Chem 122: Introduction to Chemistry — Reference Sheet
Illinois Institute of Technology

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Last Edited: September 15, 2020

Contents

1 Introduction

1.1 Basic Chemistry Things

Defn 1 (Chemistry)**.** *Chemistry* is the study of Matter and its changes.

Remark 1.1. We tend to use the macroscopic world to visualize the microscopic world.

Defn 2 (Matter)**.** *Matter* is "stuff" that has both mass and volume.

Defn 3 (Scientific Method)**.** The *scientific method* is a systematic approach to research that utilizes qualitative or quantitative measurements.

Defn 4 (Hypothesis)**.** A *hypothesis* is a tentative explanation that will be tested using the Scientific Method.

Defn 5 (Law)**.** A *law* is a statement of a relationship between phenomena that is alwasy the same, under the same conditions.

Remark 5.1. These tend to be drawn from large amounts of data.

Example 1.1: Law 1.

Chlorine (Cl) is a highly reactive gas.

Example 1.2: Law 2.

Matter is neither created nor destroyed.

Defn 6 (Theory)**.** A *theory* is a unifying principle that explains a body of facts based on facts and laws. These are constantly tested for validity.

Remark 6.1. A Hypothesis can turn into a Theory with enough experimentation and acceptance.

Example 1.3: Theory 1.

Reactivity of elements depends on the element's electron (e^-) configuration.

Example 1.4: Theory 2.

All matter is made up of tiny, indestructible particles, called atoms.

1.2 Matter

As Definition 2 said, matter must have both volume and mass. Matter can have several Matter States.

Defn 7 (Matter State)**.** A *matter state* or *state of matter* is just the configuration of atoms in a particular material. There are 3 common states:

1. Solid
2. Liquid
3. Gas

But, Matter can be categorized in different ways as well. The Matter Tree is one way to categorize them.

Figure 1.1: Matter Tree

Others include:

- Atomic Weight
- Chemical Properties
- Physical Properties
- The Periodic Table
- And many others

1.3 Significant Figures

Defn 8 (Significant Figures)**.** *Significant Figures* or *Sig Figs* are ways to handle uncertainty in our measurements. In general, we treat the data that we receive as inexact numbers, thus we must confirm our suspicions several times. Additionally, Precision and Accuracy are used interchangably when they shouldn't.

Defn 9 (Precision)**.** *Precision* is defined as the closeness of data points to each other. If you think about a dartboard, this would be all the darts landing right next to each other.

Defn 10 (Accuracy)**.** *Accuracy* is defined as how close your data is to the predicted true real value.

Remark 10.1. Generally, this must be done with a minimum of 3 trials, but more will yield more accurate data.

1.3.1 Rules for Significant Figures

1. 0s between any non-zero digit is significant (100, both 0s are significant)
2. 0s at the beginning of an integer are not significant (010 = 10)
3. 0s at the end are significant is the number is a decimal (0.003050 has 4 sig figs)
4. 0s at the end, if there is no decimal/fractional portion, are not significant (16000 has 2 sig figs)

Example 1.5: Addition and Subtraction of Significant Figures.

Add 20.3056, 1.34, and 54.2 and keeping in mind significant figures.

You want to find the least precise number first, in this case it is 54.2 because it only has one decimal place. This also determines how many decimal places to go past on the solution. Adding these 3 together gives 75.8456, but because of 54.2, it becomes 75.8.

Example 1.6: Multiplication and Division of Significant Figures.

Multiply 3.4456 and 2.15 keeping in mind significant figures.

You find the number with the least number of significant figures and use that. So, 2.15 has 3 sig figs, that's the same amount your answer must have.

$$3.4456 \times 2.15 = 7.40804$$

But because we can only have 3 sig figs in our answer, 7.41 is our solution.

2 History of Chemistry

2.1 Dalton

Dalton created the first meaningful definition of an atom. He made several claims:

1. Atoms are very small
2. The same element's atoms are identical, but different elements have different atoms.
3. Atoms are neither created, nor destroyed (Law of Conservation of Matter)
4. Compounds are 2 or more elements together.

Defn 11 (Law of Conservation of Matter)**.** Matter is neither created, nor destroyed. It can *only* change forms.

2.2 Thomson

Thomson made several discoveries about atoms and their constituent particles. For his experimentation, he used a cathode ray (a beam of positively charged ions) and magnets. He discovered the Electron.

Defn 12 (Electron)**.** The *electron* is one of 3 particles that make up an atom. Electrons are negatively charged particles that are contained *outside* of the nucleus. An electron's position and velocity can not be known simultaneously. This is known as the Heisenberg's Uncertainty Principle

The cathode ray deflected from the "negative" magnetic plate to the "positive". From this he calculated the Magnetic Deflection.

Defn 13 (Magnetic Deflection)**.** When Thomson deflected his cathode ray with magnets, he measured how far it deviated from the starting line.

$$1.76 \times 10^8 \mathrm{C/g} \tag{2.1}$$

2.3 Millikan

Millikan made 2 significant contributions to the model of the atom. He discovered the charge of a single Electron and the mass of a single Electron.

$$1.602 \times 10^{-19}\text{C} \tag{2.2}$$

$$9.10938 \times 10^{-28}\text{g} \tag{2.3}$$

Both the Millikan and Millikan were drawn from Thomson's work with Magnetic Deflection.

2.4 Becquerel

Becquerel did his work with high energy radiation caused by radioactivity. He found that there were 3 types of particles released by radioactive decay.

