i.e. a taut string held by a wall.

Location of Nodes and Antinodes

- Nodes - $x = n \frac{\lambda}{2}$ for $n = 0, 1, 2, \dots$
 - Always at closed ends of tubes
- Antinodes - $x = (n + \frac{1}{2}) \frac{\lambda}{2}$ for $n = 0, 1, 2, \dots$
 - Always at open ends of tubes

3.2.1 Resonant Frequencies/Harmonics

These can also be called harmonics. There is a resonant frequency for every number of nodes/antinodes on the standing wave.

$$f = \frac{v}{\lambda} = n \frac{v}{2L}$$

- L is the length of the medium (The String).
- λ is the wavelength of the wave formed.

This can be extended to find the base resonant frequency, if you know how many node levels are between the two resonant frequencies given, i.e. they say that the **NEXT** frequency, means $n + 1$.

$$f_{n+m} - f_n = (n + m) \frac{v}{2L} - n \frac{v}{2L} = m \frac{v}{2L}$$

3.3 Reflecting Sound

- $D = (n + 1) d = vt$
 - n is the number of reflections that occurred
 - $n + 1$ is used when we want the distance the wave covers

3.4 Sound in Different Mediums

Frequency is a property of a wave, and **CANNOT BE ALTERED**. This means that:

$$\begin{aligned}v &= \lambda f \\v_{Sound, Material1} &= \lambda_{Material1} f_{Unique, Material1} \\v_{Sound, Material2} &= \lambda_{Material2} f_{Unique, Material2} \\f_{Unique, Material1} &= f_{Unique, Material2} \\\frac{v_{Sound, Material1}}{\lambda_{Material1}} &= \frac{v_{Sound, Material2}}{\lambda_{Material2}}\end{aligned}$$

3.5 Doppler Effect

$$f' = f \frac{v \pm v_D}{v \pm v_S}$$

- Moving **TOWARDS** each other: Frequency Increase
- Moving **AWAY** from each other: Frequency Decrease
- f - Initial Frequency, Hz
- v - Sound Speed, m/s
- v_D - Detector Speed, m/s
- v_S - Source Speed, m/s
- **For Numerator:**
 - If detector is moving towards the source, +
 - If detector is moving away from the source, –
- **For Denominator:**
 - If source is moving away from detector, +
 - If source is moving towards the detector, –

4 Thermodynamics

Defn 1 (Thermodynamics). *Thermodynamics* is the study of energy transfer between two macroscopic bodies driven by temperature differences.

Defn 2 (Temperature). *Temperature* is a direct measurement of internal energy of a system

4.1 Laws of Thermodynamics

Defn 3 (0th Law of Thermodynamics). If 2 bodies, A and B are in thermal equilibrium with a third body “T”, then they are in thermal equilibrium with each other.

Defn 4 (1st Law of Thermodynamics).

$$\begin{aligned}dE_{Internal} &= dQ - dT, dQ \text{ and } dT \text{ are inexact (path-dependent) differentials.} \\ \Delta E_{Internal} &= Q - T\end{aligned}\tag{4.1}$$

Note 4.1. Special Cases for the 1st Law of Thermodynamics:

1. Adiabatic Processes - $dE_{Int} = -dW$
 - No heat exchange
 - Insulating
 - Something happens too quickly for system to keep up
2. Isothermal Processes - $dT = 0 \rightarrow dE_{Int} = 0 \rightarrow dQ = dW$
3. Isobaric Processes - $dW = pdV$, p (pressure) is constant
4. Constant Volume - $W = 0$

5. Cyclical Processes - $dE_{Int} = 0 \rightarrow dQ = dW$

- You end a cycle with the same internal energy when the cycle started

Defn 5 (2nd Law of Thermodynamics). If a *cyclical process occurs in a CLOSED system*, the entropy of the system increases for irreversible processes and remains constant for reversible processes. **IT NEVER DECREASES!!**

$$\Delta S \geq 0 \quad (4.2)$$

$$\Delta S = \int_a^b \frac{dQ(T)}{T} \quad (4.3)$$

Note 5.1. ΔS is a state-function, meaning it is path-independent.

4.2 Heat and Work

- $dW = \vec{F} \cdot d\vec{s}$
- $\vec{F} = p(V, T) dV$
- $W = \int p(V, T) dV$
- $W = \frac{dQ}{dt}$
- $W = \frac{\Delta Q}{\Delta t}$

Work done by thermal energy is path independent.

4.3 Thermal Expansion

Occurs because the “springs” between each of the atoms in a lattice have energy applied by Heat (Temperature Change).

- $\frac{\Delta L}{L_0} = \alpha \Delta T$ (One-Dimensional Expansion)
- $\frac{dL}{L_0} = \alpha dT$ (One-Dimensional Expansion)
 - α is a material-specific constant

4.4 Specific Heat/Heat Capacity

- $C = \frac{dQ}{dT} \leftarrow$ Specific Heat
- $c = \frac{dQ}{mdT} \leftarrow$ Mass Specific Heat

4.5 Heat of Phase Transitions

This is a constant unique to the material and the phase transition it is going through.

- $Q = Lm$
- $Q = \int_{m_i}^{m_f} L_f dm$
 - l_{Fusion} = Heat required to turn things from **SOLID TO LIQUID**
 - l_{Vapor} = Heat required to turn things from **LIQUID TO GAS**

4.6 Conduction Heat Transfer

$$P_{Conduction} = \frac{(T_H - T_C)}{L} Ak \quad (4.4)$$

- L = Length
- A = Cross-Sectional Area
- k = Material’s Thermal Conductivity

For multiple materials between 2 thermal reservoirs:

- $P_1 = P_2 = \dots = P_n$
- Heat will only flow as fast as the slowest thermal conductor

5 Kinetic Theory of Ideal Gases

Defn 6 (Ideal Gas). An *ideal gas* is a gas that obeys the ideal gas law.

$$pV = nRT, R \approx 8.31 \text{ J/mol K} \quad (5.1)$$

$$E_{\text{Internal}} = K_{\text{Translate}} + K_{\text{Rotate}} \quad (5.2)$$

$$\Delta E_{\text{Internal}} = \Delta K_{\text{Translate}} + \Delta K_{\text{Rotate}} \quad (5.3)$$

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} \quad \text{M=molar mass of gas} \quad (5.4)$$

5.1 Mean Free Path

Defn 7 (Mean Free Path). The *mean free path*, λ is the average distance traversed by a molecule between collisions.

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 \frac{N}{V}} \quad (5.5)$$

Where:

- d is the diameter of the atoms, or distance between centers during collision (m)
- N is the number of molecules (mol)
- V is the volume of gas being handled (L)
- λ is the wavelength, the same as in $v = f\lambda$

Note 7.1. If all the particles, except 1 are stationary, then you can use:

$$\lambda = \frac{1}{\pi d^2 \frac{N}{V}} \quad (5.6)$$

5.2 Work Done by Ideal Gases

5.2.1 Isothermally

$$\begin{aligned} W &= \int \vec{F} d\vec{s} \\ W &= \int_{V_1}^{V_2} p(V, T) dV \\ W &= \int_{V_1}^{V_2} \frac{nRT}{V} dV \\ W &= nRT \int_{V_1}^{V_2} \frac{1}{V} dV \\ W &= nRT \ln \left(\frac{V_2}{V_1} \right) \end{aligned} \quad (5.7)$$

Constant Pressure

$$W = p(V_{\text{final}} - V_{\text{init}}) \quad (5.8)$$

5.3 Translational Kinetic Energy

Defn 8 (Degrees of Freedom). *Degrees of freedom* represent the number of variables that are needed to describe a system. Represented with d , occasionally.

