EITF75: Systems and Signals — Reference Sheet
Lund University

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1 Sinusoids

There are several ways to characterize Sinusoids. The first is by dimension:

1. Multidimensional/Multichannel Signals
2. Monodimensional/Monochannel Signals

You can also classify sinusoids by their independent variable (usually time) and the values they take.

1. Continuous-Time Signals or Analog Signals
2. Discrete-Time Signals
3. There is a third way to classify sinusoids and their signals: Digital Signals

Defn 1 (Continuous-Time Signals)**.** *Continuous-time signals* or *Analog signals* are defined for every value of time and they take on values in the continuous interval (a, b), where a can be $-\infty$ and b can be ∞. Mathematically, these signals can be described by functions of a continuous variable.

For example,

$$x_1(t) = \cos \pi t,\ x_2(t) = e^{-|t|},\ -\infty < t < \infty$$

Defn 2 (Discrete-Time Signals)**.** *Discrete-time signals* are defined only at certain specifed values of time. These time instants ***need not*** be equidistant, but in practice, they are usually taken at equally speced intervals for computation convenience and mathematical tractability.

For example,

$$x(t_n) = e^{-|t_n|},\ n = 0, \pm 1, \pm 2, \ldots$$

A Discrete-Time Signals can be represented mathematically by a sequence of real or complex numbers.

Remark 2.1. To emphasize the discrete-time nature of the signal, we shall denote the signal as $x(n)$, rather than $x(t)$.

Remark 2.2. If the time instants t_n are equally spaced (i.e., $t_n = nT$), the notation $x(nT)$ is also used.

1.1 Continuous-Time Signals

1.1.1 Frequency in Continuous-Time Signals

A simple harmonic oscillation is mathematically described by Equation (1.1).

$$x_a(t) = A \cos\left(\Omega t + \theta\right),\ -\infty < t < \infty \tag{1.1}$$

Remark. The subscript a is used with $x(t)$ to denote an analog signal.

This signal is completely characterized by three parameters:

1. A, the *amplitude* of the sinusoid
2. Ω, the *frequency* in radians per second (rad/s)
3. θ, the *phase* in radians.

Instead of Ω, the frequency F in cycles per second or hertz (Hz) is used.

$$\Omega = 2\pi F \tag{1.2}$$

Plugging (1.2) into (1.1), yields

$$x_a(t) = A \cos\left(2\pi F t + \theta\right),\ -\infty < t < \infty \tag{1.3}$$

1.1.2 Properties of Continuous-Time Sinusoidal Signals

The analog sinusoidal signal in equation (1.3) is characterized by the following properties:

(i) For every fixed value of the frequency F, $x_a(t)$ is periodic.

$$x_a(t + T_p) = x_a(t)$$

where $T_p = \frac{1}{F}$ is the fundamental period.

(ii) Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.

(iii) Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in the given time interval.

1.2 Discrete-Time Signals

These are usually found by sampling analog signals or Continuous-Time Signals. There are 2 ways to express this, both are shown in Equation (1.4).

$$\begin{aligned} x(n) &= x(t|t = nT_S) \\ x(n) &= x\left(t|t = \frac{n}{F_S}\right) \end{aligned} \tag{1.4}$$

1.2.1 Frequency in Discrete-Time Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta)\,,\; n \in \mathbb{Z}\,,\; -\infty < n < \infty \tag{1.5}$$

The signal is characterized by these parameters:

1. n, the sample number. MUST be an integer.
2. A, the *amplitude* of the sinusoid
3. ω, the *angular frequency* in radians per sample
4. θ, is the *phase*, in radians.

Instead of ω, we use the frequency variable f defined by

$$\omega \equiv 2\pi f \tag{1.6}$$

Using (1.5) and (1.6) yields

$$x(n) = A\cos(2\pi f n + \theta)\,,\; n \in \mathbb{Z}\,,\; -\infty < n < \infty \tag{1.7}$$

1.2.2 Properties of Discrete-Time Sinusoidal Signals

(i) A discrete-time sinusoid is periodic ***ONLY*** if its frequency is a rational number.

(ii) Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical. This leads us to the idea of a Frequency Alias.

(iii) The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pm\pi$ or, equivalently, $f = \pm\frac{1}{2}$.

1.2.3 Frequency Aliases

The concept of a Frequency Alias is drawn from the idea that discrete-time sinusoids whose frequencies are separated by an integer mutliple of 2π are identical and that frequencies $|f| > \frac{1}{2}$ are identical. (Properties **(ii)** and **(iii)**)

Defn 3 (Frequency Alias)**.** A *frequency alias* is a sinusoid having a frequency $|\omega| > \pi$ or $|f| > \frac{1}{2}$. This is because this sinusoid is *indistinguishable* (*identical*) to one with frequency $|\omega| < \pi$ or $|f| < \frac{1}{2}$.

A *frequency alias* is a sequence resulting from the following assertion based on the sinusoid $\cos(\omega_0 n + \theta)$. It follows that

$$\cos\left[(\omega_0 + 2\pi)\,n + \theta\right] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta$$

As a result, all sinusoidal sequences

$$x_k(n) = A\cos(\omega_k n + \theta),\; k = 0, 1, 2, \ldots$$

where

$$\omega k = \omega_0 + 2k\pi,\; -\pi \le \omega_0 \le \pi$$

are *indistinguishable* (i.e., *identical*).

Because of this, we regard frequencies in the range of $-\pi \le \omega \le \pi$ or $-\frac{1}{2} \le f \le \frac{1}{2}$ as unique, and all frequencies that fall outside of these ranges as aliases.

Remark 3.1. It should be noted that there is a difference between discrete-time sinusoids and continuous-time sinusoids here. Continuous-time sinusoids have distinct signals for Ω or F in the entire range $-\infty < \Omega < \infty$ or $-\infty < F < \infty$.

1.3 Sampling Rates and Sampling Frequency

Most signals of interest are analog. To process these signals, they must be collected and converted to a digital form, that is, to convert them to a sequence of numbers having finite precision. This is called *analog-to-digital (A/D) conversion*. Conceptually, we view this conversion as a 3-step process.

1. Sampling
2. Quantization
3. Coding

1.3.1 Nyquist Rate

1.3.2 Nyquist Frequency

1.4 Digital Signals

Defn 4 (Digital Signals)**.** *Digital signals* are a subset of Discrete-Time Signals. In this case, not only are the values being measured occuring at fixed points in time, the values themselves can only take certain, fixed values.

1.4.1 Quantization

Defn 5 (Quantization)**.** This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-value (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the Quantization Error.

1.4.1.1 Quantization Levels

1.4.1.2 Quantization Error

Defn 6 (Quantization Error)**.** The *quantization error* of something.

1.4.1.3 Bit Requirements

1.4.1.4 Bit Rate

2 Discrete-Time Systems

As discussed in Section 1.2, $x(n)$ is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is *not defined* at instants between the samples. Also, if n is not an integer, $x(n)$ is not defined.

Besides graphical representation of a discrete-time system, there are 3 ways to represent a discrete-time signal.

1. Functional Representation
2. Tabular Representation
3. Sequence Representation

2.1 Representing Discrete-Time Systems

2.1.1 Functional Representation

This representation of a discrete-time system is done as a mathematical function.

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1}$$

2.1.2 Tabular Representation

This representation of a discrete-time sysem is done as a table of corresponding values.

n	...	-2	-1	0	1	2	3	4	5	...
$x(n)$	...	0	0	0	1	4	1	0	0	...

2.1.3 Sequence Representation

There are 2 methods of representation for this. The first includes all values for $-\infty < n < \infty$. In all cases, $n = 0$ is marked in the sequence, somehow. I will do this with an underline.

$$x(n) = \{\ldots, 0, \underline{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.2}$$

The second only works if all $x(n)$ values for $n < 0$ are 0.

$$x(n) = \{\underline{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.3}$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underline{-2}, 5, 0, 4, -1\} \tag{2.4}$$

This is identified as a seven-point sequence.

A finite-duration sequence where $x(n) = 0$ for all $n < 0$ is represented as

$$x(n) = \{\underline{0}, 1, 4, 1\} \tag{2.5}$$

This is identified as a four-point sequence.

2.2 Elementary Discrete-Time Signals

The following signals are basic signals that appear often and play an important role in signal processing.

2.2.1 Unit Impulse Signal

Defn 7 (Unit Impulse Signal)**.** The *unit impulse signal* or *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv= \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.6}$$

This function is a signal that is zero everywhere, except at $n = 0$, where its value is 1.

Remark 7.1. This signal is different that the analog signal $\delta(t)$, which is also called a unit impulse, and is defined to be 0 everywhere except $t = 0$. The discrete unit impulse sequence is much less mathematically complicated.

2.2.2 Unit Step Signal

Defn 8 (Unit Step Signal)**.** The *unit step signal* is denoted as $u(n)$ or as $\mathcal{U}(n)$ and is defined as

$$\mathcal{U}(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.7}$$

2.2.3 Unit Ramp Signal

Defn 9 (Unit Ramp Signal)**.** The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.8}$$

2.2.4 Exponential Signal

Defn 10 (Exponential Signal)**.** The *exponential signal* is a sequence of the form

$$x(n) = a^n \text{ for all } n \tag{2.9}$$

If a is real, then $x(n)$ is a real signal. When a is complex valued ($a \equiv b \pm cj$), it can be expressed as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n (\cos\theta n + j \sin\theta n) \end{aligned} \tag{2.10}$$

This can be expressed by graphing the real and imaginary parts

$$\begin{aligned} x_R(n) &\equiv r^n \cos\theta n \\ x_I(n) &\equiv r^n j \sin\theta n \end{aligned} \tag{2.11}$$

or by graphing the amplitude function and phase function.

$$\begin{aligned} |x(n)| &= A(n) \equiv r^n \\ \angle x(n) &= \phi(n) \equiv \theta n \end{aligned} \tag{2.12}$$

2.3 Classification of Discrete-Time Signals

In order to apply some mathematical methods to discrete-time signals, we must characterize these signals.

2.3.1 Energy Signal

Defn 11 (Energy Signal)**.** The energy E of a signal $x(n)$ is defined as

$$\begin{aligned} E &\equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \\ &\equiv \sum_{n=-\infty}^{\infty} x(n)x^*(n) \end{aligned} \tag{2.13}$$

The energy of a signal can be finite or infinite. If E is finite $(0 < E < \infty)$, then $x(n)$ is called an *energy signal.*

2.3.2 Power Signal

Defn 12 (Power Signal)**.** The average power of a discrete time signal $x(n)$ is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \tag{2.14}$$

This means that there are 2 potential outcomes:

1. If E is finite, $P = 0$
2. If E is infinite, P may be either finite or infinite

If P is finite and nonzero, the signal is called a *power signal.*

2.3.3 Periodic and Aperiodic Signals

A signal $x(n)$ is periodic with period N $(N > 0)$ if and only if

$$x(n+N) = x(n) \text{for all } n \tag{2.15}$$

The smallest value of N for which (2.15) holds is called the fundamental period. If there is no value of N that satisfies (2.15), the signal is called *nonperiodic* or *aperiodic.*

2.3.4 Symmetric and Antisymmetric Signals

A real-valued signal $x(n)$ is called *symmetric* or *even* if

$$x(n) = x(-n) \tag{2.16}$$

On the other hand, a signal $x(n)$ is called *antisymmetric* or *odd* if

$$x(n) = -x(-n) \tag{2.17}$$

2.4 Classification of Discrete-Time Systems

2.4.1 Static versus Dynamic Systems

Defn 13 (Static)**.** A discrete-time system is called *static* or *memoryless* if its output at any instant n depends only on the input sample at the same time, but not on past or future samples of the input.

Defn 14 (Dynamic)**.** A discrete-time system is called *dynamic* if its output at any instant n depends not only on the input sample at the same time, but **also** on past and/or future samples of the input.

If the output of s system at tine n is completely determined by the input samples in the interval from $n - N$ to $n (N \geq 0)$, the system is said to have a *memory* of duration N. If $N = 0$, then the system is Static, whereas if $N = \infty$, the system is said to have *infinite memory.*

2.4.2 Time-Invariant versus Time-Variant Systems

Defn 15 (Time-Invariant)**.** A *time-invariant* system in one whose output is affected only in time, if the input's time is changed. A relaxed system $\mathcal{T}$ is *time-invariant* or *shift invariant* is and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

implies that

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k)$$

for every input signal $x(n)$ and every time shift k.

To determine if any given system is Time-Invariant, we need to perform a test drawn from Definition 15.

1. Excite the system with an arbitrary input sequence $x(n)$, which produces an output $y(n)$.
2. Delay the input sequence by some amount k and recompute the output.
3. If $y(n,k) = y(n-k)$ for all possible values of k, the system is Time-Invariant.