1. Alpha Particles (α) - Positively charged particles that are charged helium atoms.
2. Beta Particles (β) - Negatively charged particles that are essentially high speed electrons.
3. Gamma particles (γ) - Uncharged particles that have next to no mass and are quite energetic.

2.5 Johnson

Johnson developed one of the first models for single atoms. This was called the Plum Pudding Model.

Defn 14 (Plum Pudding Model)**.** The *Plum Pudding Model* is a visualization of an atom. It is based off the plum pudding desert, which was one of Johnson's favorites. The positive charges were held together in a "soft" shell. The electrons were evenly distributed on the outer surface of the positively charged "plum." One of the hallmarks of this model was that the entire atom was *not* empty space.

2.6 Rutherford

Rutherford performed experiments with α-particles. He "shot" these particles at a piece of gold foil and observed what happened with after the particles passed through.

Defn 15 (Rutherford Model)**.** This is, more or less, the next model of the atom. Rutherford challenged Johnson's Plum Pudding Model of the atom. When Rutherford sent the beam of alpha particles through the gold foil, he found most of them didn't deflect, i.e. hit any thing. However some did, and were scattered in all directions. Rutherford proved that atoms are mostly empty space, with the positive charges being held in a small dense area he called the *nucleus*.

Remark 15.1. One thing to note about the *nucleus* in the Rutherford Model is that there was no concept of the neutron yet. Since neutrons are uncharged particles, they were not discovered until much later.

2.7 Atomic Mass Units, u

Eventually the neutron was discovered and the current understanding of the fundamental particles in atoms was completed. These include the:

- Proton (p^+)
- Neutron (N^0)
- Electron (e^-)

The mass of each of these particles was found and the Atomic Mass Unit was developed to make calculations easier.

$$1\text{u} = 1.66054 \times 10^{-24}\text{g} \tag{2.4}$$

Thus, the atomic mass for each particles is as follows:

- Proton (p^+) - $1.672623 \times 10^{-24}\text{g} = 1.0074\text{u}$
- Neutron (N^0) - $1.674927 \times 10^{-24}\text{g} = 1.0087\text{u}$
- Electron (e^-) - $9.109383 \times 10^{-28}\text{g} = 5.486 \times 10^{-4}\text{u}$

3 The Periodic Table

The Periodic Table was developed as a way to categorize the many chemical elements in the world. Most of the elements on the table are naturally occurring, but some of the heaviest elements have been synthesized in laboratories.

Defn 16 (The Periodic Table)**.** *The Periodic Table* is an arrangement of elements by their atomic number, Z, or the number of protons in the nucleus.

Remark 16.1. The number of protons is the *ONLY* thing that determines what an element is and where it is located on The Periodic Table. If an element has a different number of neutrons, that is an Isotope. If an element has a different number of electrons, that is an Ion.

Remark 16.2. On The Periodic Table, the elements are always in their electroneutral form. This means they have the same number of protons and electrons.

A single element drawn out of The Periodic Table will look like Figure.

Figure 3.1: Example Element from Periodic Table

There are 4 parts of each element in The Periodic Table.

1. The element's name
2. The element's atomic number, Z, which is also the number of protons in the element
3. The element's symbol
4. The element's *AVERAGE* atomic mass (g/mol), A

An element is usually written as such: ${}^{A}_{Z}\text{X}$.

Example 3.1: Element Notation.

What is Copper's (Cu) notation in text?

${}^{63.546}_{29}\text{Cu}$

3.1 Variations of Atoms

Defn 17 (Isotope)**.** An *isotope* is the same element, but with a different number of neutrons. While this does *NOT* change the element, it may change some of the properties of this element.

Example 3.2: Number of Neutrons.

What is the number of neutrons in the average isotope of these elements: ${}^{28}_{14}\text{Si}$, ${}^{29}_{14}\text{Si}$, ${}^{30}_{14}\text{Si}$?

Since Silicon, Si has 14 protons, so subtract 14 from each of the total nuclei weight.

1. ${}^{28}_{14}\text{Si}$ has 14 neutrons
2. ${}^{29}_{14}\text{Si}$ has 15 neutrons
3. ${}^{30}_{14}\text{Si}$ has 16 neutrons

Defn 18 (Ion)**.** An *ion* is the same element as the non-ionic form, however, the number of electrons and protons is different. This means that an ion can be positively or negatively charged.

Remark 18.1. The number of electrons in relation to protons determines the type of ion it is.

- If there are more electrons than protons, it is a negatively charged ion.
- If there are more protons than electrons, it is a positively charged ion.

Remark 18.2. In general:

- Metals are usually cations
- Non-Metals are usually anions

Example 3.3: What is that Element?.

Given that an element has 38 protons, 50 neutrons, and 36 electrons, what is the element?

Since there are 38 protons, that makes it Strontium (Sr). The atomic weight will be $38 + 50 = 88$. The ion is $38 + -36 = +2$. Therefore, the written atom would be ${}^{88}_{38}\text{Sr}^{+2}$.

Example 3.4: An Ion's Protons Neutrons and Electrons.

Given the element ${}^{75}_{33}\text{As}^{-3}$, how many protons, neutrons and electrons are there?

Astatine has 33 protons. The number of neutrons is $75 - 33 = 42$. Since this is Astatine, the number of electrons is $|-33 - -3| = 36$.

3.2 Types of Materials

Defn 19 (Metal)**.** A *metal* is a type of material that:
- Conducts heat well
- Conducts electricity well by freely losing electrons
- Their nuclei are congregated and their electrons form an outer electron cloud

Remark 19.1. For Metals, there are variations of some metals. For example, Iron (II) and Iron (III). These are *NOT* different ions. Rather, they are Iron in different Oxidation States.