Note 8.1. In an ideal gas, these are a means to store energy.

$$\begin{aligned} K_{\text{Translate}} &= \frac{3}{2}nRT \\ \Delta K_{\text{Translate}} &= \frac{3}{2}nR\Delta T \end{aligned} \quad (5.9)$$

	Translational	Rotational	Total
Monatomic	3	0	3
Diatomic	3	2	5
Polyatomic	3	3	6

Table 5.1: Degrees of Freedom Table for Gases

- An ideal gas has **ONLY** kinetic energy
- Completely elastic collisions

$$E_{int} = \frac{DoF}{2} nRT, \text{ where } DoF = \text{Degrees of Freedom} \quad (5.10)$$

5.4 Molar Specific Heats of Ideal Gases

5.4.1 Molar Specific Heat @ Constant Volume

$$C_V = \frac{\Delta E}{n\Delta T}$$

$$C_V = \frac{dE}{dT} \quad (5.11)$$

$$C_V = \left(\frac{DoF}{2} \right) R$$

$$Q = nC_V \Delta T \quad (5.12)$$

5.4.2 Molar Specific Heat @ Constant Pressure

$$C_P = C_V + R \quad (5.13)$$

$$Q = nC_P \Delta T \quad (5.14)$$

5.5 Adiabatic Processes in Ideal Gases

Defn 9 (Adiabatic Process). An *adiabatic process* is one in which no heat exchange occurs, namely:

$$dE_{Internal} = dQ - dW$$

$$dQ = 0 \quad (5.15)$$

$$dE_{Internal} = -dW$$

This leads to:

$$pV^\gamma = \text{Constant}, \gamma = \frac{C_P}{C_V} \quad (5.16)$$

$$p_1 V_1^\gamma = p_2 V_2^\gamma \quad (5.17)$$

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \quad (5.18)$$

You get Equation (5.18) by plugging $p = \frac{nRT}{V}$ into Equation (5.17).

6 Entropy

There is a heavy relationship between this and the 2nd Law of Thermodynamics.

Defn 10 (Entropy). *Entropy* is a measure of the number of available states/configurations. *Entropy* can be thought of as a measure of the probability of an energy distribution in materials. Entropy states that energy will want to end in its most probable state, where it is most even distributed. The *units for entropy* are J/K.

$$S = k_B \ln(\Omega) \quad (6.1)$$

Note 10.1. **Entropy is NOT disorder.**

Defn 11 (Change in Entropy). *Change in entropy* is the change in the number of available states/configurations for energy in a system.

$$\begin{aligned}
\Delta S &= S_f - S_i \\
&= \int_a^b \frac{dQ(T)}{T} \\
&= nR \ln \left(\frac{V_f}{V_i} \right) \\
&= mC \ln \left(\frac{T_f}{T_i} \right)
\end{aligned} \tag{6.2}$$

Note 11.1. Because ΔS is based off of heat,

$$\Delta S_{Total} = \sum_{states} \Delta S \tag{6.3}$$

6.1 Engines

Engines are inherently cyclical.

Defn 12 (Efficiency). *Efficiency* is a measure of how effect an engine is in turning the heat taken in and turning it into work out.

$$\begin{aligned}
\text{Efficiency} &= \frac{W_{Out}}{Q_{In}} \\
\text{Eff} &= \frac{Q_{In} - Q_{Out}}{Q_{In}} \\
\text{Eff} &= \frac{T_{Hot} - T_{Cold}}{T_{Hot}}
\end{aligned} \tag{6.4}$$

6.1.1 Stirling Engine

Based on the Ideal Gas. You can find Efficiency with Equation (6.4).

6.1.2 Carnot Engine

Uses Isothermal expansion and adiabatic expansion of gases to achieve work. The total work done is the area of the Carnot figure. You can find Efficiency with Equation (6.4).

Defn 13 (Coefficient of Performance). *Coefficient of performance* is a measurement of the amount of energy a Carnot engine moves in comparison to its cold reservoir.

$$\begin{aligned}
K_C &= \frac{T_L}{T_H - T_L} \\
K_C &= \frac{|Q_L|}{|Q_H| - |Q_L|} \\
K &= \frac{|Q_L|}{|W|}
\end{aligned} \tag{6.5}$$

7 Light

Defn 14 (Photon). A *photon* is an electric field (wave) propagating through another electric field.

The various θ used in Equations (7.1), (7.2) are measured from the surface normal.

Chromatic Dispersion is the breaking up of polychromatic light by spectra. Think of Pink Floyd's *Dark Side of the Moon* album cover. This happens because shorter wavelength, higher frequency light has a slightly higher index of refraction.

7.1 Reflection

$$\theta_{reflected} = \theta_{incident} \tag{7.1}$$

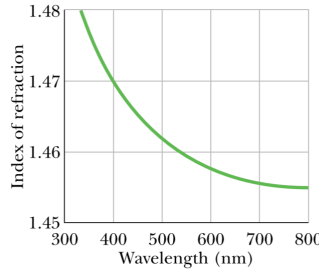


Fig. 33-18 The index of refraction as a function of wavelength for fused quartz. The graph indicates

Figure 7.1: Indices of Refraction for Polychromatic Light

7.2 Refraction

Defn 15 (Snell's Law).

$$n_{refract} \sin(\theta_{refract}) = n_{incident} \sin(\theta_{incident}) \quad (7.2)$$

Defn 16 (Index of Refraction).

$$\begin{aligned} n_i &= \frac{c}{v_i} \\ \lambda &= \frac{\lambda_0}{n} \end{aligned} \quad (7.3)$$

The total number of waves that pass through a material is given in Equation (7.4).

$$\begin{aligned} N &= \frac{L}{\lambda_n} \\ &= \frac{L}{\frac{\lambda_0}{n}} N = \frac{nL}{\lambda_0} \end{aligned} \quad (7.4)$$

7.3 Total Internal Reflection

Defn 17 (Total Internal Reflection). Total Internal Reflection occurs when the *refracted light's angle* is $\frac{\pi}{2}$.

$$n_{refract} \sin(\theta_{refract}) = n_{incident} \sin(\theta_{incident}), \text{ where } \theta_{refract} = \frac{\pi}{2} \quad (7.5)$$

7.4 Interference and Diffraction

Defn 18 (Huygen's Principle). Any point on a plane wavefront can be treated as a source of outgoing spherical waves.

Note 18.1. This is a mathematical construct/model.

Defn 19 (Phase Difference). Waves from the same source, but measured in such a way that there is a Path Length Difference (PLD) between them will have a *phase difference*.

$$\varphi = \frac{2\pi}{\lambda} (\text{PLD}) \quad (7.6)$$

All Interference and Diffraction equations used from here on out are based on Equation (7.6).

