

# EITF75: Systems and Signals - Reference Sheet

Karl Hallsby

Last Edited: September 5, 2019

## Contents

<b>1</b>	<b>Sinusoids</b>	<b>1</b>
1.1	Continuous-Time Signals . . . . .	1
1.1.1	Frequency in Continuous-Time Signals . . . . .	1
1.1.2	Properties of Continuous-Time Sinusoidal Signals . . . . .	1
1.2	Discrete-Time Signals . . . . .	2
1.2.1	Frequency in Discrete-Time Signals . . . . .	2
1.2.2	Properties of Discrete-Time Sinusoidal Signals . . . . .	2
1.2.3	Frequency Aliases . . . . .	2
1.3	Sampling Rates and Sampling Frequency . . . . .	2
1.3.1	Nyquist Rate . . . . .	3
1.3.2	Nyquist Frequency . . . . .	3
1.4	Digital Signals . . . . .	3
1.4.1	Quantization . . . . .	3
1.4.1.1	Quantization Levels . . . . .	3
1.4.1.2	Quantization Error . . . . .	3
1.4.1.3	Bit Requirements . . . . .	3
1.4.1.4	Bit Rate . . . . .	3
<b>2</b>	<b>Discrete-Time Systems</b>	<b>3</b>
2.1	Representing Discrete-Time Systems . . . . .	3
2.1.1	Functional Representation . . . . .	3
2.1.2	Tabular Representation . . . . .	3
2.1.3	Sequence Representation . . . . .	3
2.2	Elementary Discrete-Time Signals . . . . .	4
2.2.1	Unit Impulse Signal . . . . .	4
2.2.2	Unit Step Signal . . . . .	4
2.2.3	Unit Ramp Signal . . . . .	4
2.2.4	Exponential Signal . . . . .	4
2.3	Classification of Discrete-Time Signals . . . . .	5
2.3.1	Energy Signal . . . . .	5
2.3.2	Power Signal . . . . .	5
2.3.3	Periodic and Aperiodic Signals . . . . .	5
2.3.4	Symmetric and Antisymmetric Signals . . . . .	5
<b>3</b>	<b>Convolutions</b>	<b>5</b>
<b>A</b>	<b>Trigonometry</b>	<b>6</b>
A.1	Trigonometric Formulas . . . . .	6
A.2	Euler Equivalents of Trigonometric Functions . . . . .	6
A.3	Angle Sum and Difference Identities . . . . .	6
A.4	Double-Angle Formulae . . . . .	6
A.5	Half-Angle Formulae . . . . .	6
A.6	Exponent Reduction Formulae . . . . .	6
A.7	Product-to-Sum Identities . . . . .	6
A.8	Sum-to-Product Identities . . . . .	7
A.9	Pythagorean Theorem for Trig . . . . .	7
A.10	Rectangular to Polar . . . . .	7
A.11	Polar to Rectangular . . . . .	7

<b>B</b>	<b>Calculus</b>	<b>8</b>
B.1	Fundamental Theorems of Calculus . . . . .	8
B.2	Rules of Calculus . . . . .	8
B.2.1	Chain Rule . . . . .	8

# 1 Sinusoids

There are several ways to characterize Sinusoids. The first is by dimension:

1. Multidimensional/Multichannel Signals
2. Monodimensional/Monochannel Signals

You can also classify sinusoids by their independent variable (usually time) and the values they take.

1. Continuous-Time Signals or Analog Signals
2. Discrete-Time Signals
3. There is a third way to classify sinusoids and their signals: Digital Signals

**Defn 1** (Continuous-Time Signals). *Continuous-time signals* or *Analog signals* are defined for every value of time and they take on values in the continuous interval  $(a, b)$ , where  $a$  can be  $-\infty$  and  $b$  can be  $\infty$ . Mathematically, these signals can be described by functions of a continuous variable.

For example,

$$x_1(t) = \cos \pi t, x_2(t) = e^{-|t|}, -\infty < t < \infty$$

**Defn 2** (Discrete-Time Signals). *Discrete-time signals* are defined only at certain specified values of time. These time instants **need not** be equidistant, but in practice, they are usually taken at equally spaced intervals for computation convenience and mathematical tractability.