Defn 20 (Non-Metal)**.** A *non-metal* is a type of material that:
- Do not conduct heat well
- Do not conduct electricity well
- Gain electrons easily due to high electronegativity

Defn 21 (Metalloid)**.** A *metalloid* is a type of material that has properties of both Metal and Non-Metal. These elements on right on the edge between Metals and Non-Metals on the periodic table. These include:
- Boron, B
- Silicon, Si
- Germanium, Ge
- Astatine, As
- Antimony, Sb
- Tellurium, Te

Example 3.5: Metal Nonmetal or Metalloid?.

Are these elements Metals, Non-Metals, or Metalloids?
1. Phosphorus, P
2. Osmium, Os
3. Selenium, Se
4. Thallium, Tl
5. Nickel, Ni
6. Argon, Ar
7. Tellurium, Te

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1. Phosphorus, P is a Non-Metal
2. Osmium, Os is a Metal
3. Selenium, Se is a Non-Metal
4. Thallium, Tl is a Metal
5. Nickel, Ni is a Metal
6. Argon, Ar is a Non-Metal
7. Tellurium, Te is a Metalloid

4 Naming Compounds

There is a specific way to name compounds, however, it depends on the type of elements in the compound.

4.1 Naming Inorganic Compounds

Inorganic compounds are non-carbon-based compounds. There are a 2 rules for naming these compounds:
1. If there are only 2 elements in the compound and the first element only has 1 atom, then *DO NOT USE A PREFIX*

2. Otherwise, use the numerical prefix for the number of atoms present
    (a) 1 atom → Mono-
    (b) 2 atoms → Di-
    (c) 3 atoms → Tri-
    (d) 4 atoms → Tetra-
    (e) so on and so forth

4.2 Naming Metallic Compounds

Metallic compounds are ones that have Metals in them. The rules for naming metallic compounds are similar to the inorganic ones. The only difference is naming the metal itself. You use the roman numerals when writing the compound and say the number out loud. For example, rust is Fe_2O_3, is written Iron (III) Oxide.

5 Chemical Equations

Chemical Equations form one of the key pillars of chemistry.

Defn 22 (Chemical Equations)**.** Just like in mathematical equations, *chemical equations* define the way certain chemical reactions take place. Just like in mathematics, there are some rules that are followed.

1. You need to have the same elements on each side of the reaction
2. There must be reactants (input) and products (output)
3. There must be a *Species* on each element/compound
4. There must be an arrow, → or ↔ showing the direction of reaction
    - ↔ is for reactions that can occur in both directions. Nearly all do not follow this.
5. Elements ***MUST*** follow the Law of Conservation of Matter. Refer to Section 6, Stoichiometry for further information.

Defn 23 (Species)**.** An element's or compound's *species* is the current state of matter. It can only be one of the following:

- Solid - (s)
- Liquid - (ℓ)
- Gas - (g)
- Aqueous - (aq)

Example 5.1: Chemical Equation 1.

What are the coefficients (a, b, c, d) for this chemical equation?
a CH_4(g) + b Cl_2(g) ⟶ c CCl_4(ℓ) + d HCl(g)

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1 CH_4(g) + 4 Cl_2(g) ⟶ 1 CCl_4(ℓ) + 4 HCl(g)

Example 5.2: Chemical Equation 2.

What are the coefficients (a, b, c, d) for this chemical equation?
a Al_4C_3(s) + b H_2O(ℓ) ⟶ c $Al(OH)_3$(s) + d CH_4(g)

---

1 Al_4C_3(s) + 12 H_2O(ℓ) ⟶ 4 $Al(OH)_3$(s) + 3 CH_4(g)

Example 5.3: Chemical Equation 3.

What are the coefficients (a, b, c, d) for this chemical equation?
a $AgNO_3$(aq) + b Na_2SO_4(aq) ⟶ c Ag_2SO_4(s) + d $NaNO_3$(aq)

---

2 $AgNO_3$(aq) + 1 Na_2SO_4(aq) ⟶ 1 Ag_2SO_4(s) + 2 $NaNO_3$(aq)

Example 5.4: Chemical Equation 4.

What are the coefficients (a, b, c, d) for this chemical equation?
a BF_3(g) + b H_2O(ℓ) ⟶ c HF(ℓ) + d H_3BO_3(aq)

$1\,BF_3(g) + 3\,H_2O(\ell) \longrightarrow 3\,HF(aq) + 1\,H_3BO_3(aq)$

Example 5.5: Chemical Equation 5.

What are the coefficients (a, b, c, d) for this chemical equation?
$a\ C_3H_8(g) + b\ SO_2(g) \longrightarrow c\ H_2O(g) + d\ CO_2(g)$

$1\,C_3H_8(g) + 5\,SO_2(g) \longrightarrow 4\,H_2O(g) + 3\,CO_2(g)$

Example 5.6: Chemical Equation 6.

What are the coefficients (a, b, c, d) for this chemical equation?
$a\ C_6H_{14}(\ell) + b\ O_2(g) \longrightarrow c\ CO_2(g) + d\ H_2O(g)$

$2\,C_6H_{14}(\ell) + 19\,O_2(g) \longrightarrow 14\,CO_2(g) + 12\,H_2O(g)$

5.1 Limiting Reactants

Defn 24 (Limiting Reactants)**.** The *limiting reactant(s)* is the reactant(s) that limits a reaction's total output. This means that there is some amount of excess reagent from the other, no limited reactant.

