Math 251: Multivariate and Vector Calculus — Reference Material
Illinois Institute of Technology

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1 Introduction

1.1 Multiple Dimensions

Throughout this course, we will be working with multidimensional objects. In this case, we will be drawing each of the terms of the multidimensional "thing" whatever it may be from the set of all real numbers.

In real-world terms, a dimension is just a way to measure something. So, the outer shape of your calculator can be described by a set of three dimensional equations.

This means that the set of numbers $\mathbb{R}^2$ is the traditional xy-plane for graphing one-dimensional equations. The set of numbers $\mathbb{R}^3$ is the traditional xyz 3-dimensional graph.

Remark. Remember that there are **no** such things as quadrants here. Instead, the only way we can describe the traditional 2-D graph is by using a plane. Also, the individual spaces that can be identified by which direction of the origin they are lying in is called an octant.

If you take the traditional Euclidean equation for a circle, shown below, and expand it to three dimensions, you end up with a cylinder.

$$x^2 + y^2 = 1$$

This is because the z term/dimension is **not** present in the equation, so z is able to vary through all possible values, so long as the other constraints are satisifed.

1.1.1 Distance in Three Dimensions

The distance of a point any other point in three dimensions is found by a general extension to the Pythagoran Theorem. The actual equation is shown in Equation (1.1)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_2)^2} \tag{1.1}$$

1.2 Vectors

Defn 1 (Vector)**.** A *vector* in mathematics behaves identically to the way it behaves in science and engineering, i.e. a vector is a quantity with **both** a direction and a magnitude.

However, we have a slightly more mathematical definition we can leverage as well, given below. A vector is an ordered list of n numbers, typically denoted $\langle x_1, x_2, \dots, x_n \rangle$.

$$v \in \mathbb{R}^n = \langle x_1, x_2, \dots, x_n \rangle \tag{1.2}$$

There are several vectors that are special:

1. $\vec{0}$, the Zero Vector.
2. $\hat{\imath}$, one of the Unit Vectors, with value $\langle 1, 0, 0 \rangle$.
3. $\hat{\jmath}$, one of the Unit Vectors, with value $\langle 0, 1, 0 \rangle$.
4. $\hat{k}$, one of the Unit Vectors, with value $\langle 0, 0, 1 \rangle$.

Defn 2 (Zero Vector)**.** The *zero vector* is a special vector that has a magnitude of zero and is directionless. It is the only vector that has both of these characteristics. The zero vector is defined as:

$$\vec{0} = \langle 0, 0, 0 \rangle \tag{1.3}$$

Defn 3 (Unit Vector)**.** A *unit vector* is a vector with a magnitude of 1, and has a direction along one of the fundamental axes. These have various definitions, depending on the number of dimensions involved in the equation. However, the three unit vectors that are useful for this course are in Equations (1.4a) to (1.4c).

$$\hat{\imath} = \langle 1, 0, 0 \rangle \tag{1.4a}$$
$$\hat{\jmath} = \langle 0, 1, 0 \rangle \tag{1.4b}$$
$$\hat{k} = \langle 0, 0, 1 \rangle \tag{1.4c}$$

1.2.1 Adding Vectors

Vectors are added and subtraced by performing the arithmetic operation "from tip to tail". With numbers and letters, this means that each of the individual terms add together. For example,

$$\begin{aligned} \langle -1, 2 \rangle + \langle 3, 4 \rangle &= \langle -1 + 3, 2 + 4 \rangle \\ &= \langle 2, 6 \rangle \end{aligned}$$

Now, according to the associativity of addition, Equation (1.5) holds true.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \tag{1.5}$$

1.2.2 Scalar Vector Multiplication

Scalar vector multiplication applies the scalar value to every element in the Vector. Equation (1.6) represents this with the c value being a scalar function to multiply the terms of the vector by.

$$c\langle x, y, z\rangle = \langle cx, cy, cz\rangle \tag{1.6}$$

1.2.3 Magnitude of Vectors

The magnitude of a vector represents the distance from the origin the end of the vector is. The equation for the magnitude of a vector is shown in Equation (1.7).

$$\|\langle x, y, z\rangle\| = \sqrt{x^2 + y^2 + z^2} \tag{1.7}$$

A nice property of the magnitude of a vector is the way constants can be simplified, shown below.

$$\|\vec{u} + \vec{v}\| = \|u\| + \|v\|$$

$$\|c\vec{u}\| = \|c\|\|\vec{u}\|$$

1.3 Dot Product

Defn 4 (Dot Product)**.** The *dot product* is a way to define the multiplication of two vectors that results in a single scalar value. The dot product is defined as shown in Equation (1.8).