For example,

$$x(t_n) = e^{-|t_n|}, n = 0, \pm 1, \pm 2, \dots$$

A Discrete-Time Signals can be represented mathematically by a sequence of real or complex numbers.

*Remark 2.1.* To emphasize the discrete-time nature of the signal, we shall denote the signal as  $x(n)$ , rather than  $x(t)$ .

*Remark 2.2.* If the time instants  $t_n$  are equally spaced (i.e.,  $t_n = nT$ ), the notation  $x(nT)$  is also used.

## 1.1 Continuous-Time Signals

### 1.1.1 Frequency in Continuous-Time Signals

A simple harmonic oscillation is mathematically described by Equation (1.1).

$$x_a(t) = A \cos(\Omega t + \theta), -\infty < t < \infty \quad (1.1)$$

*Remark.* The subscript  $a$  is used with  $x(t)$  to denote an analog signal.

This signal is completely characterized by three parameters:

1.  $A$ , the *amplitude* of the sinusoid
2.  $\Omega$ , the *frequency* in radians per second (rad/s)
3.  $\theta$ , the *phase* in radians.

Instead of  $\Omega$ , the frequency  $F$  in cycles per second or hertz (Hz) is used.

$$\Omega = 2\pi F \quad (1.2)$$

Plugging (1.2) into (1.1), yields

$$x_a(t) = A \cos(2\pi F t + \theta), -\infty < t < \infty \quad (1.3)$$

### 1.1.2 Properties of Continuous-Time Sinusoidal Signals

The analog sinusoidal signal in equation (1.3) is characterized by the following properties:

- (i) For every fixed value of the frequency  $F$ ,  $x_a(t)$  is periodic.

$$x_a(t + T_p) = x_a(t)$$

where  $T_p = \frac{1}{F}$  is the fundamental period.

- (ii) Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.
- (iii) Increasing the frequency  $F$  results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in the given time interval.

## 1.2 Discrete-Time Signals

### 1.2.1 Frequency in Discrete-Time Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A \cos(\omega n + \theta), n \in \mathbb{Z}, -\infty < n < \infty \quad (1.4)$$

The signal is characterized by these parameters:

1.  $n$ , the sample number. MUST be an integer.
2.  $A$ , the *amplitude* of the sinusoid
3.  $\omega$ , the *angular frequency* in radians per sample
4.  $\theta$ , is the *phase*, in radians.

Instead of  $\omega$ , we use the frequency variable  $f$  defined by

$$\omega \equiv 2\pi f \quad (1.5)$$

Using (1.4) and (1.5) yields

$$x(n) = A \cos(2\pi f n + \theta), n \in \mathbb{Z}, -\infty < n < \infty \quad (1.6)$$

### 1.2.2 Properties of Discrete-Time Sinusoidal Signals

- (i) A discrete-time sinusoid is periodic **ONLY** if its frequency is a rational number.
- (ii) Discrete-time sinusoids whose frequencies are separated by an integer multiple of  $2\pi$  are identical. This leads us to the idea of a Frequency Alias.
- (iii) The highest rate of oscillation in a discrete-time sinusoid is attained when  $\omega = \pm\pi$  or, equivalently,  $f = \pm\frac{1}{2}$ .

### 1.2.3 Frequency Aliases

The concept of a Frequency Alias is drawn from the idea that discrete-time sinusoids whose frequencies are separated by an integer multiple of  $2\pi$  are identical and that frequencies  $|f| > \frac{1}{2}$  are identical. (Properties (ii) and (iii))

**Defn 3** (Frequency Alias). A *frequency alias* is a sinusoid having a frequency  $|\omega| > \pi$  or  $|f| > \frac{1}{2}$ . This is because this sinusoid is *indistinguishable* (*identical*) to one with frequency  $|\omega| < \pi$  or  $|f| < \frac{1}{2}$ .

A *frequency alias* is a sequence resulting from the following assertion based on the sinusoid  $\cos(\omega_0 n + \theta)$ .

