ETSN10: Network Architecture and Performance - Reference Sheet

Karl Hallsby

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1 Probability Review

This section is meant to quick review and introduce the equations that will be used throughout this course. It is not meant to be comprehensive and/or in-depth. For more information about the topic of probability and statistics, refer to the Math 374 - Probability and Statistics document.

1.1 Axioms of Probability

Defn 1 (Sample Space)**.** The *sample space* is the set of all possible outcomes in a random experiment. It is denoted with the capital Greek omega.

$$\Omega \tag{1.1}$$

Defn 2 (Event)**.** An *event* is a subset of the Sample Space that we are interested in. These are generally denoted with capital letters.

$$A \subseteq \Omega \tag{1.2}$$

Defn 3 (Mutually Exclusive)**.** Any two Events are *mutually exclusive* if the equation below holds.

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) \tag{1.3}$$

Laws that follow from the above definitions (Definitions 1 to 3).

1. The conjugate of the Event occurring, i.e. the Event **not** occurring is:

$$\mathrm{P}(\bar{A}) = 1 - \mathrm{P}(A) \tag{1.4}$$

2. The probability of the union of 2 Events is:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) \tag{1.5}$$

   - If A and B are Mutually Exclusive, then $\mathrm{P}(A \cup B) = 0$.

1.2 Conditional Probability

Defn 4 (Conditional Probability)**.** *Conditional probability* is the probability of an Event occurring when it is known that another Event occurred.

$$\mathrm{P}(A \,|\, B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \tag{1.6}$$

Defn 5 (Independent)**.** Events are *independent* if the probability of the events' intersection is the same as their probabilities multipled together.

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B) \tag{1.7}$$

Remark 5.1 (Conditional Probability and Independent Events)*.* If A and B are Events and are Independent, then

$$\begin{aligned} \mathrm{P}(A \,|\, B) &= \mathrm{P}(A) \\ \mathrm{P}(B \,|\, A) &= \mathrm{P}(B) \end{aligned} \tag{1.8}$$

1.3 Random Variables

Defn 6 (Random Variable)**.** A *random variable* is a mapping from an Event's outcome to a real number.

There are 2 types of random variables, based on what the mapping ends up with:

1. Discrete Random Variables are mapped to integers, $\mathbb{Z}$.
2. Continuous Random Variables are mapped to the real numbers, $\mathbb{R}$.

1.3.1 Discrete Random Variables

Defn 7 (Discrete Random Variable)**.** A *Discrete Random Variable* is one whose values are mapped from an Event's outcome to the integer numbers ($\mathbb{Z}$). These Random Variables are drawn from outcomes that are finite (sides on a die) or countably infinite.

The probability of a single value of the discrete random variable is denoted differently here than in the course material. The subscript refers to which discrete random variable we are working with (in this case X) and the variable in parentheses is the value we are calculating for (in this case $x \in X$).

$$p_X(x) \tag{1.9}$$

The sum of all probabilities for values that the discrete random variable can take **must** sum to 1.

$$\sum_{x \in X} p_X(x) = 1 \tag{1.10}$$

The mean or expected value of a discrete random variables is shown below:

$$\begin{aligned} \mu &= \sum_{x \in X} x p_X(x) \\ \mathbb{E}[X] &= \sum_{x \in X} x p_X(x) \end{aligned} \tag{1.11}$$

The variance of a discrete random variable is how "off" a value from the random variable is from the mean/expected value.

$$\begin{aligned} \sigma^2 &= \sum_{x \in X} (x - \mu)^2 p_X(x) \\ \text{VAR}[X] &= \sum_{x \in X} \left(x - \mathbb{E}[X]\right)^2 p_X(x) \end{aligned} \tag{1.12}$$

The standard deviation is the square root of the variance.

$$\begin{aligned} \sigma = \sqrt{\sigma^2} &= \sqrt{\sum_{x \in X} (x - \mu)^2 p_X(x)} \\ \text{STD}[X] = \sqrt{\text{VAR}[X]} &= \sqrt{\sum_{x \in X} \left(x - \mathbb{E}[X]\right)^2 p_X(x)} \end{aligned} \tag{1.13}$$

There are 5 different Discrete Random Variable distributions that we will be heavily utilizing in this course.

1.3.1.1 Uniform Random Variable

Defn 8 (Uniform Random Variable)**.** The *uniform random variable* is a Discrete Random Variable whose probabilities for each outcome is equal.