Defn 20 (Intensity). *Intensity* of an electromagnetic wave (light) is proportional to the square of the amplitude of the electric field.

$$\begin{aligned} I &= E^2 = \sin^2(\alpha) \\ I_0 &= \alpha^2 \end{aligned} \quad (7.7)$$

7.4.1 Single Slit Interference

Defn 21 (Single Slit Interference). When waves propagate through a single slit, there is a Probability Distribution Function that describes where intensity of light is greatest and smallest. The location of the *minima* are given in Equation (7.8). This phenomena occurs because Huygen's Principle says a wave has infinite sources of light that radiate spherically.

$$m\lambda = a \sin(\theta_m) \quad (7.8)$$

- m is the number of minima from the central major distribution of intensity.
 1. First minima from central intensity point means $m = 1$
 2. Second minima from central intensity point means $m = 2$
 3. etc.
- λ is the wavelength of the incoming wave
- a is the width of the single slit
- θ_m is the angle formed by the diffracted light from the normal

Thus, the equation for the Single Slit Interference is given in Equation (7.9).

$$\varphi_{\text{Single Slit}} = \frac{2\pi}{\lambda} a \sin(\theta) \text{ rad} \quad (7.9)$$

There is a connection between the angle θ that locates a point on the screen with the light intensity $I(\theta)$ at that point.

$$\begin{aligned} \alpha &= \frac{1}{2}\phi \\ \alpha &= m\pi \\ \alpha &= \frac{\pi}{\lambda} a \sin(\theta) \end{aligned} \quad (7.10)$$

7.4.2 Double Slit Diffraction

Defn 22 (Double Slit Diffraction). The *maxima* of the interference is given in Equation.

$$n\lambda = d \sin(\theta_n) \quad (7.11)$$

- n is the number of maxima from the central major distribution of intensity.
 1. Central maxima means $n = 0$
 2. First maxima means $n = \pm 1$
 3. etc.
- d is the distance between the slits
- λ is the incoming wave's wavelength
- θ_n is the phase difference from the normal by the diffracted light

Thus, the equation for the Double Slit Diffraction is given in Equation (7.12).

$$\varphi_{\text{Double Slit}} = \frac{2\pi}{\lambda} d \sin(\theta) \text{ rad} \quad (7.12)$$

8 Quantum Mechanics

Quantum mechanics may seem strange or weird, but it really isn't.

Defn 23 (Hamiltonian). The *hamiltonian* is a function that contains all knowable information about a particle or system in classical mechanics.

$$\text{Hamiltonian} = K + U \quad (8.1)$$

- K is the Kinetic Energy of the particle or system of particles.
 - $K = \frac{1}{2}mv^2$
 - This contains the velocity of the system, which includes direction.
- U is the Potential Energy of the particle or system of particles.
 - $U = mg(y_2 - y_1)$
 - Contains the force of the system, its location, etc.

8.1 Photoelectric Effect

Defn 24 (Photoelectric Effect). Occurs when light shines on an object. The object will lose electrons e^- if the light has a great enough frequency. This *is not* a function of intensity or brightness, only frequency.

$$\begin{aligned} K_{\text{Max}} &= hf - \Phi \\ K_{\text{Max}} &= E - \Phi \Rightarrow K_{\text{Max}} + \Phi = E \end{aligned} \quad (8.2)$$

- $K_{\text{Max}} = eV_{\text{Stop}}$ is Kinetic Energy of released electrons (J)
 - e is charge of electron
 - V_{Stop} is the stopping potential
- h is Planck's Constant
- f is Frequency of the incident wave (Hz)
- Φ is the Work Function (eV)

The is given in Equation (8.6).

8.2 Schrödinger's Equation

Defn 25 (de Broglie Wavelength). The *de Broglie wavelength*, λ , is the wavelength that is *associated* with a particle. In other words, a particle does **not** have a wavelength, but the equations describing the particle's actions have a wavelength.

$$\lambda = \frac{h}{p} \quad (8.3)$$

Note 25.1. As a direction consequence of the de Broglie Wavelength, Equation (8.3):

$$p = \frac{h}{\lambda} \quad (8.4)$$

- h is Planck's Constant
- λ is the de Broglie Wavelength

For electromagnetic waves, Equation (8.5) holds true.

$$p = \frac{E}{c} \quad (8.5)$$

For matter waves, Equation (8.6) holds true.

$$K = \frac{p^2}{2m} \quad (8.6)$$

Defn 26 (Schrödinger's Equation). *Schrödinger's equation* is a 3-dimensional Cartesian, time-independent way to express the equivalent of a Hamiltonian for quantum effects, namely in terms of waves (Denoted with Ψ or ψ). There are 2 equivalent ways to express Schrödinger's Equation. The first way, Equation (8.7), is the most general form. The second way, Equation (8.8), has expanded the nabla operator, ∇ .

$$\frac{-\hbar^2}{2m} \cdot \nabla^2 \Psi + V\Psi = E\Psi \quad (8.7)$$

$$\frac{-\hbar}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + V\Psi = E\Psi \quad (8.8)$$

- E is the **total** energy of the system
- V is the **potential energy** of the system. It may be:
 1. Constant
 2. A function of time and/or space
- Ψ is a solution to Schrödinger's Equation, and contains all that can be known about the system
- $\frac{-\hbar^2}{2m} \cdot \nabla^2 \Psi$ is the **kinetic energy** operator
 - $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is in the Cartesian coordinate system (x, y, z) , and is called “del squared”, or “2nd-degree nabla.”
- For each observable system, there exists a corresponding mathematical operator.

Note 26.1. The probability distribution function of the position of a particle in the system can be found according to Equation (8.9). This is the multiplication of Ψ with its complex conjugate, Ψ^* .

$$\text{PDF} = \Psi \cdot \Psi^* \quad (8.9)$$

8.2.1 Properties of Schrödinger's Equation

Solutions, Ψ , to Schrödinger's Equation have the following characteristics:

1. They are single-valued
2. They are piecewise continuous values
3. $\frac{\partial \Psi}{\partial x}$, $\frac{\partial \Psi}{\partial y}$, $\frac{\partial \Psi}{\partial z}$ are all continuous
4. $\iiint \Psi \Psi^* dx dy dz$ is square integrable, or $\iiint \Psi \Psi^* dx dy dz \neq \pm \infty$

Rough Justification of Schrödinger's Equation. If Ψ is to be a wave function, it must satisfy the following differential equation:

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$$

Since we are looking for Schrödinger's Equation, in which all the dynamical observables are contained within Ψ , we will set $v = c$, the upper limit of information transfer. We will first consider (without loss of generality) one dimension, the x -dimension. Then,

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$$

As a test solution, we will propose

$$\Psi = A e^{-i(kx - \omega t)}$$

With respect to x :

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= A (-ik) e^{-i(kx - \omega t)} \\ \frac{\partial^2 \Psi}{\partial x^2} &= A (-ik)^2 e^{-i(kx - \omega t)} = (-ik)^2 \Psi = k^2 \Psi \end{aligned}$$

Therefore, Ψ is a satisfactory solution in x . With respect to t :

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= A (i\omega) e^{-i(kx - \omega t)} \\ \frac{\partial^2 \Psi}{\partial t^2} &= A (i\omega)^2 e^{-i(kx - \omega t)} \end{aligned}$$

Therefore, Ψ is a satisfactory solution in t . Thus, Ψ is a solution to $\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2}$. (Also, $\frac{1}{c^2} = \frac{k^2}{\omega^2}$). In our solution, $\Psi = A e^{-i(kx - \omega t)}$, k has its usual meaning:

$$k = \frac{2\pi}{\lambda}$$

Where the wavelength, λ , is to be the de Broglie Wavelength, i.e. the wavelength that is associated with a particle. In otherwords, a particle does not have a wavelength, but the equations describing the particle's action have a wavelength.