It follows that

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta)$$

As a result, all sinusoidal sequences

$$x_k(n) = A \cos(\omega_k n + \theta), k = 0, 1, 2, \dots$$

where

$$\omega_k = \omega_0 + 2k\pi, -\pi \leq \omega_0 \leq \pi$$

are *indistinguishable* (i.e., *identical*).

Because of this, we regard frequencies in the range of  $-\pi \leq \omega \leq \pi$  or  $-\frac{1}{2} \leq f \leq \frac{1}{2}$  as unique, and all frequencies that fall outside of these ranges as aliases.

*Remark 3.1.* It should be noted that there is a difference between discrete-time sinusoids and continuous-time sinusoids here. Continuous-time sinusoids have distinct signals for  $\Omega$  or  $F$  in the entire range  $-\infty < \Omega < \infty$  or  $-\infty < F < \infty$ .

## 1.3 Sampling Rates and Sampling Frequency

Most signals of interest are analog. To process these signals, they must be collected and converted to a digital form, that is, to convert them to a sequence of numbers having finite precision. This is called *analog-to-digital* (*A/D*) *conversion*. Conceptually, we view this conversion as a 3-step process.

1. Sampling
2. Quantization
3. Coding

### 1.3.1 Nyquist Rate

### 1.3.2 Nyquist Frequency

## 1.4 Digital Signals

**Defn 4** (Digital Signals). *Digital signals* are a subset of Discrete-Time Signals. In this case, not only are the values being measured occurring at fixed points in time, the values themselves can only take certain, fixed values.

### 1.4.1 Quantization

**Defn 5** (Quantization). This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-value (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample  $x(n)$  and the quantized output  $x_q(n)$  is called the Quantization Error.

#### 1.4.1.1 Quantization Levels

#### 1.4.1.2 Quantization Error

**Defn 6** (Quantization Error). The *quantization error* of something.

#### 1.4.1.3 Bit Requirements

#### 1.4.1.4 Bit Rate

## 2 Discrete-Time Systems

As discussed in Section 1.2,  $x(n)$  is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is *not defined* at instants between the samples. Also, if  $n$  is not an integer,  $x(n)$  is not defined.

Besides graphical representation of a discrete-time system, there are 3 ways to represent a discrete-time signal.

1. Functional Representation
2. Tabular Representation
3. Sequence Representation

### 2.1 Representing Discrete-Time Systems

#### 2.1.1 Functional Representation

This representation of a discrete-time system is done as a mathematical function.

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \quad (2.1)$$

#### 2.1.2 Tabular Representation

This representation of a discrete-time system is done as a table of corresponding values.

$n$		...	-2	-1	0	1	2	3	4	5	...
$x(n)$		...	0	0	0	1	4	1	0	0	...

#### 2.1.3 Sequence Representation

There are 2 methods of representation for this. The first includes all values for  $-\infty < n < \infty$ . In all cases,  $n = 0$  is marked in the sequence, somehow. I will do this with an underline.

$$x(n) = \{\dots, 0, \underline{0}, 1, 4, 1, 0, 0, \dots\} \quad (2.2)$$

The second only works if all  $x(n)$  values for  $n < 0$  are 0.

$$x(n) = \{\underline{0}, 1, 4, 1, 0, 0, \dots\} \quad (2.3)$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underline{-2}, 5, 0, 4, -1\} \quad (2.4)$$

This is identified as a seven-point sequence.

A finite-duration sequence where  $x(n) = 0$  for all  $n < 0$  is represented as

$$x(n) = \{\underline{0}, 1, 4, 1\} \quad (2.5)$$

This is identified as a four-point sequence.

## 2.2 Elementary Discrete-Time Signals

The following signals are basic signals that appear often and play an important role in signal processing.

### 2.2.1 Unit Impulse Signal

**Defn 7** (Unit Impulse Signal). The *unit impulse signal* or *unit sample sequence* is denoted as  $\delta(n)$  and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \quad (2.6)$$

This function is a signal that is zero everywhere, except at  $n = 0$ , where its value is 1.