Example 5.7: Limiting Reactants 1.

Given the chemical equation $2\,Al(s) + 3\,Cl_2(g) \longrightarrow 2\,AlCl_3(s)$ and 1.50 mol of Al and 3.0 mol of Cl_2. Which reactant limits the reaction?

$$1.50\text{molAl}\left(\frac{3\text{molCl}_2}{2\text{molAl}}\right) = 2.25\text{molCl}_2$$
$$3.00\text{molCl}_2\left(\frac{2\text{molCl}_2}{3\text{molCl}_2}\right) = 2.00\text{molAl}$$

Since 2.25 mol of Cl_2 would be consumed when we have 1.50 mol of Al, and 2.00 mol Al would be consumed when we have 3.00 mol Cl_2, the Al is the limiting reactant.
This means that 1.50 mol of $AlCl_3$ will be created from this reaction.

$$1.50\text{molAl}\left(\frac{2\text{molAlCl}_3}{2\text{molAl}}\right) = 1.50\text{molAlCl}_3$$

We will have .75 mol Cl_2 left after the reaction takes place.

$$3.00\text{molCl}_2 - 2.25\text{molCl}_2 = .75\text{molCl}_2$$

Example 5.8: Limiting Reactants 2.

Given the chemical equation $Zn(s) + 2\,AgNO_3(aq) \longrightarrow 2\,Ag(s) + Zn(NO_3)_2(aq)$ and 2.00 g of Zn and 2.50 g of $AgNO_3$. Which reactant limits the reaction?

6 Stoichiometry

Defn 25 (Stoichiometry)**.** *Stoichiometry* is the calculation of reactants and products in chemical reactions. *Stoichiometry* is founded on the Law of Conservation of Matter, where the total mass of the reactants equals the total mass of the products.

Stoichiometry fits together with Chemical Equations very closely.

Some terms that are thrown around a lot, seemingly interchangeably are Formula Weight and Molecular Weight.

Defn 26 (Formula Weight)**.** *Formula weight* is the sum of atomic weights in a *chemical formula*. Since chemical formulae might be in a reduced form, it's formula weight might be different.

Defn 27 (Molecular Weight)**.** *Molecular Weight* is the sum of atomic weights in the *molecule*. This is the molecule as would be found in nature, not the reduced formula, like in the chemical formula. Therefore, the Molecular Weight might be different than the Formula Weight.

One question that can be asked is what is the percent composition of a molecule based on weight. This is generally referred to as Percent Composition.

Defn 28 (Percent Composition)**.** *Percent Composition* is the percentage of weight that certain atoms are a aprt of in a molecule. The equation for Percent Composition is shown below.

$$\%\text{ Element} = \frac{(\#\text{ of atoms})\,(\text{Atomic Weight of Element})}{\text{Molecular Weight}} \times 100 \tag{6.1}$$

Example 6.1: Percent Composition of Carbon in Ethane.

What is the percentage of carbon in ethane, by weight?

Ethane has the chemical formula C_2H_6. Therefore, the Percent Composition of carbon in ethane is as shown:

$$\%\text{ C} = \frac{2 * 12.0107}{30.0690} \times 100 = 79.8876\%$$

One other thing that could be asked is to calculate the Empirical Formula.

Defn 29 (Empirical Formula)**.** The *empirical formula* of a molecule is the smallest possible whole number ratio on each of the molecule's constituent elements. For example the Molecular Weight of B_2H_6 is B_2H_6. However, it's Empirical Formula is BH_3.

Calculating the Empirical Formula is done with the steps and equation below.

1. You can assume that you have a 100 g sample, to make things easier.
2. $$\frac{\text{Sample Mass}}{\text{Sample Molar Mass}} = \text{Empirical Formula Factor} \tag{6.2}$$
3. Divide each Empirical Formula factor by the smallest factor, and found to relatively nice numbers.
4. Multiply your resultant, roughly nice, numbers to whole integers.

Example 6.2: Calculate Empirical Formula 1.

Calculate the empirical formula if the molecule's percent composition is as follows:

- 75.69% Carbon
- 8.80% Hydrogen
- 15.51% Oxygen

First, assume that you have a 100g sample, to make calculating the percentages into mass more easily.

$$\frac{75.69\text{g}}{12.0\text{g/mol}} = 6.31\text{mol}$$
$$\frac{8.80\text{g}}{1.0\text{g/mol}} = 8.80\text{mol}$$
$$\frac{15.51\text{g}}{16.0\text{g/mol}} = .969\text{mol}$$

Then, you have to divide each mole by the lowest number, in this case .969mol.

$$\frac{6.31\text{mol}}{.969\text{mol}} = 6.5$$
$$\frac{8.80\text{mol}}{.969\text{mol}} = 9$$
$$\frac{.969\text{mol}}{.969\text{mol}} = 1$$

Then, multiply each number until you reach a whole integer, in this case, 2.

$$6.5 \times 2 = 13$$
$$9 \times 2 = 18$$
$$1 \times 2 = 2$$

So, we end up with $C_{13}H_{18}O_2$ as the empirical formula.

Example 6.3: Calculate Empirical Formula 2.

Calculate the empirical formula if the molecule's percent composition is as follows:

- 38.7% Carbon
- 9.7% Hydrogen
- 51.6% Oxygen

6.1 Theoretical Yield

Defn 30 (Theoretical Yield)**.** *Theoretical Yield* is the theoretical maximum amount of a product that can be made from the input reagents.