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= A (-ik) e^{-i(kx - \omega t)} \\ &= A (-ik) e^{-i\left(\frac{2\pi}{\lambda}x - 2\pi f t\right)} \\ &= A (-ik) e^{-i\frac{h}{\hbar}\left(\frac{2\pi}{\lambda}x - 2\pi f t\right)} \\ &= A (-ik) e^{-i\frac{2\pi}{\hbar}\left(\frac{h}{\lambda}x - h f t\right)} \end{aligned}$$

According to the de Broglie Wavelength, $p = \frac{h}{\lambda}$; according to Einstein, $E = hf$; substituting:

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= -i \left(\frac{2\pi}{h} \right) \left(\frac{h}{\lambda} \right) A e^{-i\frac{2\pi}{\hbar}\left(\frac{h}{\lambda}x - h f t\right)} \\ \frac{\partial \Psi}{\partial x} &= -i \left(\frac{2\pi}{h} \right) (p) A e^{-i\frac{2\pi}{\hbar}(px - Et)} \end{aligned}$$

Define (\hbar is pronounced "h-bar"):

$$\hbar = \frac{h}{2\pi} \tag{8.10}$$

This leads us to:

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= \frac{i}{\hbar} (p) \Psi \\ \frac{-\hbar}{i} \frac{\partial \Psi}{\partial x} &= p \Psi \end{aligned}$$

Here, it is clear that p does not equal the classical mechanical $p = mv$. Rather, p is an *operator* such that:

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

It is possible to construct other operators that correspond to observable dynamical variables in a similar way. (Operators are expressed with a “hat” above the variable name).

In classical mechanics, the kinetic energy of a particle can be written as:

$$K = \frac{\vec{p} \cdot \vec{p}}{2m}$$

Since there exists a quantum mechanical operator that corresponds to any observable, we can write the kinetic energy operator for a one-dimensional system as:

$$\hat{K} = \frac{\hat{p} \cdot \hat{p}}{2m} = \frac{\frac{-\hbar}{i} \frac{\partial}{\partial x} \cdot \frac{-\hbar}{i} \frac{\partial}{\partial x}}{2m} = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

8.2.1.1 Extension to Three Dimensions Recall that the “del-squared” can be written as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So, the kinetic energy part of the 3-dimensional Hamiltonian for a wave can be written as:

$$\hat{K} = \frac{-\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

Define $\Psi_{x,y,z}(x, y, z)$:

$$\Psi_{x,y,z}(x, y, z) = \langle \Psi_x(x), \Psi_y(y), \Psi_z(z) \rangle$$

Therefore,

$$\begin{aligned} \hat{K} &= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Psi_{x,y,z}(x, y, z) \\ &= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \langle \Psi_x(x), \Psi_y(y), \Psi_z(z) \rangle \\ &= -\frac{\hbar^2}{2m} \left[\left(\frac{\partial^2 \Psi_x(x)}{\partial x^2} \right) + \left(\frac{\partial^2 \Psi_y(y)}{\partial y^2} \right) + \left(\frac{\partial^2 \Psi_z(z)}{\partial z^2} \right) \right] \end{aligned}$$

■

This means that the 3-Dimensional problem is broken down into 3 1-Dimensional problems! Each dimension can be approached exactly the same way as above.

8.3 Operators

Defn 27 (Operator). In quantum mechanics, for each observable dynamical variable, there exists a corresponding *operator*, which, if applied to the solution of the Schrödinger’s Equation, will provide a value for that dynamical variable. Operator do something to the function immediately after the operator. An operator is nothing other than shorthand for instruction to do something to a function or number. For example, some common operators are:

1. +
2. −
3. ×
4. ÷
5. $\frac{\partial^2}{\partial t^2}$
6. \int
7. \sum
8. ∇^2

Operators have some vector-like properties, but they are not vectors. Further, they do not always always *commute*. Part of the main confusion arising from the public press about quantum mechanics is as statement like “... $A \times B$ is equal to $B \times A$.” This is simply sensationalizing the fact that some operators have commutative properties. One example is:

$$y \times \frac{\partial f(y)}{\partial y} \neq \frac{\partial [x \times f(y)]}{\partial y}$$

For each classical observable, there exists a corresponding quantum mechanical operator. Some examples of simple quantum mechanical operators are:

- Kinetic Energy: $\frac{-\hbar^2}{2m} \nabla^2$
- Momentum: $\frac{-\hbar}{i} \nabla$
- Potential Energy: V is simply multiplication by V
- Angular Momentum: $L_z = \mathbf{r} \times \frac{-\hbar}{i} \nabla^2$

Further, some, but not all, operators are linear. Fortunately, the quantum mechanics kinetic energy operators, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a linear operator. This allows us to *reduce a 3-dimensional problem into 3 1-dimensional problems*.

8.4 Eigenvalues

Defn 28 (Eigenvalue). Let \hat{O} be some quantum mechanical operator. If O and Ψ are *eigenvalues* and eigenfunctions of $\hat{O}\Psi = o\Psi$, the result o will always be a precise number. *Eigenvalues* represent the value of precise, exact values of the dynamical variables corresponding to the eigenfunction of the operator.

The *easy* way to find the value of a dynamical variable o , given the corresponding operator, \hat{O} is as follows.

Solve for Eigenvalue. Let Ψ be a normalized function, i.e.

$$\iiint \Psi\Psi^* dx dy dz = 1$$

Then,

$$\hat{O}\Psi = o\Psi$$

Multiply both sides by Ψ^* , the complex conjugate of Ψ .

$$\Psi^*\hat{O}\Psi = \Psi^*o\Psi$$

Integrate both sides over the space Ω .

$$\int \Psi^*\hat{O}\Psi d\Omega = \int \Psi^*o\Psi d\Omega = o \int \Psi^*\Psi d\Omega = o, \text{ since } \int \Psi^*\Psi d\Omega = 1$$

■

8.5 Expectation Values

Defn 29 (Expectation Values). If o and Ψ are **not** eigenvalues and eigenfunctions of \hat{O} , e.g. $\hat{O}\Psi = z\phi$, where z is some number, and ϕ is some function not equal to Ψ , the result z will be the average of many possible observations. This is usually written as $\langle z \rangle$ and is called an *expectation value*. The expectation value is the expected average value of a number of measurements for a dynamical variable described by the operator \hat{O} .

8.6 Determine Dynamical Variables in Quantum Mechanics

This section has an example where we solve for dynamical variables. Finding Ψ , the wave function that satisfies the Schrödinger's Equation for the one-dimensional infinite square well potential (particle in a box) is presented in the physics textbooks. Now we wish to find values for dynamical observables. Starting with the energy E .

