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# 1 Probability Models

## 1.1 Relative Frequency

**Defn 1** (Relative Frequency). *Relative frequency* is defined in Equation (1.1):

$$f_k(n) = \frac{N_k(n)}{n} \quad (1.1)$$

- $k$  is the outcome
- $N_k(n)$  is the number of times outcome  $k$

### 1.1.1 Properties of Relative Frequencies

(i)

$$f_k(n) = \frac{N_k(n)}{n} \quad (1.2)$$

(ii)

$$0 \leq N_k(n) \leq n \quad (1.3)$$

(iii)

$$0 \leq f_k(n) \leq 1 = \frac{0}{n} \leq \frac{N_k(n)}{n} \leq \frac{n}{n} \quad (1.4)$$

(iv)

$$\sum_{k=1}^k f_k(n) = \sum_{k=1}^k \frac{N_k(n)}{n} = \frac{\sum_{k=1}^k N_k(n)}{n} = \frac{n}{n} = 1 \quad (1.5)$$

(v)

$$\sum_{k=1}^k f_k(n) = 1 \quad (1.6)$$

(vi) If events  $A$  and  $B$  are disjoint and event  $C$  is " $A$  or  $B$ ", then

$$F_C = F_A(n) + F_B(n) \quad (1.7)$$

## 1.2 Statistical Regularity

**Defn 2.** The averages obtained in long sequences of trials that lead to approximately the same value have a property called *statistical regularity*. This is defined in Equation (1.8).

$$\lim_{n \rightarrow \infty} f_k(n) = p_k \quad (1.8)$$

- $p_k$  is the probability of event  $k$  occurring

# 2 Set Theory

1. A *set* is a collection of objects, denoted by capital letters
2. Denote the *universal set*,  $U$ ; consisting of all possible objects of interest in a given setting/application
3. For any set  $A$ , we say that " $x$  is an element of  $A$ ", denoted  $x \in A$  if object  $x$  of the universal set  $U$  is contained in  $A$
4. We say that " $x$  is not an element of  $A$ ", denoted  $x \notin A$  if object  $x$  of the universal set  $U$  is not contained in  $A$
5. We say that " $A$  is a subset of  $B$ ", denoted  $A \subset B$  if every element in  $A$  also belongs to  $B$ ,  $x \in A \rightarrow x \in B$
6. The *empty set*,  $\emptyset$  is defined as the set with no elements
  - The empty set is a subset of every set
7. Sets  $A$  and  $B$  are equal if they contain the same elements. To show this:
  - (a) Enumerate the elements of each set
  - (b) Thm:  $A = B \iff A \subset B \text{ AND } B \subset A$
8. The *union of 2 sets*  $A, B$ , denoted  $A \cup B$  is defined as the set of outcomes that are either in  $A$ , or in  $B$ , or both
9. The *intersection of 2 sets*,  $A, B$ , denoted  $A \cap B$  is defined as the set of outcomes in  $A$  and  $B$
10. The 2 sets  $A, B$  are said to be *disjoint or mutually exclusive* if  $A \cap B = \emptyset$
11. The *complement of a set*  $A$ , denoted  $A^C$  is defined as the set of elements of  $U$  not in  $A$ 
  - $A^C = \{x \in U | x \notin A\}$

12. *Relative complement* or *difference*, denoted  $A - B$ , is the set of elements in  $A$  that are not in  $B$

- $A - B = A \cap B^C$
- $A^C = U - A$

## 2.1 Properties of Set Operations

Set Operators are:

1. Commutative, Equation (2.1)

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned} \tag{2.1}$$

2. Associative, Equation (2.2)

$$\begin{aligned} A \cup (B \cup C) &= (A \cup B) \cup C \\ A \cap (B \cap C) &= (A \cap B) \cap C \end{aligned} \tag{2.2}$$

3. Distributive, Equation (2.3)

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned} \tag{2.3}$$

4. Set Operations obey De Morgan's Laws, Equation (2.4)

$$\begin{aligned} (A \cup B)^C &= A^C \cap B^C \\ (A \cap B)^C &= A^C \cup B^C \end{aligned} \tag{2.4}$$

Additionally,

**Defn 3** (Union of  $n$  Sets). The *union of  $n$  sets*  $\bigcup_{k=1}^n A_k = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$  is the set consisting of all elements such that  $x \in A_k$  for some  $1 \leq k \leq n$ .

- All sets need to be empty to make  $\bigcup_{k=1}^n A_k = \emptyset$

**Defn 4** (Intersection of  $n$  Sets). The *intersection of  $n$  sets*  $\bigcap_{k=1}^n A_k = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$  is the set consisting of all elements such that  $x \in A_k$  for all  $1 \leq k \leq n$

- Just one set needs to be empty to make  $\bigcap_{k=1}^n A_k = \emptyset$

## 3 Probability Theory

There are 3 main components to Probability Theory.

1. Set Theory
2. Probability Law Corollaries
3. Conditional Probability and Event Independence

### 3.1 Random Experiments

**Defn 5** (Random Experiment). A *random experiment* is an experiment whose outcome varies in an unpredictable fashion when performed under the same conditions.

**Defn 6** (Sample Space). A *sample space*,  $S$  of a random experiment is the set of all possible experiments.

**Defn 7** (Outcome/Sample Point). An *outcome*, or *sample point* of a random experiment is a result that cannot be decomposed into other results.

**Defn 8** (Event). An *event* corresponds to a subset of the sample space. We say an event occurs if and only if (iff) the outcome of the experiment is in the subset representing the event.

**Defn 9** (Event Classes). An *event class*  $\mathcal{F}$  is the collection of the all the events' sets.  $\mathcal{F}$  should be closed under unions, intersections, and complements.

- For  $S$  finite, or countably infinite, then we can let  $\mathcal{F}$  be all subsets of  $S$ .

- For  $S$  uncountably infinite, instead we can let  $\mathcal{F}$  consist of the subsets that can be obtained as countable unions and intersections of some sets of  $\mathcal{F}$ .

**Defn 10** (Probability Law). A *probability law* for a random experiment  $E$ , with sample space  $S$ , and an event class  $\mathcal{F}$  is a rule that assigns to each event  $A \in \mathcal{F}$  a number  $P[A]$ , called the probability of  $A$  that satisfies the axioms:

Axiom I:  $0 \leq P[A]$

Axiom II:  $P[S] = 1$

Axiom III: If  $A \cap B = \emptyset$ , then  $P[A \cup B] = P[A] + P[B]$

Axiom III': If  $A_1, A_2, \dots$  is a sequence of events such that  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ , then  $P[\bigcup_{k=1}^{\infty} A_k] = \sum_{k=1}^{\infty} P[A_k]$

## 3.2 Probability Law Corollaries

Axiom I:  $0 \leq P[A]$

Axiom II:  $P[S] = 1$

Axiom III: If  $A \cap B = \emptyset$ , then  $P[A \cup B] = P[A] + P[B]$

Axiom III': If  $A_1, A_2, \dots$  is a sequence of events such that  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ , then  $P[\bigcup_{k=1}^{\infty} A_k] = \sum_{k=1}^{\infty} P[A_k]$

**Corollary 3.1.**  $P[A^C] = 1 - P[A]$

**Corollary 3.2.**  $P[A] \leq 1$

**Corollary 3.3.**  $P[\emptyset] = 0$

**Corollary 3.4.** If  $A_1, A_2, \dots, A_n$  are pairwise mutually exclusive ( $A_1 \cap A_2 \cap \dots \cap A_n = \emptyset$ ), then  $P[\bigcup_{k=1}^n A_k] = \sum_{k=1}^n P[A_k]$  for  $n \geq 2$

**Corollary 3.5.**  $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

**Corollary 3.6.**  $P[A \cup B] = \sum_{j=1}^n P[A_j] - \sum_{j < k} P[A_j \cap A_k] + \dots + (-1)^{n+1} P[A_1 \cap \dots \cap A_n]$

**Corollary 3.7.** If  $A \subset B$ , then  $P[A] \leq P[B]$

## 3.3 Conditional Probability

**Defn 11** (Conditional Probability). The *conditional probability* of event  $A$  **GIVEN THAT** event  $B$  occurred is denoted  $P[A|B]$  and is defined as

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad (3.1)$$

**Theorem 3.1** (Theorem of Total Probability). Let  $B_1, B_2, \dots, B_n$  be mutually exclusive events whose union equals the sample space  $S$ , i.e.  $B_1, B_2, \dots, B_n$  is a partition of  $S$ .

**Defn 12** (Baye's Rule). Let  $B_1, B_2, \dots, B_n$  be a partition of sample space  $S$ .

$$P[B_j|A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P[A|B_j] * P[B_j]}{\sum_{k=1}^n P[A|B_k] * P[B_k]} \quad (3.2)$$

### Example 3.1: Baye's Rule. Exam 1, Problem 5

An urn contains 9 balls, identical in every way, except that they are labeled with numbers 1 through 9. Two balls are selected at random, without replacement, and the sequence of labels observed are recorded.

- Give the formula for the conditional probability of event  $A$  given that event  $B$  occurred (where  $A$  and  $B$  are arbitrary events).
- What is the probability that the label of the second ball is even?
- What is the probability that the label of the first ball was odd given that the second was even?

