Math 333: Matrix Algebra and Complex Variables — Reference Material
Illinois Institute of Technology

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Last Edited: October 8, 2020

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1 Complex Numbers

Defn 1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted $\mathbb{C}$.

A complex number is written, in Cartesian form, as shown in Equation (1.1) below.

$$z = a \pm ib \tag{1.1}$$

where

$$i = \sqrt{-1} \tag{1.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

1.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn 2 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\mathrm{Re}\{z\} = x \tag{1.3}$$

Remark 2.1 (Alternative Notation). The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re$$

Defn 3 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\mathrm{Im}\{z\} = y \tag{1.4}$$

Remark 3.1 (Alternative Notation). The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im$$

1.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$\begin{aligned} z &= x_1 + iy_1 \\ w &= x_2 + iy_2 \end{aligned}$$

1.2.1 Addition

The addition operation, still denoted with the + symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w \coloneqq (x_1 + x_2) + i(y_1 + y_2) \tag{1.5}$$

1.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$\begin{aligned} zw &\coloneqq (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\ &= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned} \tag{1.6}$$

1.3 Complex Conjugates

Defn 4 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{1.7}$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{1.8}$$

1.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$
2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - xyi + xyi - i^2y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2 \end{aligned}$$

Thus,

$$z\overline{z} = x^2 + y^2 \tag{1.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}$$

Using our solution from Equation (1.9):

$$\begin{aligned} &= \frac{(x + iy)^2}{x^2 + y^2} \\ &= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2} \end{aligned}$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$\begin{aligned} &= \frac{\left(x^2 - y^2\right) + 2xyi}{x^2 + y^2} \\ &= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \end{aligned}$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{1.10}$$

1.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.
First, start by defining some expressions so that we can prove some of these properties:

$$\begin{aligned} z &= x + iy \\ \overline{z} &= x - iy \end{aligned}$$

(i) The conjugation operation is commutative.
(ii) The conjugation operation can be distributed over addition and multiplication.

$$\begin{aligned} \overline{z + w} &= \overline{z} + \overline{w} \\ \overline{zw} &= \overline{zw} \end{aligned}$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers $(z, w \in \mathbb{C})$ where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.
We start by simplifying the left-hand side of the equation $(\overline{z + w})$.

$$\begin{aligned} \overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

Now, we simplify the other side $(\overline{z} + \overline{w})$.

$$\begin{aligned} \overline{z} + \overline{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

■

1.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn 5 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in \mathbb{C}$. This is taken from Descarte's definition of a completely real plane.
The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

1.4.1 Modulus of a Complex Number

Defn 6 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\overline{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{1.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in \mathbb{C}$). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (1.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\overline{zw})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \end{aligned}$$

Note how we never needed to define z or w, so this is as general a result as possible. ■

1.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right) \\ &= |z|^2|w|^2 \end{aligned}$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

1.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

1.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (1.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{1.12}$$

Example 1.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$\begin{aligned}
|5iz + 10| &= 7 \\
|5i(z + \frac{10}{5i})| &= 7 \\
|5i(z + \frac{2}{i})| &= 7 \\
|5i(z + \frac{2}{i}\frac{-i}{-i})| &= 7 \\
|5i(z - 2i)| &= 7
\end{aligned}$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$\begin{aligned}
|5i||z - 2i| &= 7 \\
5|z - 2i| &= 7 \\
|z - 2i| &= \frac{7}{5}
\end{aligned}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

1.5.1 Annulus

Defn 7 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (1.13).

$$r_1 \leq |z - a| \leq r_2 \tag{1.13}$$

In Equation (1.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

1.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (1.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{1.14}$$

Remark. Note that in the definition of polar form (Equation (1.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$\begin{aligned}
r &= |z| \\
\theta &= \arg(z)
\end{aligned}$$

Example 1.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned} \cos(\theta) &= \frac{-\sqrt{3}}{2} \\ \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned} r &= 2 \\ \theta &= \frac{5\pi}{6} \\ \arg(z) &= \frac{5\pi}{6} + 2\pi k \end{aligned}$$

1.6.1 Converting Between Cartesian and Polar Forms

Using Equation (1.14) and Equation (1.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{1.15}$$

1.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

1.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \tag{1.16}$$

1.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big) \tag{1.17}$$

1.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph 1.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn 8 (de Moiver's Law)**.** Given a complex number z, $z \in \mathbb{C}$ and a rational number n, $n \in \mathbb{Q}$, the exponentiation of z^n is defined as Equation (1.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{1.18}$$

1.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{1.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (1.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{1.20}$$

A w that satisfies Equation (1.20) is an nth root of z.

Example 1.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example 1.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (1.19), we can find the general form of the roots.

$$\begin{aligned}
z &= 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)\\
z^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right)\\
&= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right)
\end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \ldots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

k = 2

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(2)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(2)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)$$

k = 3

$$\begin{aligned}\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(3)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(3)}{18}\right)\right) &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + \frac{36\pi}{18}\right) + i\sin\left(\frac{\pi}{18} + \frac{36\pi}{18}\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + 2\pi\right) + i\sin\left(\frac{\pi}{18} + 2\pi\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)\end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned} z_1^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right) \\ z_2^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right) \\ z_3^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)\end{aligned}$$

1.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn 9 (Argument)**.** The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example 1.4: Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) &= \tan(\theta) = \frac{-i}{-1} \\ &= \frac{\pi}{4}\end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z) = \frac{5\pi}{4} + 2\pi k$$

Thus, the argument of $z = -1 - i$ is $\arg(z) = \frac{5\pi}{4} + 2\pi k$.

Defn 10 (Principal Argument)**.** The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi < \text{Arg}(z) \leq \pi$.

Example 1.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) = \tan(\theta) &= \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\text{Arg}(z) = \frac{5\pi}{4}$.

1.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (1.21).

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big) \tag{1.21}$$

If instead of e as the base, we have some value a, then we have Equation (1.22).

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ &= e^{\text{Re}\{z\ln(a)\}}\Big(\cos\big(\text{Im}\{z\ln(a)\}\big) + i\sin\big(\text{Im}\{z\ln(a)\}\big)\Big) \end{aligned} \tag{1.22}$$

In the case of Equation (1.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example 1.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$\begin{aligned} x &= -1 \\ y &= \sqrt{3} \end{aligned}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\begin{aligned} \left|e^{-1+i\sqrt{3}}\right| &= \left|e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right| \\ &= e^{-1} \\ \arg\left(e^{-1+i\sqrt{3}}\right) &= \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} + 2\pi k \\ \text{Arg}\left(e^{-1+i\sqrt{3}}\right) &= \text{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} \end{aligned}$$

Example 1.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^{0}\big(\cos(-1) + i\sin(-1)\big) \\ &= 1\big(\cos(-1) + i\sin(-1)\big) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1\big(\cos(-1)+i\sin(-1)\big)} \\ &= e^{-1\big(\cos(1)-i\sin(1)\big)} \\ &= e^{-\cos(1)+i\sin(1)} \end{aligned}$$

If we refer back to Equation (1.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big) \end{aligned}$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} \left|e^{-e^{-i}}\right| &= \left|e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right| \\ &= e^{-\cos(1)} \\ \arg\left(e^{-e^{-i}}\right) &= \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) + 2\pi k \\ \operatorname{Arg}\left(e^{-e^{-i}}\right) &= \operatorname{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) \end{aligned}$$

Example 1.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^{i}$?

Use Equation (1.22) to simplify this to a base of e, where we can use the usual Equation (1.21) to solve this.

$$\begin{aligned} a^{z} &= e^{z\ln(a)} \\ 1^{i} &= e^{i\ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^{2}} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

1.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{1.23}$$

1.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn 11 (Complex Logarithm)**.** The *complex logarithm* is defined in Equation (1.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e |w| + i \arg(w) \end{aligned} \tag{1.24}$$

Remark 11.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example 1.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (1.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e |z| + i \arg(z) \\ &= \log_e |-1| + i \arg(-1) \\ &= \log_e(1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \ldots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example 1.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (1.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e |w| + i \arg(w) \\ \ln(1) &= \log_e |1| + i \arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e |2\pi k i| + i \arg(2\pi k i) \\ &= \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases} \right) \end{aligned}$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases} \right)$.

1.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\overline{z}) \tag{1.25}$$

1.11 Complex Trigonometry

For the equations below, $z \in \mathbb{C}$. These equations are based on Euler's relationship, Appendix A.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{1.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{1.27}$$

Example 1.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

We start by using the defintion of complex cosine Equation (1.26).

$$\begin{aligned} \cos(z) &= 5 \\ \frac{e^{iz} + e^{-iz}}{2} &= 5 \\ e^{iz} + e^{-iz} &= 10 \\ e^{iz}\left(e^{iz} + e^{-iz}\right) &= e^{iz}(10) \\ e^{iz^2} + 1 &= 10e^{iz} \\ e^{iz^2} - 10e^{iz} + 1 = 0 \end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned} e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 4}}{2} \\ &= \frac{10 \pm \sqrt{96}}{2} \\ &= \frac{10 \pm 4\sqrt{6}}{2} \\ &= 5 \pm 2\sqrt{6} \end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned} iz &= \ln(5 \pm 2\sqrt{6}) \\ &= \log_e\left|5 \pm 2\sqrt{6}\right| + i\arg(5 \pm 2\sqrt{6}) \\ &= \log_e\left|5 \pm 2\sqrt{6}\right| + i(0 + 2\pi k) \\ &= \log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi k i \\ z &= \frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right| + 2\pi k i\right) \\ &= \frac{-i}{-i}\frac{1}{i}\left(\log_e\left|5 \pm 2\sqrt{6}\right|\right) + 2\pi k \\ &= 2\pi k - i\log_e\left|5 \pm 2\sqrt{6}\right| \end{aligned}$$

Thus, $z = 2\pi k - i\log_e\left|5 \pm 2\sqrt{6}\right|$.

1.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x \pm iy) = \big(\cos(x)\cosh(y)\big) \mp i\big(\sin(x)\sinh(y)\big) \tag{1.28}$$

$$\sin(x \pm iy) = \big(\sin(x)\cosh(y)\big) \pm i\big(\cos(x)\sinh(y)\big) \tag{1.29}$$

Example 1.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (1.28).

$$\begin{aligned}\cos(x + iy) &= \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big)\\ \cos(2 + 3i) &= \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)\end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (1.21), we can solve this.

$$\begin{aligned}e^z = e^{x+iy} &= e^x\big(\cos(y) + i\sin(y)\big)\\ x &= \cos(2)\cosh(3)\\ y &= -\sin(2)\sinh(3)\\ e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} &= e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)\end{aligned}$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned}|z| &= \Big|e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)\Big|\\ &= e^{\cos(2)\cosh(3)}\\ \arg(z) &= \arg(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3) + 2\pi k\\ \operatorname{Arg}(z) &= \operatorname{Arg}(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3)\end{aligned}$$

1.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix A.2, we could calculate their complex conjugate.

$$\begin{aligned}\overline{\cos(x)} &= \cos(x)\\ &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\end{aligned} \tag{1.30}$$

$$\begin{aligned}\overline{\sin(x)} &= \sin(x)\\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\end{aligned} \tag{1.31}$$

2 Complex Functions

Defn 12 (Complex Function)**.** A *complex function* is a like their purely real-valued brothers, but instead of mapping real number inputs to real number outputs ($\mathbb{R} \mapsto \mathbb{R}$), they map Complex Number inputs to complex outputs ($\mathbb{C} \mapsto \mathbb{C}$).

Complex functions, like their real-valued counterparts behave in much the same way.

$$f(z) = w \tag{2.1}$$

f: The function or Mapping that corresponds the input to the output.
z: The input to the complex function/Mapping.
w: The output of the complex function/Mapping.

Sometimes a Complex Function, like all other functions, is referred to as a Mapping.

Defn 13 (Mapping)**.** A *mapping* is synonym for a function in mathematics. The term comes from set theory, where the input set is mapped to an output set by some operations. The conventional way to denote a mapping is with the $\mapsto$ symbol.

An example of a mapping is shown in Equation (2.2)

$$z \mapsto z^2 \tag{2.2}$$

A complex function can only accept and will only return values in **Cartesian** or **polar** form. Because the output of a complex function is also a complex value, Equation (2.3) makes sense.

$$f(z) = U(x, y) + iV(x, y) \tag{2.3}$$

$U(x, y)$ and $V(x, y)$ can be as general as we want in x and y. This means both could be constants, both could be polynomials, one could be transcendental, and anything in between.

The functions $U(x, y)$ and $V(x, y)$ are functions that yield real-values u, v. This means that u, v can also be graphed on an Argand Plane. By our definition of $U(x, y)$ and $V(x, y)$, $U(x, y), V(x, y)$ are parametric functions.

Example 2.1: Find Output Functions. Lecture 4

Given the Mapping $z \mapsto z^2$, where $z = x + iy$, find the output functions for each term $U(x, y)$ and $V(x, y)$?