Recall the Properties of Schrödinger's Equation, Section 8.2.1. The solution for the Schrödinger's Equation in a box of length L is

$$\Psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \text{ where } n \text{ is the quantum number of the wave function}$$

First, we operate on Ψ_n with the kinetic energy operator \hat{K} :

$$\hat{K}\Psi_n = E\Psi_n = \varepsilon_n\Psi_n$$

ε_n is the kinetic energy since E ; the total energy is all kinetic with quantized values ε_n . Rewrite $\hat{K}\Psi_n$.

$$\begin{aligned}\hat{K}\Psi_n &= -\frac{\hbar^2}{2m} \frac{d^2\Psi_n}{dx^2} = \varepsilon_n \Psi_n \\ &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right] = \varepsilon_n \Psi_n\end{aligned}$$

Multiply both sides by Ψ_n^* , the complex conjugate of Ψ_n (which in this case $\Psi_n^* = \Psi_n$ since Ψ_n is real) **before the operator** (since operators may not commute nor be distributive):

$$\begin{aligned}\Psi_n^* \hat{K} \Psi_n &= \Psi_n^* \left(-\frac{\hbar^2}{2m} \frac{d^2\Psi_n}{dx^2} \right) = \Psi_n^* \varepsilon_n \Psi_n \\ &= -\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \cdot \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right] \right) \\ &= \varepsilon_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \frac{n^2\pi^2}{L^2} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\end{aligned}$$

Now integrate over both sides:

$$\begin{aligned}\int \Psi_n \left(-\frac{\hbar^2}{2m} \frac{d^2\Psi_n}{dx^2} \right) dx &= \int \Psi_n^* \varepsilon_n \Psi_n dx \\ &= \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \cdot \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right] \right) dx\end{aligned}$$

Take the second derivative of Ψ_n with respect to x :

$$\frac{d^2}{dx^2} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right] = \frac{n^2\pi^2}{L^2} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Substitute into the integral

$$\int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \left(-\frac{\hbar^2}{2m} \right) \left(-\frac{n^2\pi^2}{L^2} \right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx = \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \varepsilon_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

Pull out the constants

$$\left(-\frac{\hbar^2}{2m} \right) \left(-\frac{n^2\pi^2}{L^2} \right) \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx = \varepsilon_n \int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

Recall that

$$\int \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx = 1$$

So, $\left(-\frac{\hbar^2}{2m} \right) \left(-\frac{n^2\pi^2}{L^2} \right) = \varepsilon_n$ is the kinetic energy of a particle with n as its quantum number.

8.7 Heisenberg's Uncertainty Principle

Defn 30. The states that measured values cannot be assigned to the position vector, \vec{r} and the momentum vector, \vec{p} simultaneously with unlimited precision.

$$\begin{aligned}\delta x \cdot \delta p_x &\geq \hbar \\ \delta y \cdot \delta p_y &\geq \hbar \\ \delta z \cdot \delta p_z &\geq \hbar\end{aligned}\tag{8.11}$$

- δx and δp_x represent the uncertainty of the value from the given values of $\vec{r}_x x$ and \vec{p}_x .
- \hbar is defined in Equation (8.10).

A Physical Constants

Constant Name	Variable Letter	Value
Boltzmann Constant	R	8.314J/mol K
Universal Gravitational	G	$6.67408 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$
Planck's Constant	h	$6.62607004 \times 10^{-34} \text{mkg/s} = 4.163 \times 10^{-15} \text{eV s}$
Speed of Light	c	$299792458 \text{m/s} = 2.998 \times 10^8 \text{ m/s}$
Charge of Electron	e	$1.602 \times 10^{-19} \text{C}$
Mass of Electron	m_{e-}	$9.11 \times 10^{-31} \text{kg}$
Mass of Neutron	m_{n^0}	$1.67 \times 10^{-31} \text{kg}$
Mass of Earth	m_{Earth}	$5.972 \times 10^{24} \text{kg}$
Diameter of Earth	d_{Earth}	12742km

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{B.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{B.2})$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j \sin(\alpha) \quad (\text{B.3})$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \quad (\text{B.4})$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \quad (\text{B.5})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{B.6})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{B.7})$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \quad (\text{B.8})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \quad (\text{B.9})$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad (\text{B.10})$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (\text{B.11})$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \quad (\text{B.12})$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \quad (\text{B.13})$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = (\sin(\alpha))^2 = \frac{1 - \cos(2\alpha)}{2} \quad (\text{B.14})$$

$$\cos^2(\alpha) = (\cos(\alpha))^2 = \frac{1 + \cos(2\alpha)}{2} \quad (\text{B.15})$$

B.7 Product-to-Sum Identities

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad (\text{B.16})$$

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad (\text{B.17})$$

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (\text{B.18})$$

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (\text{B.19})$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{B.20})$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{B.21})$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (\text{B.22})$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \quad (\text{B.23})$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \quad (\text{B.24})$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2} e^{j\theta} = r e^{j\theta} \quad (\text{B.25})$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \quad (\text{B.26})$$

B.11 Polar to Rectangular

$$r e^{j\theta} = r \cos(\theta) + j r \sin(\theta) \quad (\text{B.27})$$

C Calculus

C.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcilable results.

Lemma C.0.1 (L'Hopital's Rule). *If the equation*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) holds.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad (\text{C.1})$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{C.2})$$

Defn C.2.2 (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I , and states that if F is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{C.3})$$

Defn C.2.3 (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule). The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot h(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \quad (\text{C.4})$$

C.4 Useful Integrals

$$\int \cos(x) dx = \sin(x) \quad (\text{C.5})$$

$$\int \sin(x) dx = -\cos(x) \quad (\text{C.6})$$

$$\int x \cos(x) dx = \cos(x) + x \sin(x) \quad (\text{C.7})$$

Equation (C.7) simplified with Integration by Parts.

$$\int x \sin(x) dx = \sin(x) - x \cos(x) \quad (\text{C.8})$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2 \cos(x) dx = 2x \cos(x) + (x^2 - 2) \sin(x) \quad (\text{C.9})$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2 \sin(x) dx = 2x \sin(x) - (x^2 - 2) \cos(x) \quad (\text{C.10})$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x} \cos(\beta x) dx = \frac{e^{\alpha x} (\alpha \cos(\beta x) + \beta \sin(\beta x))}{\alpha^2 + \beta^2} + C \quad (\text{C.11})$$

$$\int e^{\alpha x} \sin(\beta x) dx = \frac{e^{\alpha x} (\alpha \sin(\beta x) - \beta \cos(\beta x))}{\alpha^2 + \beta^2} + C \quad (\text{C.12})$$

$$\int e^{\alpha x} dx = \frac{e^{\alpha x}}{\alpha} \quad (\text{C.13})$$

$$\int x e^{\alpha x} dx = e^{\alpha x} \left(\frac{x}{\alpha} - \frac{1}{\alpha^2} \right) \quad (\text{C.14})$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha + \beta x} = \int \frac{1}{\alpha + \beta x} dx = \frac{1}{\beta} \ln(\alpha + \beta x) \quad (\text{C.15})$$

$$\int \frac{dx}{\alpha^2 + \beta^2 x^2} = \int \frac{1}{\alpha^2 + \beta^2 x^2} dx = \frac{1}{\alpha \beta} \arctan \left(\frac{\beta x}{\alpha} \right) \quad (\text{C.16})$$

$$\int \alpha^x dx = \frac{\alpha^x}{\ln(\alpha)} \quad (\text{C.17})$$

$$\frac{d}{dx} \alpha^x = \frac{d\alpha^x}{dx} = \alpha^x \ln(\alpha) \quad (\text{C.18})$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule). *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x, t) dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x, t)}{\partial t}$ continuous in both t and x , then

$$\frac{d}{dt} g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx + f[b(t), t] \frac{db(t)}{dt} - f[a(t), t] \frac{da(t)}{dt} \quad (\text{C.19})$$

D Complex Numbers

Defn D.0.1 (Complex Number). A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted \mathbb{C} .