2.4.3 Linear versus Non-Linear Systems

A linear system is one that satisfies the *superposition principle.*

Defn 16 (Linear)**.** A system is *linear* if and only if

$$\mathcal{T}\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 \mathcal{T}\left[x_1(n)\right] + a_2 \mathcal{T}\left[x_2(n)\right] \tag{2.18}$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

The Linearity property can be broken down into 2 parts:

1. Multiplicative Property
2. Additive Property

Remark 16.1. The Linearity property can be extended to any number of terms.

2.4.3.1 Multiplicative Property

Defn 17 (Multiplicative Property)**.** The *multiplicative* or *scaling property* is one requirement of a Linear system and is part of the definition of the superposition principle. If the input is scaled, the output is scaled by a proportional amount.

$$\begin{aligned} \mathcal{T}\left[a_1 x_1(n)\right] &= a_1 \mathcal{T}\left[x_1(n)\right] \\ &= a_1 y_1(n) \end{aligned} \tag{2.19}$$

2.4.3.2 Additive Property

Defn 18 (Additive Property)**.** The *additive property* is one requirement of a Linear system and is part of the definition of the superposition principle.

$$\begin{aligned} \mathcal{T}\left[x_1(n) + x_2(n)\right] &= \mathcal{T}\left[x_1(n)\right] + \mathcal{T}\left[x_2(n)\right] \\ &= y_1(n) + y_2(n) \end{aligned} \tag{2.20}$$

Defn 19 (Nonlinear)**.** If a relaxed system does no satisfy the superposition principle, or the definition of a Linear system, it is *nonlinear.*

2.4.4 Causal versus Noncausal Systems

Defn 20 (Causal)**.** A system is said to *causal* if the output of the system, $y(n)$, at any time n depends only on present and past inputs [i.e., $x(n), x(n-1), x(n-2), \ldots$], but does not depends on future inputs [i.e., $x(n+1), x(n+2), \ldots$].

Mathematically, the output of a causal system satisfies an equation of the form

$$y(n) = F\left[x(n), x(n-1), x(n-2), \ldots\right] \tag{2.21}$$

where $F[\cdot]$ is some arbitrary function.

Defn 21 (Noncausal)**.** If a system does not satisfy the definition of a Causal system, then it is *noncausal.* A noncausal system depends not just on present and past inputs, but also on future inputs.

Remark 21.1. You can never have a noncausal system in real-time signal processing applications. However, if the signal has been recorded and will be processed offline, then a noncausal system can be constructed.

2.4.5 Stable versus Unstable Systems

Stability is incredibly important. Unstable stystems usually have erratic and extreme behavior.

Defn 22 (Stable)**.** An arbitrary relaxed system is said to be *Bounded Input-Bounded Output Stable (BIBO)* if and only if every bounded input produces a bounded output.

Mathematically, this means the input sequence $x(n)$ and the output sequence $y(n)$ are bounded, where there are some finite numbers M_x and M_y such that

$$|x(n)| \leq M_x < \infty \ \ |y(n)| \leq M_y < \infty \, \forall n \tag{2.22}$$

Defn 23 (Unstable)**.** If the some bounded input $x(n)$, the output is unbounded (infinite), the system is *unstable*.

2.4.6 Linear Time-Invariant Systems

Defn 24 (Linear Time-Invariant)**.** A *Linear Time-Invariant (LTI)* signal or system is one that is:

(i) Linear

(ii) Time-Invariant

2.5 Discrete-Time Signal Manipulations

2.5.1 Transformation of the Independent Variable (Time)

It is important to note that Shifting in Time and Folding are not commutative. For example,

$$\mathrm{TD_k}\{\mathrm{FD}\left[x(n)\right]\} = \mathrm{TD_k}\left[x(-n)\right] = x(-n+k) \tag{2.23}$$

whereas

$$\mathrm{FD}\{\mathrm{TD_k}\left[x(n)\right]\} = \mathrm{FD}\left[x(n-k)\right] = x(-n-k) \tag{2.24}$$

2.5.1.1 Shifting in Time

A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time (moves left). If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units of time (moves right).

This could be denoted by

$$\mathrm{TD_k}\left[x(n)\right] = x(n-k) \tag{2.25}$$

You cannot advance a signal that is being generated in real-time. Because that would involve signal samples that haven't been generated yet. So, you can only advance a signal that is stored on something. However, you can always introduce a delay to a signal.

2.5.1.2 Folding

Another useful modification of the time base is to replace n with $-n$. The result is a *folding* or *reflection* of the original signal around $n = 0$.

This could be denoted by

$$\mathrm{FD}\left[x(n)\right] = x(-n) \tag{2.26}$$

2.5.2 Addition, Multiplication, and Scaling

Amplitude modifications include Addition, Multiplication, and Amplitude Scaling.

2.5.2.1 Addition

The *sum* of 2 signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the sum of the values of these two signals at that instant.

$$y(n) = x_1(n) + x_2(n), \ -\infty < n < \infty \tag{2.27}$$

2.5.2.2 Multiplication

The *product* of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the product of the values of these two signals at that instant.

$$y(n) = x_1(n)x_2(n), \ -\infty < n < \infty \tag{2.28}$$

2.5.2.3 Amplitude Scaling

Amplitude scaling of a signal by a constant A is accomplished by multiplying every signal sample by A. Consequently, we obtain

$$y(n) = Ax(n), \ -\infty < n < \infty \tag{2.29}$$

2.6 Discrete-Time System Difference Equation

There exists an equation that describes any Linear Time-Invariant discrete-time system. This equation works for both Infinite Impulse Response and Finite Impulse Response filters.

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{l=0}^{L} b_l x(n-l) \tag{2.30}$$

Occasionally, Equation (2.30) will be written like below.

$$y(n) = \sum_{l=0}^{L} b_l x(n-l) - \sum_{k=1}^{N} a_k y(n-k)$$

Defn 25 (Infinite Impulse Response)**.** An *Infinite Impulse Response (IIR)* filter is one that has an impulse response which does not become exactly zero past a certain point, but continues indefinitely. This is opposite to a Finite Impulse Response Filter.

Defn 26 (Finite Impulse Response)**.** A *Finite Impulse Response (FIR)* filter is a filter whose impulse response (or response to any finite length input) is of finite duration, because it settles to zero in finite time. This is opposite to a Infinite Impulse Response Filter.

3 Convolutions

Defn 27 (Linear Convolution)**.** The *linear convolution* is more commonly called a *convolution*. It is a mathematical operation that involves infinite sums. It defines the relationship between 2 signals to produce an output.

$$\begin{aligned} y(n) &= x(k) * h(n-k) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \end{aligned} \tag{3.1}$$

Because of associativity, commutativity, and distributivity, Equation (3.1) can be equivalently rewritten as

$$\begin{aligned} y(n) &= h(n) * x(n-k) \\ &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \end{aligned} \tag{3.2}$$

The length of the resulting sequence from a linear convolution is

$$2L - 1 \tag{3.3}$$

where L is the length of the input sequences.

Remark 27.1 (Alternate Convolution Symbol). There is no single defined symbol for Linear Convolutions. In this text, and personally, I use the $*$ symbol. However, other texts may use:

1. $\cdot$ (Centered dot)
2. $\bullet$
3. etc.

To compute the Linear Convolution of Equations (3.1) to (3.2):

1. Perform a Folding on one of the two signals
2. If necessary, pad a signal with 0s to ensure the 2 signals are the same length, L
3. "Run" both signals by each other.
    - This is illustrated in Example 3.1.
4. Perform this for all values of n.

Example 3.1: Linear Convolution. Problem 2.16b, Part 2

Compute the Linear Convolution $y(n) = h(n) * x(n)$ of the following signal. Check your result with this formula

$\sum_{y \in Y} = \sum_{h \in H} \sum_{x \in X}$.

$$x(n) = \{\underline{1}, 2, -1\}$$
$$h(n) = \{\underline{1}, 2, -1\}$$

Start by Folding a signal, I chose $h(n)$ to get

$$h(-n) = \{-1, 2, \underline{1}\}$$

Now we "run" each signal past each other. The $x(n)$ signal is the left operand and the $h(-n)$ signal is the right operand in the multiplications below.

$$\begin{aligned}
y(0) &= (0 \cdot -1) + (0 \cdot 2) + (1 \cdot 1) + (2 \cdot 0) + (-1 \cdot 0) = 1 \\
y(1) &= (0 \cdot -1) + (1 \cdot 2) + (2 \cdot 1) + (-1 \cdot 0) = 4 \\
y(2) &= (1 \cdot -1) + (2 \cdot 2) + (-1 \cdot 1) = 2 \\
y(3) &= (1 \cdot 0) + (2 \cdot -1) + (-1 \cdot 2) + (0 \cdot 1) = -4 \\
y(4) &= (1 \cdot 0) + (2 \cdot 0) + (-1 \cdot -1) + (0 \cdot 2) + (0 \cdot 1) = 1 \\
y(5) &= (1 \cdot 0) + (2 \cdot 0) + (-1 \cdot 0) + (0 \cdot -1) + (0 \cdot 2) + (0 \cdot 1) = 0
\end{aligned}$$

Thus, our output sequence is

$$y(n) = \{\underline{1}, 4, 2, -4, 1\}$$

We can verify the length of the output to be $2L - 1$. Since $2(3) - 1 = 5$, and the convolution is of length 5, we are correct here.
Now we check our solution

$$\sum_{y \in Y} = 4$$
$$\sum_{x \in X} = 2$$
$$\sum_{h \in H} = 2$$

so, according to the equation provided

$$4 = 2 \cdot 2$$

is true and correct.
So, our answer is: $y(n) = \{\underline{1}, 4, 2, -4, 1\}$.

Defn 28 (Linear Time-Invariant System Convolution)**.** If there is a relaxed Linear Time-Invariant system to an input $x(n)$, then the output can be found by computing the Linear Convolution of the input with the sample response on the system. This results in the equation shown below.

$$y(n) = x(n) * h(n) \tag{3.4}$$

3.1 Properties of the Convolution

Identity Property	$y(n) = x(n) * \delta(n) = x(n)$
Shifting Property	$x(n) * \delta(n-k) = y(n-k) = x(n-k)$
Commutative Law	$x(n) * h(n) = h(n) * x(n)$
Associative Law	$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
Distributive Law	$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

Table 3.1: Properties of the Convolution

3.1.1 Identity Property

Defn 29 (Identity Property)**.** The Unit Impulse Signal is the identity element for the Linear Convolution.

$$y(n) = x(n) * \delta(n) = x(n) \tag{3.5}$$

3.1.2 Shifting Property

Defn 30 (Shifting Property)**.** Since the $\delta(n)$ function is the Identity function, if we shift $\delta(n)$ by k, the convolution sequence is also shifted by k.

$$x(n) * \delta(n-k) = y(n-k) = x(n-k) \tag{3.6}$$

3.1.3 Commutative Law

Defn 31 (Commutative Law for Convolutions)**.** The *commutative law for Linear Convolutions* is just like many other operations.

$$x(n) * h(n) = h(n) * x(n) \tag{3.7}$$

3.1.4 Associative Law

Defn 32 (Associative Law for Convolutions)**.** The *associative law for Linear Convolutions* is just like many other operations.

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)] \tag{3.8}$$

3.1.5 Distributive Law

Defn 33 (Distributive Law for Convolutions)**.** The *distributive law for Linear Convolutions* is just like many other operations.

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n) \tag{3.9}$$

3.2 Correlation

Defn 34 (Correlation)**.** *Correlation* measures the similarity between two signals. The greater the correlation, the more similar they are.

There are 2 types of Correlation.

1. Cross Correlation
2. Auto Correlation

3.2.1 Cross Correlation

Defn 35 (Cross Correlation)**.** *Cross correlation* measures the similarity between time shifted versions of ***different*** signals.

The defining equation is shown below:

$$r_{y,x}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) \tag{3.10}$$

However, there is a way to express Equation (3.10) in terms of a Linear Convolution.

$$r_{y,x}(k) = y(n) * x(-n) \tag{3.11}$$

3.2.2 Auto Correlation

Defn 36 (Auto Correlation)**.** *Auto correlation* measures the similarity between time shifted version of the ***same*** signal.

The defining equation is:

$$r_{x,x}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) \tag{3.12}$$

However, there is a way to express Equation (3.12) in terms of a Linear Convolution.

$$r_{x,x}(k) = x(n) * x(-n) \tag{3.13}$$

Remark 36.1. It is good to note that the Auto Correlation is technically a type of Cross Correlation where the second function is the same as the first.

4 The $\mathcal{Z}$-Transform

The $\mathcal{Z}$-Transform plays the same role in the analysis of Discrete-Time Signals and LTI systems as the Laplace Transform does in the analysis of Continuous-Time Signals and LTI systems.