### 3.4 Event Independence

**Defn 13** (Independent). Two events  $A$  and  $B$  are *independent* if

$$P[A \cap B] = P[A] * P[B], P[A] \neq 0, P[B] \neq 0 \quad (3.3)$$

- If  $A \cap B = \emptyset$ , the  $A$  and  $B$  are **dependent**.
- If checking for independence between more than 2 events, you must check each pair, each triple, etc. until you check the independence of each event against each other. For 3 events,  $A, B, C$ :
  - Check  $P[A \cap B \cap C] = P[A] * P[B] * P[C]$
  - Also need to check:
    1.  $P[A \cap B] = P[A] * P[B]$
    2.  $P[B \cap C] = P[B] * P[C]$
    3.  $P[A \cap C] = P[A] * P[C]$

#### Example 3.2: Event Independence. Exam 1, Problem 4

Let  $S = \{1, 2, 3, 4\}$ , and  $A = \{1, 2\}$ ,  $B = \{1, 3\}$ ,  $C = \{1, 4\}$ ,  $D = \{3, 4\}$ . Assume the outcomes are equiprobable. Are the following events independent?

1.  $A$  and  $B$
2.  $A$  and  $D$
3.  $A, B$ , and  $C$

If 2 events  $A$  and  $B$  are independent, then their complements are also independent. This is shown in Event Independence.

*Independence of Complements of Events.* We assumed that  $A$  and  $B$  were independent, so  $P[A \cap B] = P[A] \cdot P[B]$ . There are 2 more facts we will need:

Fact 1:  $P[B] + P[B^C] = 1$

Fact 2:  $P[A \cap B^C] + P[A \cap B] = P[A]$

From Fact 1, we have:

$$P[A \cap B] = P[A] \cdot (1 - P[B^C])$$

From Fact 2, we have  $P[A \cap B] = P[A] - P[A \cap B^C]$ . Substituting these into the equation above:

$$P[A] - P[A \cap B^C] = P[A] \cdot (1 - P[B^C])$$

$$P[A] - P[A \cap B^C] = P[A] - P[A] \cdot P[B^C]$$

$$-P[A \cap B^C] = -P[A] \cdot P[B^C]$$

$$P[A \cap B^C] = P[A] \cdot P[B^C]$$

$\therefore A$  and  $B^C$  are independent, according to the definition of Independent events in Equation (3.3). ■

## 4 Counting

### 4.1 Sampling *with* Replacement *with* Order

**Defn 14.** Choose  $k$  elements in succession with replacement between selections, from a population of  $n$  distinct objects, where  $k$  needs to have no relation to  $n$ .

$$\frac{n}{\text{First}} * \frac{n}{\text{Second}} * \frac{n}{\text{Third}} * \dots * \frac{n}{\text{kth Item}} = n^k \quad (4.1)$$

#### Example 4.1: Sampling with Replacement with Order. Problem 2.42

Problem 2.42 from Homework 3.

### 4.2 Sampling *without* Replacement *with* Order

**Defn 15.** Choose  $k$  elements in succession without replacement from a population of  $n$  distinct objects, where  $k \leq n$

$$\frac{n}{\text{First}} * \frac{n-1}{\text{Second}} * \frac{n-2}{\text{Third}} * \dots * \frac{n-k+1}{\text{kth Item}} \quad (4.2)$$

### 4.2.1 Permutations

**Defn 16** (Permutation). *Permutations* are special cases of Sampling *without* Replacement *with* Order, where  $k = n$

$$\frac{n}{\text{First}} * \frac{n-1}{\text{Second}} * \frac{n-2}{\text{Third}} * \dots * \frac{2}{\text{ } } * \frac{1}{\text{ } } * \dots * \frac{n-k-1}{\text{ } } = n! \quad (4.3)$$

#### Example 4.2: Password Permutations. Exam 1, Problem 2

1. How many unique case-sensitive passwords, of length 8 characters, can be constructed using number (0-9), lower and upper-case letters (a-z, A-Z), and a set of 15 special characters and no spaces?
2. How many unique passwords if the user must also use at least one integer?

### 4.3 Sampling *with* Replacement *without* Order

**Defn 17.** Pick  $k$  objects from a set of  $n$  distinct object with replacement. Record the result without order. The total number of ways to do this is given in Equation (4.4).

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1} \quad (4.4)$$

### 4.4 Sampling *without* Replacement *without* Ordering

**Defn 18.** Pick  $k$  objects from a set of  $n$  distinct objects without replacement. Record the results with without order. We call the resulting subset of  $k$  selected objects a “combination of size  $k$ .” The number of ways to choose  $k$  items out of  $n$  items is given in Equation (4.5). Also said  $n$  choose  $k$ :

$$\binom{n}{k} = \frac{n * (n-1) * (n-2) * \dots * (n-k+1)}{k!} = \frac{n!}{k! (n-k)!} \quad (4.5)$$

$$\binom{n}{k} = \binom{n}{n-k} \quad (4.6)$$

## 5 Single Discrete Random Variables

**Defn 19** (Random Variable). A *random variable*  $X$  is a function that assigns a real number  $X(\zeta)$  to each outcome  $\zeta$  in the sample space of the random experiment.

**Defn 20** (Discrete Random Variable). A *discrete random variable* is a random variable that assumes values in a countable set. For example, the number of heads in 3 coin flips is a discrete random variable.

### 5.1 Probability Mass Function (PMF)

**Defn 21** (Probability Mass Function). The *probability mass function (PMF)* of a discrete random variable  $X$  is defined as:

$$p_X(x) = P[X = x] \quad (5.1)$$

Using the coin example from the definition of a Discrete Random Variable,

$$p_X(x) = \begin{cases} \frac{1}{8} & x = 0 \\ \frac{3}{8} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{1}{8} & x = 3 \end{cases} \quad (5.2)$$

#### Example 5.1: Probability Mass Function. Problem 3.2

Problem 3.2 from Homework 4

### 5.1.1 Properties of Probability Mass Functions

(i)

$$p_X(x) \geq 0, \forall x \in \mathbb{R} \quad (5.3)$$

(ii)

$$\sum_{x \in S_X} p_X(x) = 1 \quad (5.4)$$

<b>Example 5.2: Find Normalizing Constant. Problem 3.13</b>
Problem 3.13 from Homework 4.

(iii)

$$P[x \in B] = \sum_{x \in B} p_X(x), \text{ where } B \subset S_X \quad (5.5)$$

## 5.2 Expected Value/Mean of Single Discrete Random Variable

**Defn 22** (Expected Value/Mean of Single Discrete Random Variable). The *expected value* or *mean* of a single discrete random variable  $X$  is defined by

$$m_X = \mathbb{E}[X] = \sum_{x \in S_X} x \cdot p_X(x) \quad (5.6)$$

*Remark 22.1.* If  $X$  is countably infinite, you will have an infinite series that exists only if

$$\sum_{s \in S_X} |x| \cdot p_X(x) \quad (5.7)$$

is absolutely convergent.

<b>Example 5.3: Expectation of Discrete Random Variable. Problem 3.27</b>
Problem 3.27 from Homework 4. ONLY THE Expectation PART.

### 5.2.1 Properties of Expected Values

**Defn 23** (Linearity of Expectation). Let  $Y = X_1 + X_2$

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] \quad (5.8)$$

This can be generalized to

$$\mathbb{E}\left[\sum_{i=1}^k x_i\right] = \sum_{i=1}^k \mathbb{E}[X_i] \quad (5.9)$$

(i)

$$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] \quad (5.10)$$

(ii)

$$\mathbb{E}[g(X)] = \sum_{s \in S_X} g(x) \cdot p_X(x) \quad (5.11)$$

(iii)

$$\mathbb{E}[cg(X)] = c \mathbb{E}[g(X)] \quad (5.12)$$

(iv)

$$\mathbb{E}[g_1(X) + g_2(X) + \dots + g_m(X)] = \sum_{i=1}^m \mathbb{E}[g_i(X)] \quad (5.13)$$



### 5.2.2 Moments of Random Variable

**Defn 24** (Moment). The *moment* of a random variable,  $X$  is defined as the expectation of the random variable raised to the moment.

$$\begin{aligned}\mathbb{E}[X^1] &= \text{First Moment} \\ \mathbb{E}[X^2] &= \text{Second Moment} \\ &\vdots \\ \mathbb{E}[X^k] &= \text{kth Moment}\end{aligned}\tag{5.14}$$

### 5.3 Variance of Single Discrete Random Variable

**Defn 25** (Variance). The *variance* of a single discrete random variable  $X$  is defined as:

$$\mathbb{E}[(X - \mathbb{E}[X])^2]\tag{5.15}$$

$$\text{VAR}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2\tag{5.16}$$

and is denoted as  $\sigma_X^2$ , or as the operator  $\text{VAR}[X]$ .

*Remark 25.1.* If  $X$  is a random variable, and  $c$  is some constant coefficient, then:

$$\text{VAR}[cX] = c^2 \text{VAR}[X]\tag{5.17}$$

<b>Example 5.4: Variance of Discrete Random Variable. Problem 3.27</b>
Problem 3.27 from Homework 4. ONLY THE Variance PART.

