MMAE 320: Thermofluid Dynamics — Reference Material
Illinois Institute of Technology

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List of Theorems

1 Introduction

Defn 1 (Energy)**.** *Energy* is the stuff in the universe that can cause changes in material states. A change in energy is the ability to do Work. Energy can never be negative, but changes in energy can.

Some examples are:

- Thermal Energy
- Electrical Energy
- Mechanical Energy

One fundamental thing about Energy that was discovered was the Law of Conservation of Energy.

Defn 2 (Law of Conservation of Energy)**.** The *law of conservation of energy* states that energy cannot be created or destroyed; it can only change form. This means that the total amount of energy in an interaction is constant.

Expressed as an equation, this is recognized as

$$\sum_{i \in I} E_i = C \tag{1.1}$$

where

I: The set of all interactions of interest.
E: The energy of this particular interaction, i.
C: The total energy of the system, a constant.

The Law of Conservation of Energy leads to Equation (1.2), shown below.

$$E_{in} - E_{out} = \Delta E \tag{1.2}$$

In a typical use, we are concerned with Heat as the form of Energy.

Defn 3 (Heat)**.** *Heat* is the form of Energy that can be transferred from one System to another as a result of temperature difference. The variable letter for the energy in a system is Q.

Defn 4 (Thermodynamics)**.** *Thermodynamics* is the science of equilibrium states and changes between these states.

Pure thermodynamic analysis will show the toal amount of Energy moving through the system, but not the rate at which it occurs. This is where Heat Transfer comes in.

Defn 5 (Heat Transfer)**.** *Heat transfer* is the science that deals with the determination of the *rates* of Heat-based Energy transfers. This science deals with systems that lack a thermal equilibrium, meaning it cannot be based on only principles of Thermodynamics.

For every flow of energy in a system, there **MUST** be a Driving Force.

Defn 6 (Driving Force)**.** *Driving force* are the conditions of the system for the transfer of Energy to occur. There are any number of driving forces in the universe, some are listed here:

- Heat Transfer requires a temperature difference.
- Electric current flow requires a voltage difference.
- Fluid flow requires a pressure difference.

For Heat Transfer, the Temperature Gradient determines the rate ofthe transfer.

Defn 7 (Temperature Gradient)**.** *Temperature gradient* is the temperature difference per unit length, or the rate of change of temperature.

Remark 7.1 (Extension)*.* The use of gradients extends to other Driving Forces as well.

1.1 Fluid Mechanics

Defn 8 (Stress)**.** *Stress* is defined as force per unit area.

$$\tau = \frac{F}{A} \tag{1.3}$$

Obeying Newton's third law, typically, there is a normal force per unit area, called the **normal stress**. In a fluid at rest, the normal stress is called **pressure**.

Dimension	Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Temperature	kelvin (K)
Electric Current	ampere (A)
Amount of Light	candela (cd)
Amount of Matter	mole (mol)

Table 1.1: The 7 Fundamental Dimensions and Units

1.2 Units/Dimensions

1.2.1 SI System

There are 7 fundamental dimensions and units in the SI system, shown in Table 1.1.

1.2.2 English System

The English system uses a very different set of units to describe these dimensions.

1.2.2.1 Mass The English unit of mass is the pound-mass.

$$\text{lbm}$$

1.2.2.2 Force The English unit of mass is the pound-force.

$$1\,\text{lbf} = 32.174\,\text{lbm ft/s}^2$$

1.2.2.3 Energy The English unit of energy is the British Thermal Unit (BTU). 1 btu raises the temperature of 1 lbm of water at 68°F by 1°F.

1.2.2.4 Work

Defn 9 (Work)**.** *Work* is defined to be force multiplied by the path distance the force was applied over.

$$W = \vec{F}d \tag{1.4}$$

The English unit of Work (energy per time) is also the watt (W).

1.2.2.5 Power

Defn 10 (Power)**.** *Power* is defined as energy per unit time.

$$P = \frac{E}{t} \tag{1.5}$$

The unit for power is typically the horsepower.

2 Basic Concepts

2.1 Systems

Defn 11 (System)**.** A *system* is defined as a quantity of matter or a region in space chosen for study.

The mass or region outside the System is the **surroundings**. The surface that separates the System from the surroundings is the **boundary**.

2.1.1 Type of Systems

There are 2 types of systems:

1. Open System
2. Closed System

Defn 12 (Open System)**.** An *open system* is a System in which the mass of the system is **not** constant. Thus, mass and Energy can flow from the system to the surroundings. Energy can flow by either Work, or it can be Heat.

Remark 12.1 (Control Volume). Sometimes an Open System is called a *control volume*, because the volume of the system is constant.

Remark 12.2. Flow through these devices is typically easier by selecting the region based on volume, rather than mass.

Defn 13 (Closed System)**.** A *closed system* is a System where the mass of the system **is** constant, but Energy can move between the system and the surroundings. This type of system can either be Adiabatic or non-adiabatic. For each case:

Adiabatic There is an Energy transfer by Heat **and** Work.
Non-adiabatic The energy transfer is done by Work **only**.

Remark 13.1 (Control Mass). Sometimes a Closed System is called a *control mass*, because mass is constant.

There is also a third type of system, the Isolated System, which is a special case of the Closed System.

Defn 14 (Isolated System)**.** An *isolated system* is a System where **neither** mass or Energy can move from the system to the surroundings. This system has the properties of being Adiabatic **and** there is no Work done.

2.1.2 Properties of Systems

Defn 15 (Property)**.** A *property* of a System is a characteristic of the system.

Some common properties are:

- P, Pressure
- T, Temperature
- V, Volume
- m, Mass

There are 2 types of properties:

1. Intensive Property
2. Extensive Property

Defn 16 (Intensive Property)**.** *Intensive properties* are properties that are **independent** of the mass of the system. These include:

- P, Pressure
- T, Temperature
- ρ, Density

Defn 17 (Extensive Property)**.** *Extensive properties* are properties that are **dependent** on the size or extent of the system. These include:

- Total mass
- Total volume
- Total momentum

Some extensive properties can be used to form a Specific Property.

Defn 18 (Specific Property)**.** A *specific property* is an Extensive Property per unit mass. Some examples are:

- Specific Volume $v_s = \frac{V}{m}$
- Specific Total Energy $e = \frac{E}{m}$

2.1.3 Generalizations

We make some generalizations about the fluids we are working with to make calculations easier. For example, we treat fluids as a **continuum**, rather than the true atomic nature of the substance. This allows us to:

- Treat properties as point functions
- Assume properties vary continually in space with no discontinuities

This assumption is valid so long as the size of the System is relatively large compared to the space between the component molecules.

2.2 Density

Defn 19 (Density)**.** *Density* is defined as mass per unit volume.

$$\rho = \frac{m}{V} \text{ kg/m}^3 \tag{2.1}$$

Defn 20 (Specific Volume)**.** *Specific volume* is the reciprocal of Density, and is the amount of volume per unit mass.

$$v_s = \frac{V}{m} = \frac{1}{\rho} \tag{2.2}$$

For substances that lack a uniform mass and volume, density can also be realized by Equation (2.3).

$$\rho = \frac{dm}{dV} \tag{2.3}$$

Sometimes the density of a substance is given relative to another substance, usually water. This is Specific Gravity.