A complex number is written, in Cartesian form, as shown in Equation (D.1) below.

$$z = a \pm ib \quad (\text{D.1})$$

where

$$i = \sqrt{-1} \quad (\text{D.2})$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j . Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instead.

D.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn D.1.1 (Real Part). The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\text{Re}\{z\} = x \quad (\text{D.3})$$

Remark D.1.1.1 (Alternative Notation). The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re$$

Defn D.1.2 (Imaginary Part). The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\text{Im}\{z\} = y \quad (\text{D.4})$$

Remark D.1.2.1 (Alternative Notation). The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im$$

D.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$\begin{aligned} z &= x_1 + iy_1 \\ w &= x_2 + iy_2 \end{aligned}$$

D.2.1 Addition

The addition operation, still denoted with the $+$ symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w := (x_1 + x_2) + i(y_1 + y_2) \quad (\text{D.5})$$

D.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$\begin{aligned}
 zw &:= (x_1 + iy_1)(x_2 + iy_2) \\
 &= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\
 &= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\
 &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)
 \end{aligned} \tag{D.6}$$

D.3 Complex Conjugates

Defn D.3.1 (Complex Conjugate). The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\bar{z} = a \mp bi \tag{D.7}$$

The Complex Conjugate can also be denoted with an asterisk (*). This is generally done for complex functions, rather than single variables.

$$z^* = \bar{z} \tag{D.8}$$

D.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\bar{z}$
2. $\frac{z}{\bar{z}}$

The first is interesting because of this simplification:

$$\begin{aligned}
 z\bar{z} &= (x + iy)(x - iy) \\
 &= x^2 - xyi + xyi - i^2y^2 \\
 &= x^2 - (-1)y^2 \\
 &= x^2 + y^2
 \end{aligned}$$

Thus,

$$z\bar{z} = x^2 + y^2 \tag{D.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\bar{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z , to cause the denominator to be completely real.

$$z \left(\frac{z}{\bar{z}} \right) = \frac{z^2}{z\bar{z}}$$

Using our solution from Equation (D.9):

$$\begin{aligned}
 &= \frac{(x + iy)^2}{x^2 + y^2} \\
 &= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2}
 \end{aligned}$$

By breaking up the fraction's numerator, we can more easily recognize this to be the Cartesian form of the Complex Number.

$$\begin{aligned} &= \frac{(x^2 - y^2) + 2xyi}{x^2 + y^2} \\ &= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \end{aligned}$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$\frac{z}{\bar{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \quad (\text{D.10})$$

D.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.

First, start by defining some expressions so that we can prove some of these properties:

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned}$$

- (i) The conjugation operation is commutative.
- (ii) The conjugation operation can be distributed over addition and multiplication.

$$\begin{aligned} \overline{z + w} &= \bar{z} + \bar{w} \\ \overline{zw} &= \bar{z}\bar{w} \end{aligned}$$

Property (ii) can be proven by just performing a simplification.

Prove Property (ii). Let z and w be complex numbers ($z, w \in \mathbb{C}$) where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \bar{z} + \bar{w}$.

We start by simplifying the left-hand side of the equation $\overline{(z + w)}$.

$$\begin{aligned} \overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

Now, we simplify the other side $(\bar{z} + \bar{w})$.

$$\begin{aligned} \bar{z} + \bar{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

We can see that both sides are equivalent, thus the addition portion of Property (ii) is correct.

Remark. The proof of the multiplication portion of Property (ii) is left as an exercise to the reader. However, that proof is quite similar to this proof of addition. ■

D.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn D.4.1 (Argand Plane). An *Argand Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in \mathbb{C}$. This is taken from Descartes's definition of a completely real plane.

The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the “ x ”-component and the Imaginary Part is the “ y ”-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, referred to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y . Those are the values that inform us **where** p is located on the Argand Plane.

D.4.1 Modulus of a Complex Number

Defn D.4.2 (Modulus). The *modulus* of a Complex Number is the distance from the origin to the complex point p . This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\bar{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{D.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$.

We can prove Property (i) using an algebraic identity.

Prove Property (i). Let z and w be complex numbers ($z, w \in \mathbb{C}$). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2 |w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = (z\bar{z})(w\bar{w})$$

Using the definition of the Modulus of a Complex Number in Equation (D.11), we can expand the modulus.

$$= (z\bar{z})(w\bar{w})$$

Using Property (ii) for multiplication allows us to do the next step.

$$= (z\bar{z})(w\bar{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\bar{z})(w\bar{w}) \\ &= |z|^2 |w|^2 \end{aligned}$$

Note how we never needed to define z or w , so this is as general a result as possible. ■

D.4.1.1 Algebraic Effects of the Modulus’ Property (i) For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= (x_1^2 + x_2^2)(y_1^2 + y_2^2) \\ &= |z|^2 |w|^2 \end{aligned}$$

However, the Law of Moduli (Property (i)) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property (i)) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

D.4.1.2 Conceptual Effects of the Modulus’ Property (i) We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1)^2 \neq X^2 + Y^2 + Z^2$$

D.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (D.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{D.12}$$

Example D.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$|5iz + 10| = 7$$

$$|5i(z + \frac{10}{5i})| = 7$$

$$|5i(z + \frac{2}{i})| = 7$$

$$|5i(z + \frac{2-i}{i-i})| = 7$$

$$|5i(z - 2i)| = 7$$

Now using the Law of Moduli (Property (i)) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$|5i||z - 2i| = 7$$

$$5|z - 2i| = 7$$

$$|z - 2i| = \frac{7}{5}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

D.5.1 Annulus

Defn D.5.1 (Annulus). An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (D.13).

$$r_1 \leq |z - a| \leq r_2 \quad (\text{D.13})$$

In Equation (D.13), each of the \leq symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are \leq , then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other \leq , then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

D.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (D.14).

$$z = r(\cos(\theta) + i \sin(\theta)) \quad (\text{D.14})$$

Remark. Note that in the definition of polar form (Equation (D.14)), there is no allowance for the radius, r , to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$r = |z|$$

$$\theta = \arg(z)$$

Example D.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned}
 r &= |z| \\
 &= \sqrt{\operatorname{Re}\{z\}^2 + \operatorname{Im}\{z\}^2} \\
 &= \sqrt{(-\sqrt{3})^2 + 1^2} \\
 &= \sqrt{3 + 1} \\
 &= \sqrt{4} \\
 &= 2
 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.

Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned}
 \cos(\theta) &= \frac{-\sqrt{3}}{2} \\
 \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\
 &= \frac{5\pi}{6}
 \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \dots$

We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned}
 r &= 2 \\
 \theta &= \frac{5\pi}{6} \\
 \arg(z) &= \frac{5\pi}{6} + 2\pi k
 \end{aligned}$$

D.6.1 Converting Between Cartesian and Polar Forms

Using Equation (D.14) and Equation (D.1), it is easy to see the relation between r , θ , x , and y .

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned}
 z &= r(\cos(\theta) + i \sin(\theta)) \\
 &= r \cos(\theta) + ir \sin(\theta)
 \end{aligned}$$

Thus,

$$\begin{aligned}
 x &= r \cos(\theta) \\
 y &= r \sin(\theta)
 \end{aligned} \tag{D.15}$$

D.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

D.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\left(r_1(\cos(\theta) + i \sin(\theta))\right)\left(r_2(\cos(\phi) + i \sin(\phi))\right) = r_1 r_2 (\cos(\theta + \phi) + i \sin(\theta + \phi)) \quad (\text{D.16})$$

D.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1(\cos(\theta) + i \sin(\theta))}{r_2(\cos(\phi) + i \sin(\phi))} = \frac{r_1}{r_2} (\cos(\theta - \phi) + i \sin(\theta - \phi)) \quad (\text{D.17})$$

D.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph D.6.2.1 applies. That this generalization is true was proven by de Moivre, and is called de Moivre's Law.

Defn D.6.1 (de Moivre's Law). Given a complex number z , $z \in \mathbb{C}$ and a rational number n , $n \in \mathbb{Q}$, the exponentiation of z^n is defined as Equation (D.18).

$$z^n = r^n (\cos(n\theta) + i \sin(n\theta)) \quad (\text{D.18})$$

D.7 Roots of a Complex Number

de Moivre's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\left(\frac{\arg z}{n}\right) + i \sin\left(\frac{\arg z}{n}\right) \right) \quad (\text{D.19})$$

Remark. As the entire $\arg z$ term is being divided by n , the $2\pi k$ is **ALSO** divided by n .

Roots of a Complex Number satisfy Equation (D.20). To demonstrate that equation, $z = r(\cos(\theta) + i \sin(\theta))$ and $w = \rho(\cos(\phi) + i \sin(\phi))$.

$$w^n = z \quad (\text{D.20})$$

A w that satisfies Equation (D.20) is an n th root of z .

Example D.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example D.2, we know that the polar form of z is

$$z = 2 \left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i \sin\left(\frac{5\pi}{6} + 2\pi k\right) \right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (D.19), we can find the general form of the roots.

$$\begin{aligned} z &= 2 \left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i \sin\left(\frac{5\pi}{6} + 2\pi k\right) \right) \\ z^{\frac{1}{3}} &= \sqrt[3]{2} \left(\cos\left(\frac{1}{3} \left(\frac{5\pi}{6} + 2\pi k \right)\right) + i \sin\left(\frac{1}{3} \left(\frac{5\pi}{6} + 2\pi k \right)\right) \right) \\ &= \sqrt[3]{2} \left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i \sin\left(\frac{\pi + 12\pi k}{18}\right) \right) \end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \dots$

$k = 0$

$$\sqrt[3]{2} \left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i \sin\left(\frac{\pi + 12\pi(0)}{18}\right) \right) = \sqrt[3]{2} \left(\cos\left(\frac{\pi}{18}\right) + i \sin\left(\frac{\pi}{18}\right) \right)$$

$k = 1$

$$\sqrt[3]{2} \left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i \sin\left(\frac{\pi + 12\pi(1)}{18}\right) \right) = \sqrt[3]{2} \left(\cos\left(\frac{13\pi}{18}\right) + i \sin\left(\frac{13\pi}{18}\right) \right)$$

$$k = 2$$

$$\sqrt[3]{2} \left(\cos \left(\frac{\pi + 12\pi(2)}{18} \right) + i \sin \left(\frac{\pi + 12\pi(2)}{18} \right) \right) = \sqrt[3]{2} \left(\cos \left(\frac{25\pi}{18} \right) + i \sin \left(\frac{25\pi}{18} \right) \right)$$

$$k = 3$$

$$\begin{aligned} \sqrt[3]{2} \left(\cos \left(\frac{\pi + 12\pi(3)}{18} \right) + i \sin \left(\frac{\pi + 12\pi(3)}{18} \right) \right) &= \sqrt[3]{2} \left(\cos \left(\frac{\pi}{18} + \frac{36\pi}{18} \right) + i \sin \left(\frac{\pi}{18} + \frac{36\pi}{18} \right) \right) \\ &= \sqrt[3]{2} \left(\cos \left(\frac{\pi}{18} + 2\pi \right) + i \sin \left(\frac{\pi}{18} + 2\pi \right) \right) \\ &= \sqrt[3]{2} \left(\cos \left(\frac{\pi}{18} \right) + i \sin \left(\frac{\pi}{18} \right) \right) \end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned} z_1^{\frac{1}{3}} &= \sqrt[3]{2} \left(\cos \left(\frac{\pi}{18} \right) + i \sin \left(\frac{\pi}{18} \right) \right) \\ z_2^{\frac{1}{3}} &= \sqrt[3]{2} \left(\cos \left(\frac{13\pi}{18} \right) + i \sin \left(\frac{13\pi}{18} \right) \right) \\ z_3^{\frac{1}{3}} &= \sqrt[3]{2} \left(\cos \left(\frac{25\pi}{18} \right) + i \sin \left(\frac{25\pi}{18} \right) \right) \end{aligned}$$

D.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn D.8.1 (Argument). The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example D.4: Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can “cheat” and use \tan^{-1} (so long as you correct for the proper quadrant). I will “cheat”, as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) &= \tan(\theta) = \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z) = \frac{5\pi}{4} + 2\pi k$$

Thus, the argument of $z = -1 - i$ is $\arg(z) = \frac{5\pi}{4} + 2\pi k$.

Defn D.8.2 (Principal Argument). The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi < \text{Arg}(z) \leq \pi$.

Example D.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can “cheat” and use \tan^{-1} (so long as you correct for the proper quadrant). I will “cheat”, as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) &= \tan(\theta) = \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\text{Arg}(z) = \frac{5\pi}{4}$.

D.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (D.21).

$$e^z = e^{x+iy} = e^x (\cos(y) + i \sin(y)) \quad (\text{D.21})$$

If instead of e as the base, we have some value a , then we have Equation (D.22).

$$\begin{aligned} a^z &= e^{z \ln(a)} \\ &= e^{\text{Re}\{z \ln(a)\}} \left(\cos(\text{Im}\{z \ln(a)\}) + i \sin(\text{Im}\{z \ln(a)\}) \right) \end{aligned} \quad (\text{D.22})$$

In the case of Equation (D.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example D.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$\begin{aligned} x &= -1 \\ y &= \sqrt{3} \end{aligned}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1} (\cos(\sqrt{3}) + i \sin(\sqrt{3}))$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\begin{aligned} |e^{-1+i\sqrt{3}}| &= |e^{-1} (\cos(\sqrt{3}) + i \sin(\sqrt{3}))| \\ &= e^{-1} \\ \arg(e^{-1+i\sqrt{3}}) &= \arg(e^{-1} (\cos(\sqrt{3}) + i \sin(\sqrt{3}))) \\ &= \sqrt{3} + 2\pi k \\ \text{Arg}(e^{-1+i\sqrt{3}}) &= \text{Arg}(e^{-1} (\cos(\sqrt{3}) + i \sin(\sqrt{3}))) \\ &= \sqrt{3} \end{aligned}$$

Example D.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i} .