**Defn 26** (Standard Deviation). The standard deviation of a random variable  $X$  is:

$$\sigma_X = \sqrt{\text{VAR}[X]}\tag{5.18}$$

### 5.4 Conditional Probability Mass Function

**Defn 27** (Conditional Probability Mass of Function). Let  $X$  be a discrete random variable, with PMF  $p_X(x)$  and let  $C$  be the event with non-zero probability, i.e.  $P[C] > 0$ . The *conditional probability mass function of  $X$  given  $C$*  (*Conditional PMF*) is defined as:

$$p_{X|C}(x|C) = P[X = x | C] \text{ for } x \in \mathbb{R}\tag{5.19}$$

*Remark 27.1.* The conditional PMF,  $p_{X|C}(x|C)$ , satisfies **all** properties of Properties of Probability Density Functions.

### 5.5 Conditional Expected Value of Single Discrete Random Variable

**Defn 28** (Conditional Expected Value of Discrete Random Variable). The *conditional expected value of the discrete random variable  $X$  given  $B$*  is defined as:

$$m_{X|B} = \mathbb{E}[X | B] = \sum_{x \in S_X} s \cdot p_X(x | B)\tag{5.20}$$

<b>Example 5.5: Conditional Expected Value. Problem 3.39</b>
Problem 3.39 from Homework 5.

### 5.6 Conditional Variance of Single Discrete Random Variable

**Defn 29** (Conditional Variance of Discrete Random Variable). The *conditional variance of a discrete random variable  $X$  given event  $B$*  as defined as:

$$\begin{aligned}\sigma_{X|B}^2 &= \text{VAR}[X | B] \\ &= \mathbb{E}[(X - \mathbb{E}[X | B])^2 | B] \\ &= \sum_{x \in S_X} (x - m_{X|B})^2 \cdot p_X(x | B) \\ \text{VAR}[X | B] &= \mathbb{E}[X^2 | B] - (\mathbb{E}[X | B])^2\end{aligned}\tag{5.21}$$

## 6 Single Continuous Random Variables

**Defn 30** (Random Variable). Consider a random experiment with sample space  $S$  and event class  $\mathcal{F}$ . A *random variable*  $X$  is a function from the sample space  $S$  to the real line  $\mathbb{R}$  with the property the set  $A_b = \{\zeta : X|\zeta \leq b\}$  is in  $\mathcal{F}$  for every  $b$  in  $\mathbb{R}$ .

**Defn 31** (Continuous Random Variable). A *continuous random variable* is a random variable whose Cumulative Distribution Function (CDF) is continuous everywhere.

### 6.1 Cumulative Distribution Function (CDF)

**Defn 32** (Cumulative Distribution Function). *Cumulative Distribution Function (CDF)* of a random variable  $X$  is defined as the probability of the event  $\{X \leq x\}$ .

$$F_X(x) = P[X \leq x] \text{ for } -\infty < x < \infty \quad (6.1)$$

#### 6.1.1 Properties of Cumulative Distribution Functions

(i)

$$0 \leq F_X(x) \leq 1 \quad (6.2)$$

(ii) If you include the whole sample space, you should end up with 1.

$$\lim_{x \rightarrow \infty} F_X(x) = 1 \quad (6.3)$$

(iii) If you exclude the whole sample space, you should end up with 0.

$$\lim_{x \rightarrow -\infty} F_X(x) = 0 \quad (6.4)$$

(iv)  $F_X(x)$  is non-decreasing.

$$F_X(a) \leq F_X(b) \text{ if } a \leq b \quad (6.5)$$

(v) The CDF is continuous from the right.

$$F_X(b) = \lim_{h \rightarrow 0} F_X(b+h) \text{ where } h > 0 \quad (6.6)$$

(vi)

$$P[a < X \leq b] = F_X(b) - F_X(a) \quad (6.7)$$

(vii) The probability at a point in a CDF. (This usually ends up being 0).

$$P[X = b] = F_X(b) - F_X(b^-) \quad (6.8)$$

(viii) The probability of the event **not** occurring.

$$P[X > x] = 1 - P[X \leq x] = 1 - F_X(x) \quad (6.9)$$

#### Example 6.1: Properties of Cumulative Distribution Functions. Problem 4.29

Let  $C$  be an event for which  $P[C] > 0$ . Show that  $F_X(X|C)$  satisfies the 8 properties of a Cumulative Distribution Function.

- (i)  $0 \leq F_X(x) \leq 1$
- (ii)  $\lim_{x \rightarrow \infty} F_X(x) = 1$
- (iii)  $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- (iv) For  $a < b$ ,  $F_X(a) \leq F_X(b)$
- (v)  $h > 0$ ,  $F_X(b) = \lim_{h \rightarrow 0^+} F_X(b+h) = F_X(b^+)$
- (vi)  $P[a < X \leq b] = F_X(b) - F_X(a)$
- (vii)  $P[X = a] = F_X(a) - F_X(a^-)$
- (viii)  $P[X > x] = 1 - F_X(x)$

### 6.1.2 Conditional Cumulative Distribution Function

**Defn 33** (Conditional Cumulative Distribution Function). The *conditional cumulative distribution function* (Conditional CDF) of  $X$  given  $C$  is defined by:

$$F_{X|C}(x|C) = \frac{P[\{X = x\} | C]}{P[C]} \quad (6.10)$$

*Remark 33.1.* The conditional CDF,  $F_{X|C}(x|C)$  satisfies **all** Properties of Cumulative Distribution Functions.

<b>Example 6.2: Conditional Cumulative Distribution Function. Problem 4.38</b>
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Problem 4.38 from Homework 7.
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## 6.2 Probability Density Function (PDF)

**Defn 34** (Probability Density Function). The *probability density function* (PDF) of a random variable  $X$ , if it exists, is defined as the derivative of the CDF of  $X$ .

$$f_X(x) = \frac{d}{dx} F_X(x) \quad (6.11)$$

*Remark 34.1.* Both discrete and continuous random variables can have PDFs, however, the discrete random variable will have a discontinuous PDF.

*Remark 34.2.* It is possible to construct a random variable that has a Cumulative Distribution Function (CDF), but an undefined Probability Density Function (PDF).

*Remark 34.3.* This is an alternate, more useful way to specify the probability law described by the Cumulative Distribution Function (CDF).

<b>Example 6.3: Find Probability Density Function. Problem 4.25</b>
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Problem 4.25 from Homework 6.
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### 6.2.1 Properties of Probability Density Functions

These properties apply to PDFs of continuous random variables, and may not hold true for other types of random variables.

(i) The associated CDF is non-decreasing, a Properties of Cumulative Distribution Functions.

$$f_X(x) \geq 0 \quad (6.12)$$

(ii) Since the definition of the PDF is that it's the derivative of the CDF, integrating the space over the PDF will yield the CDF.

$$P[a \leq X \leq b] = \int_a^b f_X(x) dx = F_X(b) - F_X(a) \quad (6.13)$$

(iii) The value of a location in CDF is the integral of the PDF over the area.

$$F_X(x) = \int_{-\infty}^x f_X(t) dt \quad (6.14)$$

(iv) Including the whole sample space should yield 1.

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 \quad (6.15)$$

<b>Example 6.4: Find Normalizing Constant <math>c</math>. Final Exam Practice Problem 4</b>
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A random variable  $X$  has Probability Distribution Function (PDF):

$$f_X(x) = \begin{cases} cx(1-x^3) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the normalizing constant  $c$ . (5 pts)
2. Find  $P[X = 0.5]$ . (3 pts)
3. Find  $P[X > 0.5]$ . (7 pts)

*Remark.* Any non-negative, piecewise continuous function  $g(x)$  with finite  $\int_{-\infty}^{\infty} g(x) dx = C$  can be used to form a PDF.

## 6.2.2 Conditional Probability Density Function

**Defn 35** (Conditional Probability Density Function). The *conditional probability density function* (*Conditional PDF*) of  $X$  given  $C$  is defined by:

$$f_{X|C}(x|C) = \frac{d}{dx} F_{X|C}(x|C) \quad (6.16)$$

*Remark 35.1.* The conditional PDF,  $f_{X|C}(x|C)$  satisfies **all** Properties of Probability Density Functions.

## 6.3 Expected Value of Single Continuous Random Variable

**Defn 36** (Expected Value/Mean of Random Variable). The *expected value* of a random variable  $X$ , denoted  $\mathbb{E}[X]$  is defined as:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} t f_X(t) dt \quad (6.17)$$

*Remark 36.1.* This works with **all** random variables, or general random variables.

*Remark 36.2.*  $\mathbb{E}[X]$  is defined if the integral in Equation (6.17) converges absolutely. This means:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} t f_X(t) dt < \infty$$

<b>Example 6.5: Conditional Expected Value of Continuous Random Variable. Problem 4.57</b>
Problem 4.57 from Homework 7.

### 6.3.1 Properties of Expected Value

(i) The expected value of a function of a random variable.

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(t) \cdot f_X(t) dt \quad (6.18)$$

(ii) Expectation of a constant,  $c$ , should be the constant itself.

$$\mathbb{E}[c] = c \quad (6.19)$$

(iii) Sum of a random variable,  $X$ , and a constant,  $c$ , is the same as the sum of the expectation of the random variable and the constant.

$$\mathbb{E}[X + c] = \mathbb{E}[X] + \mathbb{E}[c] \quad (6.20)$$

(iv) Linearity of Expectations for random variables

$$\mathbb{E}[a_0 + a_1 X + a_2 X^2 + \dots + a_n X^n] = a_0 + a_1 \mathbb{E}[X] + a_2 \mathbb{E}[X^2] + \dots + a_n \mathbb{E}[X^n] \quad (6.21)$$

## 6.4 Variance of Single Continuous Random Variable

**Defn 37** (Variance of Random Variable). The *variance* of the random variable  $X$  is defined by:

$$\sigma^2 = \text{VAR}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] \quad (6.22)$$

*Remark 37.1.* This holds true for **all** types of random variables; discrete, continuous, and mixed.