Defn 21 (Specific Gravity)**.** *Specific gravity* is the Density of a substance as a ratio to another substance, usually water. This is expressed as

$$\text{SG} = \frac{\rho}{\rho_{\text{H}_2\text{O}}} \tag{2.4}$$

Defn 22 (Specific Weight)**.** *Specific weight* is the weight of a unit volume of a substance. It is expressed as:

$$\gamma_s = \rho g \text{ N/m}^3 \tag{2.5}$$

2.3 States

Defn 23 (State)**.** The *state* of the system includes all properties that can be measured or calculated which can completely describe the condition of the system.

In any given State, all the properties of a system will have fixed values. If any single property changes, all the others will change in accordance. We are typically interested in Equilibrium states.

Defn 24 (Equilibrium)**.** *Equilibrium* is a State of a System which does not change when isolated from it surroundings. A system will only leave equilibrium when disturbed by an outside Energy, or an unbalanced Driving Force.

There are several different types of Equilibrium:

Thermal Equilibrium If the temperature throughout the entire System is constant.

Mechanical Equilibrium If there is no change in pressure at any point of the System with regards to time. However, pressure can vary within the system, but for most of our concerns, this isn't a problem.

Phase Equilibrium If there is more than one phase of matter, this type of equilibrium is reached with each phase reaches its equilibrium level and stays there.

Chemical Equilibrium If the chemical composition of a System does not change with time, i.e. no chemical reactions could occur.

2.3.1 The State Postulate

Defn 25 (State Postulate)**.** The *state postulate* states: the State of a simple compressible system is **completely** specific by 2 independent, intensive properties.

Remark 25.1 (Simple Compressible System)*.* A system is called a simple compressible system in the absence of electrical, magnetic, gravitational, motion, and surface tension effects.

Properties are **independent** if one property can be varied and the other one remains constant.

2.4 Processes

Defn 26 (Process)**.** A *process* is the change in a System from one Equilibrium state to another. The **path** of the process is the set of Quasi-equilibrium processes.

It is easiest to deal with paths where if one of the variables is changed just a small amount, the change becomes linear. This is similar to the concept of a derivative, because if we sample at a small/short enough rate, the changes introduced to the System will be linear.

Defn 27 (Quasi-equilibrium)**.** A *quasi-equilibrium* process is one that is sufficiently slow so that the System can adjust itself internally so that properties change constantly **throughout** the system.

Lastly, if the Process is a Cycle, then the initial and final States are the same.

Defn 28 (Cycle)**.** A *cycle* is a Process that returns to its initital state at the end of the process.

2.4.1 *iso-* Processes

Some Processes are special in that one Property remains constant. There are several of these:

- Isothermal
- Isobaric
- Isometric

Defn 29 (Isothermal)**.** *Isothermal* Processes are ones where the temperature T remains constant.

Defn 30 (Isobaric)**.** *Isobaric* Processes are ones where the pressure P remains constant.

Defn 31 (Isometric)**.** *Isometric* (*Isochoric*) Processes are ones where the volume V remains constant.

2.5 Temperature

Defn 32 (0th Law of Thermodynamics)**.** The *0th law of thermodynamics* states that if two bodies are in thermal Equilibrium with a third body, they are also in thermal equilibrium with each other.

2.5.1 Subjective Temperature Scales

There are 2 subjective temperature scales:

1. Celsius (°C)
2. Fahrenheit (°F)

These are based off of physical temperatures at certain points on Earth regarding water freezing and water boiling. These are useful for regular use, but not as well-suited for thermodynamic use.

$$\frac{5}{9}(T(^{\circ}\mathrm{F}) - 32) = T(^{\circ}\mathrm{C}) \tag{2.6}$$

2.5.2 Objective Temperature Scales

Objective temperature scales are ones that are based off of physical, universal constants. There are 2 objective temperature scales:

1. Kelvin (K)
2. Rankine (R)

Both of these have placed 0 at the point where all molecular motion stops.

The equations to convertion between Celsius and Kelvin is shown in Equation (2.7).

$$T(\mathrm{K}) = T(^{\circ}\mathrm{C}) + 273.15 \tag{2.7}$$

The equations to convertion between Fahrenheit and Rankine is shown in Equation (2.8).

$$T(\mathrm{R}) = T(^{\circ}\mathrm{F}) + 459.67 \tag{2.8}$$

2.6 Pressure

Defn 33 (Pressure)**.** *Pressure* is the normal **scalar** force exerted per unit area. Because it is a scalar, it has no dependency on the direction of the normal force. It can be expressed using Density, gravity, and depth/length.

$$P = \rho g d \tag{2.9}$$

The SI unit of pressure is the pascal:

$$\mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2} \tag{2.10}$$

The bar is also used:

$$1\,\mathrm{bar} = 100\,\mathrm{kPa}$$

The English unit of pressure is the pound-force per square inch.

$$\mathrm{lbf/in}^2 = \mathrm{psi} \tag{2.11}$$

We exist, roughly, at one atmosphere of pressure, or 1 atm. The equivalents of this are:

$$\begin{aligned} 1\,\mathrm{atm} &= 101\,325\,\mathrm{kPa} \\ 1\,\mathrm{atm} &= 14.696\,\mathrm{psi} \end{aligned}$$

Using this, we can measure pressure, typically with either Gage Pressure or Vacuum Pressure.

Defn 34 (Gage Pressure)**.** *Gage pressure* is the Pressure that a gage reads after being calibrated to some standard pressure, typically one atmosphere. Thus, the actual pressure of something read by the gage is represented by Equation (2.12) below.

$$P_{\mathrm{abs}} = P_{\mathrm{atm}} + P_{\mathrm{gage}} \tag{2.12}$$

Defn 35 (Vacuum Pressure)**.** *Vacuum pressure* is the Pressure that a gage reads after being calibrated to some standard pressure, typically one atmosphere. Thus, the actual pressure of something read by the gage is represented by Equation (2.13) below.

$$P_{\mathrm{abs}} = P_{\mathrm{atm}} - P_{\mathrm{vacuum}} \tag{2.13}$$

2.6.1 Pressure Inside a Fluid

Pressure inside a fluid increases linearly with depth. This is seen in Equation (2.14), where Δz is the distance between the two measured points.

$$\begin{aligned} P_{\mathrm{below}} &= P_{\mathrm{above}} + \rho g |\Delta z| \\ &= P_{\mathrm{above}} + \gamma_s |\Delta z| \end{aligned} \tag{2.14}$$

ρ: The density of the fluid ($\mathrm{kg/m^3}$)

Defn 36 (Pascal's Law)**.** *Pascal's Law* states that the Pressure along any horizontal plane in the same fluid is the same pressure. This leads to Equation (2.15).

$$\begin{aligned} P_1 &= P_2 \\ \frac{\vec{F}_1}{A_1} &= \frac{\vec{F}_2}{A_2} \end{aligned} \tag{2.15}$$

3 Energy, and Energy Transfer

Defn 37 (Macroscopic Energy Form)**.** *Macroscopic energy forms* are typically ones that have to deal with objects on a macroscopic level. These energies are:

1. Kinetic
2. Potential

Remark 37.1. This definition is included because the textbook makes use of it.

Defn 38 (Microscopic Energy Form)**.** *Microscopic energy forms* are energies that act on non-macroscopic levels. Namely, they affect their systems on microscopic levels. These energies include:

1. Sensible

    - Heat
    - Kinetic energy of molecules

2. Latent
    - Phase Changes
3. Chemical
    - Combustion
4. Nuclear

Remark 38.1. This definition is included because the textbook makes use of it.