For a Discrete Random Variable X, which has $|X|$ possible values,

$$p_X(x) = \frac{1}{|X|} \tag{1.14}$$

Example 1.1: Uniform Random Variable. Lecture 1

For example, the roll of a die is typically modelled as a uniform random variable. Find the probability distribution function, the expected value, and the variance.

Let's assume this is a 6-sided die. And let's map each side's number to a value in the range of $X \in [1, 6]$.

Using Equation (1.14), we can find the probability distribution easily.

$$p_X(x) = \begin{cases} \frac{1}{6} & x = 1 \\ \frac{1}{6} & x = 2 \\ \frac{1}{6} & x = 3 \\ \frac{1}{6} & x = 4 \\ \frac{1}{6} & x = 5 \\ \frac{1}{6} & x = 6 \end{cases}$$

Using Equation (1.11), we can find the the expected value/mean.

$$\begin{aligned} \mu = \mathbb{E}[X] &= \sum_{x=1}^{6} x p_X(x) \\ &= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} \\ &= \frac{1+2+3+4+5+6}{6} \\ &= \frac{21}{6} = 3.5 \end{aligned}$$

Using Equation (1.12), we can find the variance.

$$\begin{aligned} \sigma^2 = \text{VAR}[X] &= \sum_{x=1}^{6} \big(x - \mathbb{E}[X]\big)^2 p_X(x) \\ &= \sum_{x=1}^{6} (x - 3.5)^2 \left(\frac{1}{6}\right) \\ &= 2.91667 \end{aligned}$$

Using Equation (1.13), we can find the standard deviation.

$$\begin{aligned} \sigma = \text{STD}[X] &= \sqrt{\sum_{x=1}^{6} \big(x - \mathbb{E}[X]\big)^2 p_X(x)} \\ &= \sqrt{\sum_{x=1}^{6} (x - 3.5)^2 \left(\frac{1}{6}\right)} \\ &= \sqrt{2.91667} \\ &= 1.70783 \end{aligned}$$

1.3.1.2 Bernoulli Random Variable

Defn 9 (Bernoulli Random Variable)**.** The *Bernoulli random variable* is one where **only one** test occurs, and there are only 2 outcomes.

The probability of success is denoted

$$p_X(\text{success}) = p \tag{1.15}$$

The probability of failure is denoted

$$p_X(\text{failure}) = 1 - p \tag{1.16}$$

The mean/expected value is:

$$\mu = \mathbb{E}[X] = p \tag{1.17}$$

The variance is:

$$\sigma^2 = \text{VAR}[X] = (1-p)p \tag{1.18}$$

1.3.1.3 Binomial Random Variable

Defn 10 (Binomial Random Variable)**.** The *binomial random variable* is one where n trials are run with no stops for a success, where the Random Variable in each run is a Bernoulli Random Variable.

The probability of k successes with n trials is

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} (1-p)^{n-k} \tag{1.19}$$

The mean/expected value after n trials is

$$\mu = \mathbb{E}[X] = np \tag{1.20}$$

The variance after n trials is

$$\sigma^2 = \mathrm{VAR}[X] = np(1-p) \tag{1.21}$$

1.3.1.4 Geometric Random Variable

Defn 11 (Geometric Random Variable)**.** The *geometric random variable* is one where n trials are run, where the nth trial is a success, meaning there are $n-1$ previous failures. The Random Variable in each run is a Bernoulli Random Variable.

This means **each trial** has a probability of success of

$$p_X(\text{success}) = p \tag{1.22}$$

And **each trial** has a probability of failure of

$$p_X(\text{failure}) = 1 - p \tag{1.23}$$

The mean/expected value is

$$\mu = \mathbb{E}[X] = \frac{1}{p} \tag{1.24}$$

The variance is

$$\sigma^2 = \mathrm{VAR}[X] = \frac{1-p}{p^2} \tag{1.25}$$

1.3.1.5 Poisson Random Variable

Defn 12 (Poisson Random Variable)**.** The *Poisson random variable* is used to model the number of independent events that occur over a given period of time.

The Poisson random variable has one parameter,

$$\lambda \tag{1.26}$$

λ is the average number of events per unit of time.