$$\begin{aligned}\langle a, b, c\rangle \dot{\langle} x, y, z\rangle &= ax + by + cz \\ &= \|\langle a, b, c\rangle\|\|\langle x, y, z\rangle\| \cos(\theta), \text{where } 0 \leq \theta \leq \pi\end{aligned} \tag{1.8}$$

1.3.1 Properties of the Dot Product

(i) Another way to define the cross product.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos(\theta)$$

(ii) Dotting a vector with itself.

$$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 \cos(0)$$

(iii) Distributivity and Associativity of the Dot Product.

$$\begin{aligned}(\vec{u} + \vec{v}) \cdot \vec{w} &\neq \vec{u} + (\vec{v} \cdot \vec{w}) \\ &= (\vec{u} \cdot \vec{w}) + (\vec{v} \cdot \vec{w})\end{aligned}$$

(iv) Associativity of scalar multiplication with the Dot Product.

$$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$

(v) Taking the Dot Product with a vector and the Zero Vector.

$$\vec{0} \cdot \vec{u} = 0$$

One interesting thing to note about the dot product is that if $\vec{u} \cdot \vec{v} = 0$, then the vectors are orthogonal, $\vec{u} \perp \vec{v}$.

1.3.2 Projections

Defn 5 (Projection)**.** *Projection*s are a way of translating a vector from its current location to some place else. They are expressed as seen in Equation (1.9).

$$\text{proj}_{\vec{v}}\, \vec{u} = \frac{\vec{v}(\vec{u} \cdot \vec{v})}{\vec{v} \cdot \vec{v}} \tag{1.9}$$

1.4 Cross Product

Defn 6 (Cross Product)**.** The *cross product* is a way to find a third, resulting, vector that is orthogonal to the two input vectors. To find the value of the cross product, use the equations below:

$$\begin{aligned}
\langle a,b,c\rangle \times \langle x,y,z\rangle &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} \\
&= \hat{\imath}\begin{vmatrix} b & c \\ e & f \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a & c \\ d & f \end{vmatrix} + \hat{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix} \\
&= (bf-ce)\hat{\imath} - (af-cd)\hat{\jmath} + (ae-bd)\hat{k} \\
&= \langle vf-ce, af-cd, ae-bd\rangle
\end{aligned} \tag{1.10}$$

The value of the cross product can also be found by Equation (1.11).

$$\vec{u} \times \vec{v} = \|\vec{u}\|\|\vec{v}\|\sin(\theta) \tag{1.11}$$

Remark 6.1 (Cross Product Orthogonality)*.* If $\vec{u} \times \vec{v} = \vec{0}$, then the orthogonal Vector is parallel to some other thing.

Example 1.1: Take Cross Product. Lecture 4

Given the vectors $\vec{u} = \langle 3,4,1\rangle$ and $\vec{v} = \langle 1,-2,-1\rangle$, find their value when they are taken as a cross product.

The first step is rewrite the Cross Product equation into the typical matrix form.

$$\langle 3,4,1\rangle \times \langle 1,-2,-1\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 1 \\ 1 & -2 & -1 \end{vmatrix}$$

Now, we take the determinant of the 3×3 matrix, and simplify that.

$$\begin{aligned}
\langle 3,4,1\rangle \times \langle 1,-2,-1\rangle &= (-4-(-2))\hat{\imath} - (-3-1)\hat{\jmath} + (-6-4)\hat{k} \\
&= (-2)\hat{\imath} - (-4)\hat{\jmath} + (-10)\hat{k} \\
&= \langle -2,4,-10\rangle
\end{aligned}$$

Thus, our solution is $\langle -2,4,-10\rangle$.

1.4.1 Cross Product of Unit Vectors

The Unit Vectors behave like every other vector given to the Cross Product, however their output is known.

1. $\hat{\imath} \times \hat{\jmath} = \hat{k}$
2. $\hat{\jmath} \times \hat{k} = \hat{\imath}$
3. $\hat{k} \times \hat{\imath} = \hat{\jmath}$

1.4.2 Properties of the Cross Product

For all of the properties listed below, assume:

- $\vec{u} = \langle a,b,c\rangle$
- $\vec{v} = \langle x,y,z\rangle$
- $\vec{w} = \vec{u} \times \vec{v}$

(i)

$$\text{If } \vec{v} \perp \vec{u}, \text{ then } \vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{0}$$

(ii) Another way to define the cross product's magnitude.