Defn 39 (Internal Energy)**.** *Internal energy* is equivalent to Microscopic Energy Forms. It means the Energy that the object in question inherently has at that point in time.

3.1 Energy Quality

Energy has quality!

- Macroscopic Energy Form
    - Structured
    - Moves as a single unit
- Microscopic Energy Form
    - **Not** structures
    - Does **not** move as a single unit

These differences mean that we measure the efficiency of each type of energy form differently.

3.2 Energy and Flows

When moving a fluid through a pipe, we can find the amount of work done by the fluid flowing, called teh Flow Energy.

$$\begin{aligned} P &= \frac{F}{\text{Area}} \\ V_{\text{Cylinder}} &= \ell \cdot \text{Area} \\ W &= F \cdot \text{Distance} \end{aligned}$$

If we substitute for the common terms in the formula for work, then we end up with Equation (3.1).

Defn 40 (Flow Energy)**.** *Flow energy* is the energy that a fluid flowing through a long, straight pipe has.

$$\begin{aligned} W &= PV \\ \text{FE} &= PV \end{aligned} \tag{3.1}$$

Remark 40.1 (Energy Form). Typically, Flow Energy is categorized with the Macroscopic Energy Forms, because it behaves more like those and can be nearly as efficient as them. This is true even though this is technically an application of microscopic energies. This is because we are not worried about the internal energy of the fluid in the pipe, but are instead interested in the mechanical movement of it.

3.3 Divisions of Energy

We are always interested in the change in energy that occurs due to something. This is seen as Equation (3.2).

$$\begin{aligned} \Delta E &= \Delta U + \Delta \text{KE} + \Delta \text{PE} + \Delta \text{FE} \\ &= \frac{U_2 - U_1}{m} + \frac{v_2^2 - v_1^2}{2} + g(h_2 - h_1) + \frac{P_2 - P_1}{\rho} \end{aligned} \tag{3.2}$$

- The internal energy cannot be completely converted into work.
- Mechanical energy is typically defined to be these types of energies. These can be completely converted into work by an ideal machine.
    - Kinetic energy (KE)

- Potential energy (PE)
- Flow energy (FE)

We are also interested in the Specific Energy of the system.

Defn 41 (Specific Energy)**.** *Specific energy* is an Intensive Property of a system. It is the total energy of a system divided by the total mass of the system.

$$\begin{aligned} e &= \frac{E}{m} \\ &= \frac{U}{m} + \frac{1}{2}v^2 + gh + \frac{PV}{m} \\ &= \frac{U}{m} + \frac{1}{2}v^2 + gh + \frac{P}{\rho} \end{aligned} \tag{3.3}$$

The Specific Energy of a system can be used to find the change in energy per unit time, or the Power.

$$\begin{aligned} \Delta\dot{E} &= \dot{m}\Delta e \\ P &= \dot{m}\Delta e \end{aligned} \tag{3.4}$$

where $\dot{m}$ is the mass flowrate, seen by Equation (3.5)

$$\dot{m} = \frac{m}{t} \tag{3.5}$$

3.4 Energy Flow and Systems

These systems have Energy interactions that cross the boundary of a System. This only has 2 options:

1. Heat transfer into or out of the System.
2. Work done on the system by the surroundings.

Example 3.1: Power of Water. Lecture 4, Problem 3.12

Consider a river flowing towards a lake at an average velocity of $v_{\text{River}} = 3\,\text{m/s}$ and a volume flow rate of $\dot{V} = 500\,\text{m}^3/\text{s}$. The $500\,\text{m}^3/\text{s}$ is at a location 90 m above the lake's surface. Determine the total mechanical energy in the river water per unit mass and the power generation potential of the entire river?

The question is really asking us to find the Specific Energy and the power generation.
Assumption: Assume the flow is constant, that we are in steady flow.
Concepts and Explore:
There is no state change, likely meaning $\Delta U = 0$. The velocity of the water within the lake should be $v_{\text{Lake}} = 0$ ($\Delta\text{KE} \neq 0$). There is a potential energy change, $\Delta\text{PE} \neq 0$. There is a flow, but the only pressure involved is P_{atm}, and the height difference is so small that the change in pressure is negligible.

$$P = \dot{m}\Delta e$$

To find $\dot{m}$, we can multiply the density of water with the volume flow rate of the river, to find the mass flow rate.

$$\dot{m} = \rho_{H_2O}\dot{V}$$

Plan:

1. Solve for Δe using $\Delta\text{KE} + \Delta\text{PE}$.
2. Solve for mass flow rate, $\dot{m} = \rho_{H_2O}\dot{V}$.
3. Solve for $P = \dot{m}\Delta e$.

Solve:

$$\begin{aligned} \Delta e &= \Delta\text{KE} + \Delta\text{PE} \\ &= \frac{(3\,\text{m/s})^2 + (0\,\text{m/s})^2}{2} + 9.81\,\text{m/s}^2(90\,\text{m} - 0\,\text{m}) \\ &= 4.5\,\text{m}^2/\text{s}^2 + 882\,\text{m}^2/\text{s}^2 = 886.5\,\text{J/kg} \end{aligned}$$

$$
\begin{aligned}
\dot{m} &= \rho \dot{V} \\
&= 1000\,\mathrm{kg/m^3}(500\,\mathrm{m^3/s}) \\
&= 500\,000\,\mathrm{kg/s}
\end{aligned}
$$

$$
\begin{aligned}
P &= \dot{m}\Delta e \\
&= 886.5\,\mathrm{J/kg}(500\,000\,\mathrm{kg/s}) \\
&= 444\,000\,000\,\mathrm{W} \\
&= 444\,\mathrm{MW}
\end{aligned}
$$

Generalize: Most of the energy in this problem came from the water falling in height. Overall, the pressure change and the velocity of the water made very little impact on the total energy in the system, in comparison to the change in potential energy.

Example 3.2: Mechanical Energy of Air. Lecture 4, Problem 3.14

Wind is blowing steadily at $v = 10\,\mathrm{m/s}$. Determine the mechanical energy of the air per unit mass and the power generation potential of a wind turbine with $d = 60\,\mathrm{m}$ diameter blades at that location? Take $\rho_{\mathrm{air}} = 1.25\,\mathrm{kg/m^3}$.

Concepts and Explore:

- The air is flowing steadily.
- There is no state change, so $\Delta U = 0$.
- There is a change in the blades' velocity, $\Delta\mathrm{KE} \neq 0$.
- There is no change in the potential energy of the air, so $\Delta\mathrm{PE} = 0$.
- There is no pressure change on the different sides of the turbine, so $\Delta\mathrm{FE} = 0$.

$$
\begin{aligned}
P &= \dot{m}\Delta e \\
\dot{m} &= \rho_{\mathrm{air}} \dot{V} \\
\dot{V} &= v\left(\pi\left(\frac{d}{2}\right)^2\right)
\end{aligned}
$$

Plan:

1. Solve for Δe using KE only.
2. Simplify $\dot{m}$.
3. Solve for $P = \dot{m}\Delta e$.

Solve:

$$
\begin{aligned}
\Delta e &= \frac{1}{2}(10\,\mathrm{m/s})^2 \\
&= 50\,\mathrm{m^2/s^2} \\
&= 50\,\mathrm{J/kg}
\end{aligned}
$$

$$
\begin{aligned}
\dot{m} &= \rho_{\mathrm{air}} v\left(\pi\left(\frac{d}{2}\right)^2\right) \\
&= 1.25\,\mathrm{kg/m^3}(10\,\mathrm{m/s})\left(\frac{\pi 60^2}{4}\right) \\
&= 35\,343\,\mathrm{kg/s}
\end{aligned}
$$

$$
\begin{aligned}
P &= \dot{m}\Delta e \\
&= 35\,343\,\mathrm{kg/s}(50\,\mathrm{J/kg}) \\
&= 1\,767\,150\,\mathrm{J/s} \\
&= 1767\,\mathrm{kW}
\end{aligned}
$$

Validate: Since we had a straightforward application of the equations, it is likely that the answers make sense. In addition, we did perform some dimensional analysis to figure out the way the units should be put together, which further reinforces the likelihood of this solution being the right one.
Generalize: Like Example 3.1, the value we received only holds true when the air is flowing steadily. If steady flow were not happening, then the received value will fluctuate.
Most of the power generated through this turbine is done because of the area of the blades that the air is moving through. This happens despite the very low density of air in this problem.
When looking through the equations, the velocity the air is moving becomes cubed, meaning a small change in velocity will have drastic changes in the power generated.

3.5 Adiabatic Processes

Heat is a form of energy transfer. However, there are two types of processes:

1. Processes that have **no** heat transfer (Adiabatic), $Q = 0$.
2. Processes that **have** heat transfer (Non-Adiabatic), $Q \neq 0$.

Defn 42 (Adiabatic)**.** A process is *adiabatic* when there is **no** transfer of Heat whatsoever. This can be achieved by a system that is heavily insulated, preventing a temperature difference from causing a heat transfer. Adiabatic processes yield the expression below:

$$\Delta Heat = 0 \tag{3.6}$$

Remark 42.1. An Adiabatic process does **not** say anything about Work in the System. A process can be both adiabatic and have work done on it, meaning there is still a change in energy in the system.

Defn 43 (1st Law of Thermodynamics)**.** The *1st law of thermodynamics* states that the total energy of a system **cannot** be created or destroyed during a Process; it can only change **forms**. Thus, the total change in energy of all energy forms must sum to zero.

Symbolically, this is represented as Equation (3.7).

$$\Delta E = 0 \tag{3.7}$$

If this law were to be violated in a problem, that means there is an energy form that is not being measured.

Using Equation (3.2), we can "derive" a new equation that is useful.

$$
\begin{gathered}
\Delta E = \Delta Q + \Delta W_{\mathrm{Mech}} \\
E_{\mathrm{In}} - E_{\mathrm{Out}} = (W_{\mathrm{In}} - W_{\mathrm{Out}}) + (Q_{\mathrm{In}} - Q_{\mathrm{Out}})
\end{gathered}
$$

ΔQ is equivalent to ΔU.
ΔW_{Mech} is equivalent to $\Delta \mathrm{KE} + \Delta \mathrm{PE} + \Delta \mathrm{FE}$.

Remark 43.1 (Change Energy Flowrate)*.* By the definition of the 1st Law of Thermodynamics, the change in energy flowrate (energy per unit time) is also defined to be zero, because the amount of energy in the system is constant.

$$\Delta \dot{E} = 0 \tag{3.8}$$

3.6 Energy Efficiency

Typically, we are concerned with how efficient a Process is at using the Energy it is provided to achieve a certain goal. This is typically defined as Equation (3.9).

$$\eta = \frac{\text{Desired Output}}{\text{Required Input}} \times 100 \tag{3.9}$$

3.6.1 Combustion

$$\eta_{\text{Combustion}} = \frac{\dot{Q}_{\text{Released}}}{\frac{\dot{Q}_{\text{Available}}}{m}\dot{m}} \tag{3.10}$$

3.6.2 Pumps

$$\eta_{\text{Pump}} = \frac{\text{Energy Increase of Fluid}}{\text{Energy Input}} \tag{3.11}$$

3.6.3 Turbines

$$\eta_{\text{Turbine}} = \frac{\text{Mechanical Energy Output}}{\text{Total Energy Decrease of Fluid}} \tag{3.12}$$

A Complex Numbers

Defn A.0.1 (Complex Number)**.** A *complex number* is a hyper real number system. This means that two real numbers, $a, b \in \mathbb{R}$, are used to construct the set of complex numbers, denoted mathbbC.

A complex number is written, in Cartesian form, as shown in Equation (A.1) below.

$$z = a \pm ib \tag{A.1}$$

where

$$i = \sqrt{-1} \tag{A.2}$$

Remark (i vs. j for Imaginary Numbers). Complex numbers are generally denoted with either i or j. Electrical engineering regularly makes use of j as the imaginary value. This is because alternating current i is already taken, so j is used as the imaginary value instad.

A.1 Parts of a Complex Number

A Complex Number is made of up 2 parts:

1. Real Part
2. Imaginary Part

Defn A.1.1 (Real Part)**.** The *real part* of an imaginary number, denoted with the Re operator, is the portion of the Complex Number with no part of the imaginary value i present.

If $z = x + iy$, then

$$\mathrm{Re}\{z\} = x \tag{A.3}$$

Remark A.1.1.1 (Alternative Notation)*.* The Real Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Re\mathfrak{e}$$

Defn A.1.2 (Imaginary Part)**.** The *imaginary part* of an imaginary number, denoted with the Im operator, is the portion of the Complex Number where the imaginary value i is present.

If $z = x + iy$, then

$$\mathrm{Im}\{z\} = y \tag{A.4}$$

Remark A.1.2.1 (Alternative Notation)*.* The Imaginary Part of a number sometimes uses a slightly different symbol for denoting the operation. It is:

$$\Im\mathfrak{m}$$

A.2 Binary Operations

The question here is if we are given 2 complex numbers, how should these binary operations work such that we end up with just one resulting complex number. There are only 2 real operations that we need to worry about, and the other 3 can be defined in terms of these two:

1. Addition
2. Multiplication

For the sections below, assume:

$$z = x_1 + iy_1$$
$$w = x_2 + iy_2$$

A.2.1 Addition

The addition operation, still denoted with the + symbol is done pairwise. You should treat i like a variable in regular algebra, and not move it around.

$$z + w \coloneqq (x_1 + x_2) + i(y_1 + y_2) \tag{A.5}$$

A.2.2 Multiplication

The multiplication operation, like in traditional algebra, usually lacks a multiplication symbol. You should treat i like a variable in regular algebra, and not move it around.

$$
\begin{aligned}
zw &:= (x_1 + iy_1)(x_2 + iy_2) \\
&= (x_1x_2) + (iy_1x_2) + (ix_1y_2) + (i^2y_1y_2) \\
&= (x_1x_2) + i(y_1x_2 + x_1y_2) + (-1y_1y_2) \\
&= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)
\end{aligned}
\tag{A.6}
$$

A.3 Complex Conjugates

Defn A.3.1 (Complex Conjugate)**.** The conjugate of a complex number is called its *complex conjugate*. The complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. If we have a complex number as shown below,

$$z = a \pm bi$$

then, the conjugate is denoted and calculated as shown below.

$$\overline{z} = a \mp bi \tag{A.7}$$

The Complex Conjugate can also be denoted with an asterisk ($*$). This is generally done for complex functions, rather than single variables.

$$z^* = \overline{z} \tag{A.8}$$

A.3.1 Notable Complex Conjugate Expressions

There are 2 interesting things that we can perform with *just* the concept of a Complex Number and a Complex Conjugate:

1. $z\overline{z}$
2. $\frac{z}{\overline{z}}$

The first is interesting because of this simplification:

$$
\begin{aligned}
z\overline{z} &= (x + iy)(x - iy) \\
&= x^2 - xyi + xyi - i^2y^2 \\
&= x^2 - (-1)y^2 \\
&= x^2 + y^2
\end{aligned}
$$

Thus,

$$z\overline{z} = x^2 + y^2 \tag{A.9}$$

which is interesting because, in comparison to the input values, the output is completely real.