4.1 The $\mathcal{Z}$-Transform

Defn 37 ($\mathcal{Z}$-Transform)**.** The *z-transform* is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.1}$$

Remark 37.1. For convenience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv \mathcal{Z}\{x(n)\} \tag{4.2}$$

and the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z) \tag{4.3}$$

4.1.1 Region of Convergence

Defn 38 (ROC)**.** The *ROC* or *region of convergence* is the region for which the infinite power series in the z-transform has a convergent solution.

Remark 38.1. ***Any time we cite a z-transform, we should also indicate its ROC***

Example 4.1: Simple $\mathcal{Z}$-Transform.

Determine the z-transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n \mathcal{U}(n)$$

The z-transform is the infinite power series

$$\begin{aligned} X(z) &= 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2}\right)^{-2} + \cdots + \left(\frac{1}{2}\right)^n z^{-n} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n \end{aligned}$$

Because this is an infinite geometric series, we can solve with with our equivalency:

$$1 + A + A^2 + \cdots + A^n + \cdots = \frac{1}{1-A} \text{ if } |A| < 1$$

Thus, $X(z)$ converges to

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad \text{ROC} : |z| > \frac{1}{2}$$

4.1.2 The One-Sided $\mathcal{Z}$-Transform

TODO

Defn 39 (One-Sided $\mathcal{Z}$-Transform)**.** The *one-sided z-transform* is the same as the $\mathcal{Z}$-Transform, but is only defined at n values greater than or equal to 0.

$$X(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n} \tag{4.4}$$

The One-Sided $\mathcal{Z}$-Transform is generally used when there are initial conditions on a causal signal. This captures the normal causal portion of the signal, while also showing the effect of the initial condition.

4.1.2.1 Application of The One-Sided $\mathcal{Z}$-Transform

Signal	ROC
Finite-Duration Signals	
Causal	Entire z-plane except $z = 0$
Anticausal	Entire z-plane except $z = \infty$
Two-Sided	Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals	
Causal	$\|z\| > r_2$
Anticausal	$\|z\| < r_1$
Two-Sided	$r_2 < \|z\| < r_1$

Table 4.1: Characteristic Familes of Signals with Their Corresponding ROCs

4.2 The Inverse $\mathcal{Z}$-Transform

This is the formal definition of The Inverse $\mathcal{Z}$-Transform.

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} \, dz \tag{4.5}$$

where the integrals is a contour integral over a closed path C that encloses the origin and lies within the region of convergence of $X(z)$.

There are 3 methods that are often used for the evaluation of the inverse z-transform in practice:

1. Direct evaluation of Equation (4.5). (Section 4.2.1)
2. Expansion into a series of terms, in the variables z and z^{-1}. (Section 4.2.2)
3. Partial-fraction expansion and table lookup. (Section 4.2.3)

4.2.1 The Inverse $\mathcal{Z}$-Transform by Contour Integration

Defn 40 (Cauchy's Integral Theorem)**.** Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the derivative $\frac{\mathrm{d}f(z)}{\mathrm{d}z}$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} \, \mathrm{d}z = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \tag{4.6}$$

More generally, if the $(k+1)$-order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k} \, \mathrm{d}z = \begin{cases} \frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1} f(z)}{\mathrm{d}z^{k-1}} \Big|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \tag{4.7}$$

4.2.2 The Inverse $\mathcal{Z}$-Transform by Power Series Expansion

TODO

4.2.3 The Inverse $\mathcal{Z}$-Transform by Partial-Fraction Expansion

TODO

4.3 Properties of the $\mathcal{Z}$-Transform

4.3.1 $\mathcal{Z}$-Transform Linearity

If

$$x_1(n) \overset{\mathrm{z}}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{\mathrm{z}}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{\mathrm{z}}{\longleftrightarrow} X(z) = a_1 X_1(z) + a_2 X_2(z) \tag{4.8}$$

for any constants a_1 and a_2.

The linearity property can be generalized to an arbitrary number of signals.

Property	Time Domain	z-Domain	ROC
	$x(n)$	$X(z)$	ROC : $r_2 < \|z\| < r_1$
Notation	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
$\mathcal{Z}$-Transform Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
$\mathcal{Z}$-Transform Time Shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
$\mathcal{Z}$-Domain Scaling	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
$\mathcal{Z}$-Transform Time Reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real Part	$\text{Re}\{x(n)\}$	$\frac{1}{2}\left[X(z) + X^*(z^*)\right]$	Includes ROC
Imaginary Part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j\left[X(z) - X^*(z^*)\right]$	Includes ROC
$\mathcal{Z}$-Domain Differentiation	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\|r_1$
$\mathcal{Z}$-Domain Convolutions	$x_1 * x_2$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
$\mathcal{Z}$-Transform 2 Sequence Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)x_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial Value Theorem for $\mathcal{Z}$-Transform	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
$\mathcal{Z}$-Transform 2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least, $r_{1l}r_{2l} < \|a\| < r_{1u}r_{2u}$
Parsevals Relation for $\mathcal{Z}$-Transform	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$= \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

Table 4.2: Properties of the $\mathcal{Z}$-Transform

Example 4.2: Simple Z-Transform Linearity Problem. Example 3.2.1

Deermine the z-transform and the ROC of the signal

$$x(n) = \left[3\left(2^n\right) - 4\left(3^n\right)\right]\mathcal{U}(n)$$

Solution on Page 158.

Example 4.3: Z-Transform Linearity on Trig Functions. Example 3.2.2

Determine the z-transform of the signals

(a) $x(n) = (\cos\omega_0 n)\,\mathcal{U}(n)$

(b) $x(n) = (\sin\omega_0 n)\,\mathcal{U}(n)$

Solution on Pages 158-159.

4.3.2 $\mathcal{Z}$-Transform Time Shifting

If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z) \tag{4.9}$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

Example 4.4: $\mathcal{Z}$-Transform Time Shifting. Example 3.2.3

By applying the time-shifting property, determine the z-transform of the signals

$$x_1(n) = \{1, 2, \underline{5}, 7, 0, 1\}$$
$$x_2(n) = \{\underline{0}, 0, 1, 2, 5, 7, 0, 1\}$$

from the z-transform of

$$x_0(n) = \{\underline{1}, 2, 5, 7, 0, 1\}$$
$$X_0(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}, \text{ROC : entire } z\text{-plane except } z = 0$$

Solution on Page 160.

4.3.3 $\mathcal{Z}$-Domain Scaling

If

$$x(n) \overset{\text{z}}{\longleftrightarrow} X(z), \text{ ROC} : r_1 < |z| < r_2$$

then

$$a^n x(n) \overset{\text{z}}{\longleftrightarrow} X\left(a^{-1}z\right), \text{ ROC} : |a|r_1 < |z| < |a|r_2 \tag{4.10}$$

4.3.4 $\mathcal{Z}$-Transform Time Reversal

If

$$x(n) \overset{\text{z}}{\longleftrightarrow} X(z), \text{ ROC} : r_1 < |z| < r_2$$

then

$$x(-n) \overset{\text{z}}{\longleftrightarrow} X(z^{-1}), \text{ROC} : \frac{1}{r_2} < |z| < \frac{1}{r_1} \tag{4.11}$$

Example 4.5: $\mathcal{Z}$-Transform Time Reversal. Example 3.2.6

Determine the z-transform of the signal

$$x(n) = \mathcal{U}(-n)$$

The transform for $\mathcal{U}(n)$ is given in Table 4.3.

$$\mathcal{U}(n) \overset{\text{z}}{\longleftrightarrow} \frac{1}{1 - x^{-1}}, \text{ ROC} : |z| > 1$$

By using (4.11), we obtain

$$\mathcal{U}(-n) \overset{\text{z}}{\longleftrightarrow} \frac{1}{1 - z}, \text{ROC} : |z| < 1$$

4.3.5 $\mathcal{Z}$-Domain Differentiation

If

$$x(n) \overset{\text{z}}{\longleftrightarrow} X(z)$$

then

$$nx(n) \overset{\text{z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} \tag{4.12}$$

Example 4.6: $\mathcal{Z}$-Domain Differentiation. Example 3.2.7

Determine the z-transform of the signal

$$x(n) = na^n\mathcal{U}(n)$$

The signal $x(n)$ can be expressed as $nx_1(n)$, where $x_1(n) = a^n \mathcal{U}(n)$. By passing this through the z-transform, we have

$$x_1(n) = a^n \mathcal{U}(n) \overset{\mathrm{z}}{\longleftrightarrow} X_1(z) = \frac{1}{1 - az^{-1}}, \text{ ROC} : |z| > |a|$$

Then by using (4.12), we obtain

$$na^n \mathcal{U}(n) \overset{\mathrm{z}}{\longleftrightarrow} X(z) = -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1 - az^{-1})^2}$$

4.3.6 $\mathcal{Z}$-Domain Convolutions

If

$$x_1(n) \overset{\mathrm{z}}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{\mathrm{z}}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \overset{\mathrm{z}}{\longleftrightarrow} X(z) = X_1(z)X_2(z) \tag{4.13}$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

Example 4.7: $\mathcal{Z}$-Domain Convolutions. Example 3.2.9

Compute the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$
$$x_2(n) = \begin{cases} 1, & 0 \le n \le 6 \\ 0, & \text{elsewhere} \end{cases}$$

When

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

According to (4.13) we carry out the multiplication of $X_1(z)$ and $X_2(z)$. Thus

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underline{1}, -1, 0, 0, 0, 0, -1, 1\}$$

4.3.7 $\mathcal{Z}$-Transform 2 Sequence Correlation

If

$$x_1(n) \overset{\mathrm{z}}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{\mathrm{z}}{\longleftrightarrow} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \overset{\mathrm{z}}{\longleftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) \tag{4.14}$$

Example 4.8: $\mathcal{Z}$-Transform 2 Sequence Correlation. Example 3.2.10

Determine the autocorrelation of the signal

$$x(n) = a^n \mathcal{U}(n), \ -1 < a < 1$$

Solution on Page 166.

4.3.8 $\mathcal{Z}$-Transform 2 Sequence Multiplication

If

$$x_1(n) \xleftrightarrow{\text{z}} X_1(z)$$
$$x_2(n) \xleftrightarrow{\text{z}} X_2(z)$$

then

$$x(n) = x_1(n)x_2(n) \xleftrightarrow{\text{z}} X_z = \frac{1}{2\pi j} \oint_C X_1(v)X_2\left(\frac{z}{v}\right) v^{-1} dv \tag{4.15}$$

where C is a closed contour that encloses the origin and lies within th eregion of convergence common to both $X_1(v)$ and $X_2(\frac{1}{v})$.

4.3.9 Parsevals Relation for $\mathcal{Z}$-Transform

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv \tag{4.16}$$

4.3.10 Initial Value Theorem for $\mathcal{Z}$-Transform

If $x(n)$ is *causal* [i.e., $x(n) = 0$ for $n < 0$], then

$$x(0) = \lim_{z \to \infty} X(z) \tag{4.17}$$

4.4 Properties of the One-Sided $\mathcal{Z}$-Transform

TODO

4.4.1 Time Delay

If

$$x(n) \xleftrightarrow{\text{z}^+} X^+(z)$$

then

$$x(n-k) \xleftrightarrow{\text{z}^+} z^{-k} \left[X^+(z) + \sum_{n=1}^{k} x(-n)z^n \right], \quad k > 0 \tag{4.18}$$

4.4.2 Time Advance

If

$$x(n) \xleftrightarrow{\text{z}^+} X^+(z)$$

then

$$x(n+k) \xleftrightarrow{\text{z}^+} z^{k} \left[X^+(k) - \sum_{n=0}^{k-1} x(n)z^{-n} \right], \quad k > 0 \tag{4.19}$$

4.5 Rational $\mathcal{Z}$-Transforms

An important family of z-transforms are those for which $X(z)$ is a rational function, a ratio of two polynomials in z^{-1} (or z).

4.5.1 Poles and Zeros of a $\mathcal{Z}$-Transform

Defn 41 (Zeros)**.** The *zeros* of a z-transform $X(z)$ are the values of z for which $X(z) = 0$.
This is analogous to "setting the numerator equal to zero."

Defn 42 (Poles)**.** The *poles* of a z transform $X(z)$ are the values of z for which $X(z) = \infty$.
This is analogous to "setting the denominator equal to zero."

If $X(z)$ is a rational function, then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid negative powers of z by factoring out the terms z^{-M} and z^{-N}.

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M}}{z^{-N}} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

Since $B(z)$ and $A(z)$ are polynomials in z, they can be expressed in factored form as

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M}}{z^{-N}} \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)} \tag{4.20}$$

Thus, $X(z)$ has M finite Zeros at $z = z_1, z_2, \ldots, z_M$ (the roots of the numerator polynomial), N finite Poles at $z = p_1, p_2, \ldots, p_N$ (the roots of the denominator polynomial, and $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$. Poles and zeroes may occur at $z = \infty$. A zero exists at $z = \infty$ if $X(\infty) = 0$ and a pole exists at $z = \infty$ if $X(\infty) = \infty$.

