

# 1 Introduction

Example 1.  
Test example.

## 1.1 Definitions and Terminology

**Defn 1** (Differential Equation). A *differential equation (DE)* is an equation with 1 or more derivatives.

*Remark 1.1.* The highest differential determines the order of the differential equation. This means that the differential equation below is of order 2.

$$y'' + y = 0$$
$$\frac{d^2 y}{dx^2} + y = 0$$

**Defn 2** (Initial Value Problem). A differential equation with one or more initial conditions is called an *initial value problem (IVP)*.

*Remark 2.1.* To solve an initial value problem, you must have the same number of initial conditions as the order of the differential equation.

*Remark 2.2* (Existence of Unique Solution).  $R$  is a rectangular region on the  $xy$ -plane  $a \leq x \leq b$ ,  $c \leq y \leq d$  that contains  $(x_0, y_0)$  interior. If  $f(x, y)$  and  $\frac{df}{dy}$  are continuous on  $R$ , then an interval exists  $I_0$  such that  $(x_0 - h, x_0 + h)$  where  $h > 0$ , on the interval  $[a, b]$ , and a unique function  $y(x)$ , defined on  $I_0$  that is a solution of the initial value problem.

## 1.2 Separable Differential Equation

**Defn 3** (Separable). A *separable* differential equation allows you to move various elements around to solve the equation. For example,

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P} dP = k dt$$
$$\ln(P) = kt + C$$
$$P = Ce^{kt}$$

*Remark 3.1.* These are used extensively in modelling phenomena with differential equations. These include: Population Growth, Radioactive Decay, Newton's Law of Cooling/Heating, and Spread of Disease.

## 1.3 Modeling with Differential Equations

### 1.3.1 Population Growth

**Defn 4** (Population Growth). *Population growth* can be modelled with a separable differential equation. Namely,

$$\frac{dP}{dt} = kP \tag{1}$$

*Remark 4.1* (Population Growth Parameters). The parameters for the Population Growth equation are given below.

- $k > 0$
- $P > 0$

### 1.3.2 Radioactive Decay

**Defn 5** (Radioactive Decay). *Radioactive decay* is the process that some particularly heavy atoms undergo.

**Defn 6** (Half-Life). The *half-life* is the usual reported metric, and is defined as the amount of time required for an element to half its mass through Radioactive Decay.

$$\frac{1}{2}A_0 = A_0 e^{kt} \tag{2}$$

*Remark 6.1* (Radioactive Decay Parameters). The parameters for the Radioactive Decay equation are given below.

- $k < 0$
- $A > 0$

### 1.3.3 Newton's Law of Cooling/Heating

**Defn 7** (Newton's Law of Cooling/Heating). *Newton's Law of Cooling/Heating* is the same equation, but some of the parameters change. This equation is defined as:

$$\frac{dT}{dt} = k(T - T_m) \quad (3)$$

*Remark 7.1.* The parameters for the Newton's Law of Cooling/Heating equation are given below.

- $\frac{dT}{dt}$ ; The rate of change of temperature in the object per unit time.
- $k < 0$ ; The cooling constant and is unique to every object.
- $T$ ; The starting temperature.
- $T_m$ ; The temperature of the surrounding medium.

### 1.3.4 Spread of Disease

**Defn 8** (Spread of Disease). This is used to model the spread of something throughout a society or group of people.

$$\frac{dx}{dt} = kxy \quad (4)$$

*Remark 8.1.* The parameters for the Spread of Disease equation are given below.

- $\frac{dx}{dt}$ ; Change in the number of infected per unit time.
- $k < 0$ ; Transmission Constant
- $x$ ; Number of Infected
- $y$ ; Number of non-infected,  $y$  is really a function of  $x$ 
  - $y = n + 1 - x$

## A Reference Material

### A.1 Trigonometry

#### A.1.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{A.2})$$

#### A.1.2 Euler Equivalents of Trigonometric Functions

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad (\text{A.3})$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (\text{A.4})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{A.5})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{A.6})$$

### A.2 Calculus

#### A.2.1 Fundamental Theorems of Calculus

**Defn 9** (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if  $f$  is continuous on the closed interval  $[a, b]$  and  $F$  is the indefinite integral of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{A.7})$$

**Defn 10** (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for  $f$  a continuous function on an open interval  $I$  and  $a$  any point in  $I$ , and states that if  $F$  is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{A.8})$$