The other interesting Complex Conjugate is dividing a Complex Number by its conjugate.

$$\frac{z}{\overline{z}} = \frac{x + iy}{x - iy}$$

We want to have this end up in a form of $a + ib$, so we multiply the entire fraction by z, to cause the denominator to be completely real.

$$z\left(\frac{z}{\overline{z}}\right) = \frac{z^2}{z\overline{z}}$$

Using our solution from Equation (A.9):

$$
\begin{aligned}
&= \frac{(x + iy)^2}{x^2 + y^2} \\
&= \frac{x^2 + 2xyi + i^2y^2}{x^2 + y^2}
\end{aligned}
$$

By breaking up the fraction's numerator, we can more easily recognize this to be teh Cartesian form of the Complex Number.

$$
\begin{aligned}
&= \frac{\left(x^2 - y^2\right) + 2xyi}{x^2 + y^2} \\
&= \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2}
\end{aligned}
$$

This is an interesting development because, unlike the multiplication of a Complex Number by its Complex Conjugate, the division of these two values does **not** yield a purely real number.

$$\frac{z}{\overline{z}} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{2xyi}{x^2 + y^2} \tag{A.10}$$

A.3.2 Properties of Complex Conjugates

Conjugation follows some of the traditional algebraic properties that you are already familiar with, namely commutativity.

First, start by defining some expressions so that we can prove some of these properties:

$$
\begin{aligned}
z &= x + iy \\
\overline{z} &= x - iy
\end{aligned}
$$

(i) The conjugation operation is commutative.

(ii) The conjugation operation can be distributed over addition and multiplication.

$$
\begin{aligned}
\overline{z + w} &= \overline{z} + \overline{w} \\
\overline{zw} &= \overline{zw}
\end{aligned}
$$

Property **(ii)** can be proven by just performing a simplification.

*Prove Property **(ii)**.* Let z and w be complex numbers ($z, w \in$ mathbbC) where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.

We start by simplifying the left-hand side of the equation ($\overline{z + w}$).

$$
\begin{aligned}
\overline{z + w} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\
&= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\
&= (x_1 + x_2) - i(y_1 + y_2)
\end{aligned}
$$

Now, we simplify the other side ($\overline{z} + \overline{w}$).

$$
\begin{aligned}
\overline{z} + \overline{w} &= \overline{(x_1 + iy_1)} + \overline{(x_2 + iy_2)} \\
&= (x_1 - iy_1) + (x_2 - iy_2) \\
&= (x_1 + x_2) - i(y_1 + y_2)
\end{aligned}
$$

We can see that both sides are equivalent, thus the addition portion of Property **(ii)** is correct.

Remark. The proof of the multiplication portion of Property **(ii)** is left as an exercise to the reader. However, that proof is quite similar to this proof of addition.

■

A.4 Geometry of Complex Numbers

So far, we have viewed Complex Numbers only algebraically. However, we can also view them geometrically as points on a 2 dimensional Argand Plane.

Defn A.4.1 (Argand Plane)**.** An *Argane Plane* is a standard two dimensional plane whose points are all elements of the complex numbers, $z \in$ mathbbC. This is taken from Descarte's definition of a completely real plane.

The Argand plane contains 2 lines that form the axes, that indicate the real component and the imaginary component of the complex number specified.

A Complex Number can be viewed as a point in the Argand Plane, where the Real Part is the "x"-component and the Imaginary Part is the "y"-component.

By plotting this, you see that we form a right triangle, so we can find the hypotenuse of that triangle. This hypotenuse is the distance the point p is from the origin, refered to as the Modulus.

Remark. When working with Complex Numbers geometrically, we refer to the points, where they are defined like so:

$$z = x + iy = p(x, y)$$

Note that p is **not** a function of x and y. Those are the values that inform us **where** p is located on the Argand Plane.

A.4.1 Modulus of a Complex Number

Defn A.4.2 (Modulus)**.** The *modulus* of a Complex Number is the distance from the origin to the complex point p. This is based off the Pythagorean Theorem.

$$\begin{aligned} |z|^2 &= x^2 + y^2 = z\overline{z} \\ |z| &= \sqrt{x^2 + y^2} \end{aligned} \tag{A.11}$$

(i) The *Law of Moduli* states that $|zw| = |z||w|$..

We can prove Property **(i)** using an algebraic identity.

*Prove Property **(i)**.* Let z and w be complex numbers ($z, w \in$ mathbbC). We are asked to prove

$$|zw| = |z||w|$$

But, it is actually easier to prove

$$|zw|^2 = |z|^2|w|^2$$

We start by simplifying the $|zw|^2$ equation above.

$$|zw|^2 = |z|^2|w|^2$$

Using the definition of the Modulus of a Complex Number in Equation (A.11), we can expand the modulus.

$$= (zw)(\overline{zw})$$

Using Property **(ii)** for multiplication allows us to do the next step.

$$= (zw)(\overline{z}\overline{w})$$

Using Multiplicative Associativity and Multiplicative Commutativity, we can simplify this further.

$$\begin{aligned} &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \end{aligned}$$

Note how we never needed to define z or w, so this is as general a result as possible. ■

A.4.1.1 Algebraic Effects of the Modulus' Property (i)

For this section, let $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now,

$$\begin{aligned} zw &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ |zw|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\ &= \left(x_1^2 + x_2^2\right)\left(x_2^2 + y_2^2\right) \\ &= |z|^2|w|^2 \end{aligned}$$

However, the Law of Moduli (Property **(i)**) does **not** hold for a hyper complex number system one that uses 2 or more imaginaries, i.e. $z = a + iy + jz$. But, the Law of Moduli (Property **(i)**) **does** hold for hyper complex number system that uses 3 imaginaries, $a = z + iy + jz + k\ell$.

A.4.1.2 Conceptual Effects of the Modulus' Property (i)

We are interested in seeing if $|zw| = (x_1^2 + y_1^2)(x_2^2 + y_2^2)$ can be extended to more complex terms (3 terms in the complex number).

However, Langrange proved that the equation below **always** holds. Note that the z below has no relation to the z above.

$$(x_1 + y_1 + z_1) \neq X^2 + Y^2 + Z^2$$

A.5 Circles and Complex Numbers

We need to define both a center and a radius, just like with regular purely real values. Equation (A.12) defines the relation required for a circle using Complex Numbers.

$$|z - a| = r \tag{A.12}$$

Example A.1: Convert to Circle. Lecture 2, Example 1

Given the expression below, find the location of the center of the circle and the radius of the circle?

$$|5iz + 10| = 7$$

This is just a matter of simplification and moving terms around.

$$\begin{aligned}
|5iz + 10| &= 7 \\
|5i(z + \frac{10}{5i})| &= 7 \\
|5i(z + \frac{2}{i})| &= 7 \\
|5i(z + \frac{2}{i}\frac{-i}{-i})| &= 7 \\
|5i(z - 2i)| &= 7
\end{aligned}$$

Now using the Law of Moduli (Property **(i)**) $|ab| = |a||b|$, we can simplify out the extra imaginary term.

$$\begin{aligned}
|5i||z - 2i| &= 7 \\
5|z - 2i| &= 7 \\
|z - 2i| &= \frac{7}{5}
\end{aligned}$$

Thus, the circle formed by the equation $|5iz + 10| = 7$ is actually $|z - 2i| = \frac{7}{5}$, with a center at $a = 2i$ and a radius of $\frac{7}{5}$.