I will choose to represent the mapping $z \mapsto z^2$ with the complex function $f(z) = z^2$.

$$\begin{aligned} z &\mapsto z^2 \\ f(z) &= z^2 \end{aligned}$$

Apply the definition of z.

$$\begin{aligned} &= (x + iy)^2 \\ &= x^2 + 2xyi + i^2y^2 \\ &= x^2 + 2xyi + (-1)y^2 \\ &= \left(x^2 - y^2\right) + 2xyi \end{aligned}$$

By out definition of $U(x, y)$ and $V(x, y)$ in Equation (2.3), we can finish solving this.

$$\begin{aligned} f(z) &= U(x, y) + iV(x, y) \\ f(z) &= \left(x^2 - y^2\right) + 2xyi \\ U(x, y) &= x^2 - y^2 \\ V(x, y) &= 2xy \end{aligned}$$

Thus, our output functions are $U(x, y) = x^2 - y^2$ and $V(x, y) = 2xy$.

2.1 Graphing Complex Functions

If we take a closer look at the complex function $f(z)$, we notice something that makes handling complex function difficult. $f(z)$ is really a function of x and y, because z depends on those 2 real-valued parameters. Thus, all our inputs lie on a 2-dimensional plane.

Now if we look at the output w, we also notice it is complex-valued, meaning it also depends on some u and v, which are equal to the value of their functions $U(x, y)$ and $V(x, y)$. This means that the output of the function $f(z)$ **also** lies on a 2-dimensional plane. Meaning, the function is 4-dimensional. The intersection of 2 planes in our 3-dimensional world never yields a point in the hyperplane, and thus, we cannot graph it.

Instead, we choose to graph the inputs and outputs separately, effectively showing the mapping and the way the Pre-Image is transformed into the Image instead.

Defn 14 (Pre-Image)**.** The *pre-image* consists of all points on the input plane. In this case, the input plane is the z-plane, being constructed out of the orthogonal intersection of the Re and Im axes.

Defn 15 (Image)**.** The *image* consists of all points on the output plane. In this case, the input plane is the w-plane, being constructed out of the orthogonal intersection of the $U(x, y)$ axis on the horizontal and the $V(x, y)$ axis on the vertical.

By graphing and viewing the input and the output simultaneously, we can see how the Mapping $f(z)$ distorts the input Pre-Image into the output Image.

If asked to graph the complex function/Mapping, you must find the expression for the output functions $U(x, y)$ and $V(x, y)$, and then graph the inputs x, y against the outputs $U(x, y), V(x, y)$.

Because the output equations are in terms of, possibly 2, other variables, they are parametric equations. If you are also asked to find the Cartesian form of the output equation, you must simplify the other terms away.

Example 2.2: Plot Simple Complex Function. Lecture 4

Find the Image of the line $y = 4$ on the map $f(z) = z^2$? Provide the Cartesian equation of the Image and the orientation of the image points as the Pre-Image points move $y = 4$ from $-\infty$ towards ∞?

We found the parametric output functions $U(x, y)$ and $V(x, y)$ in Example 2.1, so we will use those here.
How to perform this Pre-Image to Image plotting:

1. Start by plotting $y = 4$ in the xy-plane (z-plane).
2. Then, start plugging values of $x, y = 4$ into $U(x, y)$ and $V(x, y)$.
3. Start with $x < 0$ and move towards $x > 0$, as that will follow the orientation of the line provided in the question.
4. Graph the Image's results.
5. Indicate the orientation of the image on its graph.

To find the Cartesian form of the parametric output equations, we can start by eliminating y from the parameters, as y was specified to be a constant $y = 4$.

$$\begin{aligned}
U(x, y) &= x^2 - y^2 \\
V(x, y) &= 2xy \\
U(x, y = 4) &= x^2 - 4^2 \\
V(x, y = 4) &= 2x(4) \\
U(x) &= x^2 - 16 \\
V(x) &= 8x
\end{aligned}$$

Now that y has been eliminated, I will simplify $V(x)$ such that x is in terms of V.

$$\begin{aligned}
V(x) &= 8x \\
x &= \frac{V}{8}
\end{aligned}$$

Now that a value for x has been found, we can plug that value back into $U(x)$, and simplify.

$$\begin{aligned}
x &= \frac{V}{8} \\
U\left(x = \frac{V}{8}\right) &= \left(\frac{V}{8}\right)^2 - 16 \\
U + 16 &= \frac{V^2}{64} \\
V^2 &= 64(U + 16) \\
V &= \sqrt{64(U + 16)}
\end{aligned}$$

Thus, the Cartesian equation of the Image is a parabola whose defining equation is $V = \sqrt{64(U + 16)}$.

Remark. I could have chosen to solve for U in terms of V, but that would have required the addition of $\pm$ in many places due to the early application of the square root.

Example 2.3: Plot Complex Trigonometric Function. Lecture 4

Find the Cartesian equation of the Image of the line $x = 4$ under the map $f(z) = \sin(z)$? Provide the Cartesian equation of the Image?

Using the Complex Angle Sum and Difference Identities for sin (Equation (1.29)), we can put sin into Cartesian form and simplify.

$$\begin{aligned} \sin(x+iy) &= \big(\sin(x)\cosh(y)\big) + i\big(\cos(x)\sinh(y)\big) \\ f(z = x+iy \vert x = 4) &= \sin(z) \\ &= \big(\sin(4)\cosh(y)\big) + i\big(\cos(4)\sinh(y)\big) \end{aligned}$$

Use the definition of the output functions.

$$\begin{aligned} U(y) &= \sin(4)\cosh(y) \\ V(y) &= \cos(4)\sinh(y) \end{aligned}$$

How to perform this Pre-Image to Image plotting:

1. Start by plotting $x = 4$ in the xy-plane (z-plane).
2. Then, start plugging values of $x = 4, y$ into $U(x, y)$ and $V(x, y)$.
3. Graph the Image's results.

If you notice, the Image that is created is a hyperbola. However, only one of the 2 arcs that is created is the correct one, as we lose information when we move between parametric and Cartesian forms. We can figure this out by looking at $U(x = 4, y) = \sin(4)\cosh(y)$.

1. We know $\sin(4) < 0$, as $4 > \pi$.
2. In addition, $\cosh \not< 0$, by definition.
3. Thus, if $(\sin(4) < 0)(\cosh \not< 0) = U(x = 4, y) < 0$.
4. Therefore, the only part of the hyperbola that should be kept is the one where $U < 0$.

To get the Cartesian equation for this shape, we need to have a relation between cosh and sinh; fortunately, we have one. The Pythagorean theorem for hyperbolic trigonometry would work perfectly, so we can substitue for cosh and sinh.

$$\begin{aligned} \cosh^2(\theta) - \sinh^2(\theta) &= 1 \\ \frac{U(y)}{\sin(4)} - \frac{V(y)}{\cos(4)} &= 1 \end{aligned}$$

2.2 Limits

Like in earlier maths, sometimes we are interested not in the exact point where a function has a value; instead, we are interested in how the function behaves as we approach that point. If the function is a Complex Function, i.e. ($\mathbb{C} \to \mathbb{C}$), then we need to change our definition of a limit from the one we are familiar with to the one in Definition 16.

Defn 16 (Limit)**.** The *limit* of a Complex Function behaves quite similarly to their purely-real counterparts.

$$\lim_{\substack{z \to a \\ z \neq a}} f(z) = \ell \tag{2.4}$$

Equation (2.4) means that as z gets approaches the point a, where $z \neq a$, $f(z)$ will approach the value ℓ.

Remark 16.1 (Solving Limit and Solving Function). However, do note that the act of finding the Limit of a function approaching a point is distinctly different than finding the value of the function **at** that point.

To solve a problem involving a Complex Function and a limit, you can perform the same steps as before.

Example 2.4: Solve Limit of Complex Function. Lecture 5

If $f(z) = z^2$, then find the solution to this Limit of a Complex Function?

$$\lim_{z \to 3i} f(z)$$

$$\begin{aligned}\lim_{z \to 3i} f(z) &= \lim_{z \to 3i} z^2 \\ &= (3i)^2 \\ &= 9i^2 = 9(-1) \\ &= -9\end{aligned}$$

Thus, $\lim_{z \to 3i} f(z) = -9$.

In addition to the properties and rules that traditional real-valued limits have, Limits of Complex Functions have additional properties, because they lie on a 2-dimensional plane. This means that there are *infinitely* many approachable directions to a point.

Thus, for a Limit of a Complex Function to exist at a point a:

(i) **ALL** path Limits **MUST** exist.

(ii) **ALL** path Limits **MUST** evaluate to the same value.

2.2.1 Limits of Complex Functions that Do Not Exist

For a Limit of a Complex Function to **not** exist, as $z \to a$, then there are at least 2 paths γ_1, γ_2 that approach a such that

$$\lim_{\substack{z \to a \\ z \text{ on } \gamma_1}} f(z) \neq \lim_{\substack{z \to a \\ z \text{ on } \gamma_2}} f(z) \tag{2.5}$$

Example 2.5: Limit of a Complex Function that DNE. Lecture 5

Give the function $f(z)$ defined below, show that its Limit as $z \to 0$ Does Not Exist (DNE)?

$$f(z) = \frac{xy}{x^2 + y^2}$$

We start by referring to the definition of the lack of existence of a Limit of a Complex Function. Namely, we must find at least 2 paths γ_1, γ_2 such that evaluating the limit will yield two different values. I choose:

$$\begin{aligned}\gamma_1 &\coloneqq y = 0 \\ \gamma_2 &\coloneqq y = 2x\end{aligned}$$

Now we evaluate the expression below, using each path.

$$\lim_{\substack{z \to 0 \\ z \text{ on } \gamma}} f(z)$$

Following Path 1, γ_1:

$$\begin{aligned}\lim_{\substack{z \to 0 \\ z \text{ on } \gamma_1}} f(z) &= \lim_{\substack{z \to 0 \\ z \text{ on } \gamma_1}} \frac{xy}{x^2 + y^2} \\ &= \lim_{\substack{x \to 0 \\ y = 0}} \frac{xy}{x^2 + y^2} \\ &= \lim_{x \to 0} \frac{0}{x^2 + 0} \\ &= 0\end{aligned}$$

Following Path 2, γ_2:

$$\begin{aligned}
\lim_{\substack{z\to 0 \\ z \text{ on } \gamma_2}} f(z) &= \lim_{\substack{z\to 0 \\ z \text{ on } \gamma_2}} \frac{xy}{x^2+y^2} \\
&= \lim_{\substack{x\to 0 \\ y=2x}} \frac{xy}{x^2+y^2} \\
&= \lim_{x\to 0} \frac{2x^2}{5x^2} \\
&= \frac{2}{5}
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{\substack{z\to 0 \\ z \text{ on } \gamma_1}} f(z) &= 0 \\
\lim_{\substack{z\to 0 \\ z \text{ on } \gamma_2}} f(z) &= \frac{2}{5} \\
\lim_{\substack{z\to 0 \\ z \text{ on } \gamma_1}} f(z) &\neq \lim_{\substack{z\to 0 \\ z \text{ on } \gamma_2}} f(z)
\end{aligned}$$

Because there exist two different paths that yield different results from the Limit of the provided function $f(z)$,

$$\lim_{z\to 0} f(z) = \text{DNE}$$

2.3 Derivatives

Defn 17 (Derivative)**.** The *derivative* of a Complex Function, like many other operations, must have its definitions slightly redefined to account for the extra dimension provided by the Argand Plane.

The **domain** of a derivative of a Complex Function $f(z)$ requires that f be defined in a Neighborhood of the point a. Once this is satisified, then Equation (2.6) is used to find the actual value.

$$f'(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z - a} \tag{2.6}$$

Remark 17.1 (Path Existence). Because the Derivative of a Complex Function is defined with a Limit, **ALL** paths must exist and have the same value.

Remark 17.2 (Uses). The definition of a Derivative is rarely used to calculate anything. Instead, it is used to prove the non-existence of a derivative of a complex function.

Defn 18 (Neighborhood)**.** A *neighborhood* is an open disk $|z - a| < r$ for some r. The radius r is unspecified, meaning that we must choose its value.

Example 2.6: Derivative of Complex Function. Lecture 5

Given $f(z)$, what is its Derivative at a point a, $f'(a)$?

$$f(z) = z^2$$

As this problem is simple, we can just apply Equation (2.6) directly.

$$\begin{aligned}
f'(a) &= \lim_{z\to a} \frac{f(z)-f(a)}{z-a} \\
&= \lim_{z\to a} \frac{z^2-a^2}{z-a} \\
&= \lim_{z\to a} \frac{(z-a)(z+a)}{z-a} \\
&= \lim_{z\to a} (z+a) \\
&= (a+a) \\
&= 2a
\end{aligned}$$

Thus, $f'(a) = 2a$.