The probability function for the value of $x \in X$ of this random variable is

$$p_X(x) = e^{-\lambda} \left(\frac{\lambda^x}{k!} \right) \tag{1.27}$$

The mean/expected value is

$$\mu = \mathbb{E}[X] = \lambda \tag{1.28}$$

The variance is

$$\sigma^2 = \mathrm{VAR}[X] = \lambda \tag{1.29}$$

1.3.2 Continuous Random Variables

Defn 13 (Continuous Random Variable)**.** A *Continuous Random Variable* is one whose values are mapped from an Event's outcome to the real numbers ($\mathbb{R}$).

Continuous random variables cannot be calculated at single points, but must be integrated over a range. This is called the Cumulative Distribution Function (CDF). The subscript refers to the random variable we are using, and x is the value being calculated for.

$$F_X(x) = \mathrm{P}(X \leq x) \tag{1.30}$$

Continuous random variables have a Probability Density Function (PDF), which is the derivative of the CDF.

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{1.31}$$

This PDF must integrate to 1

$$\int_{x \in X} f_X(x) = 1 \tag{1.32}$$

The expected value/mean is

$$\begin{aligned} \mu &= \int_{x \in X} x f_X(x)\, dx \\ \mathbb{E}[X] &= \int_{x \in X} x f_X(x)\, dx \end{aligned} \tag{1.33}$$

The variance is

$$\begin{aligned} \sigma^2 &= \int_{x \in X} (x-\mu)^2 f_X(x)\, dx \\ \mathrm{VAR}[X] &= \int_{x \in X} \left(x - \mathbb{E}[X]\right)^2 f_X(x)\, dx \end{aligned} \tag{1.34}$$

1.3.2.1 Negative Exponential Random Variable

Defn 14 (Negative Exponential Random Variable)**.** The *negative exponential random variable*, sometimes shortened to exponential random variable, is a Continuous Random Variable that has 1 parameter μ, the rate of decay.

The negative exponential random variable has a PDF of

$$f_X(x) = \mu e^{-\mu x} \tag{1.35}$$

The negative exponential random variable has a CDF of

$$F_X(x) = 1 - e^{-\mu x} \tag{1.36}$$

Its mean/expected value is

$$\begin{aligned} \mu &= \frac{1}{\mu} \\ \mathbb{E}[X] &= \frac{1}{\mu} \end{aligned} \tag{1.37}$$

Its variance is

$$\begin{aligned} \sigma^2 &= \left(\frac{1}{\mu}\right)^2 \\ \mathrm{VAR}[X] &= \left(\frac{1}{\mu}\right)^2 \end{aligned} \tag{1.38}$$

Its standard deviation is

$$\begin{aligned} \sigma &= \sqrt{\sigma^2} = \frac{1}{\mu} \\ \mathrm{STD}[X] &= \sqrt{\mathrm{VAR}[X]} = \frac{1}{\mu} \end{aligned} \tag{1.39}$$

1.3.2.2 Gaussian Random Variable

Defn 15 (Gaussian Random Variable)**.** The *Gaussian random variable*, sometimes called the normal random variable, has 2 parameters, μ and σ.

The Probability Density Function (PDF) is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x-\mu^2}{2\sigma^2}} \tag{1.40}$$

The mean/expected value is

$$\begin{aligned} \mu &= \mu \\ \mathbb{E}[X] &= \mu \end{aligned} \tag{1.41}$$

The variance is

$$\begin{aligned} \sigma^2 &= \sigma^2 \\ \mathrm{VAR}[X] &= \sigma^2 \end{aligned} \tag{1.42}$$

The standard deviation is

$$\begin{aligned} \sigma &= \sqrt{\sigma^2} = \sigma \\ \mathrm{STD}[X] &= \sqrt{\mathrm{VAR}[X]} = \sigma \end{aligned} \tag{1.43}$$

1.4 Multiple Random Variables

Defn 16 (Joint Probability Function)**.** If X, Y are Discrete Random Variables, the probability of the joint event occurring is

$$p_{X,Y}(x, y) \tag{1.44}$$

where the subscripted X, Y refers to the Discrete Random Variables in use and x, y refers to the values being calculated for.

$$\mathrm{P}(x, y) = \mathrm{P}(X = x, Y = y) \tag{1.45}$$

Defn 17 (Joint Cumulative Distribution Function)**.** If X, Y are Continuous Random Variables, then their *joint cumulative distribution function* is

$$F_{X,Y}(x, y) = \mathrm{P}(X \leq x, Y \leq y) \tag{1.46}$$

Remark 17.1 (Independent Multiple Random Variables)*.* X, Y are independent if and only if

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \tag{1.47}$$

Defn 18 (Joint Probability Density Function)**.** If X, Y are Continuous Random Variables, then their *joint probability density function* is

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) \tag{1.48}$$

1.5 Properties of Expected Value

(i) Expected Value obeys the laws of linearity.