**Defn 38** (Standard Deviation). The *standard deviation* of a random variable  $X$ , denoted by:

$$\sigma = \text{STD}[X] = \sqrt{\text{VAR}[X]} \quad (6.23)$$

*Remark 38.1.* This holds true for **all** types of random variables; discrete, continuous, and mixed.

## 6.5 Gaussian/Normal Random Variable

**Defn 39** (Gaussian/Normal Random Variable). The *Gaussian or normal random variable* is the classic “bell curve” probability distribution. It is usually described as  $X \sim N(\mu, \sigma^2)$ .  $\mu$  is  $\mathbb{E}[X]$  and  $\sigma^2$  is how narrow/sharp the bell is. A Gaussian Random Variable has a PDF of:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R} \quad (6.24)$$

**Defn 40** (Standard Normal Distribution). The *standard normal distribution* is just a specific Gaussian/Normal Random Variable. The standard normal distribution is a Gaussian/Normal Random Variable with  $\mu = 0, \sigma^2 = 1$ .

*Remark 40.1.* The CDF of the Standard Normal Distribution is denoted with  $\Phi$ .

To find the probability of something for a Gaussian Random Variable, you would end up converting it to the Standard Normal Distribution. If  $X \sim N(\mu, \sigma^2)$  and  $Y \sim N(0, 1)$ ,

$$\begin{aligned} P[a \leq x \leq b] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{1}{2}y^2} dy \\ &= P\left[\frac{a-\mu}{\sigma} \leq Y \leq \frac{b-\mu}{\sigma}\right] \\ &= F_Y\left(\frac{b-\mu}{\sigma}\right) - F_Y\left(\frac{a-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned} \quad (6.25)$$

### 6.5.1 Q-Function

**Defn 41** (Q-Function). The *Q-Function* is primarily used in electrical engineering. It is defined as:

$$\begin{aligned} Q &= 1 - \Phi(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt \end{aligned} \quad (6.26)$$

*Remark 41.1.*

$$Q(Z) = 1 - f_Z(z) \quad (6.27)$$

<b>Example 6.6: Q-Function Application. Problem 4.62</b>
Problem 4.62 from Homework 7.

## 6.6 Markov Inequality

**Defn 42** (Markov Inequality). Let  $X$  be a non-negative random variable with  $\mathbb{E}[X] < \infty$ . The *Markov Inequality* states that:

$$P[X \geq a] \leq \frac{\mathbb{E}[X]}{a} \quad (6.28)$$

*Proving the Markov Inequality.*

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

Because we defined  $X \geq 0$ , we change the lower bound to 0.

$$\mathbb{E}[X] = \int_0^{\infty} x f_X(x) dx$$

We then split the integral up around some point,  $a$ .

$$\mathbb{E}[X] = \int_0^a x f_X(x) dx + \int_a^{\infty} x f_X(x) dx$$

Since the first integral is integrating over a non-negative function, the integral is also non-negative.

$$\int_0^a x f_X(x) dx + \int_a^\infty x f_X(x) dx \geq \int_a^\infty x f_X(x) dx$$

$$\mathbb{E}[X] \geq \int_a^\infty x f_X(x) dx$$

Because  $x > a$ , we can pull a term out of  $f_X(x)$

$$\mathbb{E}[X] \geq \int_a^\infty a f_X(x) dx$$

Because  $a$  is a constant, we pull it out of the integral,

$$\mathbb{E}[X] \geq a \int_a^\infty f_X(x) dx$$

Then, we end up with an integral that is the definition of the probability of a continuous random variable.

$$\mathbb{E}[X] \geq a P[X \geq a]$$

$$\therefore \mathbb{E}[X] \geq a P[X \geq a]$$

■

<b>Example 6.7: Markov Inequality. Problem 4.97</b>
Problem 4.97 from Homework 7.

## 6.7 Chebychev Inequality

**Defn 43** (Chebychev Inequality). Let  $X$  be a non-negative random variable with  $\mathbb{E}[X] < \infty$ . The *Chebychev Inequality* states that:

$$P[|X - \mu| \geq a] \leq \frac{\sigma^2}{a^2} \quad (6.29)$$

*Proving the Chebychev Inequality.*

$$P[(X - \mu)^2 \geq a^2] \leq \frac{\mathbb{E}[(X - \mu)^2]}{a^2}$$

Because  $X - \mu = \sigma$ , we replace it.

$$P[(X - \mu)^2 \geq a^2] \leq \frac{\mathbb{E}[\sigma^2]}{a^2}$$

■

<b>Example 6.8: Chebychev Inequality. Problem 4.100</b>
Problem 4.100 from Homework 7.

## 7 Multiple Random Variables

### 7.1 Joint Probability Mass Function

**Defn 44** (Joint Probability Mass Function). The *joint probability mass function (joint PMF)* of 2 discrete random variables  $X, Y$  is defined as:

$$p_{X,Y} = P[\{X = x\} \cap \{Y = y\}] \text{ for all } x, y \in S_{X,Y} \quad (7.1)$$

- This satisfies ALL properties of single random variable PMFs

<b>Example 7.1: Joint PMF. Problem 5.1</b>
Problem 5.1, Part a & Part B.

### 7.1.1 Marginal Probability Mass Function

**Defn 45** (Marginal Probability Mass Function). Given a joint PMF of discrete random variables  $X, Y$ , the *Marginal Probability Mass Function (Marginal PMF)* of  $X$  is defined as:

$$p_X(x_i) = P[X = x_i] \text{ for } x_i \in S_X \quad (7.2)$$

and is calculated as:

$$p(x_i) = \sum_{y \in S_Y} p_{X,Y}(x_i, y) \quad (7.3)$$

<b>Example 7.2: Marginal PMFs. Problem 5.11</b>
Problem 5.11, Part a & Part b & Part c.

## 7.2 Joint Cumulative Distribution Function

**Defn 46** (Joint Cumulative Distribution Function). The *Joint Cumulative Distribution Function (Joint CDF)* of  $X$  and  $Y$  is defined as the probability of the event  $\{X \leq x\} \cap \{Y \leq y\}$

$$\begin{aligned} F_{X,Y}(x, y) &= P[\{X \leq x\} \cap \{Y \leq y\}] \text{ for all } (x, y) \in \mathbb{R}^2 \\ &= P[\{X \leq x\}, \{Y \leq y\}] \end{aligned} \quad (7.4)$$

### 7.2.1 Properties of Joint Cumulative Distribution Functions

(i)  $F_{X,Y}(x, y)$  is non decreasing.

$$F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2) \text{ if } x_1 \leq x_2 \text{ and } y_1 \leq y_2 \quad (7.5)$$

(ii)

$$\begin{aligned} \lim_{y \rightarrow -\infty} F_{X,Y}(x, y) &= 0 \\ \lim_{x \rightarrow -\infty} F_{X,Y}(x, y) &= 0 \\ \lim_{(x,y) \rightarrow (\infty, \infty)} F_{X,Y}(x, y) &= 1 \end{aligned} \quad (7.6)$$

(iii) The Marginal CDFs can be obtained from the Joint CDF by removing restrictions for all but one variable.

$$\begin{aligned} F_X(x) &= P[\{X \leq x\}, \{Y \text{ is anything}\}] \\ &= P[\{X \leq x\}, \{-\infty \leq y \leq \infty\}] \\ &= \lim_{y \rightarrow \infty} F_{X,Y}(x, y) \\ F_Y(y) &= \lim_{x \rightarrow \infty} F_{X,Y}(x, y) \end{aligned} \quad (7.7)$$

(iv) The Joint CDF is continuous from  $\infty$  to  $-\infty$ .

$$\begin{aligned} \lim_{x \rightarrow a^+} F_{X,Y}(x, y) &= F_{X,Y}(a, y) \\ \lim_{y \rightarrow b^+} F_{X,Y}(x, y) &= F_{X,Y}(x, b) \end{aligned} \quad (7.8)$$

(v) The probability of the “rectangle”  $\{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\}$

$$\begin{aligned} P[\{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\}] &= P[\{X \leq x_2, Y \leq y_2\}] - P[\{X \leq x_1, Y \leq y_2\}] - \\ &\quad P[\{X \leq x_2, Y \leq y_1\}] + P[\{X \leq x_1, Y \leq y_1\}] \\ &= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1) \end{aligned} \quad (7.9)$$

## 7.2.2 Marginal Cumulative Distribution Function

**Defn 47** (Marginal Cumulative Distribution Function). We obtain the *Marginal Cumulative Distribution Functions* (*Marginal CDFs*) by removing the constraint on one of the variables.

$$\begin{aligned} F_X(x) &= P[\{X \leq x\}, \{Y \text{ is anything}\}] \\ &= P[\{X \leq x\}, \{-\infty \leq y \leq \infty\}] \\ &= \lim_{y \rightarrow \infty} F_{X,Y}(x, y) \\ F_Y(y) &= \lim_{x \rightarrow \infty} F_{X,Y}(x, y) \end{aligned} \quad (7.10)$$

<b>Example 7.3: Marginal CDFs. Problem 5.20</b>
Problem 5.20, Part b from Homework 8.