A.5.1 Annulus

Defn A.5.1 (Annulus)**.** An *annulus* is a region that is bounded by 2 concentric circles. This takes the form of Equation (A.13).

$$r_1 \leq |z - a| \leq r_2 \tag{A.13}$$

In Equation (A.13), each of the $\leq$ symbols could also be replaced with $<$. This leads to 3 different possibilities for the annulus:

1. If both inequality symbols are $\leq$, then it is a **Closed Annulus**.
2. If both inequality symbols are $<$, then it is an **Open Annulus**.
3. If **only one** inequality symbol $<$ and the other $\leq$, then it is not an **Open Annulus**.

The concept of an Annulus can be extended to angles and arguments of a Complex Number. A general example of this is shown below.

$$\theta_1 \leq \arg(z) \leq \theta_2$$

Angular Annuli follow all the same rules as regular annuli.

A.6 Polar Form

The polar form of a Complex Number is an alternative, but equally useful way to express a complex number. In polar form, we express the distance the complex number is from the origin and the angle it sits at from the real axis. This is seen in Equation (A.14).

$$z = r\big(\cos(\theta) + i\sin(\theta)\big) \tag{A.14}$$

Remark. Note that in the definition of polar form (Equation (A.14)), there is no allowance for the radius, r, to be negative. You must fix this by figuring out the angle change that is required for the radius to become positive.

Thus,

$$\begin{aligned}
r &= |z| \\
\theta &= \arg(z)
\end{aligned}$$

Example A.2: Find Polar Coordinates from Cartesian Coordinates. Lecture 2, Example 1

Find the complex number's $z = -\sqrt{3} + i$ polar coordinates?

We start by finding the radius of z (modulus of z).

$$\begin{aligned} r &= |z| \\ &= \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Thus, the point is 2 units away from the origin, the radius is 2 $r = 2$.
Now, we need to find the angle, the argument, of the Complex Number.

$$\begin{aligned} \cos(\theta) &= \frac{-\sqrt{3}}{2} \\ \theta &= \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

Now that we have one angle for the point, we also need to consider the possibility that there have been an unknown amount of rotations around the entire plane, meaning there have been $2\pi k$, where $k = 0, 1, \ldots$.
We now have all the information required to reconstruct this point using polar coordinates:

$$\begin{aligned} r &= 2 \\ \theta &= \frac{5\pi}{6} \\ \arg(z) &= \frac{5\pi}{6} + 2\pi k \end{aligned}$$

A.6.1 Converting Between Cartesian and Polar Forms

Using Equation (A.14) and Equation (A.1), it is easy to see the relation between r, θ, x, and y.

Definition of a Complex Number in Cartesian form.

$$z = x + iy$$

Definition of a Complex Number in polar form.

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

Thus,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{A.15}$$

A.6.2 Benefits of Polar Form

Polar form is good for multiplication of Complex Numbers because of the way sin and cos multiply together. The Cartesian form is good for addition and subtraction. Take the examples below to show what I mean.

A.6.2.1 Multiplication For multiplication, the radii are multiplied together, and the angles are added.

$$\Big(r_1\big(\cos(\theta) + i\sin(\theta)\big)\Big)\Big(r_2\big(\cos(\phi) + i\sin(\phi)\big)\Big) = r_1 r_2\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big) \tag{A.16}$$

A.6.2.2 Division For division, the radii are divided by each other, and the angles are subtracted.

$$\frac{r_1\big(\cos(\theta) + i\sin(\theta)\big)}{r_2\big(\cos(\phi) + i\sin(\phi)\big)} = \frac{r_1}{r_2}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big) \tag{A.17}$$

A.6.2.3 Exponentiation Because exponentiation is defined to be repeated multiplication, Paragraph A.6.2.1 applies. That this generalization is true was proven by de Moiver, and is called de Moiver's Law.

Defn A.6.1 (de Moiver's Law)**.** Given a complex number z, $z \in$ mathbbC and a rational number n, $n \in$ mathbbQ, the exponentiation of z^n is defined as Equation (A.18).

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \tag{A.18}$$

A.7 Roots of a Complex Number

de Moiver's Law also applies to finding **roots** of a Complex Number.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right) \tag{A.19}$$

Remark. As the entire $\arg z$ term is being divided by n, the $2\pi k$ is **ALSO** divided by n.

Roots of a Complex Number satisfy Equation (A.20). To demonstrate that equation, $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ and $w = \rho\big(\cos(\phi) + i\sin(\phi)\big)$.

$$w^n = z \tag{A.20}$$

A w that satisfies Equation (A.20) is an nth root of z.

Example A.3: Roots of a Complex Number. Lecture 2, Example 2

Find the cube roots of $z = -\sqrt{3} + i$?

From Example A.2, we know that the polar form of z is

$$z = 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right)$$

Because the question is asking for **cube** roots, that means there are 3 roots. Using Equation (A.19), we can find the general form of the roots.

$$\begin{aligned} z &= 2\left(\cos\left(\frac{5\pi}{6} + 2\pi k\right) + i\sin\left(\frac{5\pi}{6} + 2\pi k\right)\right) \\ z^{\frac{1}{3}} &= \sqrt[3]{2}\left(\cos\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{5\pi}{6} + 2\pi k\right)\right)\right) \\ &= \sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi k}{18}\right) + i\sin\left(\frac{\pi + 12\pi k}{18}\right)\right) \end{aligned}$$

Now that we have a general equation for **all** possible cube roots, we need to find all the unique ones. This is because after $k = n$ roots, the roots start to repeat themselves, because the $2\pi k$ part of the expression becomes effective, making the angle a full rotation. We simply enumerate $k \in \mathbb{Z}^+$, so $k = 0, 1, 2, \ldots$.

k = 0

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(0)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(0)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right) + i\sin\left(\frac{\pi}{18}\right)\right)$$

k = 1

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi + 12\pi(1)}{18}\right) + i\sin\left(\frac{\pi + 12\pi(1)}{18}\right)\right) = \sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)$$

k = 2

$$\sqrt[3]{2}\left(\cos\left(\frac{\pi+12\pi(2)}{18}\right)+i\sin\left(\frac{\pi+12\pi(2)}{18}\right)\right)=\sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right)+i\sin\left(\frac{25\pi}{18}\right)\right)$$

k = 3

$$\begin{aligned}\sqrt[3]{2}\left(\cos\left(\frac{\pi+12\pi(3)}{18}\right)+i\sin\left(\frac{\pi+12\pi(3)}{18}\right)\right)&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}+\frac{36\pi}{18}\right)+i\sin\left(\frac{\pi}{18}+\frac{36\pi}{18}\right)\right)\\&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}+2\pi\right)+i\sin\left(\frac{\pi}{18}+2\pi\right)\right)\\&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right)+i\sin\left(\frac{\pi}{18}\right)\right)\end{aligned}$$

Thus, the 3 cube roots of z are:

$$\begin{aligned}z_1^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{\pi}{18}\right)+i\sin\left(\frac{\pi}{18}\right)\right)\\z_2^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{13\pi}{18}\right)+i\sin\left(\frac{13\pi}{18}\right)\right)\\z_3^{\frac{1}{3}}&=\sqrt[3]{2}\left(\cos\left(\frac{25\pi}{18}\right)+i\sin\left(\frac{25\pi}{18}\right)\right)\end{aligned}$$

A.8 Arguments

There are 2 types of arguments that we can talk about for a Complex Number.