Defn 43 (Pole-Zero Plot)**.** Poles and Zeros of a z-transform can be shown graphically by a *pole-zero plot* in the complex plane, which shows the location of poles by crosses ($\times$) and the location of zeros by circles ($\circ$). Multiplicity is shown by a number close to the corresponding cross or circle. The ROC of a z-transform should not contain any poles, by definition of a Stable signal.

4.5.2 Decomposition of Rational $\mathcal{Z}$-Transforms

The short end of this story is that you should group complex-conjugate pairs together.

4.6 Application of the $\mathcal{Z}$-Transform

TODO. Include an example of how to perform the $\mathcal{Z}$ to solve a system when the output is noncausal. There is a good example of this in the October 2018 exam.

4.7 Analysis of LTI Systems in the $\mathcal{Z}$-Domain

There are a few steps to move from the time-domain to the $\mathcal{Z}$-domain to perform analysis.

1. Convert all your time-based to terms to the $\mathcal{Z}$-domain.
   - $y(n-k) \to z^{-k}Y(z)$
   - $x(n-k) \to z^{-k}X(z)$
2. Express $H(z)$ as $\frac{Y(z)}{X(z)}$, the System Function.
3. Find the roots of the numerator and the denominator.
   - When solving for the roots, you should solve in terms of z^k, not z^{-k}. Factor our z^{-k} to achieve this.
   - If the degree of the numerator is greater than or equal to the degree of the numerators, you have to reduce the degree of the numerator.
     (a) Use Long Polynomial Division
     (b) Use Partial-Fraction Expansion on the Remainder

   (a) The roots of the numerator is/are Zeros. This is where the $\mathcal{Z}$-plane becomes 0.
   (b) The roots of the denominator is/are Poles. This is where the $\mathcal{Z}$-plane tends towards ∞.

Defn 44 (System Function)**.** The *system function* or *system equation* is the $\mathcal{Z}$-transform of the filter response.

$$H(z) = \mathcal{Z}\{h(n)\} \tag{4.21}$$

Because of Equation (4.13), and the relation shown in Equation (3.4), we can write the system equation like so

$$\begin{aligned} Y(z) &= X(z)H(z) \\ H(z) &= \frac{Y(z)}{X(z)} \end{aligned} \tag{4.22}$$

This forms the basis for Rational $\mathcal{Z}$-Transforms and Analysis of LTI Systems in the $\mathcal{Z}$-Domain

Signal, $x(n)$	z-Transform, $X(z)$	ROC
$\delta(n)$	1	All z
$\mathcal{U}(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n \mathcal{U}(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n \mathcal{U}(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n \mathcal{U}(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$-na^n \mathcal{U}(-n-1)$	$\frac{az^{-z}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$(\cos\omega_0 n)\mathcal{U}(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(\sin\omega_0 n)\mathcal{U}(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(a^n\cos\omega_0 n)\mathcal{U}(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$(a^n\sin\omega_0 n)\mathcal{U}(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 4.3: Common $\mathcal{Z}$-Transforms

4.8 Common $\mathcal{Z}$-Transforms

The most common $\mathcal{Z}$-transforms are shown in Table 4.3.

5 The Fourier Transform and Fourier Series

When a signal is decomposed with either the Fourier Transform or the Fourier Series, you receive either sinusoids or complex-valued exponentials. This decomposition is said to be represented in the *frequency domain*.

Defn 45 (Fourier Transform)**.** When decomposing the class of signals with finite energy, you perform a *Fourier transform*. This is generally shown as the function

$$c_k = \mathcal{F}\{x(t)\}$$

There are 2 possible equations for the Fourier Transform, depending of the function is continuous-time or discrete-time.

1. Continuous-Time: Equation (5.1)
2. Discrete-Time: Equation (5.2)

The Fourier Transform is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt \tag{5.1}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \tag{5.2}$$

Remark 45.1. Sometimes $X(F)$ and $X(f)$ will be denoted with Ω and ω ($X(\Omega)$ and $X(\omega)$) respectively. In both cases, Ω and ω mean something similar.

$$\Omega = 2\pi F$$
$$\omega = 2\pi f$$

This means that we can rewrite Equations (5.1) to (5.2) as

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{5.3}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.4}$$

Remark 45.2. Generally, when people say the Fourier Transform, they are referring to the transform on Continuous-Time Signals. There is a distinction that occurs with the *DTFT* or *Discrete-Time Fourier Transform*.

This document explains them side-by-side, but will primarily focus on the Discrete-Time Fourier Transform.

Defn 46 (Fourier Series)**.** When decomposing the class of periodic signals, you are returned a *Fourier series.* This is generally shown as the function

$$X(F) = \mathcal{F}\{x(t)\}$$

Defn 47 (Discrete-Time Fourier Transform)**.** The *Discrete-Time Fourier Transform, DTFT* is a special case of the Fourier Transform that occurs when the input function $x(n)$ is a case of Discrete-Time Signals.

The transformation (analysis) equations are:

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \tag{5.5a}$$

$$\omega = 2\pi f$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.5b}$$

The reverse (synthesis) equations are:

$$x(n) = \int_{-\infty}^{\infty} X(f)e^{j2\pi fn}df \tag{5.6a}$$

$$x(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega n}d\omega \tag{5.6b}$$

These equations are expanded more upon in Section 5.2, The Inverse Fourier Transform.

5.1 Fourier Transform Relations

Each of these relations is just a side-note, the only relation of real importance is Equation (5.7). The Fourier Transform is just a special case in each of these scenarios. The Fourier Transform is evaluated around the unit circle on the real-imaginary plane.

5.1.1 Laplace Transform Fourier Transform Relation

There is a correlation between the Laplace Transform and the Fourier Transform. The Fourier Transform is a more specific case of the Laplace Transform, when

$$e^{-st} = e^{-j2\pi ft}$$

5.1.2 $\mathcal{Z}$-Transform Discrete-Time Fourier Transform Relation

There is a relationship between the $\mathcal{Z}$-Transform and the Discrete-Time Fourier Transform.

$$\begin{aligned} z &= e^{j2\pi f} \\ z &= e^{j2\pi n} \end{aligned} \tag{5.7}$$

The Discrete-Time Fourier Transform can be viewed as the $\mathcal{Z}$-transform of the sequence evaluated at the unit circle. If $X(z)$ does not converge in the region $|z| = 1$ (i.e., if the unit circle is not contained within the ROC of $X(z)$), the Fourier Transform $X(f)$ does not exist.

The existence of the $\mathcal{Z}$-transform requires that the sequence $\{x(n)r^{-n}\}$ be absolutely summable for some value of r, that is,

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \tag{5.8}$$

Therefore, if Equation (5.8) converges only for values of $r < r_0 < 1$, the ***$\mathcal{Z}$-transform exists***, but the ***Discrete-Time Fourier Transform DOES NOT EXIST***. This is the case for causal sequences of the form $x(n) = a^n\,\mathcal{U}(n)$, where $|a| > 1$.

There are sequences that do not satisfy Equation (5.8), for example

$$x(n) = \frac{\sin \omega_c n}{\pi n},\ -\infty < n < \infty$$

This sequences does not have a $\mathcal{Z}$-transform. However, since it is a finite Energy Signal, it has a Discrete-Time Fourier Transform that converges to

$$X(f) = \begin{cases} 1, & |f| < f_c \\ 0, & f_c < |f| \le \frac{1}{2} \end{cases}$$

The existence of the $\mathcal{Z}$-transform requires that Equation (5.8) be satisfied for some region in the z-plane. If this region contains the unit circle, the Discrete-Time Fourier Transform, $X(f)$ exists. However, the existence of the Discrete-Time Fourier Transform, which is defined for finite Energy Signals, does not necessarily ensure the existence of the $\mathcal{Z}$-transform.

5.2 The Inverse Fourier Transform

Defn 48 (Inverse Fourier Transform)**.** Since the Fourier Transform is a "lossless" function (the definition of a transformation), the *inverse fourier transform* is just the opposite setup of Equations (5.1) to (5.2).

In both cases, a Continuous-Time signal and a Discrete-Time signal, you use the below synthesis equations (Equations (5.9) to (5.10)).

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF \\ x(n) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi fn} df \end{aligned} \tag{5.9}$$

If you're calculating with Ω or ω instead of F or f, then use these synthesis equations.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \\ x(n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega n} d\omega \end{aligned} \tag{5.10}$$

5.3 Properties of the Discrete-Time Fourier Transform

One thing to keep in mind with all of these properties is that $\omega = 2\pi f$.

Property	Time Domain $x(n)$	Frequency Domain $X(f)$ or $X(\omega)$
	$x(n)$	$X(\omega)$
Notation	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time Shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time Reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1,x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1,x_2}(\omega) = X_1(\omega)X_2(\omega)$
		$= X_1(\omega)X_2^*(\omega)$
		[if $x_2(n)$ is real]
Wiener-Khintchine Theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency Shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos(\omega_0 n)$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication in Time Domain	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in Frequency Domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

Table 5.1: Properties of the Fourier Transform for Discrete-Time Signals

5.3.1 Linearity

If

$$x_1(n) \xleftrightarrow{\mathcal{F}} X_1(f)$$
$$x_2(n) \xleftrightarrow{\mathcal{F}} X_2(f)$$

then

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{\mathcal{F}} a_1 X_1(f) + a_2 X_2(f) \tag{5.11}$$

5.3.2 Time Shifting

If

$$x(n) \xleftrightarrow{\mathcal{F}} X(f)$$

then

$$x(n-k) \xleftrightarrow{\mathcal{F}} e^{-j\omega k} X(f) \tag{5.12}$$

5.3.3 Time Reversal

If

$$x(n) \xleftrightarrow{\mathcal{F}} X(f)$$

then

$$x(-n) \xleftrightarrow{\mathcal{F}} X(-f) \tag{5.13}$$

5.3.4 Convolution

If

$$x_1(n) \xleftrightarrow{\mathcal{F}} X_1(f)$$
$$x_2(n) \xleftrightarrow{\mathcal{F}} X_2(f)$$

then

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{\mathcal{F}} X(f) = X_1(f)X_2(f) \tag{5.14}$$

Remark. There is one thing to note here. Both $x_1(n)$ and $x_2(n)$ must be reasonably well-behaved and have be BIBO-stable for this relation to hold.

5.3.5 Correlation

If

$$x_1(n) \xleftrightarrow{\mathcal{F}} X_1(f)$$
$$x_2(n) \xleftrightarrow{\mathcal{F}} X_2(f)$$

then

$$r_{x_1 x_2}(m) \xleftrightarrow{\mathcal{F}} S_{x_1 x_2}(f) = X_1(f)X_2(-f) \tag{5.15}$$

5.3.6 Wiener-Khintchine Theorem

Let $x(n)$ be a real signal. Then

$$r_{xx}(l) \xleftrightarrow{\mathcal{F}} S_{xx}(f) \tag{5.16}$$

That is, the energy spectral density of an energy signal is the Fourier Transform of its autocorrelation sequence. This is a special case of Equation (5.15).

5.3.7 Frequency Shifting

If

$$x(n) \xleftrightarrow{\mathcal{F}} X(f)$$

then

$$e^{-i2\pi f_0 n} x(n) \xleftrightarrow{\mathcal{F}} X(f - f_0) \tag{5.17}$$

5.3.8 Modulation

If

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$

then

$$x(n)\cos(2\pi f_0 n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(f+f_0) + X(f-f_0)\right] \tag{5.18}$$

5.3.9 Multiplication in Time Domain

This is also called the *Windowing Theorem*.

If

$$x_1(n) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f)$$
$$x_2(n) \overset{\mathcal{F}}{\longleftrightarrow} X_2(f)$$

then

$$x_3(n) \equiv x_1(n)x_2(n) \overset{\mathcal{F}}{\longleftrightarrow} X_3(f) = \int_{\frac{1}{2}}^{\frac{1}{2}} X_1(\lambda)X_2(f-\lambda)d\lambda \tag{5.19}$$

5.3.10 Differentiation in Frequency Domain

If

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$

then

$$nx(n) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(f)}{df} \tag{5.20}$$

5.3.11 Parseval's Theorem

If

$$x_1(n) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f)$$
$$x_2(n) \overset{\mathcal{F}}{\longleftrightarrow} X_2(f)$$

then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \int\limits_{-0.5}^{0.5} X_1(f)X_2^*(f)df \tag{5.21}$$

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega \tag{5.22}$$

Both Equations (5.21) to (5.22) can be expressed in another format.