## 7.3 Joint Probability Density Function

**Defn 48** (Joint Probability Density Function). We say that  $X, Y$  are jointly continuous if the probabilities of events involving  $X$  and  $Y$  can be expressed as an integral of a *Joint Probability Density Function* (*Joint PDF*).

i.e. There exists some nonnegative function  $f_{X,Y}(x, y)$ , which we call the joint PDF, that is defined on the real plane such that for every event  $B$  which is a subset of the  $xy$  plane

$$P[(X, Y) \text{ in } B] = \iint_B f_{X,Y}(x, y) dx dy \quad (7.11)$$

*Remark 48.1.* The probability mass of an event is found by integrating the PDF over the region in the  $xy$  plane corresponding to your event.

### 7.3.1 Properties of Joint Probability Density Functions

(i)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1 \quad (7.12)$$

<b>Example 7.4: Find Normalizing Constant 1. Example 5.16</b>
Example 5.16 from the textbook.

<b>Example 7.5: Find Normalizing Constant 2. Problem 5.27</b>
Problem 5.27, Part a from Homework 8.

(ii)

$$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(s, t) dt ds \quad (7.13)$$

<b>Example 7.6: Find Joint CDF. Problem 5.27</b>
Problem 5.27, Part b from Homework 8.

(iii)

$$f_{X,Y} = \frac{\partial^2 f_{X,Y}(x, y)}{\partial x \partial y} \quad (7.14)$$



### 7.3.2 Marginal PDF

**Defn 49** (Marginal Probability Density Function). The *Marginal Probability Density Functions (Marginal PDFs)*  $f_X(x)$  and  $f_Y(y)$  are obtained by taking the derivative of the marginal CDFs.

$$\begin{aligned}
 f_X(x) &= \frac{d}{dx} F_X(x) \\
 &= \frac{d}{dx} \int_{-\infty}^x \left[ \int_{-\infty}^{\infty} f_{X,Y}(s,t) dt ds \right] \\
 &= \frac{d}{dx} \int_{-\infty}^x \int_{-\infty}^{\infty} f_{X,Y}(s,t) dt ds \\
 &\quad \text{Simplified with Second Fundamental Theorem of Calculus} \\
 &= \int_{-\infty}^{\infty} f_{X,Y}(x,t) dt \\
 f_X &= \int_{-\infty}^{\infty} f_{X,Y}(x,t) dt
 \end{aligned} \tag{7.15}$$

#### Example 7.7: Find Marginal PDFs. Problem 5.27

Problem 5.27, Part c from Homework 8.

## 7.4 Independence of Multiple Random Variables

**Defn 50** (Independent Random Variables).  $X$  and  $Y$  are *independent random variables* if **ANY** event  $A_1$  defined in terms of  $S$  is independent of **ANY** event  $A_2$  defined in terms of  $Y$ .

$$P[X \in A_1, Y \in A_2] = P[X \in A_1] * P[Y \in A_2] \tag{7.16}$$

There are 3 ways to phrase this:

1. For discrete random variables  $X$  and  $Y$ ,  $X$  and  $Y$  are independent if and only if:

$$p_{X,Y}(x,y) = p_X(x) * p_Y(y) \tag{7.17}$$

2. For general random variables  $X$  and  $Y$ ,  $X$  and  $Y$  are independent if and only if:

$$F_{X,Y}(x,y) = F_X(x) * F_Y(y) \tag{7.18}$$

#### Example 7.8: Confirm Independence by PDF. Final Exam Practice, Problem 7

Let  $X$  and  $Y$  have joint Probability Density Function (PDF):

$$f_{X,Y}(x,y) = \begin{cases} 12x(1-x)y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Are  $X$  and  $Y$  independent?

3. For (continuous) random variables  $X$  and  $Y$ ,  $X$  and  $Y$  are independent if and only if:

$$f_{X,Y}(x,y) = f_X(x) * f_Y(y) \tag{7.19}$$

#### Example 7.9: Confirm Independence by CDF. Final Exam Practice, Problem 7

Let  $X$  and  $Y$  have joint Cumulative Distribution Function (CDF):

$$F_{X,Y}(x,y) = \begin{cases} 6y^2 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Are  $X$  and  $Y$  independent?

You can prove Independence of Multiple Random Variables, Equation (7.17).

**Theorem 7.1** (Independence of Random Functions). *If random variables  $X, Y$  are independent, then  $g(X)$  and  $h(Y)$  are also independent.*

## 7.5 Expected Value of Functions with 2 Random Variables

**Defn 51** (Expectation of a Function with 2 Random Variables). Let  $Z$  be a random variable described by the function  $Z = g(X, Y)$ .

$$\mathbb{E} = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f_{X,Y}(x, y) dx dy & \text{if } X \text{ and } Y \text{ are jointly continuous} \\ \sum_{i \in S_X} \sum_{j \in S_Y} g(x_i, y_j) \cdot p_{X,Y}(x, y) & \text{if } X \text{ and } Y \text{ are both discrete} \end{cases} \quad (7.20)$$

*Remark 51.1* (Expected Value of Sum of Random Variables). You **do not** need to assume independence to say:

$$\mathbb{E}[X_1 + X_2 + \dots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n] \quad (7.21)$$

*Remark 51.2* (Expected Value of Product of Random Variables). If  $X$  and  $Y$  are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)] \cdot \mathbb{E}[h(Y)] \quad (7.22)$$

## 7.6 Joint Moments, Correlation, and Covariance

### 7.6.1 Joint Moments

**Defn 52** (The  $j, k$ th Moment). The  $j, k$ th moment of  $X$  and  $Y$  is:

$$\mathbb{E}[X^j Y^k] = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k \cdot f_{X,Y}(x, y) dx dy & \text{if } X, Y \text{ are jointly continuous} \\ \sum_{i \in S_X} \sum_{\ell \in S_Y} x_i^j y_\ell^k \cdot p_{X,Y}(x_i, y_\ell) & \text{if } X, Y \text{ are discrete} \end{cases} \quad (7.23)$$

### 7.6.2 Correlation

**Defn 53** (Correlation). The *Correlation* of  $X$  and  $Y$  is defined as the 1, 1 moment, i.e.  $\mathbb{E}[X^1 Y^1]$ .

*Remark 53.1.* If  $X, Y$  are such that  $\mathbb{E}[X^1 Y^1] = 0$ , then we say that  $X, Y$  are *orthogonal*.

*Remark 53.2* (Uncorrelated). If  $X, Y$  are such that  $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ , then  $X$  and  $Y$  are *uncorrelated*.

*Remark 53.3.* If  $X, Y$  are independent, then they are uncorrelated; but if  $X$  and  $Y$  are uncorrelated, **they are not always independent**.

**Defn 54** (Correlation Coefficient). The *correlation coefficient* of  $X, Y$  is defined as

$$\rho_{X,Y} = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y} \quad (7.24)$$

*Remark 54.1.*  $\rho_{X,Y}$  only ranges  $-1 \leq \rho_{X,Y} \leq 1$

*Remark 54.2.* The closer  $\rho_{X,Y}$  is to +1, the closer  $X$  and  $Y$  are to having a positive linear relationship (Positive slope).

The closer  $\rho_{X,Y}$  is to -1, the closer  $X$  and  $Y$  are to having a negative linear relationship (Negative slope).

If  $\rho_{X,Y} = 0$ , the  $\text{Cov}[X, Y] = 0$ , which means that  $X$  and  $Y$  are *uncorrelated*.

### 7.6.3 Covariance

**Defn 55** (Covariance). The *covariance* of  $X$  and  $Y$  is denoted:

$$\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \quad (7.25)$$

$$\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \quad (7.26)$$

## 7.7 Conditional Probability Functions

There are 3 major cases for these:

1. 2 Discrete Random Variables
2. 1 Discrete and 1 Continuous Random Variable
3. 2 Continuous Random Variables

### 7.7.1 2 Discrete Random Variables

**Defn 56** (Conditional Probability Mass Function). The *conditional Probability Mass Function (Conditional PMF)* of  $Y$  given that  $X = x$  is:

$$p_Y(y|x) = \frac{P[\{Y = y\} \cap \{X = x\}]}{P[X = x]} = \frac{p_{X,Y}(x, y)}{p_X(x)} \quad (7.27)$$

*Remark 56.1.* This also implies that

$$p_{X,Y}(x, y) = p_Y(y|x) \cdot p_X(x) \quad (7.28)$$

*Remark 56.2.* If  $X$  and  $Y$  are *independent*, then:

$$p_X(y|x) = \frac{p_{X,Y}(x, y)}{p_X(x)} = \frac{p_X(x) p_Y(y)}{p_X(x)} = p_Y(y) \quad (7.29)$$

*Remark 56.3.* The Conditional Probability Mass Function of 2 discrete random variables satisfies all Properties of Probability Mass Functions.

### 7.7.2 1 Discrete and 1 Continuous Random Variable

For this section, let  $X$  be a discrete random variable and  $Y$  a continuous random variable.