1. The Argument
2. The Principal Argument

Defn A.8.1 (Argument)**.** The *argument* of a Complex Number refers to **all** possible angles that can satisfy the angle requirement of a Complex Number.

Example A.4: Argument of Complex Number. Lecture 3, Example 1

If $z=-1-i$, then what is its **Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned}z&=-1-i\\\arg(z)=\tan(\theta)&=\frac{-i}{-1}\\&=\frac{\pi}{4}\end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$=\frac{5\pi}{4}$$

Now, we have to account for **all** possible angles that form this angle.

$$\arg(z)=\frac{5\pi}{4}+2\pi k$$

Thus, the argument of $z=-1-i$ is $\arg(z)=\frac{5\pi}{4}+2\pi k$.

Defn A.8.2 (Principal Argument)**.** The *principal argument* is the exact or reference angle of the Complex Number. By convention, the principal Argument of a complex number z is defined to be bounded like so: $-\pi<\operatorname{Arg}(z)\leq\pi$.

Example A.5: Principal Argument of Complex Number. Lecture 3, Example 1

If $z = -1 - i$, then what is its **Principal Argument**?

You can plot this value on the Argand Plane and find the angle graphically/geometrically, or you can "cheat" and use $\tan^{-1}$ (so long as you correct for the proper quadrant). I will "cheat", as I cannot plot easily.

$$\begin{aligned} z &= -1 - i \\ \arg(z) = \tan(\theta) &= \frac{-i}{-1} \\ &= \frac{\pi}{4} \end{aligned}$$

Remember to correct for the proper quadrant. We are in quadrant IV.

$$= \frac{5\pi}{4}$$

Thus, the Principal Argument of $z = -1 - i$ is $\operatorname{Arg}(z) = \frac{5\pi}{4}$.

A.9 Complex Exponentials

The definition of an exponential with a Complex Number as its exponent is defined in Equation (A.21).

$$e^z = e^{x+iy} = e^x\big(\cos(y) + i\sin(y)\big) \tag{A.21}$$

If instead of e as the base, we have some value a, then we have Equation (A.22).

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ &= e^{\operatorname{Re}\{z\ln(a)\}}\Big(\cos\big(\operatorname{Im}\{z\ln(a)\}\big) + i\sin\big(\operatorname{Im}\{z\ln(a)\}\big)\Big) \end{aligned} \tag{A.22}$$

In the case of Equation (A.21), z can be presented in either Cartesian or polar form, they are equivalent.

Example A.6: Simplify Simple Complex Exponential. Lecture 3

Simplify the expression below, then find its Modulus, Argument, and its Principal Argument?

$$e^{-1+i\sqrt{3}}$$

If we look at the exponent on the exponential, we see

$$z = -1 + i\sqrt{3}$$

which means

$$\begin{aligned} x &= -1 \\ y &= \sqrt{3} \end{aligned}$$

With this information, we can simplify the expression **just** by observation, with no calculations required.

$$e^{-1+i\sqrt{3}} = e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)$$

Now, we can solve the other 3 parts of this example **by observation**.

$$\begin{aligned} |e^{-1+i\sqrt{3}}| &= |e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)| \\ &= e^{-1} \\ \arg\left(e^{-1+i\sqrt{3}}\right) &= \arg\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} + 2\pi k \\ \operatorname{Arg}\left(e^{-1+i\sqrt{3}}\right) &= \operatorname{Arg}\left(e^{-1}\big(\cos(\sqrt{3}) + i\sin(\sqrt{3})\big)\right) \\ &= \sqrt{3} \end{aligned}$$

Example A.7: Simplify Complex Exponential Exponent. Lecture 3

Given $z = e^{-e^{-i}}$, what is this expression in polar form, what is its Modulus, its Argument, and its Principal Argument?

We start by simplifying the exponent of the base exponential, i.e. e^{-i}.

$$\begin{aligned} e^{-i} &= e^{0-i} \\ &= e^0\big(\cos(-1) + i\sin(-1)\big) \\ &= 1\big(\cos(-1) + i\sin(-1)\big) \end{aligned}$$

Now, with that exponent simplified, we can solve the main question.

$$\begin{aligned} e^{-e^{-i}} &= e^{-1\big(\cos(-1)+i\sin(-1)\big)} \\ &= e^{-1\big(\cos(1)-i\sin(1)\big)} \\ &= e^{-\cos(1)+i\sin(1)} \end{aligned}$$

If we refer back to Equation (A.21), then it becomes obvious what x and y are.

$$\begin{aligned} x &= -\cos(1) \\ y &= \sin(1) \\ e^{-e^{-i}} &= e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big) \end{aligned}$$

Now that we have "simplified" this exponential, we can solve the other 3 questions by **observation**.

$$\begin{aligned} |e^{-e^{-i}}| &= |e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)| \\ &= e^{-\cos(1)} \\ \arg\left(e^{-e^{-i}}\right) &= \arg\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) + 2\pi k \\ \operatorname{Arg}\left(e^{-e^{-i}}\right) &= \operatorname{Arg}\left(e^{-\cos(1)}\Big(\cos\big(\sin(1)\big) + i\sin\big(\sin(1)\big)\Big)\right) \\ &= \sin(1) \end{aligned}$$

Example A.8: Non-e Complex Exponential. Lecture 3

Find all values of $z = 1^i$?

Use Equation (A.22) to simplify this to a base of e, where we can use the usual Equation (A.21) to solve this.

$$\begin{aligned} a^z &= e^{z\ln(a)} \\ 1^i &= e^{i\ln(1)} \end{aligned}$$

Simplify the logarithm in the exponent first, $\ln(1)$.

$$\begin{aligned} \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e(1) + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now, plug $\ln(1)$ back into the exponent, and solve the exponential.

$$\begin{aligned} e^{i(2\pi k i)} &= e^{2\pi k i^2} \\ &= e^{2\pi k(-1)} \\ z &= e^{-2\pi k} \end{aligned}$$

Thus, all values of $z = e^{-2\pi k}$ where $k = 0, 1, \ldots$.

A.9.1 Complex Conjugates of Exponentials

$$\overline{e^z} = e^{\overline{z}} \tag{A.23}$$

A.10 Complex Logarithms

There are some denotational changes that need to be made for this to work. The traditional real-number natural logarithm ln needs to be redefined to its defining form $\log_e$.

With that denotational change, we can now use ln for the Complex Logarithm.

Defn A.10.1 (Complex Logarithm)**.** The *complex logarithm* is defined in Equation (A.24). The only requirement for this equation to hold true is that $w \neq 0$.

$$\begin{aligned} e^z &= w \\ z &= \ln(w) \\ &= \log_e|w| + i\arg(w) \end{aligned} \tag{A.24}$$

Remark A.10.1.1. The Complex Logarithm is different than it's purely-real cousin because we allow negative numbers to be input. This means it is more general, but we must lose the precision of the purely-real logarithm. This means that each nonzero number has infinitely many logarithms.