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \int\limits_{-0.5}^{0.5} |X_1(f)|^2 df \tag{5.23}$$

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |X_1(\omega)|^2 d\omega \tag{5.24}$$

6 The Fourier Transform and LTI Systems

6.1 Magnitude Response

Defn 49 (Magnitude Reponse)**.** The *magnitude response* of a Fourier Transform is commonly denoted as $|H(\omega)|$ or $|H(f)|$. However, in this text, it is denoted as $\|H(\omega)\|$ or $\|H(f)\|$.

The equation that defines a Fourier Transform's Magnitude Reponse is

$$\begin{aligned}\|H(\omega)\| &= \sqrt{\Big(\mathrm{Re}\{H(\omega)\}\Big)^2 + \Big(\mathrm{Im}\{H(\omega)\}\Big)^2} \\ \|H(f)\| &= \sqrt{\Big(\mathrm{Re}\{H(f)\}\Big)^2 + \Big(\mathrm{Im}\{H(f)\}\Big)^2}\end{aligned} \tag{6.1}$$

Remark 49.1. It is important to note that the numerator will ***NOT*** have any imaginary terms (i or j) in it!

Remark. It is important to note that in this reference guide, the magnitude is denoted with double bars. For example, if a complex function $a(n)$ exists, then I will denote its magnitude as $\|a(n)\|$. This helps distinguish between magnitude of a function and its absolute value. These sometimes have different values, so it is useful to differentiate between the two.

6.1.1 Methods of Finding Magnitude Response

There are 2 easy ways to find the Magnitude Reponse.

1. Solve $H(\omega)$ or $H(f)$ and make sinusoids
2. Multiply $H(\omega)$ or $H(f)$ by $e^{-j\omega}$ or $e^{-j2\pi f}$ and cancel terms out

Example 6.1: Find Magnitude Response-Method 2.

Compute the amplitude function $\|H(f)\|$?

$$H(z) = \frac{1}{5}\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right)$$

Substitute $z = e^{j2\pi f}$.

$$H(f) = \frac{1}{5}\left(1 + e^{-1j2\pi f} + e^{-2j2\pi f} + e^{-3j2\pi f} + e^{-4j2\pi f}\right)$$

You could try to find a common exponential term(s) to factor out, but there this second method is easier for series of exponentials.

$$\begin{aligned}H(f) &= \frac{1}{5}\left(1 + e^{-1j2\pi f} + e^{-2j2\pi f} + e^{-3j2\pi f} + e^{-4j2\pi f}\right) \\ e^{-j2\pi f}H(f) &= \frac{1}{5}\left(e^{-1j2\pi f} + e^{-2j2\pi f} + e^{-3j2\pi f} + e^{-4j2\pi f} + e^{-5j2\pi f}\right) \\ H(f) - e^{-j2\pi f}H(f) &= \left(\frac{1}{5}\left(1 + e^{-1j2\pi f} + e^{-2j2\pi f} + e^{-3j2\pi f} + e^{-4j2\pi f}\right)\right) \\ &\quad - \left(\frac{1}{5}\left(e^{-1j2\pi f} + e^{-2j2\pi f} + e^{-3j2\pi f} + e^{-4j2\pi f} + e^{-5j2\pi f}\right)\right) \\ H(f)\left(1 - e^{-j2\pi f}\right) &= \frac{1}{5}\left(1 - e^{-5j2\pi f}\right) \\ H(f) &= \frac{1}{5}\frac{1 - e^{-5j2\pi f}}{1 - e^{-j2\pi f}}\end{aligned}$$

Now we can factor terms out to make complex exponentials that can form sinusoids.

$$\begin{aligned}
H(f) &= \frac{1}{5}\frac{e^{-\frac{5}{2}j2\pi f}\left(e^{\frac{5}{2}j2\pi f} - e^{-\frac{5}{2}j2\pi f}\right)}{e^{-\frac{1}{2}j2\pi f}\left(e^{\frac{1}{2}j2\pi f} - e^{-\frac{1}{2}j2\pi f}\right)} \\
&= \frac{1}{5}\frac{e^{-\frac{5}{2}j2\pi f}}{e^{-\frac{1}{2}j2\pi f}}\frac{2\cos\left(\frac{5}{2}\cdot 2\pi f\right)}{2\cos\left(\frac{1}{2}\cdot 2\pi f\right)} \\
&= \frac{1}{5}e^{-2j2\pi f}\frac{\cos(5\pi f)}{\cos(\pi f)} \\
\|H(f)\| &= \left\|\frac{1}{5}e^{-2j2\pi f}\frac{\cos(5\pi f)}{\cos(\pi f)}\right\| \\
&= \frac{1}{5}(1)\left\|\frac{\cos(5\pi f)}{\cos(\pi f)}\right\| \\
&= \frac{1}{5}\left\|\frac{\cos(5\pi f)}{\cos(\pi f)}\right\|
\end{aligned}$$

Our solution is

$$\|H(f)\| = \frac{1}{5}\left\|\frac{\cos(5\pi f)}{\cos(\pi f)}\right\|$$

6.2 Phase Response

Defn 50 (Phase Response)**.** The *phase response* of a Fourier Transform is commonly denoted as $\measuredangle H(\omega)$ or $\measuredangle H(f)$. This is a function that defines a Fourier Transform's Phase Response is defined in the equation below.

$$\begin{aligned}
\measuredangle H(\omega) = \Theta(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(\omega)\}}{\mathrm{Re}\{H(\omega)\}}\right) \\
\measuredangle H(f) = \Theta(f) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(f)\}}{\mathrm{Re}\{H(f)\}}\right)
\end{aligned} \tag{6.2}$$

Remark 50.1. It is important to note that the numerator will ***NOT*** have any imaginary terms (i or j) in it!

Remark 50.2. Note that real positive values have argument $= 0$

$$\Theta(\omega) = 0, \quad \Theta(f) = 0 \tag{6.3}$$

Real negative values have argument $= \pm\pi$

$$\Theta(\omega) = \pm\pi, \quad \Theta(f) = \pm\pi \tag{6.4}$$

Remark 50.3 (Complex Exponential to Unit Circle). ***REMEMBER:***

$$\begin{aligned}
e^{\pm j\omega} &= \cos(\omega) \pm j\sin(\omega) \\
e^{\pm j2\pi f} &= \cos(2\pi f) \pm j\sin(2\pi f)
\end{aligned} \tag{6.5}$$

This is also defined in Equation (B.3) on Page 49.

6.3 Frequency Response

The Frequency Response of a function is defined in 2 parts.

1. Magnitude Response
2. Phase Response

$$\begin{aligned}
H(\omega) = \|H(\omega)\|\,\Theta(\omega) \\
H(f) = \|H(f)\|\,\Theta(f)
\end{aligned} \tag{6.6}$$

Remark. This is similar to the Rectangular to Polar conversion shown on Page 50.

7 Sampling and Reconstruction

Defn 51 (Optimal Sampling)**.** *Optimal sampling* (*lossless sampling*) of a signal $x(t)$ occurs when

$$x_a\left(\frac{n}{F_S}\right) = x(n) \tag{7.1}$$

If there is a noisy signal $x(t) = s(t) + n(t)$, then so long as we can recover the interesting signal $s(t)$, then the sampling was lossless or optimal.

We can recover $x(t)$ from $x(n)$ if $X_a(F)$ looks the same as $X(f)$. (If the Fourier Transform of the analog signal is the same as the Discrete-Time Fourier Transform of the discrete signal).

$$X_a(F) \approx X(f) \tag{7.2}$$

Remark 51.1 (Shape of $X(f)$)*.* The sampling of $X(f)$ is optimal even if $X(f)$ is:

- Flipped
- Scaled
- Negative
- Compressed
- Expanded
- etc.

In summary, the sampling is *optimal* or *lossless* if:

1. $X_a(F)$ contains all information about $x(t)$
2. $A_a(F)$ is aperiodic, but has finite bandwidth
3. $x(n)$ is a sampled version of $x(t)$
4. $X(f)$ is sufficient to recover $X_a(F) \longrightarrow x(n)$ sufficient to recover $x(t)$

There exists a relationship between $x(n)$ and $x\left(\frac{n}{F_S}\right)$ in the frequency domain.

$$x(n) = \frac{1}{F_s}\int_{\frac{-F_S}{2}}^{\frac{F_S}{2}} X\left(\frac{F}{F_S}\right) e^{j2\pi n\frac{F}{F_S}}\,dF \tag{7.3a}$$

$$x(n) = x(t|t = \frac{n}{F_S}) = \int_{\frac{-F_S}{2}}^{\frac{F_S}{2}} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_S)\right] e^{j2\pi n\frac{F}{F_S}}\,dF \tag{7.3b}$$

- Equation (7.3a) is the Discrete-Time Fourier Transform of $x(n)$, the discrete signal
- Equation (7.3b) is the Fourier representation of the sampled continuous-time signal, $x(t|t = \frac{n}{F_S}) = x(n)$

Equations (7.3a) to (7.3b) can be combined and simplified to form Equation (7.4).

$$X(f) = F_S \sum_{k=-\infty}^{\infty} X_a\Big((f-k)F_S\Big) \tag{7.4}$$

7.1 Sampling

There exists a sampling theory presented by Shannon in 1948 that describes a sufficient condition for complete signal recovery.

Theorem 7.1 (Sampling Theorem (Shannon 1948))**.** *If $F_S > 2B$, where B is the highest frequency of the analog signal, then the analog signal can be recovered from its sampled version.*

Remark. Note that Sampling Theorem (Shannon 1948) is a *sufficient* condition. It does not necessarily represent the smallest sampling frequency possible. If the conditions frequency conditions are correct, and later signal processing possible, you can fold signals into empty bands.

7.1.1 Aliasing

Defn 52 (Aliasing)**.** *Aliasing* occurs when the sampling frequency $2F_S < F$. When this happens, some of the values on each side of the origin ($X_a(0)$ in Equation (7.4)) modify each other.

If Equation (7.4) is applied directly, the values for each point in the sampled frequency domain are found directly. If done graphically, you can employ Folding.

Remark 52.1 (Optimal Sampling)*.* Optimal Sampling occurs when there is *no* aliasing.

Defn 53 (Folding)**.** *Folding* is a method of realizing an Aliasing. There are several steps to perform a folding:

1. Identify $\frac{F_S}{2}$
2. Fold at $\frac{F_S}{2}$. (Turn the values past $\frac{F_S}{2}$ backwards, towards the origin).
3. Add the turned value to the original
4. If your folded signal goes past the origin, fold the remainaing signal at the origin.
5. Add this secondly-turned value to the current amount
6. Repeat Steps 1-6 until all of the original signal is contained between 0 and $\frac{F_S}{2}$.
7. Change the bounds from F to $\frac{-1}{2} \leq f \leq \frac{1}{2}$

7.2 Reconstruction

The equation for reconstruction of a signal from its sampled counterpart is shown in Equation (7.5)

$$x(t) = \sum_{n=-\infty}^{\infty} x(n) \operatorname{sinc}\Big(F_S \left(\frac{t-n}{F_S}\right)\Big) \tag{7.5}$$

$$x(n) = x(t|t = \frac{n}{F_S}) \tag{7.6}$$

7.3 Interconnection of Systems with Different Sampling Frequencies

Frequently, various elements in a system will have different sampling frequencies. This means that your A/D converter and D/A converter will have different sampling frequencies. This could affect the output of your system, by changing the frequency characteristics through Aliasing.

There are 2 main ways that an interconnection of elements in a system will affect the output:

1. Decimation
2. Interpolation

7.3.1 Decimation

Defn 54 (Decimation)**.** *Decimation* takes an input signal and compresses it. This effectively "downsamples" the signal. Decimation uses the symbol $D \in \mathbb{Z}^+$.

$$y(m) = x(mD)$$

If decimation occurs later in the system, then if just the input and output are compared, $y(m)$ appears it was sampled at

$$f = \frac{F_S}{D} \tag{7.7}$$

Thus, when we perform sampling on the input signal, then there is folding at

$$f = \frac{F_S}{2D} \tag{7.8}$$

7.3.2 Interpolation

Defn 55 (Interpolation)**.** *Interpolation* is the act of putting zeros in between each of the original signal values, while maintaining the original signal profile. This effectively "upsamples" the signal. Interpolation uses the symbol $I \in \mathbb{Z}^+$.

The input/output relationship of a signal when interpolated is more easily shown in the frequency domain.

$$Y(f) = X(If) \tag{7.9}$$

But, in the time domain, the interpolated output signal would look like

$$y(n) = \{\underline{x(0)}, 0, x(1), 0, x(2), 0, \ldots\}$$

8 Discrete Fourier Transform

Defn 56 (Discrete Fourier Transform)**.** The *Discrete Fourier Transform* or *DFT* can be the Discrete-Time Fourier Transform sampled at certain values. This is only true if $N >$ Length of Signal DTFT.