Example 2.7: Non-Existence of Derivative. Lecture 5

Given $f(z)$, $a = 2i$, show that $f'(2i)$ does not exist?

$$f(z) = \overline{z}$$

We start by applying the definition of a Derivative of a Complex Function.

$$\begin{aligned}
f'(a) &= \lim_{z\to 2i} \frac{f(z)-f(a)}{z-a} \\
&= \lim_{z\to 2i} \frac{\overline{z}-\overline{a}}{z-a} \\
f'(2i) &= \lim_{z\to 2i} \frac{\overline{z}-(-2i)}{z-a} \\
&= \lim_{z\to 2i} \frac{\overline{z}+2i}{z-2i}
\end{aligned}$$

Now, we can fall back to the definition of the non-existence of a Limit of a Complex Function. If the value of the function approaching the point a by two different paths (γ_1, γ_2) have two different values, the limit does not exist. If we choose our paths to be:

$$\begin{aligned}
\gamma_1 &\coloneqq x = 0 \\
\gamma_2 &\coloneqq y = x + 2i
\end{aligned}$$

Now solving the limit above using Path 1 (γ_1):

$$\begin{aligned}
f'(2i) &= \lim_{\substack{z\to 2i \\ z \text{ on } \gamma_1}} \frac{\overline{z}+2i}{z-2i} \\
&= \lim_{\substack{x=0 \\ y\to 2}} \frac{-yi+2i}{yi-2i} \\
&= \lim_{y\to 2} \frac{-i(y-2)}{i(y-2)} \\
&= -1
\end{aligned}$$

Now solve the above limit using Path 2 (γ_2):

$$\begin{aligned}
f'(2i) &= \lim_{\substack{z\to 2i \\ z \text{ on } \gamma_2}} \frac{\overline{z}+2i}{z-2i} \\
&= \lim_{\substack{x\to 0 \\ y=2 \\ z=x+2i}} \frac{\overline{z}+2i}{z-2i} \\
&= \lim_{\substack{x\to 0 \\ y=2}} \frac{x-2i+2i}{x+2i-2i} \\
&= \lim_{x\to 0} \frac{x}{x} \\
&= 1
\end{aligned}$$

Now, like the definition of a Limit of a Complex Function states:

$$\lim_{\substack{z\to a \\ z \text{ on } \gamma_1}} \frac{\overline{z}+2i}{z-2i} \neq \lim_{\substack{z\to a \\ z \text{ on } \gamma_2}} \frac{\overline{z}+2i}{z-2i}$$
$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = \text{DNE}$$

Thus, because the backing Limit does not exist, the entire Derivative does not exist.

2.3.1 Nicities

Derivatives of Complex Functions obey the same rules as purely-real calculus. This includes:

- Product Rule
- Quotient Rule
- Chain Rule
- etc.

2.3.2 Cauchy-Riemann Equations

Because we know the Derivative of a Complex Function **can** exist, and that $f(z) = U(x,y) + iV(x,y)$, we need to know just how special $U(x,y)$ and $V(x,y)$ really are.

If f has a derivative at $z = a$, then the Cauchy-Riemann Equations **MUST** also hold true for $f'(z)$ to exist. In the Cartesian form, these equations are seen as Equation (2.7).

$$\begin{aligned}
\frac{\partial U}{\partial x}(a) &= \frac{\partial V}{\partial y}(a) \\
U_x(a) &= V_y(a)
\end{aligned} \tag{2.7a}$$

$$\begin{aligned}
\frac{\partial U}{\partial y}(a) &= -\frac{\partial V}{\partial x}(a) \\
U_y(a) &= -V_x(a)
\end{aligned} \tag{2.7b}$$

In the polar form, the Cauchy-Riemann Equations are seen as Equation (2.8):

$$\begin{aligned}
\frac{\partial U}{\partial r}(a) &= \frac{1}{r}\frac{\partial V}{\partial \theta}(a) \\
U_r(a) &= \frac{1}{r}V_\theta(a)
\end{aligned} \tag{2.8a}$$

$$\begin{aligned}
\frac{\partial V}{\partial r}(a) &= \frac{-1}{r}\frac{\partial U}{\partial \theta}(a) \\
V_r(a) &= \frac{-1}{r}V_\theta(a)
\end{aligned} \tag{2.8b}$$

Example 2.8: Existence of Derivative using Cauchy-Riemann Equations. Lecture 5

Given the function $f(z)$, find its Derivative?

$$f(z) = \overline{z} = x - iy$$

First, we identify the functions $U(x,y)$ and $V(x,y)$.

$$U(x,y) = x$$
$$V(x,y) = -y$$

Now we start by checking Equation (2.7a).

$$\frac{\partial U}{\partial x} = 1$$
$$\frac{\partial V}{\partial y} = -1$$
$$\frac{\partial U}{\partial x} \neq \frac{\partial V}{\partial y}$$
$$U_x \neq V_y$$

Because the Cauchy-Riemann Equations are **not** satisified, $f(z) = \overline{z}$ has **NO** Derivative at any point z.

Theorem 2.1. *Suppose f is a Complex Function defined in the Neighborhood $|z - a| < r$ for some r. Suppose the Cauchy-Riemann Equations hold at a point, a, and that the 4 partial derivatives U_x, U_y, V_x, and V_y exist and are continuous at $z = a$. Then the Derivative of f at $z = a$ is defined to be:*

$$\begin{aligned} f'(z) &= \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} \\ f'(a) &= \frac{\partial V}{\partial y}(a) - i\frac{\partial U}{\partial y}(a) \end{aligned} \tag{2.9}$$

Theorem 2.2. *Suppose f is a Complex Function defined in the Neighborhood $|z - a| < r$ for some r. Suppose the polar form of the Cauchy-Riemann Equations (Equation (2.8)) hold at a point, a, and that the 4 partial derivatives U_r, U_θ, V_r, and V_θ exist and are continuous at $z = a$. Then the Derivative of f at $z = a$ is defined to be:*

$$\begin{aligned} f'(z) &= e^{-i\theta}\left(\frac{\partial U}{\partial r} + i\frac{\partial V}{\partial r}\right) \\ f'(a) &= e^{-i\theta}\left(\frac{\partial V}{\partial \theta}(a) - i\frac{\partial U}{\partial \theta}(a)\right) \end{aligned} \tag{2.10}$$

Remark. If the Cauchy-Riemann Equations fail to hold at $z = a$, then $f(z)$ fails to have a Derivative at $z = a$.

Remark. There *are* functions where the Cauchy-Riemann Equations hold at a point a, but the function does **NOT** have a Derivative at that point. However, these are rare pathological examples, so we will not discuss these functions.

Example 2.9: Differentiate Complex Function using Cauchy-Riemann Equations. Lecture 5

Given the function $f(z)$, use the Cauchy-Riemann Equations to find $f'(z)$?

$$f(z) = z^2 = \left(x^2 - y^2\right) + 2xyi$$

We identify $U(x,y)$ and $V(x,y)$ first.

$$U(x,y) = x^2 - y^2$$
$$V(x,y) = 2xyi$$

Use Equation (2.7).

$$\frac{\partial U}{\partial x} = 2x \qquad \frac{\partial V}{\partial y} = 2x$$
$$\frac{\partial U}{\partial y} = -2y \qquad \frac{\partial V}{\partial x} = 2y$$

$$\begin{aligned}\frac{\partial U}{\partial x} &= 2x = \frac{\partial V}{\partial y}\\ \frac{\partial U}{\partial y} &= -2y = -\frac{\partial V}{\partial x}\end{aligned}$$

U_x, V_y, U_y, V_x are polynomials. Hence, they are continuous. Hence f has a derivative at all points. Because this function passes the requirements placed on $f(z)$ by the Cauchy-Riemann Equations, we can find the Derivative of $f(z)$.

$$\begin{aligned}f'(z) &= \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}\\ &= 2x + 2yi\end{aligned}$$

Thus, $f'(z) = 2x + 2yi$.

Example 2.10: Differentiate Complex Trig using Cauchy-Riemann Equations. Lecture 5

Given the function $f(z) = \cos(z)$, verify that $f'(z) = -\sin(z)$?

Start by identifying $U(x, y)$ and $V(x, y)$.

$$\begin{aligned}f(z) &= \cos(z)\\ \cos(z) &= \cos(x + iy)\\ &= \cos(x)\cosh(y) - i\sin(x)\sinh(y)\\ U(x, y) &= \cos(x)\cosh(y)\\ V(x, y) &= -\sin(x)\sinh(y)\end{aligned}$$

Now use Equation (2.7).

$$\frac{\partial U}{\partial x} = -\sin(x)\cosh(y) \qquad \frac{\partial V}{\partial y} = -\sin(x)\cosh(y)$$
$$\frac{\partial U}{\partial y} = \cos(x)\sinh(y) \qquad \frac{\partial V}{\partial x} = -\cos(x)\sinh(y)$$

$$\begin{aligned}\frac{\partial U}{\partial x} &= -\sin(x)\cosh(y) = \frac{\partial V}{\partial y}\\ \frac{\partial U}{\partial y} &= \cos(x)\sinh(y) = -\frac{\partial V}{\partial x}\end{aligned}$$

Thus, the Cauchy-Riemann Equations are satisified. In addition, the 4 partial derivatives are continuous at all points. Therefore, $f(z) = \cos(z)$ **has** a derivative at all points. According to Equation (2.10), the solution is:

$$\begin{aligned}f'(z) &= \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}\\ &= -\sin(x)\cosh(y) + i\big(-\cos(x)\sinh(y)\big)\end{aligned}$$

Factor out the negative.

$$= -\big(\sin(x)\cosh(y) + i\cos(x)\sinh(y)\big)$$

By the definition of $\sin(z)$ in Cartesian form, we can simplify everything in the parentheses.

$$= -\sin(x + iy)$$
$$= -\sin(z)$$

Defn 19 (Open)**.** An *open* set means that the boundary edge is **not** included with the set. This also means that for every point a, within the set, we can define a disk with radius r, where the **entire** disk is contained within the set. Mathematically,

$$\forall a \in \Omega,\ \exists r > 0,\ \forall z, |z - a| < r \text{ are in } \Omega$$

Defn 20 (Connected)**.** A *connected* set is possible when a union of two points, with their disks, occurs inside this set. To form this union, there is a **continuous** path between them that lies entirely within the set's boundaries.

Defn 21 (Open Connected Set)**.** An *Open Connected set* is a special type of set that we use to visualize on the Argand Plane.

Theorem 2.3. *Let Ω be an Open Connected Set. A complex-valued function $f(z)$ on Ω ($f : \Omega \to \mathbb{C}$) is said to be Analytic if f has a Derivative at* ***all*** *points of Ω.*

Defn 22 (Analytic)**.** Let f be a Complex Function ($f : \Omega \to \mathbb{C}$) which has a Derivative at **all** points of an Open Connected Set Ω. Then we say f is *analytic* in Ω.

This means that f has a derivative at all points in the Open Connected Set.

However, our definition of an Analytic function is unsatisfactory, because we don't know what an analytic function looks like. Theorem 2.4 shows and then motivates what kind of functions $U(x, y)$ and $V(x, y)$ must be, and how they are related to one another.

Theorem 2.4. *Define a function f like so:*

$$f(z) = U(x, y) + iV(x, y)$$

If the function f is Analytic on Ω, then f has a derivative at all points in Ω, meaning

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$
$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

This means that the Cauchy-Riemann Equations hold, and the partial derivatives are valid.
Then, according to our work in Section 2.3.3, $U(x, y)$ and $V(x, y)$ are Harmonic.

2.3.3 Specialties of U, V, and their Interdependency

We start by using Equation (2.7).

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$
$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Now, if we take the partial derivative of $U_x = V_y$ with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right)$$
$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 V}{\partial y^2}$$

If we take the partial derivative of $U_y = -V_x$ with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial x}\right)$$
$$\frac{\partial^2 U}{\partial x \partial y} = -\frac{\partial^2 V}{\partial x^2}$$

For the class of functions we are concerned with, the order of differentiation does not matter, meaning $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$. Now, if we add $\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 V}{\partial y^2}$ and $\frac{\partial^2 U}{\partial x \partial y} = -\frac{\partial^2 V}{\partial x^2}$, then:

$$\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} = 0 \tag{2.11a}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \tag{2.11b}$$

The class of equations that satisfy Equation (2.11) are called Harmonic functions.

Defn 23 (Harmonic)**.** *Harmonic* functions are real-valued functions that are twice continuously differentiable function $f : U \to R$, where U is an open subset of $\mathbb{R}^n$, that satisfies Laplace's Equation (Equation (B.20)).

Example 2.11: Prove Function Terms are Harmonic. Lecture 5

Given $f(z)$, verify $U(x, y)$ is Harmonic?

$$f(z) = z^3$$

We start by needing to find $U(x, y)$ and $V(x, y)$.

$$\begin{aligned}
f(z) &= z^3 \\
&= z^2 z \\
&= \left(x^2 - y^2 + 2xyi\right)(x + iy) \\
&= x^3 - xy^2 + ix^2y - iy^3 - 2xy^2 + 2x^2yi \\
&= x^3 - 3xy^2 + i\left(3x^2y - y^3\right) \\
U(x, y) &= x^3 - 3xy^2 \\
V(x, y) &= 3x^2y - y^3
\end{aligned}$$

Now that we have $U(x, y)$, we can find the partial derivatives.

$$\begin{aligned}
\frac{\partial U}{\partial x} &= 3x^2 - 3y^2 \\
\frac{\partial^2 U}{\partial x^2} &= 6x \\
\frac{\partial U}{\partial y} &= -6xy \\
\frac{\partial^2 U}{\partial y^2} &= -6x
\end{aligned}$$

Now, we use Equation (2.11b) to check the validity of the Harmonic relation.

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= 0 \\
6x + -6x &= 0 \\
0 &\stackrel{\checkmark}{=} 0
\end{aligned}$$

Thus, $U(x, y) = x^3 - 3xy^2$ **is** Harmonic. Similarly, $V(x, y) = 3x^2y - y^3$ is also harmonic.

2.3.4 Analytic vs. Non-Analytic

A Complex Function can either be Analytic or non-analytic.

2.3.4.1 Analytic The functions below are **usually** Analytic, although, if the region Ω is defined in particular ways, they may not be.