*Remark 7.1.* This signal is different than the analog signal  $\delta(t)$ , which is also called a unit impulse, and is defined to be 0 everywhere except  $t = 0$ . The discrete unit impulse sequence is much less mathematically complicated.

### 2.2.2 Unit Step Signal

**Defn 8** (Unit Step Signal). The *unit step signal* is denoted as  $u(n)$  and is defined as

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \quad (2.7)$$

### 2.2.3 Unit Ramp Signal

**Defn 9** (Unit Ramp Signal). The *unit ramp signal* is denoted as  $u_r(n)$  and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \quad (2.8)$$

### 2.2.4 Exponential Signal

**Defn 10** (Exponential Signal). The *exponential signal* is a sequence of the form

$$x(n) = a^n \text{ for all } n \quad (2.9)$$

If  $a$  is real, then  $x(n)$  is a real signal. When  $a$  is complex valued ( $a \equiv b \pm cj$ ), it can be expressed as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n (\cos \theta n + j \sin \theta n) \end{aligned} \quad (2.10)$$

This can be expressed by graphing the real and imaginary parts

$$\begin{aligned} x_R(n) &\equiv r^n \cos \theta n \\ x_I(n) &\equiv r^n j \sin \theta n \end{aligned} \quad (2.11)$$

or by graphing the amplitude function and phase function.

$$\begin{aligned} |x(n)| &= A(n) \equiv r^n \\ \angle x(n) &= \phi(n) \equiv \theta n \end{aligned} \quad (2.12)$$

## 2.3 Classification of Discrete-Time Signals

### 2.3.1 Energy Signal

### 2.3.2 Power Signal

### 2.3.3 Periodic and Aperiodic Signals

### 2.3.4 Symmetric and Antisymmetric Signals

## 3 Convolutions

**Defn 11** (Convolution). The *convolution* operator.

$$y(t) = \sum_{k=-\infty}^{\infty} x(k) * h(n - k) \quad (3.1)$$

## A Trigonometry

### A.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.1})$$

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (\text{A.2})$$

### A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm i\alpha} = \cos(\alpha) \pm i \sin(\alpha) \quad (\text{A.3})$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad (\text{A.4})$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (\text{A.5})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (\text{A.6})$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (\text{A.7})$$

### A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \quad (\text{A.8})$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \quad (\text{A.9})$$

### A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad (\text{A.10})$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (\text{A.11})$$

### A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \quad (\text{A.12})$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \quad (\text{A.13})$$

### A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2} \quad (\text{A.14})$$

$$\cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2} \quad (\text{A.15})$$

### A.7 Product-to-Sum Identities

$$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad (\text{A.16})$$

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad (\text{A.17})$$

$$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \quad (\text{A.18})$$

$$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (\text{A.19})$$



## A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \quad (\text{A.20})$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.21})$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (\text{A.22})$$

## A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \quad (\text{A.23})$$

## A.10 Rectangular to Polar

$$a + ib = \sqrt{a^2 + b^2} e^{i\theta} = r e^{i\theta} \quad (\text{A.24})$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \quad (\text{A.25})$$

## A.11 Polar to Rectangular

$$r e^{i\theta} = r \cos(\theta) + ir \sin(\theta) \quad (\text{A.26})$$

## B Calculus

### B.1 Fundamental Theorems of Calculus

**Defn B.1.1** (First Fundamental Theorem of Calculus). The *first fundamental theorem of calculus* states that, if  $f$  is continuous on the closed interval  $[a, b]$  and  $F$  is the indefinite integral of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a) \quad (\text{B.1})$$

**Defn B.1.2** (Second Fundamental Theorem of Calculus). The *second fundamental theorem of calculus* holds for  $f$  a continuous function on an open interval  $I$  and  $a$  any point in  $I$ , and states that if  $F$  is defined by

$$F(x) = \int_a^x f(t) dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t) dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \quad (\text{B.2})$$

**Defn B.1.3** (argmax). The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname{argmax}_x$$

### B.2 Rules of Calculus

#### B.2.1 Chain Rule

**Defn B.2.1** (Chain Rule). The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot h(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \quad (\text{B.3})$$