The N-point DFT is shown as:

$$X_{DFT}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi \frac{k}{N} n} \text{ for } k = 0, 1, 2, \ldots, N-1 \tag{8.1}$$

If N is specified, then replace all occurrences of N in Equation (8.1) with that value.

Remark 56.1. If the length, N of the DFT is not specified, it is assumed that $N =$ length of the signal. If the length of the DFT N is greater than the length of the signal, you are sampling the Discrete-Time Fourier Transform of the signal.

Remark 56.2 (Discrete Fourier Transform Resolution). The *resolution* of the Discrete Fourier Transform is tied to the length of the signal used. The longer the signal is, the greater the resolution of the Discrete Fourier Transform.

- Padding a signal with 0s will increase the resolution of a signal without changing it too much

Remark 56.3 (Time Complexity to Compute). If we are interested in the time complexity (Big-O) $O(n)$ of the Discrete Fourier Transform, it is $O(n^2)$.

1. N values of $X_{DFT}(k)$ to be computed
2. Each value of $X_{DFT}(k)$ requires N multiplications of $x(n)e^{-j2\pi \frac{k}{N} n}$
3. This means the time complexity is $O(n^2)$.

The Discrete Fourier Transform is used for many reasons, some of which are listed below:

1. Computers have limited memory, and the Discrete-Time Fourier Transform of a Discrete-Time Signals is a Continuous-Time Signals. Thus, the Discrete-Time Fourier Transform cannot be stored in memory.
2. The Fast Fourier Transform can be used to compute Circular Convolutions relatively quickly.

Defn 57 (Fast Fourier Transform)**.** This is a special case of the Discrete Fourier Transform that is only useful for computers. If $N = 2^k$, where k is an integer, then the Discrete Fourier Transform can be performed in $O(n \log_2(n))$ time. This can be used to calculate Linear Convolutions relatively quickly, especially when the number of terms in the sequence is quite large.

These 2 pieces of MATLAB/GNU Octave source produce the same output, but through different methods.

```
x = [1 2 3 4]
h = [2 2 1 1]
y = conv(x, h)
y = 2 6 11 17 13 7 4
```

```
x = [1 2 3 4 0 0 0 0]
h = [2 2 1 1 0 0 0 0]
Y_k = ifft(fft(x) .* fft(h))
Y_k = 2.0000 6.0000 11.0000 17.0000 13.0000 7.0000 4.0000 -0.0000
```

Remark 57.1 (Zero-Padding). Note that the padding with zeros to a length greater than the output length of the Linear Convolution is required. Also, to take advantage of the Fast Fourier Transform's quick calculation property, the length of the inputs **MUST** be a power of 2, 2^k.

So, the below code produces something different because it calculates the Circular Convolution directly.

```
x = [1 2 3 4]
h = [2 2 1 1]
Y_k_bad = ifft(fft(x) .* fft(h))
Y_k_bad = 15 13 15 17
```

Remark. We now have several very different possible representations of the same signal with only slight variations in the function description. We will list them out to ensure clarity.

- $x(t)$, Continuous-Time Signals
- $x(n)$, Discrete-Time Signals
- $X(z)$, the The $\mathcal{Z}$-Transform of $x(n)$
- $X(F)$, the Fourier Transform of $x(t)$
- $X(f)$, the Discrete-Time Fourier Transform of $x(n)$
- $X(k)$, the Discrete Fourier Transform of $x(n)$

8.1 Discrete Fourier Transform of Sinusoids

The 2 equations presented below are important for performing the Discrete Fourier Transform on sinusoids. Both Equations (8.2) to (8.3) work in both directions.

$$\begin{aligned}
x(n) &= A\cos\left(2\pi\frac{k_0}{N}n\right),\ 0 < k_0 \le N-1 \\
&= \frac{A}{2}\left(e^{j\frac{2\pi k_0}{N}n} + e^{-j\frac{2\pi k_0}{N}n}\right) \\
X(k) &= \sum_{n=0}^{N-1}\frac{A}{2}e^{j\frac{2\pi k_0 n}{N}}e^{-j\frac{2\pi kn}{N}} + \sum_{n=0}^{N-1}\frac{A}{2}e^{-j\frac{2\pi k_0 n}{N}}e^{-j\frac{2\pi kn}{N}} \\
&= \frac{A}{2}\sum_{n=0}^{N-1}e^{\frac{-j2\pi(k-1)}{N}} + \frac{A}{2}\sum_{n=0}^{N-1}e^{\frac{-j2\pi(k+1)}{N}} \\
&\text{This is a Geometric Series} \\
X(k) &= \frac{AN}{2}\left[(\delta(k-k_0) \bmod N) + (\delta(k+k_0) \bmod N)\right]
\end{aligned} \tag{8.2}$$

$$\begin{aligned}
x(n) &= A\sin\left(2\pi\frac{k_0}{N}n\right),\ 0 < k_0 \le N-1 \\
&= \frac{A}{2j}\left(e^{j\frac{2\pi k_0}{N}n} - e^{-j\frac{2\pi k_0}{N}n}\right) \\
X(k) &= \sum_{n=0}^{N-1}\frac{A}{2j}e^{j\frac{2\pi k_0 n}{N}}e^{-j\frac{2\pi kn}{N}} - \sum_{n=0}^{N-1}\frac{A}{2j}e^{-j\frac{2\pi k_0 n}{N}}e^{-j\frac{2\pi kn}{N}} \\
&= \frac{A}{2j}\sum_{n=0}^{N-1}e^{\frac{-j2\pi(k-1)}{N}} - \frac{A}{2j}\sum_{n=0}^{N-1}e^{\frac{-j2\pi(k+1)}{N}} \\
&\text{This is a Geometric Series} \\
X(k) &= \frac{AN}{2j}\left[(\delta(k-k_0) \bmod N) - (\delta(k+k_0) \bmod N)\right]
\end{aligned} \tag{8.3}$$

8.2 Inverse Discrete Fourier Transform

Defn 58 (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the inverse of the Discrete Fourier Transform.

$$x_{IDFT}(n) = \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j2\pi\frac{k}{N}n} \text{ for } n = 0, 1, \dots, N-1 \tag{8.4}$$

8.3 The Discrete Fourier Transform Expressed with Matrices

We start with Equation (8.5) to simplify the writing of our equations.

$$W_N = e^{-j\frac{2\pi}{N}} \tag{8.5}$$

W_N is a complex variable used to replace the exponential function. You will see W_N^{kn}, which evaluates to $e^{-j\frac{2\pi}{N}kn}$.
If we define 2 matrices, where $\mathbf{x}(n)$ is a discretely-valued function and $\mathbf{X}_N(k)$ is its Discrete Fourier Transform:

$$\mathbf{x}_N(n) = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix},\quad \mathbf{X}_N(k) = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \tag{8.6}$$

Now, we extend our definition of W_N from Equation (8.5) to this matrix format as well.

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{kn}|_{k=1,n=1} & W_N^{kn}|_{k=1,n=2} & \cdots & W_N^{kn}|_{k=1,n=N-1} \\ \vdots & W_N^{kn}|_{k=2,n=1} & W_N^{kn}|_{k=2,n=2} & \cdots & W_N^{kn}|_{k=2,n=N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{kn}|_{k=N-1,n=1} & W_N^{kn}|_{k=N-1,n=2} & \cdots & W_N^{kn}|_{k=N-1,n=N-1} \end{bmatrix} \tag{8.7}$$

With these matrices defined, we can express the N-point Discrete Fourier Transform as

$$\mathbf{X}_N(k) = \mathbf{W}_N \mathbf{x}_n(n) \tag{8.8}$$

and, because of matrix inversion, we can express the N-point Inverse Discrete Fourier Transform as

$$\mathbf{x}_N(n) = \mathbf{W}_N^{-1} \mathbf{X}_N(k) \tag{8.9}$$

Because W_N is a complex-valued matrix, $\mathbf{x}_N(n)$ can also be expressed as the Inverse Discrete Fourier Transform with Equation (8.10)

$$\mathbf{x}_N(n) = \frac{1}{N}\mathbf{W}_N^* \mathbf{X}_N(k) \tag{8.10}$$

The duality presented from Equations (8.9) to (8.10) means

$$\begin{aligned} \mathbf{W}_N^{-1} &= \frac{1}{N}\mathbf{W}_N^* \\ \mathbf{W}_N \mathbf{W}_N^* &= N\mathbf{I}_N \end{aligned} \tag{8.11}$$

where $\mathbf{I}_N$ is an $N \times N$ identity matrix.

8.4 Properties of the Discrete Fourier Transform

With the Discrete Fourier Transform, the properties that we have grown used to do not apply.

- $x(n) * y(n) \neq X(k)Y(k)$
- $x(n-n_0) \not\leftrightarrow X(k)e^{j2\pi \frac{k}{N} n_0}$

They must be modified for the circular nature of our new transformation.

Property	Time Domain $x(n)$	DFT Domain $X(k)$
Notation	$x(n)$, $y(n)$	$X(k)$, $Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time Reversal	$x(N-n)$	$X(N-k)$
Circular Time Shifting	$x(n - n_0 \bmod N)$	$X(k)e^{-j2\pi \frac{k}{N} n_0}$
Circular Frequency Shift	$x(n)e^{j2\pi ln/N}$	$X(k - l \bmod N)$
Complex Conjugate	$X^*(n)$	$X^*(N-k)$
Circular Convolution	$x(n) \circledast y(n)$	$X(k)Y(k)$
Circular Correlation	$x(n) \circledast y^*(-n)$	$X(k)Y^*(k)$
Multiplication of 2 Sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \circledast X_2(k)$
Parseval's Theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Table 8.1: Properties of the Discrete Fourier Transform

8.4.1 Periodicity

If $x(n)$ and $X(k)$ are an N-point Discrete Fourier Transform pair, then

$$x(n+N) = x(n) \tag{8.12}$$

$$X(k+N) = X(k) \tag{8.13}$$

The periodicities in $x(n)$ and $X(k)$ follow immediately from Equation (8.1) and Equation (8.4), respectively.

8.4.2 Linearity

If

$$x_1(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_1(k)$$
$$x_2(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1x_1(n) + a_2x_2(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k) \tag{8.14}$$

8.4.3 Time Reversal

If

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

then

$$x(-n \bmod N) = x(N-n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(-k \bmod N) = X(N-k) \tag{8.15}$$

Hence, reversing the N-point sequence in time is equivalent to reversing the Discrete Fourier Transform values.

8.4.4 Circular Time Shifting

The definition of time shifting needs to be modified from the definition of time shifting in Section 5.3.2.

Defn 59 (Discrete Fourier Transform Time Shifting)**.** If a signal $x(n)$ is shifted by n_0, then the Discrete Fourier Transform is also shifted.

$$x(n - n_0 \bmod N) = X(k)e^{-j2\pi \frac{k}{N} n_0} \tag{8.16}$$

The modulus is because the signal is now circular.

Remark 59.1. This is closely related to Section 4.3.2 and Section 5.3.2.

8.4.5 Circular Frequency Shift

If

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

then

$$x(n)e^{j2\pi ln/N} \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X((k-l) \bmod N) \tag{8.17}$$

Hence, the multiplication of the sequence $x(n)$ with the complex exponential sequence $e^{k2\pi kn/N}$ is equivalent to the circular shift of the Discrete Fourier Transform by l units in frequency. This is the dual to the Circular Time Shifting property.

8.4.6 Complex Conjugate

If

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

then

$$x^*(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X^*(-k \bmod N) = X^*(N-k) \tag{8.18}$$

The Inverse Discrete Fourier Transform of $X^*(k)$ is

$$\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N} = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]$$

therefore,

$$x^*(-n \bmod N) = x^*(N-n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X^*(k) \tag{8.19}$$

8.4.7 Circular Convolution

Defn 60 (Circular Convolution)**.** The *circular convolution* is similar to the Linear Convolution. The key difference is that the circular convolution repeats its sequence indefinitely. Computing a Circular Convolution is shown in Example 8.1.

Mathematically, this type of convolution is represented as

$$x_N(n) \circledast h(n) = \sum_{m=-\infty}^{\infty} h(m) \sum_{k=-\infty}^{\infty} x(n-m-kN) \tag{8.20}$$

This can be simplified a little bit to

$$x_1(n) \circledast x_2(n) = \sum_{k=0}^{N-1} x_1(k) x_2(n-k \bmod N) \tag{8.21}$$

It is important to remember that the modulus (mod) operator yields 0 when the input is a multiple of the divisor.