- Polynomials, $4z^3 + 2z^2 + z + 45$
- Rational Functions, $\frac{z^2+9}{z+1}$
- Circular Functions, $\cos(z)$, $\sin(z^2)$
- Exponential Functions, e^z

2.3.4.2 Non-Analytic The Complex Functions below are non-Analytic, some of them very much not so. If a function is non-analytic, then it will not satisfy the Cauchy-Riemann Equations. These include:

- The real operator, Re
- The imaginary operator, Im
- The magnitude function, $|z|^2$

2.4 Harmonic Conjugates

Now that we have shown and proven that a function can be Harmonic in Example 2.11, we are curious to see that if one function is known, can we find the other one.

Defn 24 (Simply Connected)**.** A *simply* Connected set is a connected set Ω that does not have any holes in it.

Remark 24.1 (Multiply-Connected). If there is one hole, the region is *doubly*-connected.
If there are two holes, the region is *triply*-connected.

Theorem 2.5 (Harmonic Conjugate)**.** *Let Ω be a Simply Connected, Connected, and Open set. Let $U : \Omega \to \mathbb{R}$, which is Harmonic. Then, there exists a function V, which is also harmonic, such that $f = U + iV$ is Analytic. Such a $V(x, y)$ would be called the harmonic conjugate of $U(x, y)$.*

Example 2.12: Verify Harmonicity and Find Harmonic Conjugates. Lecture 6, Example 1

Given $U(x, y) = x^4 - 6x^2y^2 + y^4$ is defined on the $\mathbb{C}$ plane, verify that $U(x, y)$ is Harmonic? Find **all** Harmonic Conjugates of $U(x, y)$?

First, we can verify that a function is Harmonic by checking that it satisfies Laplace's Equation, (Equation (B.20)).

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

So, we start by taking the partial derivatives of $U(x, y)$.

$$\frac{\partial U}{\partial x} = 4x^3 - 12xy^2 \qquad \frac{\partial U}{\partial y} = -12x^2y + 4y^3$$
$$\frac{\partial^2 U}{\partial x^2} = 12x^2 - 12y^2 \qquad \frac{\partial^2 U}{\partial y^2} = -12x^2 + 12y^2$$

Now, plugging these into Equation (B.20):

$$\begin{aligned} 0 &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \\ &= 12x^2 - 12y^2 + \left(-12x^2 + 12y^2\right) \\ &= \left(12x^2 - 12x^2\right) + \left(12y^2 - 12y^2\right) \\ &\overset{\checkmark}{=} 0 \end{aligned}$$

$\therefore U(x, y)$ satisfies Laplace's Equation, and thus, is Harmonic.

Now, we want $f = U + iV$ to be Analytic, so what is V? For f to be analytic, the Cauchy-Riemann Equations **must** hold.

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} \\
\frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x} \\
4x^3 - 12xy^2 &= V_y \\
12x^2y - 4y^3 &= V_x
\end{aligned}$$

Now, we want to find just V, so we integrate V_y with respect to y.

$$\begin{aligned}
V(x, y) &= \int V_y \partial y \\
&= \int 4x^3 - 12xy^2 \partial y \\
&= 4x^3y - 4xy^3 + C(x)
\end{aligned}$$

Now, we need to solve for $C(x)$, a potential constant within the x domain.

$$\frac{\partial}{\partial x} V(x, y) = 12x^2y - 4y^3 + \frac{dC}{dx}$$

Using the value for V_x we found earlier, we can set each of these equal to each other.

$$\begin{aligned}
12x^2y - 4y^3 &= 12x^2y - 4y^3 + \frac{dC}{dx} \\
\frac{dC}{dx} &= 0
\end{aligned}$$

Because we said $C(x)$ was a function solely in x, if $\frac{dC}{dx} = 0$, then $C(x)$ **MUST** be equal to only a constant. Thus, $C(x) = C$. So, all possible Harmonic Conjugates of $U(x, y)$ are represented by a complete $V(x, y)$, shown below.

$$V(x, y) = 4x^3y - 4xy^3 + C$$

Therefore, f, according to our definition, is:

$$\begin{aligned}
f &= U(x, y) + iV(x, y) \\
&= x^4 - 6x^2y^2 + y^2 + i\left(4x^3y - 4xy^3 + C\right) \\
&= x^4 + 4x^3yi - 6x^2y^2 - 4xy^3i + y^4 \\
&= (x + iy)^4 + iC
\end{aligned}$$

2.5 Paths

In this course, Paths, curve, and contour are synonymous. However, in general, they are not.

Defn 25 (Path)**.** A *path* is static Image of a line on a plane. However, when we say we are interested in the path of an image, we are not really talking about the static image we see, but rather **HOW** we got that image, what function generated it, etc.

In general, if we are given a Path, we can find the any point on the path by parameterizing the beginning and end of the path. This is the function $z : [a, b] \to \mathbb{C}$ where $z = z(t)$ where $a \leq t \leq b$.

Example 2.13: Parameterize a Path. Lecture 6, Example 2

Given a line segment $[0, 2 + 3i]$ in the $\mathbb{C}$ plane, what is the parameterized function that creates **every** point for the segment?

Technically, there are infinitely many solutions. Just a few are presented below:

$$z(t) = (2+3i)t \;\; 0 \leq t \leq 1$$
$$z(t) = (2+3i)2t \;\; 0 \leq t \leq \frac{1}{2}$$
$$z(t) = (2+3i)(1-t) \;\; 0 \leq t \leq 1$$

If $z = x + iy$:

$$y = \frac{3}{2}x \;\; 0 \leq x \leq 2$$

The general form for Parameterizing a Path is shown in Equation (2.12).

Defn 26 (Parameterizing)**.** The act of *parameterizing* a Path is the act of finding a parametric equation that completely describes the Path in the Image. The general equation for a path defined from $[a, b]$ where $a, b \in \mathbb{C}$ is shown below.

$$z(t) = a(1-t) + b(t) \;\; 0 \leq t \leq 1 \tag{2.12}$$

Example 2.14: Parameterize a Circle. Lecture 6, Example 4

Given the circle centered at point a, where $a \in \mathbb{C}$ with a radius $r > 0$, find a parameterized equation?

Start by stating the definition of a circle in the complex plane.

$$|z - a| = r$$

By thinking about this a little, we can see that we can generate every point on the circle's edge by keeping r constant and varying only $\theta = \arg(z)$, the argument of the circle. Now, to simplify things, we will also write some of the subsequent equations in polar form.

$$z - a = r\big(\cos(\theta) + i\sin(\theta)\big)$$

Equation (A.3) can simplify this further.

$$= a + re^{i\theta} \;\; 0 \leq \theta \leq 2\pi$$

Thus, we have parameterized the equation for a circle, in 2 forms: Equation (2.13a) and Equation (2.13b). Parameterization of a circle in Counter Clockwise Direction (CCD)

$$z = a + re^{i\theta} \;\; 0 \leq \theta \leq 2\pi \tag{2.13a}$$

$$z = a + re^{-i\theta} \;\; 0 \leq \theta \leq 2\pi \tag{2.13b}$$

2.6 Integration

2.6.1 Integration on Paths

The integration method that we will present in this section is Path-agnostic. This means that regardless of how you go about Parameterizing the Path, the integral returns the same result.

The functions we will be integrating are complex-valued functions. We will be integrating these functions along one of their many Paths. These paths are typically denoted with either C, Γ, or γ. I will choose γ in this document, to match up with the use of γ in Limits.

Defn 27 (Integration on Paths)**.** Let f be a continuous function on the image of the Path. We denote this integral similarly to how we denote it regularly.

$$\begin{aligned} \int_\gamma f &= \int_\gamma f(z)dz \\ &:= \int_a^b f\big(z(t)\big)z'(t)dt \end{aligned} \tag{2.14}$$

Remark 27.1. You can **always** perform an integration by parameterizing the path and integrating. However, this can become very complex sometimes.

Example 2.15: Calculate the Integral on any Path. Lecture 7

Given the line segment $\ell = [-4+5i, 3-2i]$, and an arbitrary point z defined by the function $z(t)$. Evaluate $\int_\ell \mathrm{Re}\{z\}dz$?

Start by parameterizing the path. I will use the general form of finding a parameterized function for $z(t)$ from Equation (2.12).

$$\begin{aligned} z(t) &= a(1-t) + bt \ \ 0 \le t \le 1 \\ &= (-4+5i)(1-t) + (3-2i)t \ \ 0 \le t \le 1 \end{aligned}$$

Now, we can evaluate the integral.

$$\begin{aligned} \int_\ell \mathrm{Re}\{z\}dz &= \int_0^1 \big(-4(1-t)+3t\big)z'(t)dt \\ &= \int_0^1 (7t-4)z'(t)dt \\ &= \int_0^1 (7t-4)\big((-4+5i)(-t)+(3-2i)\big)dt \\ &= \int_0^1 (7t-4)(7-7i)dt \\ &= (7-7i)\int_0^1 (7t-4)dt \\ &= (7-7i)\left(\left[\frac{7}{2}t^2-4t\right]_{t=0}^{t=1}\right) \\ &= \frac{-7}{2}(1-i) \end{aligned}$$

2.6.2 Antiderivatives

Theorem 2.6 (Complex Antiderivative)**.** *Let Ω be a Simply Connected, Connected, and Open region. Let f be an Analytic Complex Function defined on Ω, $f : \Omega \to \mathbb{C}$ on Ω.*

Then, there exists another Analytic function ($\exists F : \Omega \to \mathbb{C}$) such that the derivative of the new function is equal to the original function.

$$F' = f$$

*Thus, F is **an** antiderivatives of f.*

$$\int_\gamma f = F(B) - F(A) \tag{2.15}$$

Remark (Use)*.* This is mainly used for evaluating integrals.

2.6.3 Using Antiderivatives

To use antiderivatives to evaluate integrals, we use the steps below:

1. Have a Simply Connected, Open, Connected region, Ω.
2. Show that C is a Path/curve fully inside the region Ω.
3. Show that f is Analytic on Ω.
   - This is the big condition and the hard one to prove.
4. Evaluate $\int_C f = [F]_A^B = F(B) - F(A)$.
   - Notice that the right-hand side $(F(B) - F(A))$ is **independent** of the Path.
   - Only the starting and end points matter.

Example 2.16: Antiderivatives. Lecture 7

Evaluate the integral $\int_\gamma z\sin\left(z^2\right)dz$ where γ is equal to the union of the line segment that stretches from the origin to $(1,2)$, **with** the lower half of the circle whose center is at $a = 3+2i$ with radius $r = 2$.

Using Theorem 2.6, we know that we don't have to actually evaluate this integral. Instead, we can find the beginning and end points of the Path, and just take the difference of those. So, $A = 0 + 0i$, and $B = 5 + 2i$. $z \sin\left(z^2\right)$ is Analytic on $\mathbb{C}$. Therefore, we are guaranteed the derivative's existence. This also means that **an** antiderivative exists.

$$F(z) = \frac{-1}{2} \cos\left(z^2\right)$$

Remember that there would be a constant term at the end of the antiderivative, but because we will be using this function and subtracting it from another value, the constants would just cancel each other out.
Using Equation (2.15), we can now solve this directly.

$$\begin{aligned}
\int_\gamma f &= \int_\gamma z \sin\left(z^2\right) dz \\
&= F(B) - F(A) \\
&= \frac{-1}{2} \cos\left((5+2i)^2\right) - \frac{-1}{2} \cos\left((0+0i)^2\right) \\
&= \frac{-1}{2} \left(\cos\left((5+2i)^2\right) - \cos(0)\right) \\
&= \frac{-1}{2} \left(\cos\left((5+2i)^2\right)\right) + \frac{1}{2}
\end{aligned}$$

The last step would be to use the Angle Sum and Difference Identities to simplify the remaining cos into a normal, Cartesian, form.

2.6.4 Which to Method to Integrate with?

You choose your integration method based on whether or not the function you are integrating is Analytic. Refer to Section 2.3.4 if you need a short list of Complex Functions that usually are/aren't analytic.

- If the function **IS** Analytic, then you can use Antiderivatives.
- If the function is **NOT** Analytic, then you must use Integration on Paths.

Example 2.17: Choose Method of Integration. Lecture 7, Problem 5

Given a Path C being the unit circle in the counter-clockwise direction, evaluate the expression below?

$$\int_C \frac{1}{z} dz$$

In this case, our $\Omega = \mathbb{C}$. If we start by looking at our given expression to integrate, $\frac{1}{z}$, we quickly notice that this is **NOT** Analytic, because there is a hole at $z = 0$, which conflicts with Ω's requirement to be Simply Connected.
This leads to us **NOT** being able to use Antiderivatives. So, we have to perform an Integration on Paths.
We start by Parameterizing the Path. Using Equation (2.13a), we can easily get the parameterization.

$$\begin{aligned}
z(\theta) &= a + re^{i\theta} \quad 0 \leq \theta \leq 2\pi \\
r &= 1 \\
a &= 0 \\
z(\theta) &= e^{i\theta}
\end{aligned}$$

Using our result for $z(\theta)$, we can find $z'(\theta)$.

$$\begin{aligned}
z(\theta) &= e^{i\theta} \\
z'(\theta) &= ie^{i\theta}
\end{aligned}$$

Now, we integrate according to Equation (2.14).

$$\begin{aligned}\int_C \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{z(\theta)} z'(\theta) d\theta \\ &= \int_0^{2\pi} \frac{1}{e^{i\theta}} \left(ie^{i\theta}\right) d\theta \\ &= \int_0^{2\pi} i d\theta \\ &= i \int_0^{2\pi} d\theta \\ &= i[\theta]_{\theta=0}^{\theta=2\pi} \\ &= 2\pi i\end{aligned}$$

2.6.5 Cauchy's Integral Theorem

Theorem 2.7 (Cauchy's Integral Theorem)**.** *Let Ω be a Simply Connected and Open set. Let f be an Analytic function in Ω. Let C be a curve/Path **INSIDE** Ω, described in a counter-clockwise direction, be Closed and a Simple. Let a be a point in the interior of C.*

Then, the below equation holds true.

$$\int_C \frac{f(z)}{(z-a)^n} dz = 2\pi i \left(\frac{1}{(n-1)!}\right) \frac{d^{n-1} f(a)}{dz^{n-1}} \tag{2.16}$$

Defn 28 (Closed)**.** A *closed* path is one whose starting point is the same as its ending point.