Defn 31 (Percent Yield)**.** *Percent Yield* is a percentage of the amount of product that you actually received from a reaction in comparison to the Theoretical Yield.

$$\%\ \text{Yield} = \frac{\text{Actual Yield}}{\text{Theoretical Yield}} \times 100 \tag{6.3}$$

Remark 31.1. The use of Percent Yield can be combined with limiting reactants to find out some other information.

Example 6.4: Percent Yield.

Given the chemical equation $Fe_2O_3 + 3\,CO \longrightarrow {}_2Fe + 3\,CO_2$ and a limiting reactant of 150 g of Fe_2O_3, what is the theoretical yield of Fe? Given that the reaction actually produced 87.9 g of Fe, what was the percent yield?

$$150\text{gFe}_2\text{O}_3\left(\frac{1\text{molFe}_2\text{O}_3}{159.688\text{gFe}_2\text{O}_3}\right)\left(\frac{2\text{molFe}}{1\text{molFe}_2\text{O}_3}\right)\left(\frac{55.845\text{gFe}}{1\text{molFe}}\right) = 105\text{gFe}$$

Therefore, the theoretical yield of Fe is 105 g.
It's percent yield is:

$$\frac{87.9\text{gFe}}{105\text{gFe}} \times 100 = 83.7\%$$

7 Eletrolytes

Defn 32 (Eletrolyte)**.** An *electrolyte* is an ion that dissociates into its constituent ions when dissolved in water. For example, salt.

Defn 33 (Strong Electrolyte)**.** A *strong electrolyte* is an Eletrolyte that *completely* dissociates into its constituent ions when dissolved in water.

Defn 34 (Weak Electrolyte)**.** A *weak electrolyte* is an Eletrolyte that *somewhat* dissociates into its constituent ions when dissolved in water.

Defn 35 (Non-Electrolyte)**.** A *non-electrolyte* is a molecule that ***DOES NOT*** dissolve into its constituent ions when dissolved in water. For example, sugar.

There are some easy and general rules for determining if something is an electrolyte.

1. *Ionic* compound means it is a Strong Electrolyte
2. *Molecular* compounds have 3 potential electrolytic states
    (a) Strong Acids/Bases are Strong Electrolytes
    (b) Weak Acids/Bases are Weak Electrolytes
    (c) All others are Non-Electrolytes

Make sure you refer to your solubility chart when referring to the strong and weak electrolytes above. Your solubility chart/table is in your textbook or online. These will give you which molecules are soluble in water.

7.1 Acids

Defn 36 (Acid)**.** An *acid* is an Eletrolyte that dissociates in water. It is a compound that when dissociated makes the solution end up with additional H^+ ions. These ions will bond with any free hydroxide molecule (OH^-).

Remark 36.1. You may hear these solutions being called "corrosive".

Remark 36.2. Acids can be dangerous chemicals! Depending on the concentration of the acid, they can eat through skin, concrete, and even glass!! **Handle these with extreme caution!!**

There are 2 types of acids:

1. Brønstad-Lowry Acid
    - Arrhenius Acids are a special form of Brønstad-Lowry Acids
2. Lewis Acid

Defn 37 (Brønstad-Lowry Acid)**.** A *Brønstad-Lowry acid* is a general name for compounds that are proton (H^+) donors.

Remark 37.1. In general, Acids refer to Brønstad-Lowry Acids.

Defn 38 (Arrhenius Acid)**.** An *Arrhenius acid* are aqueous Brønstad-Lowry Acids that are proton donors. Usually when they donate their proton, they form a Hydronium ion H_3O^+.

Defn 39 (Lewis Acid)**.** A *Lewis acid* is a molecule that forms a covalent bond with an electron pair. This generalizes such that an acid is a chemical species that accepts electron pairs directly or by releasing protons.

Remark 39.1. Lewis Acids will usually be referred to as Lewis acids.

Defn 40 (Hydronium)**.** *Hydronium* is a water molecule with an extra proton (H_3O^+). In Reduction-Oxidation Reactions that utilize Arrhenius Acids the proton donor gives the water solution the extra proton. The water becomes an Hydronium ion.

7.2 Bases

Defn 41 (Base)**.** A *base* is an Eletrolyte that dissociates in water. It is a compound that when dissociated makes the solution end up with additional OH^- ions. These ions will bond with any free hydrogen ions (H^+).

Remark 41.1. You may hear these solutions being called "caustic".

Remark 41.2. Bases can be dangerous chemicals! Depending on the concentration of the base, they can eat through plastic, skin, and many other items!! **Handle these with extreme caution!!**

7.3 Dilution

There is a simple formula to follow when diluting acids and bases.

$$M_1V_1 = M_2V_2 \tag{7.1}$$

- M is the molarity of the solution
- V is the volume of the solution

8 Reduction-Oxidation Reactions

These reactions are somewhat unique because the electrons *ONLY* move around but do not change the output of the reaction.

Defn 42 (Reduction-Oxidation Reactions)**.** *Reduction-Oxidation Reactions* or *Redox Reactions* are a type of chemical reaction. These reactions take place due to differences in electronegativity and each element's Oxidation State. The compounds that undergo a Redox Reaction are either Reduced/Oxidizing Agent or Oxidized/Reducing Agent.

Defn 43 (Oxidation State)**.** An *oxidation state* describes the degree of loss of electrons in a chemical compound.

There are some rules for how elements in a compound are assigned their Oxidation State.