Remark 60.1 (Alternate Circular Convolution Symbol)*.* There is no single defined symbol for Circular Convolutions and Linear Convolutions. In this text, and personally, I use the $\circledast$ symbol, however, I occasionally use the $\otimes$ symbol. However, other texts may use (In order of likelihood):

1. Ⓝ
2. $\circledast$
3. $\otimes$
4. $*$
5. $\cdot$ (Centered Dot)
6. $\bullet$
7. etc.

Remark 60.2 (Circular Convolution Length)*.* The length of the resulting sequence from a Circular Convolution is

$$L \tag{8.22}$$

where L is the length of the input sequences.

Remark 60.3 (Linear vs. Circular Convolution Length)*.* The length of a Linear Convolution is $2L-1$, whereas the length of a Circular Convolution is L; given that the input signal lengths are L.

Example 8.1: Circular Convolution. Lecture 10, Slide 90

Perform a Circular Convolution on these signals.

$$x(n) = \{\underline{1}, 2, 3, 4\}$$
$$h(n) = \{\underline{2}, 2, 1, 1\}$$

There are 2 realistic ways to solve this by hand.

1. Perform a convolution where one of the signals is repeated periodically
2. Perform a normal convolution, but pad with 0s to get input signals that are longer than $2L-1$ of the original signals. Then, add the first term where 0s were padded to the first where they weren't.
   - This is the basis of performing a Linear Convolution with the Fast Fourier Transform and Circular Convolutions

For both methods, I will perform a Folding on $h(n)$ to get $h(-n) = \{1, 1, 2, \underline{2}\}$.

Method 1

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$					
$x(k)$	2	3	4	$\underline{1}$	2	3	4	1	2	3
$y(k)$				$\underline{15}$	13	15	17			

Method 2

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$									
$x(k)$	0	0	0	$\underline{1}$	2	3	4	0	0	0	0	1	2	3
$y(k)$				$\underline{2}$	6	11	17	13	7	4	0			

Then, $\underline{2} + 13 = 15$, $6 + 7 = 13$, $11 + 4 = 15$, $17 + 0 = 17$.

Both methods yield $y(k) = \{\underline{15}, 13, 15, 17\}$.

Remark. You can also solve this with MATLAB/GNU Octave by following the code examples in the definition of the Fast Fourier Transform, Definition 57.

Remark. Personally, I use Method 1 shown in Example 8.1 to calculate Circular Convolutions by hand.

8.4.8 Circular Correlation

In general, for complex-valued sequences $x(n)$ and $y(n)$, if

$$x(n) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} X(k)$$
$$y(n) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} Y(k)$$

then

$$x(n) \circledast y^*(n) = \tilde{r}_{xy}(l) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} \tilde{R}_{xy}(k) = X(k)Y^*(k) \tag{8.23}$$

Remark. This is closely related to the linear Cross Correlation.

8.4.8.1 Circular Autocorrelation If $x(n) = y(n)$, then an autocorrelation is performed. This is closely related to the linear Auto Correlation.

$$x(n) \circledast x^*(n) = \tilde{r}_{xx}(l) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} \tilde{R}_{XX}(k) = X(k)X^*(k) \tag{8.24}$$

8.4.9 Multiplication of 2 Sequences

If

$$x_1(n) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} X_1(k)$$
$$x_2(n) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} X_2(K)$$

then

$$x_1(n)x_2(n) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} \frac{1}{N} X_1(k) \circledast X_2(k) \tag{8.25}$$

8.4.10 Parseval's Theorem

For complex-valued sequences $x(n)$ and $y(n)$, in general, if

$$x(n) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} X(k)$$
$$y(n) \overset{\text{DFT}}{\underset{\text{N}}{\longleftrightarrow}} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k) \tag{8.26}$$

If $x(n) = y(n)$

$$\begin{aligned} \sum_{n=0}^{N-1} x(n)x^*(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k)X^*(k) \\ \sum_{n=0}^{N-1} \|x(n)\|^2 &= \frac{1}{N} \sum_{k=0}^{N-1} \|X(k)\|^2 \end{aligned} \tag{8.27}$$

8.5 Application of the Discrete Fourier Transform

If the signal that is being filtered by a finite filter with equation $h(n)$ has length M is generating an infinite number of discrete values, then the standard Linear Convolution does not work. This is the case with things like data streams, where there is a constant flow of data that is discretely valued.

We need to have a way to perform convolutions and other operations on, what is essentially, an infinite signal. There are 3 ways discussed in the textbook, but I will only heavily discuss the first 2. These **do not** work for Infinite Impulse Response filters.

1. Overlap-Add
2. Overlap-Save
3. Overlap-Discard

8.5.1 Overlap-Add

This is a way to perform a Linear Convolution on an infinite discretely-valued signal with a filter $h(n)$ that has a finite length of M.

The steps to perform Overlap-Add are:

1. Partition your "infinite" discrete signal into blocks of length L
   - This will give you A blocks, where $0 \leq A < \infty$.
   - This means you have blocks with sequences $x_a(n)$, $a \in A$ that are of length L.
2. Compute the Discrete Fourier Transform of your filter $h(n)$ to get $H(k)$.
   (a) Zero-pad $h(n)$ to a length of $N = L + M - 1$. Remember that $h(n)$ has length L.
   (b) Perform the Discrete Fourier Transform on the zero-padded sequence to get $H(k)$
   (c) You can disregard $h(n)$ now
      - If you're programming this, this means you can precalculate $H(k)$ somewhere else.
3. Compute the Discrete Fourier Transform of one of the blocks from the partitioned input signal
   (a) Zero-Pad $x_a(n)$ to a length of $N = L + M - 1$
   (b) Perform the Discrete Fourier Transform on the zero padded sequence to get $X_a(k)$
4. Multiply $X_a(k)$ and $H(k)$ to get $Y_a(k)$
   - $Y_a(k) = X_a(k)H(k)$
5. Perform the Inverse Discrete Fourier Transform on $Y_a(k)$ to get $y_a(n)$.
   - $y_a(n)$ is of length $N = L + M - 1$
6. Take the first L terms of $y(n)$ and make that your real output $z(n)$
7. Save the other $M - 1$ terms in memory (leftovers), call this $\ell_a(n)$
8. Take the next block, find $y_{a+1}(n)$, and add $\ell_a(n)$ to the output of the next block, $y_{a+1}(n)$
   (a) The first term of $\ell_a(n)$, while not the first term from block a, is now the first term when adding it to $y_{a+1}(n)$.
   (b) Perform element-wise addition.
   (c) $z_{a+1}(n) = y_{a+1}(n) + \ell_a(n)$
9. Repeat this for the entire length of the infinitely long input signal.

Remark. In case the steps above were confusing, the variables used and their meanings are below.

- $x(n)$, the infinitely long input signal
- a, the block index from the partitioning of the input signal
- A, the total number of partitions of $x(n)$ that are present
- L, the length of each block
- $x_a(n)$, the signal sequence in block a
- $X_a(k)$, the Discrete Fourier Transform of the signal sequence in block a
- $h(n)$, the finite length filter's sequence
- $H(k)$, the Discrete Fourier Transform of the filter's sequence
- M, the length of the filter's sequence
- N, the length used in the circular convolution, equal to $L + M - 1$
- $Y_a(k)$, the output of the multiplication of $X_a(k)$ and $H(k)$
- $y_a(n)$, the Inverse Discrete Fourier Transform of $Y_a(k)$
- $\ell_a(n)$, the part of $y_a(n)$ after pulling off the first L terms
- $z_a(n)$, the final output of performing the convolution of block a and adding $\ell_{a-1}(n)$

The trick for Overlap-Add is to get each block's convolution and multiplication fast enough, and it becomes efficient. To make it fast enough, N should be a power of 2 (2^k). This turns the calculation of the Discrete Fourier Transforms into Fast Fourier Transforms, which are really fast (relatively speaking). Then multiplication, while lengthy, is a singular operation, unlike convolutions which are many. Because we zero-pad smartly, while we are technically calculating the Circular Convolution of the signal, we are actually performing the Linear Convolution.

8.5.2 Overlap-Save

This is another way to perform a Linear Convolution on an infinitely long discretely-valued signal with a filter $h(n)$ that has finite length of M.

The steps to perform Overlap-Save are:

1. Partition your infinite discrete input signal, $x(n)$, into blocks of length L to get $x_a(n)$
   - This will give you A block, where $0 \leq A < \infty$.
   - This means you have blocks with sequences $x_a(n)$, $a \in A$ that are of length L.
2. Compute the Discrete Fourier Transform of your filter $h(n)$ to get $H(k)$.
   (a) Perform the Discrete Fourier Transform on the sequence
   (b) You can disregard $h(n)$ now
      - If you're programming this, this means you can precalculate $H(k)$ somewhere else.
3. Compute the Discrete Fourier Transform of one of the blocks from the partitioned input signal to get $X_a(k)$
   (a) Zero-pad the ***FRONT*** of the block until the length of the block's sequences with the padding is $N = L + M - 1$.
4. Perform the Circular Convolution of $h(n)$ and zero-padded block signal $x_a(n)$ by multiplying $X_a(k)$ and $H(k)$, which yields $Y_a(k)$
5. Make your output $Z_a(k)$ the last $M - 1$ terms in the $Y_a(k)$ sequences. (Leave out the first L terms).
6. Take the Inverse Discrete Fourier Transform of $Z_a(k)$ to get $z_a(n)$.
7. Move onto the $(a + 1)$th block, and prepend (attach to the front) the last $L - M$ terms from $x_a(n)$, to $x_{a+1}(n)$.
8. Perform the Discrete Fourier Transform of this sequence, and take the last $M - 1$ terms from $Y_{a+1}(k)$ to the output $Z(k)$

8.5.3 Overlap-Discard

This was not discussed in this class and is not necessary to pass the course or take the course's exam. Thus, this section will **NOT** be completed.

9 Implementation of Filters

9.1 Finite Impulse Response Filters

TODO

9.2 Infinite Impulse Response Filters

TODO

9.3 Transposing a System

To transpose any system's block diagram, you need to follow these steps.

1. Reverse the direction of each interconnection
2. Reverse direction of each multiplier
3. Change junctions to adders and adders to junctions
   - Adders become junctions
   - Junctions become adders
4. Interchange the input and output

9.4 Numerical Precision Issues

The main issues here are:

1. Our coefficients, α, β, etc. are stored in hardware with finite precision, if not a power of 2.
2. Arithmetic is done with finite precision

To deal with this, use typically say "Our system is perfect, and the above issues are just noise in the system." These imprecisions cause ***HUGE*** issues. Wilkinson's Polynomial illustrates this.

The solution to these issues is to not implement the filter's whole $H(z)$ at once. Instead, we create a cascade of smaller filters with Biquads. You implement a cascade of Biquad filters until you have achieved the $H(z)$ that you desire.

Defn 61 (Biquad)**.** A *biquad* filter is one that has only 2 poles and 2 zeros.

You pick the setup of the cascade in the following order:

1. Find the optimal pole-zero combinations
   (a) Plot the magnitude response
   (b) Choose the Biquad that minimizes noise and maximizes the signal frequency desired
2. Find the optimal arrangement of cascaded filter Biquads
   (a) Minimize the average output power of the noise signal
   (b) This means the filter with the lowest output power is **LAST** in the cascade.

A Complex Numbers

Defn A.0.1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted mathbbC.

A complex number is written, in Cartesian form, as shown in Equation (A.1) below.

$$z = a \pm ib \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

A.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn A.1.1 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\mathrm{Re}\{z\} = x \tag{A.3}$$

Remark A.1.1.1 (Alternative Notation). The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re\mathfrak{e}$$

Defn A.1.2 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\mathrm{Im}\{z\} = y \tag{A.4}$$

Remark A.1.2.1 (Alternative Notation). The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im\mathfrak{m}$$

A.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$\begin{aligned} z &= x_1 + iy_1 \\ w &= x_2 + iy_2 \end{aligned}$$

A.2.1 Addition

The addition operation, still denoted with the + symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w \coloneqq (x_1 + x_2) + i(y_1 + y_2) \tag{A.5}$$

A.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$\begin{aligned} zw &\coloneqq (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\ &= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned} \tag{A.6}$$

A.3 Complex Conjugates

Defn A.3.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{A.7}$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{A.8}$$

A.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$
2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - xyi + xyi - i^2y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2 \end{aligned}$$

Thus,

$$z\overline{z} = x^2 + y^2 \tag{A.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}$$

Using our solution from Equation (A.9):

$$\begin{aligned} &= \frac{(x + iy)^2}{x^2 + y^2} \\ &= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2} \end{aligned}$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$
\begin{aligned}
&= \frac{\left(x^2 - y^2\right) + 2xyi}{x^2 + y^2} \\
&= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2}
\end{aligned}
$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$
\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{A.10}
$$

A.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.

First, start by defining some expressions so that we can prove some of these properties:

$$
\begin{aligned}
z &= x + iy \\
\overline{z} &= x - iy
\end{aligned}
$$

(i) The conjugation operation is commutative.