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^0 (\cos(-1) + i \sin(-1)) \\ &= 1(\cos(-1) + i \sin(-1)) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1(\cos(-1) + i \sin(-1))} \\ &= e^{-1(\cos(1) - i \sin(1))} \\ &= e^{-\cos(1) + i \sin(1)} \end{aligned}$$

If we refer back to Equation (D.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)} (\cos(\sin(1)) + i \sin(\sin(1))) \end{aligned}$$

Now that we have “simplified” this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} |e^{-e^{-i}}| &= |e^{-\cos(1)} (\cos(\sin(1)) + i \sin(\sin(1)))| \\ &= e^{-\cos(1)} \\ \arg(e^{-e^{-i}}) &= \arg(e^{-\cos(1)} (\cos(\sin(1)) + i \sin(\sin(1)))) \\ &= \sin(1) + 2\pi k \\ \text{Arg}(e^{-e^{-i}}) &= \text{Arg}(e^{-\cos(1)} (\cos(\sin(1)) + i \sin(\sin(1)))) \\ &= \sin(1) \end{aligned}$$

Example D.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^i$?

Use Equation (D.22) to simplify this to a base of e , where we can use the usual Equation (D.21) to solve this.

$$\begin{aligned} a^z &= e^{z \ln(a)} \\ 1^i &= e^{i \ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e |1| + i \arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^2} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \dots$

D.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\bar{z}} \quad (\text{D.23})$$

D.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm \ln needs to be redefined to its defining form \log_e .

With that denotational change, we can now use \ln for the Complex Logarithm.

Defn D.10.1 (Complex Logarithm). The *complex logarithm* is defined in Equation (D.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e |w| + i \arg(w) \end{aligned} \quad (\text{D.24})$$

Remark D.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example D.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (D.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e |z| + i \arg(z) \\ &= \log_e |-1| + i \arg(-1) \\ &= \log_e (1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \dots$

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example D.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z , before finding $\ln(z)$. We can make use of Equation (D.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e |w| + i \arg(w) \\ \ln(1) &= \log_e |1| + i \arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z , we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e |2\pi k i| + i \arg(2\pi k i) \\ &= \log_e (2\pi |k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ -\frac{\pi}{2} + 2\pi m & k < 0 \end{cases} \right) \end{aligned}$$

The $|k|$ is the **absolute value** of k , because k is an integer.

Thus, our solution of $\ln(\ln(1)) = \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ -\frac{\pi}{2} + 2\pi m & k < 0 \end{cases}\right)$.

D.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\bar{z}) \quad (\text{D.25})$$

D.11 Complex Trigonometry

For the equations below, $z \in \text{mathbbC}$. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad (\text{D.26})$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad (\text{D.27})$$

Example D.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

We start by using the definition of complex cosine Equation (D.26).

$$\begin{aligned} \cos(z) &= 5 \\ \frac{e^{iz} + e^{-iz}}{2} &= 5 \\ e^{iz} + e^{-iz} &= 10 \\ e^{iz} (e^{iz} + e^{-iz}) &= e^{iz}(10) \\ e^{iz^2} + 1 &= 10e^{iz} \\ e^{iz^2} - 10e^{iz} + 1 &= 0 \end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned} e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 4}}{2} \\ &= \frac{10 \pm \sqrt{96}}{2} \\ &= \frac{10 \pm 4\sqrt{6}}{2} \\ &= 5 \pm 2\sqrt{6} \end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned} iz &= \ln(5 \pm 2\sqrt{6}) \\ &= \log_e|5 \pm 2\sqrt{6}| + i \arg(5 \pm 2\sqrt{6}) \\ &= \log_e|5 \pm 2\sqrt{6}| + i(0 + 2\pi k) \\ &= \log_e|5 \pm 2\sqrt{6}| + 2\pi ki \\ z &= \frac{1}{i} \left(\log_e|5 \pm 2\sqrt{6}| + 2\pi ki \right) \\ &= \frac{-i}{-i} \frac{1}{i} \left(\log_e|5 \pm 2\sqrt{6}| \right) + 2\pi k \\ &= 2\pi k - i \log_e|5 \pm 2\sqrt{6}| \end{aligned}$$

Thus, $z = 2\pi k - i \log_e|5 \pm 2\sqrt{6}|$.

D.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x + iy) = (\cos(x) \cosh(y)) - i(\sin(x) \sinh(y)) \quad (D.28)$$

$$\sin(x + iy) = (\sin(x) \cosh(y)) + i(\cos(x) \sinh(y)) \quad (D.29)$$

Example D.12: Simplify Trigonometric Exponential. Lecture 3

Simplify $z = e^{\cos(2+3i)}$, and find z 's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (D.28).

$$\begin{aligned} \cos(x + iy) &= (\cos(x) \cosh(y)) - i(\sin(x) \sinh(y)) \\ \cos(2 + 3i) &= (\cos(2) \cosh(3)) - i(\sin(2) \sinh(3)) \end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (D.21), we can solve this.

$$\begin{aligned} e^z &= e^{x+iy} = e^x (\cos(y) + i \sin(y)) \\ x &= \cos(2) \cosh(3) \\ y &= -\sin(2) \sinh(3) \\ e^{\cos(2) \cosh(3) - i \sin(2) \sinh(3)} &= e^{\cos(2) \cosh(3)} \left(\cos(-\sin(2) \sinh(3)) + i \sin(-\sin(2) \sinh(3)) \right) \end{aligned}$$

Now that we have simplified z , we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned} |z| &= |e^{\cos(2) \cosh(3)} (\cos(-\sin(2) \sinh(3)) + i \sin(-\sin(2) \sinh(3)))| \\ &= e^{\cos(2) \cosh(3)} \\ \arg(z) &= \arg(e^{\cos(2) \cosh(3)} (\cos(-\sin(2) \sinh(3)) + i \sin(-\sin(2) \sinh(3)))) \\ &= -\sin(2) \sinh(3) + 2\pi k \\ \text{Arg}(z) &= \text{Arg}(e^{\cos(2) \cosh(3)} (\cos(-\sin(2) \sinh(3)) + i \sin(-\sin(2) \sinh(3)))) \\ &= -\sin(2) \sinh(3) \end{aligned}$$

D.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned} \overline{\cos(x)} &= \cos(x) \\ &= \frac{1}{2} (e^{ix} + e^{-ix}) \end{aligned} \quad (D.30)$$

$$\begin{aligned} \overline{\sin(x)} &= \sin(x) \\ &= \frac{1}{2i} (e^{ix} - e^{-ix}) \end{aligned} \quad (D.31)$$