Defn 29 (Simple)**.** A *simple* path is one that doesn't cross itself. For example, the figure 8 is an example of a curve that is **not** simple.

Example 2.18: Use Cauchy's Integral Theorem. Lecture 7, Problem 6

Let a curve C be the rectangle with points $(4, 1), (-4, 1), (-4, -1), (4, -1)$. Use Cauchy's Integral Theorem to solve the below expression?

$$\int_C \frac{\sin(z)}{3z - 4(z-5)} dz$$

In this case, our $\Omega = \mathbb{C}$. In addition, from inspection of Equation (2.16), we see that the denominator **must** be in the form of $z - a$. However, we are given the product of $3z - 4(z - 5)$.
But, we can "eliminate" one of them, and treat it as part of $f(z)$ by remembering the fourth assumption of Theorem 2.7, that a must be **inside** C. If we look at the value $z - 5$, we notice that the value for $a = 5$, which is outside C. So, we "bring up" that expression and have it all divided by $3z - 4$.

$$\begin{aligned}\int_C \frac{\sin(z)}{3z - 4(z-5)} dz &= \int_C \frac{\frac{\sin(z)}{z-5}}{3z-4} dz \\ f(z) &= \frac{\sin(z)}{z-5}\end{aligned}$$

$f(z)$ is Analytic **inside** C. However, it is non-analytic at $z = 5$.

From Equation (2.16), we see that the denominator $z - a$ **must** be in that form. So, we can factor the 3 out.

$$\int_C \frac{\frac{\sin(z)}{z-5}}{3z-4} dz = \frac{1}{3} \int_C \frac{\frac{\sin(z)}{z-5}}{z - \frac{4}{3}} dz$$

Use Equation (2.16) now.

$$
\begin{aligned}
&= \frac{1}{3}\left(2\pi i\left(\frac{1}{(1-1)!}\right)\right)\frac{d^{1-1}f(a)}{dz^{1-1}} \\
&= \frac{1}{3}\left(2\pi i\left(\frac{1}{0!}\right)\right)f(a) \\
&= \frac{1}{3}(2\pi i(1))\left(\frac{\sin\left(\frac{4}{3}\right)}{\frac{4}{3}-5}\right) \\
&= \frac{-2\pi i}{11}\sin\left(\frac{4}{3}\right)
\end{aligned}
$$

2.6.5.1 Generalizing Cauchy's Integral Theorem We want to remove requirements on C to be Simple from Cauchy's Integral Theorem. This means that a non-Simple is one with at least two loops, meaning there is no single uniform direction the path travels. However, if we were to break the path into the union of two simple paths, then we could use Cauchy's Integral Theorem. In fact, this is exactly what we will do!

Remark. If you remember back to Calculus of a single variable, we had something similar that we used frequently. We were allowed to break a single integral into two separate ones that were added together, which allowed us to evaluate many different expressions. Likewise, we were also able to flib the integral's bounds and change teh direction of integration, which is also used here.

Example 2.19: Cauchy's Integral Theorem on a Non-Simple Path. Lecture 8, Example 1

Let C be the union of two loops, which cross at $z = 0$, forming a figure-8 on the real-axis. The right-hand portion is oriented in the counterclockwise direction, and the left-hand portion is oriented in the clockwise direction. Then, evaluate the following expression?

$$\int_C \frac{2z+3}{(z-1)(z+2)}dz$$

The first step here is to "break" the union into two separate loops that we can then use Cauchy's Integral Theorem on. Let C_1 be the right-hand portion and C_2 be the left-hand side.

$$\int_C \frac{2z+3}{(z-1)(z+2)}dz = \int_{C_1} \frac{2z+3}{(z-1)(z+2)}dz + \int_{C_2} \frac{2z+3}{(z-1)(z+2)}dz$$

We were told that C_2 is oriented in the clockwise direction, but Cauchy's Integral Theorem states we must have the path be in the counterclockwise direction. This is easily solved by just negating the entire C_2 integral. This is a natural extension of the idea of "flipping the bounds" of a regular Riemann integral.

$$\int_C \frac{2z+3}{(z-1)(z+2)}dz = \int_{C_1} \frac{2z+3}{(z-1)(z+2)}dz - \int_{C_2,\ \text{CCD}} \frac{2z+3}{(z-1)(z+2)}dz$$

Now, we need to determine what $f(z)$ from Theorem 2.7 is in both parts of the loops. In this case, $a_1 = 1$ inside C_1 and $a_2 = -2$ in C_2. That means those 2 values must end up in the denominator of their respective integral.

$$\int_C \frac{2z+3}{(z-1)(z+2)}dz = \int_{C_1} \frac{\frac{2z+3}{z+2}}{z-1}dz - \int_{C_2,\ \text{CCD}} \frac{\frac{2z+3}{z-1}}{z+2}dz$$

Now that both integrals are on Simple paths, are inside $\Omega = \mathbb{C}$, and have a inside them, we can use Theorem 2.7.

$$\begin{aligned}
\int_C \frac{2z+3}{(z-1)(z+2)}dz &= \int_{C_1} \frac{\frac{2z+3}{z+2}}{z-1}dz - \int_{C_2,\text{ CCD}} \frac{\frac{2z+3}{z-1}}{z+2}dz \\
&= 2\pi i\left(\frac{1}{(1-1)!}\right)\frac{d^{1-1}f(a_1)}{dz^{1-1}} - 2\pi i\left(\frac{1}{(1-1)!}\right)\frac{d^{1-1}f(a_2)}{dz^{1-1}} \\
&= 2\pi i(1)f(a_1) - 2\pi i(1)f(a_2) \\
&= 2\pi i\left(\frac{2+3}{1+2}\right) - 2\pi i\left(\frac{-4+3}{-2-1}\right) \\
&= 2\pi i\left(\frac{5}{3}\right) - 2\pi i\left(\frac{-1}{-3}\right) \\
&= 2\pi i\left(\frac{5}{3}\right) - 2\pi i\left(\frac{1}{3}\right)
\end{aligned}$$

And now, for completion's sake, we have an example using a sinusoid.

Example 2.20: Using Cauchy's Integreal Theorem with a Sinusoid. Lecture 8, Example 2

Given a rectangle R with vertices $A = (3,1), B = (-2,1), C = (-2,-1), D = (3,-1)$ inscribed on the complex plane, evaluate the integral?

$$\int_R \frac{\sin(z)}{(z-2)^2(z+3)}dz$$

We know that $z+3$ cannot be the denominator for Cauchy's Integral Theorem, because $a = -3$ is **not** inside the rectangle R ($a = -3 \notin R$).

$$\int_R \frac{\sin(z)}{(z-2)^2(z+3)}dz = \int_R \frac{\frac{\sin(z)}{z+3}}{(z-2)^2}dz$$

Because R is already described in the counterclockwise direction, there is nothing more we have to do,, other than realize that we can apply Cauchy's Integral Theorem.

$$\begin{aligned}
\int_R \frac{\sin(z)}{(z-2)^2(z+3)}dz &= \int_R \frac{\frac{\sin(z)}{z+3}}{(z-2)^2}dz \\
&= 2\pi i\left(\frac{1}{(2-1)!}\right)\frac{d^{2-1}f(a)}{dz^{2-1}} \\
&= 2\pi i(1)f'(a)
\end{aligned}$$

This time, we need to find $f'(z)$ before we can finish our evaluation.

$$\begin{aligned}
f(z) &= \frac{\sin(z)}{z+3} \\
f'(z) &= \frac{d}{dz}\frac{\sin(z)}{z+3}
\end{aligned}$$

Use the Quotient Rule.

$$= \frac{\cos(z)(z+3)+\sin(z)(1)}{(z+3)^2}$$

Now, we finish.

$$\begin{aligned}
\int_R \frac{\sin(z)}{(z-2)^2(z+3)}dz &= 2\pi i(1)f'(z=2) \\
&= 2\pi i\left(\frac{\cos(2)(2+3)+\sin(2)(1)}{(2+3)^2}\right) \\
&= 2\pi i\left(\frac{5\cos(2)+\sin(2)}{25}\right)
\end{aligned}$$

3 Complex Series

We start our discussion of complex series by talking about Power Series.

Defn 30 (Power Series)**.** A *power series* is an infinite summation of terms that form infinitely long polynomials. These are defined around a center a, $a \in \mathbb{C}$.

$$\sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \tag{3.1}$$

Because our Power Series are infinite, we now have to ask about its convergence. There is a theorem for this, Theorem 3.1, called the Region of Convergence.

Theorem 3.1 (Region of Convergence)**.** *There exists a number R, $0 \leq R \leq \infty$, such that $\forall z, |z-a| < R$ the series is convergent, and $\forall z, |z-a| > R$ is divergent. $forall z, |z-a| = R$ does not have guaranteed divergence or convergence. This value R is called the* Region of Convergence*, or the* RoC.

Example 3.1: Finding the Region of Convergence. Lecture 8, Example 3

Compute the radius of convergence of the power series shown below.

$$\sum_{n=0}^{\infty} \frac{3^{2n+5}}{2n+3} (z-i)^{7n+2}$$

We can solve this by using d'Alembert's Ratio Test. This test states that for an infinite series to be convergent, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = C$, where C is some constant value.
Applying d'Alembert's Ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{\frac{3^{2(n+1)+5}}{2(n+1)+3}(z-i)^{7(n+1)+2}}{\frac{3^{2n+5}}{2n+3}(z-i)^{7n+2}} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{3^{2n+7}}{2n+5}(z-i)^{7n+9}}{\frac{3^{2n+5}}{2n+3}(z-i)^{7n+2}} \right|
\end{aligned}$$

We can cancel the values with exponents raised to the n.

$$= \lim_{n\to\infty} \frac{2n+3}{2n+5} 3^2 \left|(z-i)^7\right|$$

Evaluating the limit, we see that the fraction with n will become $\frac{1}{1}$.

$$= 3^2 \left|(z-i)^7\right|$$

Using the Law of Moduli, we can simplify this.

$$= 3^2 |z-i|^7$$

Now, we have to use Theorem 3.1.

$$\begin{aligned}
&\forall z, 3^2|z-i|^7 < 1, \text{ Series is convergent} \\
&\forall z, 3^2|z-i|^7 > 1, \text{ Series is divergent}
\end{aligned}$$

Now, we move terms and exponents to the other side, leaving just $|z-i|$ on the left.

$$\begin{aligned}
&\forall z, |z-i| < \frac{1}{3^{\frac{2}{7}}}, \text{ Series is convergent} \\
&\forall z, |z-i| > \frac{1}{3^{\frac{2}{7}}}, \text{ Series is divergent}
\end{aligned}$$

Theorem 3.2. *If the Power Series $\sum_{n=0}^{\infty} a_n(z-a)^n$ has a Region of Convergence RoC > 0, we can define $f(z)$ to be the summation of all the terms in the series, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, $|z-a| < R$.*

Then, f is Analytic in $|z-a|<R$ and $f'(z)$ is term-by-term differentiable.

$$\begin{aligned} f(z) &= a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 \cdots \\ f'(z) &= 0 + a_1 + 2a_2(z-a) + 3a_3(z-a)^2 + \cdots \\ &= \sum_{n=1}^{\infty} n a_n (z-a)^{n-1},\ \forall z,\ |z-a|<R \end{aligned}$$

Likewise, the converse of Theorem 3.2 holds true, seen in Theorem 3.3

Theorem 3.3. *Let f be an Analytic Complex Function on the disk $|z-a|<R$.*
Then there exists a Power Series with the same center a, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

3.1 Power Series of Common Functions

All of the common transcendental functions below are Analytic on the entire complex place, $|z|<\infty$.

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= 1 + \frac{1}{1!}z + \frac{1}{2!}z + \frac{1}{3!}z^3 + \cdots \end{aligned} \tag{3.2}$$

$$\begin{aligned} \cos(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \end{aligned} \tag{3.3}$$

$$\begin{aligned} \sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \end{aligned} \tag{3.4}$$

Now, some examples on how to use these series to solve problems.

Example 3.2: Exponential not Centered on Origin. Lecture 9 Example 1

The mapping $z \to e^z$ is Analytic on $|z-2|<\infty$. Determine its Power Series?