**Defn 57** (Conditional Cumulative Distribution Function). The *conditional Cumulative Distribution Function (Conditional CDF)* of  $Y$  given that  $X = x$  is:

$$F_Y(y|x) = P[Y \leq y | X = x] = \frac{P[\{Y \leq y\} \cap \{X = x\}]}{P[X = x]} \quad (7.30)$$

*Remark 57.1.* If  $X$  and  $Y$  are *independent*, then:

$$F_Y(y|x) = \frac{F_{X,Y}(x, y)}{p_X(x)} = \frac{F_Y(y) p_X(x)}{p_X(x)} = F_Y(y) \quad (7.31)$$

This also means that:

$$P[Y \leq y | X = x] = P[Y \leq y] \cdot P[X = x]$$

*Remark 57.2.* The similar relations for independent random variables with their conditional and marginal probability functions does not hold true with this.

*Remark 57.3.* The Conditional Cumulative Distribution Function of 1 discrete random variable and 1 continuous random variable satisfies all Properties of Cumulative Distribution Functions.

**Defn 58** (Conditional Probability Density Function). The *conditional Probability Distribution Function (Conditional PDF)* of  $Y$  given  $X = x$  is

$$f_Y(y|x) = \frac{d}{dy} F_Y(y|x) \quad (7.32)$$

This also means,

$$P[Y \leq y | X = x] = \int_{y \in A} f_Y(y|x) dy$$

*Remark 58.1.* The Conditional Probability Density Function of 1 discrete random variable and 1 continuous random variable satisfies all Properties of Probability Density Functions.

### 7.7.3 2 Continuous Random Variables

**Defn 59** (Conditional Cumulative Distribution Function). The *conditional Cumulative Distribution Function (Conditional CDF)* of  $Y$  given  $X = x$  for  $X$  and  $Y$  continuous random variables is:

$$F_Y(y|x) = \lim_{h \rightarrow 0} P_Y(y|x < X \leq (x+h)) = \frac{\int_{-\infty}^y f_{X,Y}(x, v) dv}{f_X(x)} \quad (7.33)$$

*Remark 59.1.* The Conditional Cumulative Distribution Function of 2 continuous random variables satisfies all Properties of Cumulative Distribution Functions.

*Remark 59.2.* The similar relations for the conditional and marginal probability functions do not hold up for 2 continuous random variables too well.

**Defn 60** (Conditional Probability Density Function). The *conditional Probability Density Function (Conditional PDF)* of  $Y$  given  $X = x$  for  $X$  and  $Y$  continuous random variables is:

$$f_Y(y|x) = \frac{d}{dy} F_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad (7.34)$$

*Remark 60.1.* If  $X$  and  $Y$  are independent, then:

$$f_X(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_X(x) f_Y(y)}{f_X(x)} = f_Y(y) \quad (7.35)$$

*Remark 60.2.* The Conditional Probability Density Function of 2 continuous random variables satisfies all Properties of Probability Density Functions.

## 7.8 Conditional Expectation of Multiple Random Variables

**Defn 61** (Conditional Expectation). The *conditional expectation* of  $Y$  given  $X$  is:

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y \cdot f_Y(y|x) dy \quad (7.36)$$

*Remark 61.1* (Special Case). There is a special case when **both**  $X$  and  $Y$  are discrete random variables.

$$\mathbb{E}[Y|X=x] = \sum_{y \in S_Y} y \cdot p_Y(y|x) \quad (7.37)$$

*Remark 61.2.* When calculating the Conditional Expectation of Multiple Random Variables, and they as for  $\mathbb{E}[Y|X=x]$ , that means you **must** consider all possible values that  $X$  can take. This can be generalized to the equation below.

$$\mathbb{E}[Y|X=x] = \sum_{x \in S_X} \left( \sum_{y \in S_Y} y \cdot p_Y(y|x) \right) \quad (7.38)$$

This can be described. You must take a single value for  $x$ , and take it over all  $y$ 's, then take the next value for  $x$ , until you have exhausted all values in both  $S_X$  and  $S_Y$ .

This can also be translated into the continuous case, but the discrete case is a little simpler to understand this generality.

*Remark 61.3.*  $\mathbb{E}[Y|X=x]$  is a function of  $X$ , so it can be written as  $g(x) = \mathbb{E}[Y|X=x]$ . Thus, we can also say

$$\mathbb{E}[g(x)] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y] \quad (7.39)$$

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \int_{-\infty}^{\infty} \mathbb{E}[Y|x] f_X(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_Y(y|x) dy f_X(x) dx \quad (7.40a)$$

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \sum_{x \in S_X} \mathbb{E}[Y|x] p_X(x) = \sum_{x_j \in S_X} \sum_{y_i \in S_Y} y_i p_Y(y_i|x_k) p_X(x_j) \quad (7.40b)$$

*Prove Expectation of Conditional Expected Value.* ■

## 8 Random Vectors

Random Vectors are usually denoted:

$$\vec{X} = \langle X_1, X_2, X_3, \dots, X_n \rangle \quad (8.1)$$

**Defn 62** (Random Vector). A *random vector* is a list of Random Variables.

*Remark 62.1.* Almost all of the material for Multiple Random Variables is applicable here. However, the 2 random variable equations and definitions must be generalized to  $n$  random variables.

### 8.1 Joint CDF of a Random Vector

$$\begin{aligned} F_{\vec{X}}(\vec{x}) &= F_{X_1, X_2, X_3, \dots, X_n}(x_1, x_2, x_3, \dots, x_n) \\ &= P[X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3, \dots, X_n \leq x_n] \end{aligned} \quad (8.2)$$

## 8.2 Joint PDF of a Random Vector

$$f_{\vec{X}}(\vec{x}) = \frac{\partial^n F_{\vec{X}}(\vec{x})}{\partial x_1 \partial x_2 \partial x_3 \cdots \partial x_n} \quad (8.3)$$

### 8.2.1 Marginal PDF of a Random Vector

Integrate out the terms that you're not interested in.

$$f_{\vec{X}} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\vec{X}}(\vec{x}) \partial x_2 \partial x_3 \cdots \partial x_n \quad (8.4)$$

For instance, say we want the marginal PDF of some function with respect to  $X_1$ ,  $X_3$ , and  $X_4$ .

$$f_{X_1, X_3, X_4}(x_1, x_3, x_4) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\vec{X}}(\vec{x}) \partial x_2 \partial x_5 \partial x_6 \cdots \partial x_n \quad (8.5)$$

## 8.3 Conditional Probability Functions of Random Vectors

This section is just an extension of Section 7.7, Conditional Probability Functions. There are 3 major cases for these:

1. Discrete Random Vectors
2. Mixed Random Vectors
3. Continuous Random Vectors

*Remark.* For the sections below, let  $\vec{Y} = \langle Y_1, Y_2, Y_3 \rangle$  and  $\vec{y} = \langle y_1, y_2, y_3 \rangle$ .

While I am using  $\vec{Y}$  and  $\vec{y}$ , these equations can be further generalized to higher dimensions. All that would be required for this is to keep track of everything.

### 8.3.1 Discrete Random Vectors

**Defn 63** (Conditional Probability Mass Function). The *conditional Probability Mass Function (Conditional PMF)* of  $Y_3$  given that  $Y = y$  is:

$$p_{Y_3}(y_3 | y_1, y_2) = \frac{P[\{Y_3 = y_3\} \cap (\{Y_1 = y_1\} \cap \{Y_2 = y_2\})]}{P[\{Y_1 = y_1\} \cap \{Y_2 = y_2\}]} = \frac{p_{\vec{Y}}(\vec{y})}{p_{Y_1, Y_2}(y_1, y_2)} \quad (8.6)$$

*Remark 63.1.* This also implies that

$$p_{\vec{Y}}(\vec{y}) = p_{Y_3}(y_3 | y_1, y_2) \cdot p_{Y_2}(y_2 | y_1) \cdot p_{Y_1}(y_1) \quad (8.7)$$

*Remark 63.2.* If all elements of  $\vec{Y}$  are *independent* (Remember that you need to check each subgroup too, like shown in Section 3.4), then:

$$p_{Y_3}(y_3 | y_1, y_2) = \frac{p_{\vec{Y}}(\vec{y})}{p_{Y_1, Y_2}(y_1, y_2)} = \frac{p_{Y_1, Y_2}(y_1, y_2) p_{Y_3}(y_3)}{p_{Y_1, Y_2}(y_1, y_2)} = p_{Y_3}(y_3) \quad (8.8)$$

*Remark 63.3.* The Conditional Probability Mass Function of 2 discrete random variables satisfies all Properties of Probability Mass Functions.

### 8.3.2 Mixed Random Vectors

### 8.3.3 Continuous Random Vectors

## 8.4 Mean Vector

**Defn 64** (Mean Vector). For  $\vec{X} = \langle X_1, X_2, \dots, X_n \rangle$ , the *mean vector* is defined as the column vector of expected values of the components of  $X_k$ :

$$\mathbf{m}_X = \mathbb{E}[\vec{X}] = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \triangleq \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix} \quad (8.9)$$

*Remark 64.1.* Note that we defined the vector of expected values as a column vector. Other texts will use row vectors for other things, but the use of column vectors here is intentional.