(ii) The conjugation operation can be distributed over addition and multiplication.

$$
\begin{aligned}
\overline{z + w} &= \overline{z} + \overline{w} \\
\overline{zw} &= \overline{zw}
\end{aligned}
$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers ($z, w \in$ mathbbC) where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.

We start by simplifying the left-hand side of the equation ($\overline{z + w}$).

$$
\begin{aligned}
\overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\
&= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\
&= (x_1 + x_2) - i(y_1 + y_2)
\end{aligned}
$$

Now, we simplify the other side ($\overline{z} + \overline{w}$).

$$
\begin{aligned}
\overline{z} + \overline{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\
&= (x_1 - iy_1) + (x_2 - iy_2) \\
&= (x_1 + x_2) - i(y_1 + y_2)
\end{aligned}
$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

■

A.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn A.4.1 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in$ mathbbC. This is taken from Descarte's definition of a completely real plane.

The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

A.4.1 Modulus of a Complex Number

Defn A.4.2 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\overline{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{A.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in \text{mathbbC}$). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (A.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\overline{z}\overline{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \end{aligned}$$

Note how we never needed to define z or w, so this is as general a result as possible. ∎

A.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right) \\ &= |z|^2|w|^2 \end{aligned}$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

A.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

A.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (A.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{A.12}$$

Example A.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$\begin{aligned}
|5iz + 10| &= 7 \\
|5i(z + \frac{10}{5i})| &= 7 \\
|5i(z + \frac{2}{i})| &= 7 \\
|5i(z + \frac{2}{i}\frac{-i}{-i})| &= 7 \\
|5i(z - 2i)| &= 7
\end{aligned}$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$\begin{aligned}
|5i||z - 2i| &= 7 \\
5|z - 2i| &= 7 \\
|z - 2i| &= \frac{7}{5}
\end{aligned}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

A.5.1 Annulus

Defn A.5.1 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (A.13).

$$r_1 \leq |z - a| \leq r_2 \tag{A.13}$$

In Equation (A.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

A.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (A.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{A.14}$$

Remark. Note that in the definition of polar form (Equation (A.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$\begin{aligned}
r &= |z| \\
\theta &= \arg(z)
\end{aligned}$$

Example A.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned} \cos(\theta) &= \frac{-\sqrt{3}}{2} \\ \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned} r &= 2 \\ \theta &= \frac{5\pi}{6} \\ \arg(z) &= \frac{5\pi}{6} + 2\pi k \end{aligned}$$

A.6.1 Converting Between Cartesian and Polar Forms

Using Equation (A.14) and Equation (A.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{A.15}$$

A.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

A.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \tag{A.16}$$

A.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big) \tag{A.17}$$

A.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph A.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn A.6.1 (de Moiver's Law)**.** Given a complex number z, $z \in$ mathbbC and a rational number n, $n \in$ mathbbQ, the exponentiation of z^n is defined as Equation (A.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{A.18}$$

A.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{A.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (A.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{A.20}$$

A w that satisfies Equation (A.20) is an nth root of z.

Example A.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example A.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (A.19), we can find the general form of the roots.

$$\begin{aligned}
z &= 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right) \\
z^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right) \\
&= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right)
\end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \ldots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

k = 2

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(2)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(2)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)$$

k = 3

$$\begin{aligned}\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(3)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(3)}{18}\right)\right) &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + \frac{36\pi}{18}\right) + i\sin\left(\frac{\pi}{18} + \frac{36\pi}{18}\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18} + 2\pi\right) + i\sin\left(\frac{\pi}{18} + 2\pi\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)\end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned} z_1^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right) \\ z_2^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right) \\ z_3^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)\end{aligned}$$

A.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn A.8.1 (Argument)**.** The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example A.4: Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) = \tan(\theta) &= \frac{-i}{-1} \\ &= \frac{\pi}{4}\end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z) = \frac{5\pi}{4} + 2\pi k$$

Thus, the argument of $z = -1 - i$ is $\arg(z) = \frac{5\pi}{4} + 2\pi k$.

Defn A.8.2 (Principal Argument)**.** The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi < \mathrm{Arg}(z) \leq \pi$.

Example A.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) = \tan(\theta) &= \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\operatorname{Arg}(z) = \frac{5\pi}{4}$.

A.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (A.21).

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big) \tag{A.21}$$

If instead of e as the base, we have some value a, then we have Equation (A.22).

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ &= e^{\mathrm{Re}\{z\ln(a)\}}\Big(\cos\big(\mathrm{Im}\{z\ln(a)\}\big) + i\sin\big(\mathrm{Im}\{z\ln(a)\}\big)\Big) \end{aligned} \tag{A.22}$$

In the case of Equation (A.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example A.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$\begin{aligned} x &= -1 \\ y &= \sqrt{3} \end{aligned}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\begin{aligned} |e^{-1+i\sqrt{3}}| &= |e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)| \\ &= e^{-1} \\ \arg\left(e^{-1+i\sqrt{3}}\right) &= \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} + 2\pi k \\ \operatorname{Arg}\left(e^{-1+i\sqrt{3}}\right) &= \operatorname{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} \end{aligned}$$

Example A.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^{0}\big(\cos(-1) + i\sin(-1)\big) \\ &= 1\big(\cos(-1) + i\sin(-1)\big) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1\big(\cos(-1)+i\sin(-1)\big)} \\ &= e^{-1\big(\cos(1)-i\sin(1)\big)} \\ &= e^{-\cos(1)+i\sin(1)} \end{aligned}$$

If we refer back to Equation (A.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big) \end{aligned}$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} |e^{-e^{-i}}| &= |e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)| \\ &= e^{-\cos(1)} \\ \arg\left(e^{-e^{-i}}\right) &= \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) + 2\pi k \\ \operatorname{Arg}\left(e^{-e^{-i}}\right) &= \operatorname{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) \end{aligned}$$

Example A.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^i$?

Use Equation (A.22) to simplify this to a base of e, where we can use the usual Equation (A.21) to solve this.

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ 1^i &= e^{i\ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^2} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

A.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.23}$$

A.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn A.10.1 (Complex Logarithm)**.** The *complex logarithm* is defined in Equation (A.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e|w| + i\arg(w) \end{aligned} \tag{A.24}$$

Remark A.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example A.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (A.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e|z| + i\arg(z) \\ &= \log_e|-1| + i\arg(-1) \\ &= \log_e(1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \ldots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example A.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (A.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e|w| + i\arg(w) \\ \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e|2\pi k i| + i\arg(2\pi k i) \\ &= \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases} \right) \end{aligned}$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases}\right)$.

A.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\overline{z}) \tag{A.25}$$

A.11 Complex Trigonometry

For the equations below, $z \in$ mathbbC. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{A.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{A.27}$$

Example A.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

We start by using the defintion of complex cosine Equation (A.26).

$$\begin{aligned} \cos(z) &= 5 \\ \frac{e^{iz} + e^{-iz}}{2} &= 5 \\ e^{iz} + e^{-iz} &= 10 \\ e^{iz}\left(e^{iz} + e^{-iz}\right) &= e^{iz}(10) \\ {e^{iz}}^2 + 1 &= 10e^{iz} \\ {e^{iz}}^2 - 10e^{iz} + 1 = 0 \end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned} e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 4}}{2} \\ &= \frac{10 \pm \sqrt{96}}{2} \\ &= \frac{10 \pm 4\sqrt{6}}{2} \\ &= 5 \pm 2\sqrt{6} \end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned} iz &= \ln(5 \pm 2\sqrt{6}) \\ &= \log_e|5 \pm 2\sqrt{6}| + i\arg(5 \pm 2\sqrt{6}) \\ &= \log_e|5 \pm 2\sqrt{6}| + i(0 + 2\pi k) \\ &= \log_e|5 \pm 2\sqrt{6}| + 2\pi k i \\ z &= \frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}| + 2\pi k i\right) \\ &= \frac{-i}{-i}\frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}|\right) + 2\pi k \\ &= 2\pi k - i\log_e|5 \pm 2\sqrt{6}| \end{aligned}$$

Thus, $z = 2\pi k - i\log_e|5 \pm 2\sqrt{6}|$.

A.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x+iy) = \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big) \tag{A.28}$$

$$\sin(x+iy) = \big(\sin(x)\cosh(y)\big) + i\big(\cos(x)\sinh(y)\big) \tag{A.29}$$

Example A.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (A.28).

$$\begin{aligned}\cos(x+iy) &= \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big)\\ \cos(2+3i) &= \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)\end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (A.21), we can solve this.

$$\begin{aligned}e^z = e^{x+iy} &= e^x\big(\cos(y) + i\sin(y)\big)\\ x &= \cos(2)\cosh(3)\\ y &= -\sin(2)\sinh(3)\\ e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} &= e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)\end{aligned}$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned}|z| &= |e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)|\\ &= e^{\cos(2)\cosh(3)}\\ \arg(z) &= \arg(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3) + 2\pi k\\ \operatorname{Arg}(z) &= \operatorname{Arg}(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big))\\ &= -\sin(2)\sinh(3)\end{aligned}$$

A.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned}\overline{\cos(x)} &= \cos(x)\\ &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\end{aligned} \tag{A.30}$$

$$\begin{aligned}\overline{\sin(x)} &= \sin(x)\\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\end{aligned} \tag{A.31}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{B.24}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.26}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.27}$$

C Calculus

C.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcialable results.

Lemma C.0.1 (L'Hopital's Rule). *If the equation*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) *holds.*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.3}$$

Defn C.2.3 (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\underset{x}{\operatorname{argmax}}$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule). The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.4}$$

C.4 Useful Integrals

$$\int \cos(x)\, dx = \sin(x) \tag{C.5}$$

$$\int \sin(x)\, dx = -\cos(x) \tag{C.6}$$

$$\int x \cos(x)\, dx = \cos(x) + x \sin(x) \tag{C.7}$$

Equation (C.7) simplified with Integration by Parts.

$$\int x\sin(x)\;dx = \sin(x) - x\cos(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2\cos(x)\;dx = 2x\cos(x) + (x^2-2)\sin(x) \tag{C.9}$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2\sin(x)\;dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\;dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.11}$$

$$\int e^{\alpha x}\sin(\beta x)\;dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x}\;dx = \frac{e^{\alpha x}}{\alpha} \tag{C.13}$$

$$\int xe^{\alpha x}\;dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{C.14}$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha+\beta x} = \int \frac{1}{\alpha+\beta x}\;dx = \frac{1}{\beta}\ln(\alpha+\beta x) \tag{C.15}$$

$$\int \frac{dx}{\alpha^2+\beta^2x^2} = \int \frac{1}{\alpha^2+\beta^2x^2}\;dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{C.16}$$

$$\int \alpha^x\;dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.17}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{C.18}$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\big[b(t),t\big]\,\frac{db(t)}{dt} - f\big[a(t),t\big]\,\frac{da(t)}{dt} \tag{C.19}$$

D Laplace Transform

D.1 Laplace Transform

Defn D.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$

D.2 Inverse Laplace Transform

Defn D.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\, ds \tag{D.2}$$

D.3 Properties of the Laplace Transform

D.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (D.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{D.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{D.3b}$$

D.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (D.4).

$$\mathcal{L}\big\{x(\alpha t)\big\} = \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \tag{D.4}$$

D.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (D.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\big\{x(t-a)\big\} = X(s)e^{-as} \tag{D.5}$$

D.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\big\{X(s-a)\big\} = x(t)e^{at} \tag{D.6}$$

D.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\, d\lambda\right\} = \frac{1}{s} X(s) \tag{D.7}$$

D.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\big\{x(t) * v(t)\big\} = X(s)V(s) \tag{D.8}$$

D.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (D.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \tag{D.9}$$

D.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem D.1 (Intial Value Theorem)**.** *The Initial Value Theorem states that when the signal is treated at its starting time, i.e.* $t = 0^+$*, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Theorem D.2 (Final Value Theorem)**.** *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

D.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t-T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n\,\mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t-T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at}\,\mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at}\,\mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)\,\mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)\,\mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)\,\mathcal{U}(t)$	$\begin{cases} a: & \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: & \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb}+\frac{re^{-j\theta}}{s+a+jb}\right) \\ c: & \frac{As+B}{s^2+2as+c} \begin{cases} r & = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta & = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$
$e^{-at}\left(A\cos(\sqrt{c-a^2}t)+\frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

D.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0)-X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0)+X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n\frac{d^n}{ds^n}X(s)\; n\in\mathbb{N}$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s)-\sum_{0}^{n-1}s^{n-1-i}\left.\frac{d^i}{dt^i}x(t)\right\vert_{t=0^-}\; n\in\mathbb{N}$