We start by remembering that we already have a general power series for e^z, but this one is centered on $z=0$.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

We can try just replacing e^z weith e^{z-2}.

$$e^{z-2} = \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!}$$

The series above is technically correct, but not in the form we want.

$$\begin{aligned} e^z e^{-2} &= \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!} \\ &= e^2 \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^n \end{aligned}$$

Example 3.3: Power Series of Sinusoid with Angle Squared. Lecture 9, Example 2

Find the Power Series for $z \to \sin(z^2)$ for $|z| < \infty$?

We already know what $\sin(z)$ is on $|z| < \infty$, so we start there.

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Now, we need to modify it to handle z^2. We can just replace all instances of z with z^2.

$$\begin{aligned}\sin(z^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(z^2\right)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!}\end{aligned}$$

Example 3.4: Power Series of Sinusoid not Centered on Origin. Lecture 9, Example 3

$z \to \sin(z)$ is Analytic on $|z-2| < \infty$. Find the Power Series of this function?

We start by taking what we already know, $\sin(z)$ on $|z| < \infty$.

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Now, we attempt to replace all instances of z with $z-2$.

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

This is if sin were centered on $z = 0$ though.

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^{2n+1}}{(2n+1)!}$$

But, that doesn't work, as we have no reasonable way to put this into our "standard" Power Series form. Instead, start from the beginning again and say that $\sin(z) = \sin(z-2+2)$. Now, we treat $\sin(z-2+2)$ as $\sin\big((z-2)+2\big)$. We use the Angle Sum and Difference Identities to simplify this, namely Equation (A.8).

$$\sin\big((z-2)+2\big) = \sin(z-2)\cos(2) + \cos(z-2)\sin(2)$$

Now we expand the sinusoids that have z terms in them.

$$= \cos(2)\left(\sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^{2n+1}}{(2n+1)!}\right) + \sin(2)\left(\sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^{2n}}{(2n)!}\right)$$

Again, this is not in our "standard" form, but we can change it so it is, using 2 cases.

$$\sin\big((z-2)+2\big) = \sum_{n=0}^{\infty} a_n (z-2)^n$$

Where

$$a_n = \begin{cases} \cos(2)(-1)^{\frac{n-1}{2}} \frac{(z-2)^n}{n!} & \text{If } n \text{ is odd} \\ \sin(2)(-1)^{\frac{n}{2}} \frac{(z-2)^n}{n!} & \text{If } n \text{ is even} \end{cases}$$

One more useful series that we frequently use is the Taylor Series.

Defn 31 (Taylor Series)**.** The *Taylor series* of a Complex Function is created out of 2 equations. The first is Equation (3.1), and the other is shown below.

$$\begin{aligned} a_n &= \frac{1}{n!}\frac{d^n}{dz^n}f(a) \\ &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

By replacing the a_n terms, we end up with the general form of a Taylor series for a function f.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \tag{3.5}$$

Remark 31.1 (Maclaurin Series). A *Maclaurin series* is a special case of a Taylor Series, where the function is centered on $a = 0$.

3.2 Power Series of Rational Functions

The key functions to know a Power Series for are $\frac{1}{1-z}$ and $\frac{1}{1+z}$.

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \\ &= 1 + z + z^2 + z^3 + \cdots, |z| < 1 \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{1}{1+z} &= \sum_{n=0}^{\infty} z^n \\ &= 1 - z + z^2 - z^3 + \cdots, |z| < 1 \end{aligned} \tag{3.7}$$

Example 3.5: Power Series of Rational Function 1. Lecture 9, Example 4

Given the rational function $f(z) = \frac{1}{1-2z^3}$ is Analytic, find its Power Series?

We start be determining the Region of Convergence.
If in the standard format, we have $\frac{1}{1-z}$ where $|z| < 1$, we can follow that format.

$$\left|2z^3\right| < 1$$

Property of Modulus, Constants can Factor out.

$$2\left|z^3\right| < 1$$

Use the Law of Moduli.

$$\begin{aligned} |z|^3 &< \frac{1}{2} \\ |z| &< \frac{1}{\sqrt[3]{2}} \end{aligned}$$

Now that we have our Region of Convergence, which is a circle with center $a = 0$ and $r = \frac{1}{\sqrt[3]{2}}$, there **MUST** be a Power Series of $f(z)$.

$$\begin{aligned} |z| &\Rightarrow \sum_{n=0}^{\infty} z^n \\ \left|2z^3\right| &\Rightarrow \sum_{n=0}^{\infty} \left(2z^3\right)^n \\ &= \sum_{n=0}^{\infty} 2^n z^{3n} \end{aligned}$$

Example 3.6: Power Series of Rational Function 2. Lecture 9, Example 4

Consider $z \to \frac{1}{2+3z}$ as a sum of Power Series of a circle with center $z = 2+5i$. Find the power series of this function?

The function is Analytic everywhere exception $z = \frac{-3}{2}$. Thus, the Region of Convergence is the distance from the center a to the hole $z = \frac{-3}{2}$.

We start by taking our function, and plugging the left-hand side of the circle, the modulus portion, into this function for z.

$$\frac{1}{2+3z} = \frac{1}{2+3\big(z-(2+5i)\big)+3(2+5i)}$$

The extra $3(2+5i)$ was needed be cancel out the added $-3(2+5i)$. This way, we are only adding by 0, which is allowed by algebra.

$$= \frac{1}{8+15i+3\big(z-(2+5i)\big)}$$

Now, we force our constants to match our standard equation, Equation (3.7), by factoring out the constants.

$$= \frac{1}{(8+15i)\left(1+\frac{3}{8+15i}\big(z-(2+5i)\big)\right)}$$

To make our lives easier, we will create a new variable that will represent the variable expression in the denominator.

$$\begin{aligned} w &= \frac{3\big(z-(2+5i)\big)}{8+15i} \\ \frac{1}{2+3z} &= \frac{1}{(8+15i)(1+w)} \\ &= \frac{1}{8+15i}\left(\frac{1}{1+w}\right) \end{aligned}$$

Now, we can use Equation (3.7) as our template.

$$= \frac{1}{8+15i}\sum_{n=0}^{\infty}(-1)^n w^n,\ |w|<1$$

Now, we have 2 derivations left to perform:

1. We need to substitute the series back into z, and simplify it.
2. We need to solve for the Region of Convergence by substituting our z expression back in for w.

I will start with the Power Series, and then complete the Region of Convergence.

$$\begin{aligned} \frac{1}{2+3z} &= \frac{1}{8+15i}\sum_{n=0}^{\infty}(-1)^n w^n \\ &= \frac{1}{8+15i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3\big(z-(2+5i)\big)}{8+15i}\right)^n \end{aligned}$$

Pull the exponent into all the sub-terms.

$$\begin{aligned} &= \frac{1}{8+15i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3^n\big(z-(2+5i)\big)^n}{(8+15i)^n}\right) \\ &= \frac{1}{8+15i}\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3^n}{(8+15i)^n}\right)\big(z-(2+5i)\big)^n \\ &= \frac{1}{8+15i}\sum_{n=0}^{\infty}\left(\frac{(-1)\,3}{(8+15i)}\right)^n\big(z-(2+5i)\big)^n \end{aligned}$$

With the Power Series found, we now focus on the Region of Convergence.

$$|w| < 1$$

$$\left|\frac{3\big(z-(2+5i)\big)}{8+15i}\right| < 1$$

Factor out the constants.

$$\left|\frac{3}{8+15i}\right| |z-(2+5i)| < 1$$

$$|z-(2+5i)| < \left|\frac{8+15i}{3}\right|$$

Use the Law of Moduli to separate the moduli in the fraction.

$$\begin{aligned}|z-(2+5i)| &< \frac{|8+15i|}{|3|}\\ &< \frac{\sqrt{64+225}}{3}\\ &< \frac{\sqrt{289}}{3}\end{aligned}$$

Thus, our solution is:

$$\frac{1}{2+3z} = \frac{1}{8+15i}\sum_{n=0}^{\infty}\left(\frac{(-1)\,3}{(8+15i)}\right)^n \big(z-(2+5i)\big)^n,\ |z-(2+5i)| < \frac{\sqrt{289}}{3}$$

3.3 Laurent Series

Laurent Series are generalizations of the Power Series that we have already been discussing.

Theorem 3.4 (Laurent Series)**.** *Suppose f is Analytic on the Annulus $r_1 < |z-a| < r_2$.*
Then there exists a Laurent series *such that* $\forall z, r_1 < |z-a| < r_2$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \cdots \tag{3.8}$$

Defn 32 (Principal Part)**.** The *principal part* of the Laurent Series $f(z)$ is the summation of all the b_n elements.

$$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \cdots \tag{3.9}$$

Before we continue any farther, we need to add some more terms to our dictionary.

Defn 33 (Deleted)**.** *Deleted* is a term that is used to describe disks. A deleted disk is one that is of the form $0 < |z-a| < r$.

3.3.1 Function Singularities

Defn 34 (Singularity)**.** A *singularity* is a point where a function is non-Analytic.

Defn 35 (Isolated)**.** In this definition, *isolated* refers to a Singularity. For a singularity to be isolated, this means that for **all** points surrounding the singularity $z = a$, the function **IS** Analytic.

Example 3.7: Simple Isolated Singularity. Lecture 10, Example 1

Given the function $f(z) = \frac{\sin(z)}{z^2}$, find its Singularity?

If we look at $f(z)$, we notice that f is undefined at $z = 0$. This also means that f is **not** differentiable at $z = 0$, making $z = 0$ a Singularity.
$\therefore$ f has a Singularity at $z = 0$. In addition, $z = 0$ is an Isolated Singularity, because at **every** other point, f is differentiable.

Example 3.8: Multiple Isolated Singularities. Lecture 10, Example 2

Given the function $f(z) = \frac{z+1}{(z-2)(z+3)}$, find the singularities?

Looking at f, we immediately notice two points where f is undefined, and therefore does not have a derivative. $z = 2$ and $z = -3$. In addition, both of these singularities are Isolated, because we can find a disk/Annulus surrounding the Singularity such that f is Analytic.

With the terms above defined, we can move onto seeing a theorem regarding the Existence of Laurent Series.

Theorem 3.5 (Existence of Laurent Series)**.** *A function f, that is Analytic on a Deleted disk, $0 < |z-a| < r$ is said to have an (Isolated) Singularity of $z = a$ if f is **not** differentiable at $z = a$.*
Then, there exists a Laurent Series of the form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots \tag{3.10}$$

Defn 36 (Removable)**.** A *removable* Singularity is one that can be canceled out, somehow. Sometimes these are obvious, and can just be factored out. Other times, these can only be found by working with the Laurent Series.

If Theorem 3.5 is satisfied, then there are 3 cases:

1. All terms $b_1, b_2, b_3, \ldots = 0$. Then the Singularity is said to be Removable. Refer to Example 3.9.
2. The number of non-zero b_n is finite. This means that the Principal Part of the Laurent Series is finite. This implies that there is a Pole with an Order. Refer to Example 3.10.
3. The number of non-zero b_n is infinite. This means that the Principal Part of the Laurent Series is infinite. This means there is an Essential Singularity present. Refer to Example 3.11.

Example 3.9: Laurent Series Removable Singularity. Lecture 10, Example 3

Given the function $f(z) = \frac{\sin(z^3)}{z^2}$, find the singularity and determine if it is Removable?

---

f has an Isolated Singularity at $z = 0$.
Now, look at the Annulus where f is defined; which also determines where f is Analytic. f is analytic on the annulus $0 < |z| < \infty$. This disk comprises the set $\mathbb{C} - \{z = 0\}$.
Now, we expand f into a Laurent Series. We start by first expanding the a_n components of the series using the definition of the Power Series.

$$\begin{aligned} \sin(z) &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots \\ \sin\left(z^3\right) &= z^3 - \frac{1}{3!}\left(z^3\right)^3 + \frac{1}{5!}\left(z^3\right)^5 - \cdots \end{aligned}$$

Now, replace our simple sin with its corresponding power series.

$$\begin{aligned} f(z) &= \frac{z^3 - \frac{1}{3!}\left(z^3\right)^3 + \frac{1}{5!}\left(z^3\right)^5 - \cdots}{z^2} \\ &= z - \frac{1}{3!}z^7 + \frac{1}{5!}z^{13} - \cdots \end{aligned}$$

Note the complete lack of negative exponents of z. This means that the Principal Part of the Laurent Series has 0 for all its $b_1, b_2, b_3, \ldots$ terms.
$\therefore$ The Singularity $z = 0$ is Removable.

Defn 37 (Pole)**.** A *pole* is the Singularity of a Complex Function when the Laurent Series has a finite number of terms in its Principal Part.
This pole would be located at $z = a$.

$$f(z) = \frac{g(z)}{z-a},\ g(a) \neq 0 \tag{3.11}$$

Defn 38 (Order)**.** The *order* of a Pole is the highest negative power in the Principal Part of a Laurent Series with a finite principal part.

Defn 39 (Residue)**.** The *residue* of a Complex Function is the coefficient of the first term of the Principal Part of the Laurent Series. This means that the residue of a function f with Laurent series $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$ is b_1.

However, if a function as a finite number of terms in the Principal Part of the Laurent Series, then there are formulas for finding the residues, based on the Order, n, of the Pole at the Singularity $z = a$.

$$b_1 = \operatorname*{Res}_{z=a}(f) = \lim_{z \to a} (z-a)^n f^{(n-1)}(z) \tag{3.12}$$

The number of residues correspond to the number of Poles in a Laurent Series. This is shown in Example 3.14.