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b \tag{1.49}$$

(ii) For any Random Variables X, Y

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] \tag{1.50}$$

(iii) For any Independent Multiple Random Variables X, Y,

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] \tag{1.51}$$

1.6 Properties of Variance

(i) Variance does not obey the laws of linearity.

$$\mathrm{VAR}[aX + b] = a^2\,\mathrm{VAR}[X] \tag{1.52}$$

(ii) For any Independent Multiple Random Variables X, Y,

$$\mathrm{VAR}[X + Y] = \mathrm{VAR}[X] + \mathrm{VAR}[Y] \tag{1.53}$$

1.7 Covariance

Defn 19 (Covariance)**.** The *covariance* of two Random Variables is defined as

$$\begin{aligned} \mathrm{Cov}[X, Y] &= \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)\big(Y - \mathbb{E}[Y]\big)\Big] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] \end{aligned} \tag{1.54}$$

Remark 19.1 (Relationship Between Covariance and Variance)*.* Variance is actually a case of Covariance of a Random Variable with itself.

$$\mathrm{VAR}[X] = \mathrm{Cov}[X, X] \tag{1.55}$$

The Covariance of two Random Variables can have 3 possible values:

1. Positive: If one Random Variable increases, the other does too.
2. Negative: If one Random Variable increases, the other decreases.
3. Zero: If one Random Variable increases, the other does nothing.

1.8 Correlation Coefficient

Defn 20 (Correlation)**.** The *correlation of X and Y* is defined as the $1, 1$ moment.

$$\mathbb{E}\left[X^1 Y^1\right] \tag{1.56}$$

Defn 21 (Correlation Coefficient)**.** The *correlation coefficient of X and Y* is a measure of the **LINEAR relationship** between X and Y. It does not say anything about nonlinear dependence. It is defined as

$$\begin{aligned} \rho_{X,Y} &= \frac{\operatorname{Cov}[X,Y]}{\sigma_X \sigma_Y} \\ &= \frac{\operatorname{Cov}[X,Y]}{\operatorname{STD}[X]\operatorname{STD}[Y]} \end{aligned} \tag{1.57}$$

Remark 21.1. $\rho_{X,Y}$ only ranges from $-1 \leq \rho_{X,Y} \leq 1$.

Remark 21.2. The closer $\rho_{X,Y}$ are to $+1$, the closer X and Y are to having a positive linear relationship. The closer $\rho_{X,Y}$ are to -1, the closer X and Y are to having a negative linear relationship. The closer $\rho_{X,Y}$ are to 0, the closer X and Y are to being *uncorrelated*.

If $\rho_{X,Y} = 0$, then

- If X, Y are Independent Multiple Random Variables, then they are uncorrelated.
- If X, Y are uncorrelated ($\rho_{X,Y} = 0$), then they may still **not** be independent.

A Complex Numbers

Complex numbers are numbers that have both a real part and an imaginary part.

$$z = a \pm bi \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Since this is an appendix section, I will denote complex numbers with i, to make it more general. However, electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

$$Ae^{-ix} = A\left[\cos(x) + i\sin(x)\right] \tag{A.3}$$

A.1 Complex Conjugates

If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{A.4}$$

Defn A.1.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign.

The complex conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{A.5}$$

A.1.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.6}$$

$$\overline{\log(z)} = \log(\overline{z}) \tag{A.7}$$

A.1.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned} \overline{\cos(x)} &= \cos(x) \\ &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \end{aligned} \tag{A.8}$$

$$\begin{aligned} \overline{\sin(x)} &= \sin(x) \\ &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \end{aligned} \tag{A.9}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.25}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.26}$$

C Calculus

C.1 Fundamental Theorems of Calculus

Defn C.1.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.1}$$

Defn C.1.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.2}$$

Defn C.1.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\underset{x}{\operatorname{argmax}}$$

C.2 Rules of Calculus

C.2.1 Chain Rule

Defn C.2.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.3}$$

D Laplace Transform

Defn D.0.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$