1. Elemental form of element $\rightarrow 0$
2. Monoatomic ion $\rightarrow$ Oxidation State = Charge
   - Peroxides O_2^{2-} are always -2
   - Hidrides H^- are always -1
3. Fluorine is always -1
4. Groups 1 and 2 are always $+1$ and $+2$, respectively
5. Hydrogen, H, will usually be $+1$
6. Oxygen, O, will usually be -2
7. The sum of all Oxidation States ***MUST*** = Charge

Defn 44 (Reduced)**.** A compound that is *reduced* is one that ***GAINS*** electrons. A Reduced compound is the one that is Oxidizing Agent.

Defn 45 (Oxidized)**.** A compound that is *oxidized* is one that ***LOSES*** electrons. A Oxidized compound is the Reducing Agent.

Defn 46 (Oxidizing Agent)**.** An *oxidizing agent* or *oxidant* is one that ***GAINS*** electrons. A Oxidizing Agent is one that is Reduced.

Defn 47 (Reducing Agent)**.** A *reducing agent* or *reductant* is one that ***LOSES*** electrons. A Reducing Agent is one that is Oxidized.

Example 8.1: Redox Reaction 1.

Given the chemical equation below, which compounds are Reduced and Oxidized, and which are the Oxidizing Agent and Reducing Agent?
$SO_2(g) + I_2(aq) \longrightarrow SO_3(g) + I^-$

Solution from Redox and More.

9 Quantum Chemistry

This section is a brief introduction to how we have discovered certain properties of atoms due to quantum mechanics and physics. One of the biggest ideas in quantum mechanics is Heisenberg's Uncertainty Principle.

Defn 48 (Heisenberg's Uncertainty Principle)**.** *Heisenberg's Uncertainty Principle* states that it is impossible to accurately know both the position and velocity of a particle in a system.

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi} \tag{9.1}$$

Remark 48.1. The parameters for Equation (9.1) are below.

- Δx is the change in position of the "thing"
- Δp is the change in the momentum of the "thing"
- h is Planck's Constant.

A Physical Constants

Constant Name	**Variable Letter**	**Value**
Boltzmann Constant	R	$8.314\mathrm{J/mol\,K}$
Universal Gravitational	G	$6.67408 \times 10^{-11}\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Planck's Constant	h	$6.62607004 \times 10^{-34}\mathrm{mkg/s} = 4.163 \times 10^{-15}\mathrm{eV\,s}$
Speed of Light	c	$299792458\mathrm{m/s} = 2.998 \times 10^{8}\ \mathrm{m/s}$
Charge of Electron	e	$1.602 \times 10^{-19}\mathrm{C}$
Mass of Electron	m_{e^-}	$9.11 \times 10^{-31}\mathrm{kg}$
Mass of Neutron	m_{n^0}	$1.67 \times 10^{-31}\mathrm{kg}$
Mass of Earth	m_{Earth}	$5.972 \times 10^{24}\mathrm{kg}$
Diameter of Earth	d_{Earth}	$12742\mathrm{km}$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{B.24}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.26}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.27}$$

C Calculus

C.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcialable results.

Lemma C.0.1 (L'Hopital's Rule)**.** *If the equation*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) *holds.*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.3}$$

Defn C.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.
If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.4}$$

C.4 Useful Integrals

$$\int \cos(x)\, dx = \sin(x) \tag{C.5}$$

$$\int \sin(x)\, dx = -\cos(x) \tag{C.6}$$

$$\int x\cos(x)\, dx = \cos(x) + x\sin(x) \tag{C.7}$$

Equation (C.7) simplified with Integration by Parts.

$$\int x \sin(x)\ dx = \sin(x) - x\cos(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2 \cos(x)\ dx = 2x\cos(x) + (x^2 - 2)\sin(x) \tag{C.9}$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2 \sin(x)\ dx = 2x\sin(x) - (x^2 - 2)\cos(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\ dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2 + \beta^2} + C \tag{C.11}$$

$$\int e^{\alpha x}\sin(\beta x)\ dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2 + \beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x}\ dx = \frac{e^{\alpha x}}{\alpha} \tag{C.13}$$

$$\int x e^{\alpha x}\ dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{C.14}$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha + \beta x} = \int \frac{1}{\alpha + \beta x}\ dx = \frac{1}{\beta}\ln(\alpha + \beta x) \tag{C.15}$$

$$\int \frac{dx}{\alpha^2 + \beta^2 x^2} = \int \frac{1}{\alpha^2 + \beta^2 x^2}\ dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{C.16}$$

$$\int \alpha^x\ dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.17}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{C.18}$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\big[b(t),t\big]\,\frac{db(t)}{dt} - f\big[a(t),t\big]\,\frac{da(t)}{dt} \tag{C.19}$$

D Complex Numbers

Defn D.0.1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted mathbbC.

A complex number is written, in Cartesian form, as shown in Equation (D.1) below.

$$z = a \pm ib \tag{D.1}$$

where

$$i = \sqrt{-1} \tag{D.2}$$

Remark (i vs. j for Imaginary Numbers)*.* Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

D.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn D.1.1 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\mathrm{Re}\{z\} = x \tag{D.3}$$

Remark D.1.1.1 (Alternative Notation)*.* The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re\mathfrak{e}$$

Defn D.1.2 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\mathrm{Im}\{z\} = y \tag{D.4}$$

Remark D.1.2.1 (Alternative Notation)*.* The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im\mathfrak{m}$$

D.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$\begin{aligned} z &= x_1 + iy_1 \\ w &= x_2 + iy_2 \end{aligned}$$

D.2.1 Addition

The addition operation, still denoted with the + symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w \coloneqq (x_1 + x_2) + i(y_1 + y_2) \tag{D.5}$$

D.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$\begin{aligned} zw &\coloneqq (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\ &= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \end{aligned} \tag{D.6}$$

D.3 Complex Conjugates

Defn D.3.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{D.7}$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{D.8}$$

D.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$
2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - xyi + xyi - i^2y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2 \end{aligned}$$

Thus,

$$z\overline{z} = x^2 + y^2 \tag{D.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}$$

Using our solution from Equation (D.9):

$$\begin{aligned} &= \frac{(x + iy)^2}{x^2 + y^2} \\ &= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2} \end{aligned}$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$\begin{aligned} &= \frac{(x^2 - y^2) + 2xyi}{x^2 + y^2} \\ &= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \end{aligned}$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{D.10}$$

D.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.