## 8.5 Correlation and Covariance Matrix

**Defn 65** (Correlation Matrix). The *correlation matrix* has the second moments of  $\vec{X}$  as its entries:

$$\bar{\mathbf{R}}_{\mathbf{X}} = \begin{bmatrix} \mathbb{E}[X_1^2] & \mathbb{E}[X_1 X_2] & \cdots & \mathbb{E}[X_1 X_n] \\ \mathbb{E}[X_2 X_1] & \mathbb{E}[X_2^2] & \cdots & \mathbb{E}[X_2 X_n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[X_n X_1] & \mathbb{E}[X_n X_2] & \cdots & \mathbb{E}[X_n^2] \end{bmatrix} \quad (8.10)$$

*Remark 65.1.*  $\bar{\mathbf{R}}_{\mathbf{X}}$  is a  $n \times n$  symmetric matrix.

**Defn 66** (Covariance Matrix). The *covariance matrix* has the second-order central moments as its entries:

$$\begin{aligned} \bar{\mathbf{K}}_{\mathbf{X}} &= \begin{bmatrix} \mathbb{E}[(X_1 - m_1)^2] & \mathbb{E}[(X_1 - m_1)(X_2 - m_2)] & \cdots & \mathbb{E}[(X_1 - m_1)(X_n - m_n)] \\ \mathbb{E}[(X_2 - m_2)(X_1 - m_1)] & \mathbb{E}[(X_2 - m_2)^2] & \cdots & \mathbb{E}[(X_2 - m_2)(X_n - m_n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[(X_n - m_n)(X_1 - m_1)] & \mathbb{E}[(X_n - m_n)(X_2 - m_2)] & \cdots & \mathbb{E}[(X_n - m_n)^2] \end{bmatrix} \\ &= \begin{bmatrix} \text{VAR}[X_1] & \text{Cov}[X_1, X_2] & \cdots & \text{Cov}[X_1, X_n] \\ \text{Cov}[X_2, X_1] & \text{VAR}[X_2] & \cdots & \text{Cov}[X_2, X_n] \\ \cdots & \cdots & \ddots & \cdots \\ \text{Cov}[X_n, X_1] & \text{Cov}[X_n, X_2] & \cdots & \text{VAR}[X_n] \end{bmatrix} \end{aligned} \quad (8.11)$$

*Remark 66.1.*  $\bar{\mathbf{K}}_{\mathbf{X}}$  is a  $n \times n$  symmetric matrix.

*Remark 66.2.* The diagonal elements of  $\bar{\mathbf{K}}_{\mathbf{X}}$  are given by the variances  $\text{VAR}[X_k] = \mathbb{E}[(X_k - m_k)^2]$  of the elements of  $\vec{X}$ .

*Remark 66.3.* If the diagonal elements of  $\bar{\mathbf{K}}_{\mathbf{X}}$  are Uncorrelated, then  $\text{Cov}[X_j, X_k] = 0$  for  $j \neq k$ , and  $\bar{\mathbf{K}}_{\mathbf{X}}$ , the Covariance Matrix is a diagonal matrix.

*Remark 66.4.* If the random variables  $X_1, X_2, \dots, X_n$  are independent, then they are uncorrelated and  $\bar{\mathbf{K}}_{\mathbf{X}}$  is diagonal.

*Remark 66.5.* If the Mean Vector is  $\vec{0}$ , that is,  $m_k = \mathbb{E}[X_k] = 0$  for all  $k$ , then  $\bar{\mathbf{R}}_{\mathbf{X}} = \bar{\mathbf{K}}_{\mathbf{X}}$ .

## 9 Sums of Random Variables

**Defn 67** (Sum of Random Variables). The definition of a *sum of random variables* is given in Equation (9.1) below. Where  $X_i$  is a random variable,

$$S_n = \sum_{i=1}^n X_i = X_1 + X_2 + \dots + X_n \quad (9.1)$$

### 9.1 Means and Variances of Sums of Random Variables

**Defn 68** (Mean of Sums of Random Variables). The *mean of sums of random variables* is the same as the *expected value of sums of random variables*.

$$\mathbb{E}[S_n] = \sum_{i=1}^n \mathbb{E}[X_i] \quad (9.2)$$

*Remark 68.1.* All the properties of Properties of Expected Values and/or Properties of Expected Value hold true here as well..

**Defn 69** (Variance of Sums of Random Variables). The definition of the *variance of sums of random variables* is the same as we have been using them previously, Variance of Single Discrete Random Variable and Variance of Single Continuous Random Variable.

$$\text{VAR}[S_n] = \text{VAR}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \text{VAR}[X_i] + \sum_{j=1}^n \sum_{\substack{k=1 \\ j \neq k}}^n \text{Cov}[X_j, X_k] \quad (9.3)$$

*Remark 69.1.* If  $X_1, X_2, \dots, X_n$  are independent, then:

$$\text{VAR}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \text{VAR}[X_i] \quad (9.4)$$

**Defn 70** (Independent and Identically Distributed). We say that  $X_1, X_2, \dots, X_n$  are *Independent and Identically Distributed (iid)* random variables if  $X_i$  are drawn independently from the same population/probability distribution.

$$\sum_{i=1}^n \mathbb{E}[X_i] = n\mu \quad (9.5a)$$

$$\text{VAR}[S_n] = n\sigma^2 \quad (9.5b)$$

- $\mu$  is the mean of a random variable  $X_i$
- $\sigma^2$  is the variance of a random variable  $X_i$ .

## 10 Statistics

In applying probability models to real situations, we perform experiments and collect data to answer questions such as:

1. What are the values of the parameters of the distribution of a random variable of interest?
  - Mean or Expected value
  - Variance
2. Is the data set consistent with some model?
  - Some assumed distribution, which must be true, otherwise the model is wrong.
3. Is the data set consistent with some parameter value of the assumed value?

**Defn 71** (Random Sample). A *random sample* is a set of  $n$  Random Variable or Statistic that are drawn with Independent and Identically Distributed.

$$\mathbf{X}_n = (X_1, X_2, \dots, X_n) \quad (10.1)$$

*Remark 71.1.* This is *similar* to the definition of a Random Vector. The difference here is that the values in a Random Sample must be Independent and Identically Distributed and *may* be related to each other somehow.

*Remark 71.2* (Random Sample Parameters). These are an additional variable that is added onto the Probability Density Function or Probability Mass Function. When we were using these functions in the previous sections, these parameters were either constant or assumed to be constant. When considering these samples in Statistics, you must also account for the Random Sample Parameters.

**Defn 72** (Statistic). A *statistic*  $W(\mathbf{X}_n)$  is a function of the random sample  $X_1, X_2, \dots, X_n$ .

$$W(\mathbf{X}_n) = g(X_1, X_2, \dots, X_n) \quad (10.2)$$

**Defn 73** (Unit Variance). The *unit variance* means that the standard deviation,  $\sigma$  of a sample, as well as the variance,  $\sigma^2$  will tend towards 1 as the sample size increases to infinity.

### 10.1 Sample Mean

**Defn 74** (Sample Mean). The *sample mean* of a sequence is denoted as,

$$\bar{X} = M_n = \frac{\sum_{i=1}^n X_i}{n} \quad (10.3)$$

**Defn 75** (Expected Value of Sample Mean). The *expected value of the sample mean* is defined as:

$$\mathbb{E}[\bar{X}] = \mathbb{E}[M_n] = \frac{\mathbb{E}[S_n]}{n} = \frac{n\mu}{n} = \mu \quad (10.4)$$

*Remark 75.1.* The sample mean  $M_n$  is an *Unbiased Estimator* of population mean  $\mu$ .

**Defn 76** (Variance of Sample Mean). The *variance of the sample mean* is denoted as:

$$\text{VAR}[\bar{X}] = \text{VAR}[M_n] = \text{VAR}\left[\frac{S_n}{n}\right] = \frac{1}{n^2} \text{VAR}[S_n] = \frac{\sigma^2}{n} \quad (10.5)$$

*Remark 76.1.* The larger  $n$  gets, the smaller  $\text{VAR}[M_n]$  gets, and the closer  $M_n$  gets to  $\mu$ .

Also, we can use the Chebychev Inequality to approximate many values. In this case, we change the Chebychev Inequality from Equation (6.29) to Equation (10.6) like so:

$$P[|M_n - \mathbb{E}[M_n]| \geq \varepsilon] \leq \frac{\text{VAR}[M_n]}{\varepsilon^2} \quad (10.6)$$

## 10.2 Important Probability and Statistics Theorems

There are 3 very important theorems that are used quite frequently in both Probability Theory and Statistics.

1. Weak Law of Large Numbers
2. Strong Law of Large Numbers
3. Central Limit Theorem

**Theorem 10.1** (Weak Law of Large Numbers). *Let  $X_1, X_2, \dots, X_n$  be a sequence of Independent and Identically Distributed random variables from a population with mean  $\mathbb{E}[X] = \mu$ , then for  $\varepsilon > 0$ ,*

$$\lim_{n \rightarrow \infty} P[|M_n - \mu| < \varepsilon] = 1 \quad (10.7)$$

*Remark.* In words this means, for large enough fixed values of  $n$ ,  $M_n$  is close to  $\mu$  with high probability.

**Theorem 10.2** (Strong Law of Large Numbers). *Let  $X_1, X_2, \dots, X_n$  be a sequence of Independent and Identically Distributed random variables from a population with mean  $\mathbb{E}[X] = \mu$  and finite variance, then*

$$P\left[\lim_{n \rightarrow \infty} M_n = \mu\right] = 1 \quad (10.8)$$

*Remark.* With probability 1, every sequence of sample mean calculations will eventually approach and stay close to the population mean.