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin(\theta)$$

(iii) Scalar multiplication associativity.

$$(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$$

(iv) Distributivity of the Cross Product.

$$\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$$

(v) Distributivity of the Cross Product.

$$(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$$

(vi) Cross Product identity.

$$\vec{0} \times \vec{u} = \vec{0}$$

(vii) Taking the cross product of a vector with itself.

$$\hat{\imath} \times \hat{\imath} = \vec{0}$$

(viii) The order of the operands in the Cross Product operation can be reversed, and negated to yield the same value.

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

1.5 Cartesian and Parametric Equations

Defn 7 (Cartesian)**.** A *Cartesian* equation is the one that most people are used to. For example, for a line:

$$y(x) = mx + b \tag{1.12}$$

In a Cartesian equation, there are the same number of variables as dimensions in the problem.

Defn 8 (Parametric)**.** A *parametric* equation is actually a set of equations that together form the resulting vector. For the same line as in Equation (1.12), we have a parametric equation as shown below.

$$\langle x(t), y(t) \rangle = \langle t, mt + b \rangle \tag{1.13}$$

In a parametric equation, there are the same number of parameterized equations as dimensions.

There are 2 standard Parametric equations that we will be using:

1. Parametric Lines, Equation (1.14)
2. Parametric Planes, Equation (1.16)

1.5.1 Parametric Lines

The Parametric equation for a line (Equation (1.14)).

$$\vec{r}(t) = t\vec{v} + \vec{p} \tag{1.14}$$

You might notice that this resembles the definition of a generic Cartesian line, seen in Equation (1.12), but:

- $\vec{r}(t)$ replaces $y(x)$
- t replaces m
- $\vec{v}$ replaces x
- $\vec{p}$ replaces b

Each of the terms used in a Cartesian equation for a line must be related to each other and t for Equation (1.14) to work.

$$t = \frac{x - p_x}{v_x} = \frac{y - p_y}{v_y} = z \vec{-} p_z v_z \tag{1.15}$$

However, if $v_z = 0$, then the equation below holds instead:

$$\begin{aligned} t &= \frac{x - p_x}{v_x} = \frac{y - p_y}{v_y} \\ z &= v_z \end{aligned}$$

Example 1.2: Convert Vector to Parametric Equation. Lecture 4

Given the system of equations below, convert it to a parametric equation for a line?

$$\begin{cases} x(t) &= 1 + 2t \\ y(t) &= 2 + 2t \\ z(t) &= 1 + -1t \end{cases}$$

Here, they ask us to convert this system of equations to a vectorized and parameterized equation for a line, Equation (1.14). From just inspection, we can see the obvious solution here.
First, the constants form the vector $\vec{p}$.

$$\vec{p} = \langle 1, 2, 1 \rangle$$

Next, the other terms form the scaled vector $t\vec{v}$.

$$t\vec{v} = \langle 2t, 2t, -1t \rangle = t\langle 2, 2, -1 \rangle$$

Thus, the parameterized equation for this line is $\vec{r}(t) = \langle 1, 2, 1 \rangle + t\langle 2, 2, -1 \rangle$.

1.5.2 Parametric Planes

The Parametric equation for a plane (Equation (1.16)).

$$\vec{r}(s, t) = t\vec{v} + s\vec{u} + \vec{p} \tag{1.16}$$

Here, we have defined two vectors, $\vec{v}$ and $\vec{u}$ that represent the 2 directions that the plane must be built out of. In addition, the scalar values t and s scale the vectors to be of any size within the three dimentionsal space. Lastly, $\vec{p}$ is the location where the plane is when both $\vec{v} = \vec{u} = \vec{0}$.

Because we have two vectors, $\vec{v}$ and $\vec{u}$, you can define a new, third vector called the Normal Vector.

Defn 9 (Normal Vector)**.** A *normal vector* is a vector that is normal (orthogonal) to the surface it was created from. If we work with Equation (1.16), then the normal vector can be defined as shown below:

$$\vec{n} = \vec{u} \times \vec{v} \tag{1.17}$$

Remark 9.1 (Component Vectors Dot Product Zero)*.* If you take the Normal Vector and dot it with any point on the **plane**, then the resulting value will be zero. This is a logical extension to Property **(i)**, as the normal vector $\vec{n}$ has a 90° angle to both component vectors, making $\cos(90°) = 0$.

If we wanted to convert the equation for a parametric plane back to a Cartesian form, then we can refer to the equations below:

$$n_x(x - p_x) + n_y(y - p_y) + n_z(z - p_z) = 0 \tag{1.18a}$$

n_x, n_y, n_z are the components of the plane's Normal Vector.
x, y, z are the variables used in a three dimensional equation.
p_x, p_y, p_z are the components of the planes $\vec{p}$.

$$n_x x + n_y y + n_z z = D \tag{1.18b}$$

D is three dimensional distance, as shown in Equation (1.1).
n_x, n_y, n_z are the components of the plane's Normal Vector.
x, y, z are the variables used in a three dimensional equation.