Example 3.10: Laurent Series Pole Singularity. Lecture 10, Example 4

Given the function $f(z) = \frac{\sin(3z^2)}{z^{17}}$, find its Principal Part, Poles, Order of the pole, and the Residue?

The only Singularity here, is again $z = 0$. f is defined on the Deleted disk $0 < |z| < \infty$.
Now, we use the definition of the sin function's Power Series, Equation (3.4).

$$\begin{aligned}
\sin(z) &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots \\
\sin\left(3z^2\right) &= 3z^2 - \frac{1}{3!}\left(3z^2\right)^3 + \frac{1}{5!}\left(3z^2\right)^5 - \frac{1}{7!}\left(3z^2\right)^7 + \frac{1}{9!}\left(3z^2\right)^9 - \cdots \\
&= 3z^2 - \frac{3^3}{3!}z^6 + \frac{3^5}{5!}z^{10} - \frac{3^7}{7!}z^{14} + \frac{3^9}{9!}z^{18} - \cdots
\end{aligned}$$

Now, plugging back into f.

$$\begin{aligned}
f(z) &= \frac{3z^2 - \frac{3^3}{3!}z^6 + \frac{3^5}{5!}z^{10} - \frac{3^7}{7!}z^{14} + \frac{3^9}{9!}z^{18} - \cdots}{z^{17}} \\
&= 3z^{-15} - \frac{3^3}{3!}z^{-11} + \frac{3^5}{5!}z^{-7} - \frac{3^7}{7!}z^{-3} + \frac{3^9}{9!}z^1 - \cdots
\end{aligned}$$

Therefore, the Principal Part of f is

$$3z^{-15} - \frac{3^3}{3!}z^{-11} + \frac{3^5}{5!}z^{-7} - \frac{3^7}{7!}z^{-3} + \frac{3^9}{9!}z^1$$

Because the Principal Part is finite, the Singularity $z = 0$ is a Pole. In addition, the pole is of Order 15.
Lastly, the residue of f is b_1, which in this case is 0.

Defn 40 (Essential Singularity)**.** An *essential singularity* only occurs when the exponent of the z terms of the b_n terms are all negative and there are infinitely many terms. This can be seen like so:

$$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \frac{b_4}{(z-a)^4} + \cdots$$

Example 3.11: Laurent Series Essential Singularity. Homework 7, Problem 5

What category is the Singularity of $f(z) = z^7 \sin\left(\frac{1}{z^2}\right)$? What is its residue?

3.3.2 Reverse Engineer Function from Laurent Series Properties

Now that we know how to find the various properties of a Complex Function with a Laurent Series, what if we are given the various properties of the function. What does the function look like then? This question is explored in Example 3.12.

Example 3.12: Laurent Series Properties Find Function. Lecture 10, Example 5

Suppose $f(z)$ is Analytic on the Annulus $0 < |z-a| < r$, and has a single Pole of Order 1. What does the Complex Function look like?

We start with the general form of a Complex Function expressed as a Laurent Series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$$

Now, we remove terms and items that we do not need. Because we are told that the only Pole has Order of 1, the highest term in the b_n portion has an exponent of 1 on the z term. This means we only go up to the b_1 term. Thus, our equation is:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$$

Example 3.12 shows us something else that is will become important in Section 3.4. We can perform some algebraic manipulations to have the numerator of some Analytic Complex Function be another analytic function. Take our solution from Example 3.12.

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} \\ &= \frac{(z-a)\left(\sum_{n=0}^{\infty} a_n (z-a)^n\right) + b_1}{z-a} \\ &= \frac{g(z)}{z-a} \end{aligned}$$

Thus, $g(z)$ is a sum of a power series with center $z = a$, meaning g is Analytic on $|z-a| < r$, where $g(a) = b_1 \neq 0$. If $g(a) = 0$, that means there would be some $z - a$ term that could cancel another.

3.3.3 Multiple Poles

Sometimes, it is tricky to determine the number of Poles present in an equation. This is explored in Example 3.13

Example 3.13: Laurent Series Non-Obvious Number of Poles. Lecture 10, Example 7

Given the function $f(z) = \frac{z^2-5z+6}{z^2-4}$, what are its Poles?

Start by factoring the equation.

$$\begin{aligned} f(z) &= \frac{z^2-5z+6}{z^2-4} \\ &= \frac{(z-2)(z-3)}{(z-2)(z+2)} \\ &= \frac{z-3}{z+2} \end{aligned}$$

Now, looking at this, we see it is obvious that there is only a single Pole of Order 1 (a Simple pole) at $z = -2$. However, $z = 2$ **IS** a Singularity for $f(z)$, but it is Removable.

Defn 41 (Simple)**.** A *simple* Pole is one that is of Order 1.

Example 3.14: Laurent Series Multiple Residues. Lecture 10, Example 8

Given the function $f(z) = \frac{z+1}{(z-2)(z+3)}$, what are the Residues of f?

We start by noting that there are 2 singularities, $z = 2$ and $z = -3$. As these singularities **cannot** be canceled out, they are Poles too. Both of these poles are Simple, as well. Because they are both of Order 1, we modify Equation (3.12) for that.

Starting with the Pole $z = 2$.

$$\begin{aligned}
\operatorname*{Res}_{z=2}(f) &= \lim_{z\to 2}(z-2)f(z) \\
&= \lim_{z\to 2}(z-2)\left(\frac{z+1}{(z-2)(z+3)}\right) \\
&= \lim_{z\to 2}\frac{z+1}{z+3} \\
&= \frac{3}{5}
\end{aligned}$$

Now, the Pole $z = -3$.

$$\begin{aligned}
\operatorname*{Res}_{z=-3}(f) &= \lim_{z\to -3}\big(z-(-3)\big)f(z) \\
&= \lim_{z\to -3}(z+3)\left(\frac{z+1}{(z-2)(z+3)}\right) \\
&= \lim_{z\to -3}\frac{z+1}{z-2} \\
&= \frac{-2}{-5} \\
&= \frac{2}{5}
\end{aligned}$$

Thus, our two Residues are:

$$\begin{aligned}
\operatorname*{Res}_{z=2}(f) &= \frac{3}{5} \\
\operatorname*{Res}_{z=-3}(f) &= \frac{2}{5}
\end{aligned}$$

Lastly, another example of finding the Residues of a Complex Function, but this time a function with a Pole of Order 2. This is demonstrated in Example 3.15

Example 3.15: Laurent Series Higher-Order Residues. Lecture 10, Example 9

Given the function $f(z) = \frac{e^z}{(z-2)^2(z+3)}$, find its Residues?

This has two singularities: $z = 2$ and $z = -3$. In addition, these are both Poles as well. The $z = 2$ pole has Order 2, the other has order 1.
Starting with the $z = 2$ Pole:

$$\begin{aligned}
\operatorname*{Res}_{z=2}(f) &= \lim_{z\to 2}\frac{d}{dz}\left\{(z-2)^2\left(\frac{e^z}{(z-2)^2(z+3)}\right)\right\} \\
&= \lim_{z\to 2}\frac{d}{dz}\left\{\frac{e^z}{z+3}\right\} \\
&= \lim_{z\to 2}\frac{e^z(z+3)-e^z(1)}{(z+3)^2} \\
&= \frac{5e^z - e^1}{5^2} \\
&= \frac{4e^z}{25}
\end{aligned}$$

Now, the $z = -3$ pole:

$$\begin{aligned}
\operatorname*{Res}_{z=-3}(f) &= \lim_{z \to -3} (z+3) \left(\frac{e^z}{(z-2)^2 (z+3)} \right) \\
&= \lim_{z \to -3} \frac{e^z}{(z-2)^2} \\
&= \frac{e^{-3}}{(-3-2)^2} \\
&= \frac{e^{-3}}{(-5)^2} \\
&= \frac{e^{-3}}{25}
\end{aligned}$$

Thus, our two Residues are:

$$\begin{aligned}
\operatorname*{Res}_{z=2}(f) &= \frac{4e^z}{25} \\
\operatorname*{Res}_{z=-3}(f) &= \frac{e^{-3}}{25}
\end{aligned}$$

3.3.4 Zeros of a Function

Defn 42 (Zero)**.** Let f be a Complex Function defined on an Open, Connected set (domain), Ω. Let $a \in \Omega$. Let $f \neq 0$, to avoid trivial cases.

We say that f has a *zero* at $z = a$ if $f(a) = 0$.

Defn 43 (Simple Zero)**.** A *simple zero* is a Zero with a Multiplicity of 1.

Example 3.16: Find Simple Zeros. Lecture 11, Example 1

Find all zeros of the function $f(z)$ below.

$$f(z) = \sin(z)$$

---

f has a zero at $z = 0, \pi, 2\pi, \ldots$, because when z is set to any of those values, the function yields a value of 0.

In addition to finding the location of a Zero, we also need to find the Multiplicity of the zero.

3.3.5 Multiplicity

Theorem 3.6 (Multiplicity)**.** *$f \neq 0$ has a Zero at $z = a$ if and only if there exists an Analytic Complex Function g on Ω and $n \geq 1$, $n \in \mathbb{N}$ such that $f(z) = (z-a)^n g(z)$.*

The value of n is the multiplicity *of the zero $z = a$.*

Example 3.17: Find Multiplicity by Observation. Lecture 11, Example 2

Given the Complex Function $f(z)$, as defined below, what are the zeros of the function and their multiplicities?

$$f(z) = (z^2 - 4)^3 (z^2 - z - 6)^5$$

The first thing to check that f is Analytic on some Ω, where I assume $\Omega = \mathbb{C}$. f is analytic, so we can move on. For this polynomial, we can find all the zeros by factoring.

$$\begin{aligned}
f(z) &= (z^2 - 4)^3 (z^2 - z - 6)^5 \\
&= (z-2)(z+2)^3 (z+2)(z-3)^5 \\
&= (z-2)^3 (z+2)^3 (z+2)^5 (z-3)^5 \\
&= (z-2)^3 (z+2)^8 (z-3)^5
\end{aligned}$$

Therefore, the three zeros are:

$z = 2$, with Multiplicity 3
$z = -2$, with Multiplicity 8
$z = 3$, with Multiplicity 5

Because we have only found the Multiplicity by observation so far, we would prefer an algorithm to figure it out. This would be particularly useful if we cannot factor the term. The algorithm is illustrated by Example 3.18, and is codified in Algorithm 3.1.

Example 3.18: Multiplicity Algorithm. Lecture 11, Example 3

Suppose f has a zero of Multiplicity 2 at $z = a$. This means that the function is of the general form $f(z) = (z-a)^2 g(z)$, where $g(a) \neq 0$.
To confirm that the Multiplicity is actually 2, we perform the derivation below.

Start by taking the derivative of f.

$$f'(z) = 2(z-a)g(z) + (z-a)^2 g'(z)$$

Now, plug $z = a$ directly into $f'(z)$.

$$\begin{aligned} f'(a) &= 2(a-a)g(a) + (a-a)^2 g'(a) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Because $f'(z) = 0$, we must repeat this procedure again.

$$f''(z) = (z-a)^2 g''(z) + 2(z-a)g(z) + 2(z-a)g'(z) + 2g(z)$$

Plug $z = a$ into $f''(z)$.

$$\begin{aligned} f''(a) &= (a-a)^2 g''(a) + 2(a-a)g(a) + 2(a-a)g'(a) + 2g(a) \\ &= 0 + 0 + 0 + 2g(z) \end{aligned}$$

We stop at the point where $f^{(n)}(a) \neq 0$, and the value of n is the Multiplicity of the zero $z = a$.

Algorithm 3.1: Multiplicity

Input : An Analytic Complex Function, $f(z)$.
Output: Multiplicity of the zero $z = a$.

Confirm that the function f is Analytic, and has a Zero at $z = a$.
$h(z) \leftarrow f(z)$
$n \longleftarrow 0$
while $h(a) = 0$ **do**
    $n \leftarrow n + 1$
    $h(z) \leftarrow \dfrac{d^n}{dz^n}\{f(z)\}$
return n, *as the Multiplicity of the Zero* $z = a$.

Sometimes, we aren't limited to using Algorithm 3.1 to find the Multiplicity of a Zero. Example 3.19 shows this.

Example 3.19: 2 Ways to find Multiplicity. Lecture 11, Example 4

Find the Multiplicity of the Zero $z = 0$ for $g(z) = \sin(z) - z$?

We start by confirming that $z = 0$ is a Zero, which it is, and that $g(z)$ is Analytic, which it also is.
The first method I will use to find the Multiplicity of $z = 0$ is Algorithm 3.1.

$$g(0) = 0 - 0$$

Take the derivative of g.

$$g'(z) = \cos(z) - 1$$

Apply $z = 0$ to g'.

$$\begin{aligned} g'(a) &= \cos(0) - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Take the derivative of g again.

$$g''(z) = -\sin(z)$$

Apply $z = 0$ to g''.

$$\begin{aligned} g''(a) &= -\sin(0) \\ &= 0 \end{aligned}$$

Take the derivative of g again.

$$g'''(z) = -\cos(z)$$

Apply $z = 0$ to g'''.

$$\begin{aligned} g'''(a) &= -\cos(0) \\ &= -1 \end{aligned}$$

Thus, the Multiplicity of $z = 0$ for $g(z)$ is 3.
Now, we can use Equation (3.4) to help simplify and factor this.