**Theorem 10.3** (Central Limit Theorem). *Let  $X_1, X_2, \dots, X_n$  be a sequence of Independent and Identically Distributed random variables from a population with mean  $\mathbb{E}[X] = \mu < \infty$  and finite variance  $\sigma^2$  and let*

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

*then,*

$$\lim_{n \rightarrow \infty} P[Z_n \leq z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{x^2}{2}} dx \quad (10.9)$$

- $S_n = X_1 + X_2 + \dots + X_n$ , The sum of the random variables
- $\mu = \mathbb{E}[X_1]$ , The mean for an individual random variable
- $\sigma = \sqrt{\text{VAR}[X_1]}$ , The variance of an individual random variable
- $n$  is the number of trials/recordings/samples/etc.
- $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ , The Sample Mean

*Remark.* This means that over time, as you gain more and more sample means, they will start to resemble the Gaussian/Normal Random Variable, or the Normal Random Variable.

<b>Example 10.1: Central Limit Theorem. Problem 7.25</b>
Problem 7.25 from Homework 11 (Extra Credit).

Central Limit Theorem is Example 10.1.

## 10.3 Estimators

- A Statistic is a function of the data  $X_1, X_2, \dots, X_n$
- An *estimator* for a parameter,  $\theta$ , usually denoted  $\hat{\theta}$ , is also a statistic

**Defn 77** (Unbiased Estimator). In general we say that a Statistic  $\Theta(X)$  (a function of data  $X_1, X_2, \dots, X_n$ ) is an *unbiased estimator* of a parameter  $\theta$  if  $\mathbb{E}[\Theta(\mathbf{X})] = \theta$ .

*Remark 77.1* (What makes a good estimator of any parameter,  $\theta$ ?). A *good estimator* of any parameter,  $\theta$ , should:

- Give the correct value of  $\theta$
- Not vary too much around  $\theta$

*Remark 77.2.* This is the definition of *unbiased*, drawn from the definition of Bias



### 10.3.1 Goodness of an Estimator

There are 4 measures we use to determine how good our estimator is.

1. Bias
2. Variance of Sample Mean
3. Mean Squared Error
4. Consistency

If our estimator is an Unbiased Estimator, then:

- Accuracy is defined as  $B[\hat{\theta}] = \mathbb{E}[\hat{\theta}] - \theta$
- Precision is defined as  $\text{VAR}[\hat{\theta}]$

**Defn 78** (Bias). *Bias* is defined as:

$$B[\hat{\Theta}] = \mathbb{E}[\hat{\Theta}] - \theta \quad (10.10)$$

*Remark 78.1.* The estimator  $\hat{\Theta}$  is *unbiased* for  $\theta$  if

$$\mathbb{E}[\hat{\Theta}] = \theta \quad (10.11)$$

**Defn 79** (Mean Squared Error). The *Mean Squared Error* of an estimator for parameter  $\hat{\theta}$  is:

$$\text{MSE}[\hat{\theta}] = \mathbb{E} \left[ \left( \hat{\Theta} - \theta \right)^2 \right] = \text{VAR} [\hat{\theta}] + \left( B [\hat{\theta}] \right)^2 \quad (10.12)$$

*Remark.* When doing statistical analysis, there is something called the *Bias-Variance Tradeoff*. When doing the analysis, if you try to minimize bias, your variance will increase and vice-versa. There is a happy medium, which is not discussed in this class.

**Defn 80** (Consistency).  $\hat{\theta}$  is a *consistent estimator* for  $\theta$  if  $\hat{\Theta}$  converges to  $\theta$  in probability.

$$\lim_{n \rightarrow \infty} P \left[ |\hat{\Theta} - \theta| > \varepsilon \right] = 0 \quad (10.13)$$

## 10.4 How to Find a Good Estimator

There are several methods, two of which are:

1. Method of Moments
  - Sample Moments and Population Moments,  $\bar{X}_n = \mu$
  - You need as many moments as parameters to get enough equations
2. Maximum Likelihood Estimation

### 10.4.1 Method of Moments

**Example 10.2: Method of Moments.**

Problem 8.6 from Homework 11 (Extra Credit).  $x + y = z$

$$\sum_{x \in S_X} p_k(x) = 1$$

### 10.4.2 Maximum Likelihood Estimation

**Defn 81** (Maximum Likelihood Estimation). Let  $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} f(x|\theta)$ .

$$\hat{\Theta}_{\text{MLE}} = \underset{\theta \in \Theta}{\operatorname{argmax}} \mathcal{L}(\theta | x_1, x_2, \dots, x_n) \quad (10.14)$$

*Remark 81.1.* Likelihood, denoted  $\mathcal{L}$  is defined as the Joint Probability Density Function of the Random Sample and its Random Sample Parameters.

$$\mathcal{L}(\theta | x_1, x_2, \dots, x_n) = f_{X_1}(x_1 | \theta) \cdot f_{X_2}(x_2 | \theta) \cdot \dots \cdot f_{X_n}(x_n | \theta) \quad (10.15)$$

*Remark 81.2.* It is often easier to maximize  $\hat{\Theta}_{\text{MLE}}$  over the log-likelihood.

$$\hat{\Theta}_{\text{MLE}} = \underset{\theta \in \Theta}{\operatorname{argmax}} \log \mathcal{L}(\theta | x_1, x_2, \dots, x_n)$$

*Remark 81.3.*

$$\underset{\theta \in \Theta}{\operatorname{argmax}}$$

is the global maxima of the function. This is further described in Definition B.1.3.

## 10.5 Confidence Intervals

Because certain estimators are not discrete, but continuous, the estimators expected value might not be quite right. This is where Confidence Intervals come in.

**Defn 82** (Confidence Interval). A *confidence interval* is an interval or set of values that is highly likely to contain the true value of the parameter.

**Defn 83** ( $1 - \alpha$  Confidence Interval). The  $1 - \alpha$  *confidence interval* is a Confidence Interval where we estimate the probability of the parameter,  $\theta$ , being in the random interval to  $1 - \alpha$ . The problem is, find a random interval  $[\ell(\mathbf{X}), u(\mathbf{X})]$  such that

$$P[\ell(\mathbf{X}_n), u(\mathbf{X}_n)] = 1 - \alpha \quad (10.16)$$

*Remark 83.1.*  $\ell(\mathbf{X})$  is the *lower bound of the random interval*.  $u(\mathbf{X})$  is the *upper bound of the random interval*. **Only ONE of these may be  $\infty$  at a time.** Otherwise, you're including the whole sample space, making the confidence interval useless.

*Remark 83.2.* If the problem says, with a confidence interval of 95%, that is the  $1 - \alpha$  portion; i.e.  $1 - \alpha = 0.95\%$ .

## 10.6 Hypothesis Testing

There are 4 parts to Hypothesis Testing

1. Identify the hypotheses.  $H_0$  is the *null hypothesis*. It is usually compared against another, complementary hypothesis,  $H_1$ .
2. The Rejection Region, which is evidence against the null hypothesis that we may or may not choose to include.  $R \subset \mathcal{X}$ , where  $\mathcal{X}$  is the sample space.
3. The Decision Rule:
  - (a) Reject  $H_0$  if  $X \in R$
  - (b) Accept/Do not reject  $H_0$  if  $X \notin R$
4. The Test Statistic

There are 2 errors that can occur in Hypothesis Testing.

**Type I Error:** Reject  $H_0$  if  $H_0$  is true.

**Type II Error:** Accept  $H_0$  if  $H_0$  is false.

Section 1

## A Trigonometry

### A.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{A.2})$$

### A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm i\alpha} = \cos(\alpha) \pm i \sin(\alpha) \quad (\text{A.3})$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad (\text{A.4})$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (\text{A.5})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{A.6})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{A.7})$$

### A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \quad (\text{A.8})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \quad (\text{A.9})$$

### A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad (\text{A.10})$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (\text{A.11})$$

### A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \quad (\text{A.12})$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \quad (\text{A.13})$$

### A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2} \quad (\text{A.14})$$

$$\cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2} \quad (\text{A.15})$$

### A.7 Product-to-Sum Identities

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad (\text{A.16})$$

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad (\text{A.17})$$

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (\text{A.18})$$

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (\text{A.19})$$

## A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{A.20})$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.21})$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.22})$$

## A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \quad (\text{A.23})$$

## A.10 Rectangular to Polar

$$a + ib = \sqrt{a^2 + b^2} e^{i\theta} = r e^{i\theta} \quad (\text{A.24})$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \quad (\text{A.25})$$

## A.11 Polar to Rectangular

$$r e^{i\theta} = r \cos(\theta) + ir \sin(\theta) \quad (\text{A.26})$$

## B Calculus

### B.1 Fundamental Theorems of Calculus

**Defn B.1.1** (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if  $f$  is continuous on the closed interval  $[a, b]$  and  $F$  is the indefinite integral of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{B.1})$$

**Defn B.1.2** (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for  $f$  a continuous function on an open interval  $I$  and  $a$  any point in  $I$ , and states that if  $F$  is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{B.2})$$

**Defn B.1.3** (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname{argmax}_x$$