First, start by defining some expressions so that we can prove some of these properties:

$$\begin{aligned} z &= x + iy \\ \overline{z} &= x - iy \end{aligned}$$

(i) The conjugation operation is commutative.

(ii) The conjugation operation can be distributed over addition and multiplication.

$$\begin{aligned} \overline{z + w} &= \overline{z} + \overline{w} \\ \overline{zw} &= \overline{zw} \end{aligned}$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers ($z, w \in$ mathbbC) where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.

We start by simplifying the left-hand side of the equation ($\overline{z + w}$).

$$\begin{aligned} \overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

Now, we simplify the other side ($\overline{z} + \overline{w}$).

$$\begin{aligned} \overline{z} + \overline{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

■

D.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn D.4.1 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in$ mathbbC. This is taken from Descarte's definition of a completely real plane.

The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

D.4.1 Modulus of a Complex Number

Defn D.4.2 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\overline{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{D.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in$ mathbbC). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (D.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\overline{z}\overline{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \end{aligned}$$

Note how we never needed to define z or w, so this is as general a result as possible. ∎

D.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right) \\ &= |z|^2|w|^2 \end{aligned}$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

D.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

D.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (D.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{D.12}$$

Example D.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$\begin{aligned} |5iz + 10| &= 7 \\ |5i(z + \frac{10}{5i})| &= 7 \\ |5i(z + \frac{2}{i})| &= 7 \\ |5i(z + \frac{2}{i}\frac{-i}{-i})| &= 7 \\ |5i(z - 2i)| &= 7 \end{aligned}$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$\begin{aligned} |5i||z - 2i| &= 7 \\ 5|z - 2i| &= 7 \\ |z - 2i| &= \frac{7}{5} \end{aligned}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

D.5.1 Annulus

Defn D.5.1 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (D.13).

$$r_1 \leq |z - a| \leq r_2 \tag{D.13}$$

In Equation (D.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

D.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (D.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{D.14}$$

Remark. Note that in the definition of polar form (Equation (D.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$\begin{aligned} r &= |z| \\ \theta &= \arg(z) \end{aligned}$$

Example D.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned} \cos(\theta) &= \frac{-\sqrt{3}}{2} \\ \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned} r &= 2 \\ \theta &= \frac{5\pi}{6} \\ \arg(z) &= \frac{5\pi}{6} + 2\pi k \end{aligned}$$

D.6.1 Converting Between Cartesian and Polar Forms

Using Equation (D.14) and Equation (D.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{D.15}$$

D.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

D.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \tag{D.16}$$

D.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big) \tag{D.17}$$

D.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph D.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn D.6.1 (de Moiver's Law)**.** Given a complex number z, $z \in$ mathbbC and a rational number n, $n \in$ mathbbQ, the exponentiation of z^n is defined as Equation (D.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{D.18}$$

D.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{D.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (D.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{D.20}$$

A w that satisfies Equation (D.20) is an nth root of z.

Example D.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example D.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (D.19), we can find the general form of the roots.

$$\begin{aligned}
z &= 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right) \\
z^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right) \\
&= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right)
\end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \ldots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

k = 2

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi+12\pi(2)}{18}\right)+i\sin\left(\frac{\pi+12\pi(2)}{18}\right)\right)=\sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right)+i\sin\left(\frac{25\pi}{18}\right)\right)$$

k = 3

$$\begin{aligned}\sqrt[3]{2}\left(\cos\left(\frac{\pi+12\pi(3)}{18}\right)+i\sin\left(\frac{\pi+12\pi(3)}{18}\right)\right)&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}+\frac{36\pi}{18}\right)+i\sin\left(\frac{\pi}{18}+\frac{36\pi}{18}\right)\right)\\&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}+2\pi\right)+i\sin\left(\frac{\pi}{18}+2\pi\right)\right)\\&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right)+i\sin\left(\frac{\pi}{18}\right)\right)\end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned}z_1^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right)+i\sin\left(\frac{\pi}{18}\right)\right)\\z_2^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right)+i\sin\left(\frac{13\pi}{18}\right)\right)\\z_3^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right)+i\sin\left(\frac{25\pi}{18}\right)\right)\end{aligned}$$

D.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn D.8.1 (Argument)**.** The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example D.4: Argument of Complex Number. Lecture 3, Example 1

If $z=-1-i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned}z&=-1-i\\\arg(z)&=\tan(\theta)=\frac{-i}{-1}\\&=\frac{\pi}{4}\end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$=\frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z)=\frac{5\pi}{4}+2\pi k$$

Thus, the argument of $z=-1-i$ is $\arg(z)=\frac{5\pi}{4}+2\pi k$.

Defn D.8.2 (Principal Argument)**.** The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi<\operatorname{Arg}(z)\leq\pi$.