Recall:

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots$$

Now replace it in $g(z)$.

$$\begin{aligned} g(z) &= \left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right) - z \\ &= -\frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots \\ &= z^3\left(\frac{1}{3!} + \frac{1}{5!}z^2 - \cdots\right) \end{aligned}$$

And again, the Multiplicity shows up again, like it did earlier, as a factor we can observe.

Because the Multiplicity of a zero shows a potential factor, we can use this to our advantage when finding another multiplicity.

Example 3.20: Use one Multiplicity for Another. Lecture 11, Example 4

Given the functions $h(z)$ and $g(z)$, find the Multiplicity of the Zero $z = 0$?

$$\begin{aligned} h(z) &= \big(g(z)\big)^{10} \\ g(z) &= \sin(z) - z \end{aligned}$$

We found the Multiplicity of $z = 0$ for $g(z)$ in Example 3.19. The multiplicity of $z = 0$ was 3.
Therefore, we can argue that $h(z) = \big(z^3 f(z)\big)^{10}$, where $f(z)$ is some function of z that we don't really care about.
Now, perform some simplifications:

$$\begin{aligned} h(z) &= \big(z^3 f(z)\big)^{10} \\ &= (z^3)^{10}\big(f(z)\big)^{10} \\ &= z^{30}\big(f(z)\big)^{10} \end{aligned}$$

Thus, the Zero $z = 0$ for $h(z)$ has Multiplicity 30.

3.3.6 Generalizing Singularities

From our earlier definition of Poles, we know $f(z) = \frac{g(z)}{z-a}$, where $g(a) \neq 0$ and g is Analytic on some disk with center $z = a$ will have a singularity at $z = a$.

But, and equivalent definition of this is to have the denominator be a Complex Function of z too. This is shown in Equation (3.13).

$$f(z) = \frac{g(z)}{h(z)},\ g(a) \neq 0 \tag{3.13}$$

Thus, h has a Simple Zero at $z = a$.

Example 3.21: Use General Pole Equation. Lecture 11, Example 5

Given the function $f(z)$, what are the Zeros of f? What are the singularities of f? Classify them?

$$f(z) = \frac{(z^2-9)^2}{(z^2-6z+9)(z-5)^3}$$

First, we factor f.

$$\begin{aligned} f(z) &= \frac{(z^2-9)^2}{(z^2-6z+9)(z-5)^3} \\ &= \frac{(z-3)^2(z+3)^2}{(z-3)(z-3)(z-5)^3} \\ &= \frac{(z-3)^2(z+3)^2}{(z-3)^2(z-5)^3} \end{aligned}$$

Thus:

Singularities are $z = 3$ and $z = 5$, and must be excluded.
Zeros are $z = -3$. The case of $z = 3$ being a zero is excluded by the fact that $z = 3$ is a singularity. Thus, $z = 3$ is prohibited from being a zero.

Now, performing some arithmetic reductions.

$$\begin{aligned} f(z) &= \frac{(z-3)^2(z+3)^2}{(z-3)^2(z-5)^3} \\ &= \frac{(z+3)^2}{(z-5)^3} \end{aligned}$$

Remember that our singularities are still $z = 3$ and $z = 5$, but $z = 3$ is a Removable Singularity, and $z = 5$ is a Pole. The Pole $z = 5$ has Order 3.

3.3.7 Generalizing Residues

Now that we have 2 ways to express a Pole, Equation (3.11) and Equation (3.13). This means that we now have 2 ways to finding Residues as well.

The first way is the same as in Equation (3.12).

The second way is shown below:

$$
\begin{aligned}
f(z) &= \frac{h(z)}{k(z)},\ k \text{ has some simple zero} \\
\operatorname*{Res}_{z=a}(f) &= \lim_{z\to a}(z-a)\left(\frac{h(z)}{k(z)}\right) \\
&= \lim_{z\to a}\frac{h(z)}{\frac{k(z)}{z-a}} \\
&= \lim_{z\to a}\frac{h(z)}{\frac{k(z)-k(a)}{z-a}} \\
&= \lim_{z\to a}\frac{h(z)}{k'(a)}
\end{aligned}
$$

The result from above is shown for posterity below.

$$
\operatorname*{Res}_{z=a}(f) = \lim_{z\to a}\frac{h(z)}{k'(a)},\ k'(a) \neq 0 \tag{3.14}
$$

You want to use Equation (3.14) when $k(z)$ and $h(z)$ cannot be factored in any meaningful way. An example of this is shown in Example 3.22.

Example 3.22: Using General Residue Equation. Lecture 11, Example 7

Given $f(z)$, find all singularities and their Residues?

$$
f(z) = \frac{\cos(z)}{e^z - 1}
$$

The Singularity is at $e^z = 1$. So, we simplify for that.

$$
\begin{aligned}
e^z &= 1 \\
z &= \ln(1) \\
&= \log_e|1| + i\arg(1) \\
&= \log_e(1) + i(2\pi k) \\
&= 0 + 2\pi k i \\
&= 2\pi k i
\end{aligned}
$$

Therefore, there are singularities at $z = 2\pi k i$, where $k = 0, 1, 2, \ldots$.
If we let $g(z) = e^z - 1$, then $g(z)$ has Zeros at $z = 2\pi k i$, where $k = 0, 1, 2, \ldots$. We can find the Multiplicity of this zero with Algorithm 3.1.

$$
\begin{aligned}
g(z) &= e^z - 1 \\
g(2\pi k i) &= e^{2\pi k i} - 1 \\
&= \big(\cos(2\pi k) + i\sin(2\pi k)\big) - 1 \\
&= (1 + 0i) - 1 \\
&= 0 \\
g'(z) &= e^z \\
g'(2\pi k i) &= e^{2\pi k i} \\
&= 1
\end{aligned}
$$

Therefore $z = 2\pi k i$ is a Simple Zero of g, and is a Simple of f.

To find the residue, we want to use Equation (3.14), because there is no reasonable factoring to help us here.

$$\begin{aligned}
\operatorname*{Res}_{z=2\pi ki}(f) &= \frac{\cos(z)}{\frac{d}{dz}\{e^z - 1\}} \\
&= \frac{\cos(z)}{e^z} \\
&= \frac{\cos(2\pi ki)}{e^{2\pi ki}} \\
&= \frac{\cosh(2\pi k)}{1} \\
&= \cosh(2\pi k)
\end{aligned}$$

The reduction for the exponential is shown above.

3.4 Cauchy's Residue Theorem

This is the 3rd and final method of integration we will discuss.

Theorem 3.7 (Cauchy's Residue Theorem)**.** *Let Ω be a Connected, Open, Simply Connected region. Let Γ be a Path in Ω which is Simple, Closed, and counterclockwise oriented. Let f be Analytic on Ω, except perhaps with finitely many singularities, with no Singularity on the image of the path Γ.*

Then,

$$\int_\Gamma f(z)dz = 2\pi i \left(\sum_{k=1}^{n} \operatorname*{\mathit{Res}}_{z=a_k}(f) \right) \tag{3.15}$$

The integral is equal to $2\pi i \times$ (Sum of residues at singularities inside Γ).

Remark. Note that Cauchy's Integral Theorem is actually a special case of Cauchy's Residue Theorem.

Example 3.23: Use Cauchy's Residue Theorem. Lecture 11, Example 7

Evaluate $\int_\Gamma \frac{\cos(z)}{e^z-1}$ where Γ is the rectangle with vertices $\pm 5\pi \pm \pi i$?

Let $f(z) = \frac{\cos(z)}{e^z-1}$. From Example 3.22, we know that the singularities for f are located at $z = 2\pi ki$, where $k = 0, 1, 2, \ldots$.

Now, we look at the singularities **inside** the Closed Path Γ. The only Singularity that is inside this region is when $k = 0$, so $z = 0$.

$$\begin{aligned}
\int_\Gamma \frac{\cos(z)}{e^z - 1} dz &= 2\pi i \left(\operatorname*{Res}_{z=0}(f) \right) \\
&= \big[2\pi i \big(\cosh(2\pi k) \big) \big]_{k=0} \\
&= 2\pi i \big(\cosh(0) \big) \\
&= 2\pi i(1) \\
&= 2\pi i
\end{aligned}$$

A Trigonometry

A.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{A.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{A.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{A.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.7}$$

A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{A.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{A.9}$$

A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{A.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{A.11}$$

A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{A.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{A.13}$$

A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{A.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{A.15}$$

A.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{A.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{A.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{A.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{A.19}$$

A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{A.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{A.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{A.22}$$

A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{A.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{A.24}$$

A.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{A.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{A.26}$$

A.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{A.27}$$

B Calculus

B.1 L'Hôpital's Rule

L'Hôpital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcilable results.

Lemma B.0.1 (L'Hôpital's Rule)**.** *If the equation*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (B.1) *holds.*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{B.1}$$

B.2 Fundamental Theorems of Calculus

Defn B.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{B.2}$$

Defn B.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then

$$\begin{aligned} \frac{d}{dx}\int_a^x f(t)\,dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{B.3}$$

Defn B.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

B.3 Rules of Calculus

B.3.1 Chain Rule

Defn B.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{B.4}$$

B.4 Useful Integrals

$$\int \cos(x)\,dx = \sin(x) \tag{B.5}$$

$$\int \sin(x)\,dx = -\cos(x) \tag{B.6}$$

$$\int x\cos(x)\,dx = \cos(x) + x\sin(x) \tag{B.7}$$

Equation (B.7) simplified with Integration by Parts.

$$\int x\sin(x)\ dx = \sin(x) - x\cos(x) \tag{B.8}$$

Equation (B.8) simplified with Integration by Parts.

$$\int x^2\cos(x)\ dx = 2x\cos(x) + (x^2-2)\sin(x) \tag{B.9}$$

Equation (B.9) simplified by using Integration by Parts twice.

$$\int x^2\sin(x)\ dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{B.10}$$

Equation (B.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\ dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{B.11}$$

$$\int e^{\alpha x}\sin(\beta x)\ dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{B.12}$$

$$\int e^{\alpha x}\ dx = \frac{e^{\alpha x}}{\alpha} \tag{B.13}$$

$$\int xe^{\alpha x}\ dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{B.14}$$

Equation (B.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha+\beta x} = \int \frac{1}{\alpha+\beta x}\ dx = \frac{1}{\beta}\ln(\alpha+\beta x) \tag{B.15}$$

$$\int \frac{dx}{\alpha^2+\beta^2x^2} = \int \frac{1}{\alpha^2+\beta^2x^2}\ dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{B.16}$$

$$\int \alpha^x\ dx = \frac{\alpha^x}{\ln(\alpha)} \tag{B.17}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{B.18}$$

B.5 Leibnitz's Rule

Lemma B.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\ dx + f\big[b(t),t\big]\ \frac{db(t)}{dt} - f\big[a(t),t\big]\ \frac{da(t)}{dt} \tag{B.19}$$

B.6 Laplace's Equation

Laplace's Equation is used to define a harmonic equation. These functions are twice continuously differentiable $f: U \to R$, where U is an open subset of $\mathbb{R}^n$, that satisfies Equation (B.20).

$$\frac{\partial^2 f}{\partial {x_1}^2} + \frac{\partial^2 f}{\partial {x_2}^2} + \cdots + \frac{\partial^2 f}{\partial {x_n}^2} = 0 \tag{B.20}$$

This is usually simplified down to

$$\nabla^2 f = 0 \tag{B.21}$$

C Laplace Transform

C.1 Laplace Transform

Defn C.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int\limits_{-\infty}^{\infty} x(t)e^{-st}dt \tag{C.1}$$

C.2 Inverse Laplace Transform

Defn C.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\, ds \tag{C.2}$$

C.3 Properties of the Laplace Transform

C.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (C.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{C.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{C.3b}$$

C.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (C.4).

$$\mathcal{L}\big\{x(\alpha t)\big\} = \frac{1}{\alpha}X\left(\frac{s}{\alpha}\right) \tag{C.4}$$

C.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (C.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\big\{x(t-a)\big\} = X(s)e^{-as} \tag{C.5}$$

C.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\big\{X(s-a)\big\} = x(t)e^{at} \tag{C.6}$$

C.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\, d\lambda\right\} = \frac{1}{s}X(s) \tag{C.7}$$

C.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\big\{x(t) * v(t)\big\} = X(s)V(s) \tag{C.8}$$

C.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (C.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \tag{C.9}$$

C.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem C.1 (Intial Value Theorem)**.** *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Theorem C.2 (Final Value Theorem)**.** *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

C.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t-T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n\,\mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t-T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at}\,\mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at}\,\mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)\,\mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)\,\mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)\,\mathcal{U}(t)$	$\begin{cases} a: & \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: & \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb}+\frac{re^{-j\theta}}{s+a+jb}\right) \\ c: & \frac{As+B}{s^2+2as+c} \begin{cases} r & = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta & = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$
$e^{-at}\left(A\cos(\sqrt{c-a^2}t)+\frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

C.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0)-X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0)+X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n \frac{d^n}{ds^n}X(s)\; n\in\mathbb{N}$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s)-\sum_{0}^{n-1} s^{n-1-i}\frac{d^i}{dt^i}x(t)\vert_{t=0^-}\; n\in\mathbb{N}$