1.5.3 Intersections in Three Dimensions

In three dimensions, $\mathbb{R}^3$, we have 3 possibilities for what can intersect with each other right now, each leading to some interesting properties:

1. Two lines can be at the point of interest at the same time.
2. One line and a plane can be at roughly the same point of interest.
3. Two planes can intersect.

A Complex Numbers

Complex numbers are numbers that have both a real part and an imaginary part.

$$z = a \pm bi \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Since this is an appendix section, I will denote complex numbers with i, to make it more general. However, electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

$$Ae^{-ix} = A\left[\cos(x) + i\sin(x)\right] \tag{A.3}$$

A.1 Complex Conjugates

If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{A.4}$$

Defn A.1.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign.

The complex conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{A.5}$$

A.1.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.6}$$

$$\overline{\log(z)} = \log(\overline{z}) \tag{A.7}$$

A.1.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned}\overline{\cos(x)} &= \cos(x) \\ &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\end{aligned} \tag{A.8}$$

$$\begin{aligned}\overline{\sin(x)} &= \sin(x) \\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\end{aligned} \tag{A.9}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.25}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.26}$$

C Calculus

C.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcialable results.

Lemma C.0.1 (L'Hopital's Rule)**.** *If the equation*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) *holds.*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.3}$$

Defn C.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.4}$$

C.4 Useful Integrals

$$\int \cos(x)\ dx = \sin(x) \tag{C.5}$$

$$\int \sin(x)\ dx = -\cos(x) \tag{C.6}$$

$$\int x\cos(x)\ dx = \cos(x) + x\sin(x) \tag{C.7}$$

Equation (C.7) simplified with Integration by Parts.

$$\int x\sin(x)\;dx = \sin(x) - x\cos(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2\cos(x)\;dx = 2x\cos(x) + (x^2-2)\sin(x) \tag{C.9}$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2\sin(x)\;dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\;dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.11}$$

$$\int e^{\alpha x}\sin(\beta x)\;dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x}\;dx = \frac{e^{\alpha x}}{\alpha} \tag{C.13}$$

$$\int xe^{\alpha x}\;dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{C.14}$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha+\beta x} = \int \frac{1}{\alpha+\beta x}\;dx = \frac{1}{\beta}\ln(\alpha+\beta x) \tag{C.15}$$

$$\int \frac{dx}{\alpha^2+\beta^2x^2} = \int \frac{1}{\alpha^2+\beta^2x^2}\;dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{C.16}$$

$$\int \alpha^x\;dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.17}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{C.18}$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\big[b(t),t\big]\,\frac{db(t)}{dt} - f\big[a(t),t\big]\,\frac{da(t)}{dt} \tag{C.19}$$

D Laplace Transform

D.1 Laplace Transform

Defn D.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$

D.2 Inverse Laplace Transform

Defn D.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi}\int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\,ds \tag{D.2}$$

D.3 Properties of the Laplace Transform

D.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (D.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{D.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{D.3b}$$

D.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (D.4).

$$\mathcal{L}\big\{x(\alpha t)\big\} = \frac{1}{\alpha}X\left(\frac{s}{\alpha}\right) \tag{D.4}$$

D.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (D.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\big\{x(t-a)\big\} = X(s)e^{-as} \tag{D.5}$$

D.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\big\{X(s-a)\big\} = x(t)e^{at} \tag{D.6}$$

D.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\,d\lambda\right\} = \frac{1}{s}X(s) \tag{D.7}$$

D.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\big\{x(t) * v(t)\big\} = X(s)V(s) \tag{D.8}$$

D.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (D.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \tag{D.9}$$

D.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem D.1 (Intial Value Theorem)**.** *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Theorem D.2 (Final Value Theorem)**.** *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

D.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t - T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n\,\mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t - T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at}\,\mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at}\,\mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)\,\mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)\,\mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)\,\mathcal{U}(t)$	$\begin{cases} a: & \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: & \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb}+\frac{re^{-j\theta}}{s+a+jb}\right) \\ c: & \frac{As+B}{s^2+2as+c}\begin{cases} r & = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta & = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)\end{cases}\end{cases}$
$e^{-at}\left(A\cos(\sqrt{c-a^2}t)+\frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

D.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0)-X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0)+X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n \frac{d^n}{ds^n}X(s)\; n\in\mathbb{N}$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s)-\sum\limits_0^{n-1} s^{n-1-i}\frac{d^i}{dt^i}x(t)\vert_{t=0^-}\; n\in\mathbb{N}$