Example D.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) &= \tan(\theta) = \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\text{Arg}(z) = \frac{5\pi}{4}$.

D.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (D.21).

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big) \tag{D.21}$$

If instead of e as the base, we have some value a, then we have Equation (D.22).

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ &= e^{\text{Re}\{z\ln(a)\}}\Big(\cos\big(\text{Im}\{z\ln(a)\}\big) + i\sin\big(\text{Im}\{z\ln(a)\}\big)\Big) \end{aligned} \tag{D.22}$$

In the case of Equation (D.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example D.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$\begin{aligned} x &= -1 \\ y &= \sqrt{3} \end{aligned}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\begin{aligned} |e^{-1+i\sqrt{3}}| &= |e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)| \\ &= e^{-1} \\ \arg\left(e^{-1+i\sqrt{3}}\right) &= \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} + 2\pi k \\ \text{Arg}\left(e^{-1+i\sqrt{3}}\right) &= \text{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} \end{aligned}$$

Example D.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^{0}\big(\cos(-1) + i\sin(-1)\big) \\ &= 1\big(\cos(-1) + i\sin(-1)\big) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1\big(\cos(-1)+i\sin(-1)\big)} \\ &= e^{-1\big(\cos(1)-i\sin(1)\big)} \\ &= e^{-\cos(1)+i\sin(1)} \end{aligned}$$

If we refer back to Equation (D.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big) \end{aligned}$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} |e^{-e^{-i}}| &= |e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)| \\ &= e^{-\cos(1)} \\ \arg\left(e^{-e^{-i}}\right) &= \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) + 2\pi k \\ \operatorname{Arg}\left(e^{-e^{-i}}\right) &= \operatorname{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) \end{aligned}$$

Example D.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^i$?

Use Equation (D.22) to simplify this to a base of e, where we can use the usual Equation (D.21) to solve this.

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ 1^i &= e^{i\ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^2} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

D.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{D.23}$$

D.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn D.10.1 (Complex Logarithm)**.** The *complex logarithm* is defined in Equation (D.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e|w| + i\arg(w) \end{aligned} \tag{D.24}$$

Remark D.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example D.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (D.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e|z| + i\arg(z) \\ &= \log_e|-1| + i\arg(-1) \\ &= \log_e(1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \ldots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example D.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (D.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e|w| + i\arg(w) \\ \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e|2\pi k i| + i\arg(2\pi k i) \\ &= \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases}\right) \end{aligned}$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases}\right)$.

D.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\overline{z}) \tag{D.25}$$

D.11 Complex Trigonometry

For the equations below, $z \in$ mathbbC. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{D.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{D.27}$$

Example D.11: Simplify Complex Sinusoid. Lecture 3

Solve for z in the equation $\cos(z) = 5$?

We start by using the defintion of complex cosine Equation (D.26).

$$\begin{aligned}
\cos(z) &= 5 \\
\frac{e^{iz} + e^{-iz}}{2} &= 5 \\
e^{iz} + e^{-iz} &= 10 \\
e^{iz}\left(e^{iz} + e^{-iz}\right) &= e^{iz}(10) \\
{e^{iz}}^2 + 1 &= 10e^{iz} \\
{e^{iz}}^2 - 10e^{iz} + 1 = 0
\end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned}
e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} \\
&= \frac{10 \pm \sqrt{100 - 4}}{2} \\
&= \frac{10 \pm \sqrt{96}}{2} \\
&= \frac{10 \pm 4\sqrt{6}}{2} \\
&= 5 \pm 2\sqrt{6}
\end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned}
iz &= \ln(5 \pm 2\sqrt{6}) \\
&= \log_e|5 \pm 2\sqrt{6}| + i\arg(5 \pm 2\sqrt{6}) \\
&= \log_e|5 \pm 2\sqrt{6}| + i(0 + 2\pi k) \\
&= \log_e|5 \pm 2\sqrt{6}| + 2\pi k i \\
z &= \frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}| + 2\pi k i\right) \\
&= \frac{-i}{-i}\frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}|\right) + 2\pi k \\
&= 2\pi k - i\log_e|5 \pm 2\sqrt{6}|
\end{aligned}$$

Thus, $z = 2\pi k - i\log_e|5 \pm 2\sqrt{6}|$.

D.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x+iy) = \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big) \tag{D.28}$$

$$\sin(x+iy) = \big(\sin(x)\cosh(y)\big) + i\big(\cos(x)\sinh(y)\big) \tag{D.29}$$

Example D.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (D.28).

$$\begin{aligned}
\cos(x+iy) &= \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big) \\
\cos(2+3i) &= \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)
\end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (D.21), we can solve this.

$$\begin{aligned}
e^z = e^{x+iy} &= e^x\big(\cos(y) + i\sin(y)\big) \\
x &= \cos(2)\cosh(3) \\
y &= -\sin(2)\sinh(3) \\
e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} &= e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)
\end{aligned}$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned}
|z| &= |e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)| \\
&= e^{\cos(2)\cosh(3)} \\
\arg(z) &= \arg(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)) \\
&= -\sin(2)\sinh(3) + 2\pi k \\
\operatorname{Arg}(z) &= \operatorname{Arg}(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)) \\
&= -\sin(2)\sinh(3)
\end{aligned}$$

D.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned}
\overline{\cos(x)} &= \cos(x) \\
&= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)
\end{aligned} \tag{D.30}$$

$$\begin{aligned}
\overline{\sin(x)} &= \sin(x) \\
&= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)
\end{aligned} \tag{D.31}$$