Example A.9: All Complex Logarithms of Simple Expression. Lecture 3

What are **all** Complex Logarithms of $z = -1$?

We can apply the definition of a Complex Logarithm (Equation (A.24)) directly.

$$\begin{aligned} \ln(z) &= \log_e|z| + i\arg(z) \\ &= \log_e|-1| + i\arg(-1) \\ &= \log_e(1) + i(\pi + 2\pi k) \\ &= 0 + i(\pi + 2\pi k) \\ &= i(\pi + 2\pi k) \end{aligned}$$

Thus, all logarithms of $z = -1$ are defined by the expression $i(\pi + 2\pi k)$, $k = 0, 1, \ldots$.

Remark. You can see the loss of specificity in the Complex Logarithm because the variable k is still present in the final answer.

Example A.10: All Complex Logarithms of Complex Logarithm. Lecture 3

What are **all** the Complex Logarithms of $z = \ln(1)$?

We start by simplifying z, before finding $\ln(z)$. We can make use of Equation (A.24), to simplify this value.

$$\begin{aligned} \ln(w) &= \log_e|w| + i\arg(w) \\ \ln(1) &= \log_e|1| + i\arg(1) \\ &= \log_e 1 + i(0 + 2\pi k) \\ &= 0 + 2\pi k i \\ &= 2\pi k i \end{aligned}$$

Now that we have simplified z, we can solve for $\ln(z)$.

$$\begin{aligned} \ln(z) &= \ln(2\pi k i) \\ &= \log_e|2\pi k i| + i\arg(2\pi k i) \\ &= \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases}\right) \end{aligned}$$

The $|k|$ is the **absolute value** of k, because k is an integer.

Thus, our solution of $\ln\big(\ln(1)\big) = \log_e(2\pi|k|) + \left(i \begin{cases} \frac{\pi}{2} + 2\pi m & k > 0 \\ \frac{-\pi}{2} + 2\pi m & k < 0 \end{cases}\right)$.

A.10.1 Complex Conjugates of Logarithms

$$\overline{\log(z)} = \log(\overline{z}) \tag{A.25}$$

A.11 Complex Trigonometry

For the equations below, $z \in$ mathbbC. These equations are based on Euler's relationship, Appendix B.2

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{A.26}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{A.27}$$

Example A.11: Simplify Complex Sinusoid. Lecture

Solve for z in the equation $\cos(z) = 5$?

We start by using the defintion of complex cosine Equation (A.26).

$$\begin{aligned} \cos(z) &= 5 \\ \frac{e^{iz} + e^{-iz}}{2} &= 5 \\ e^{iz} + e^{-iz} &= 10 \\ e^{iz}\left(e^{iz} + e^{-iz}\right) &= e^{iz}(10) \\ e^{iz^2} + 1 &= 10e^{iz} \\ e^{iz^2} - 10e^{iz} + 1 = 0 \end{aligned}$$

Solve this quadratic equation by using the Quadratic Equation.

$$\begin{aligned} e^{iz} &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 4}}{2} \\ &= \frac{10 \pm \sqrt{96}}{2} \\ &= \frac{10 \pm 4\sqrt{6}}{2} \\ &= 5 \pm 2\sqrt{6} \end{aligned}$$

Use the definition of complex logarithms to simplify the exponential.

$$\begin{aligned} iz &= \ln(5 \pm 2\sqrt{6}) \\ &= \log_e|5 \pm 2\sqrt{6}| + i\arg(5 \pm 2\sqrt{6}) \\ &= \log_e|5 \pm 2\sqrt{6}| + i(0 + 2\pi k) \\ &= \log_e|5 \pm 2\sqrt{6}| + 2\pi k i \\ z &= \frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}| + 2\pi k i\right) \\ &= \frac{-i}{-i}\frac{1}{i}\left(\log_e|5 \pm 2\sqrt{6}|\right) + 2\pi k \\ &= 2\pi k - i\log_e|5 \pm 2\sqrt{6}| \end{aligned}$$

Thus, $z = 2\pi k - i\log_e|5 \pm 2\sqrt{6}|$.

A.11.1 Complex Angle Sum and Difference Identities

Because the definitions of sine and cosine are unsatisfactory in their Euler definitions, we can use angle sum and difference formulas and their Euler definitions to yield a set of Cartesian equations.

$$\cos(x+iy) = \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big) \tag{A.28}$$

$$\sin(x+iy) = \big(\sin(x)\cosh(y)\big) + i\big(\cos(x)\sinh(y)\big) \tag{A.29}$$

Example A.12: Simplify Trigonometric Exponential. Lecture 3

Simpify $z = e^{\cos(2+3i)}$, and find z's Modulus, Argument, and Principal Argument?

We start by simplifying the cos using Equation (A.28).

$$\begin{aligned}
\cos(x+iy) &= \big(\cos(x)\cosh(y)\big) - i\big(\sin(x)\sinh(y)\big) \\
\cos(2+3i) &= \big(\cos(2)\cosh(3)\big) - i\big(\sin(2)\sinh(3)\big)
\end{aligned}$$

Now that we have put the cos into a Cartesian form, one that is usable with Equation (A.21), we can solve this.

$$\begin{aligned}
e^{z} = e^{x+iy} &= e^{x}\big(\cos(y) + i\sin(y)\big) \\
x &= \cos(2)\cosh(3) \\
y &= -\sin(2)\sinh(3) \\
e^{\cos(2)\cosh(3) - i\sin(2)\sinh(3)} &= e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)
\end{aligned}$$

Now that we have simplified z, we can solve for the modulus, argument, and principal argument **by observation**.

$$\begin{aligned}
|z| &= |e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)| \\
&= e^{\cos(2)\cosh(3)} \\
\arg(z) &= \arg(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)) \\
&= -\sin(2)\sinh(3) + 2\pi k \\
\operatorname{Arg}(z) &= \operatorname{Arg}(e^{\cos(2)\cosh(3)}\Big(\cos\big(-\sin(2)\sinh(3)\big) + i\sin\big(-\sin(2)\sinh(3)\big)\Big)) \\
&= -\sin(2)\sinh(3)
\end{aligned}$$

A.11.2 Complex Conjugates of Sinusoids

Since sinusoids can be represented by complex exponentials, as shown in Appendix B.2, we could calculate their complex conjugate.

$$\begin{aligned}
\overline{\cos(x)} &= \cos(x) \\
&= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)
\end{aligned} \tag{A.30}$$

$$\begin{aligned}
\overline{\sin(x)} &= \sin(x) \\
&= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)
\end{aligned} \tag{A.31}$$

B Trigonometry

B.1 Trigonometric Formulas

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{B.2}$$

B.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{B.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{B.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{B.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{B.7}$$

B.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{B.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{B.9}$$

B.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{B.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{B.11}$$

B.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\alpha)}{2}} \tag{B.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 + \cos(\alpha)}{2}} \tag{B.13}$$

B.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \big(\sin(\alpha)\big) = \frac{1 - \cos(2\alpha)}{2} \tag{B.14}$$

$$\cos^2(\alpha) = \big(\cos(\alpha)\big) = \frac{1 + \cos(2\alpha)}{2} \tag{B.15}$$

B.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta) \tag{B.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \tag{B.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta) \tag{B.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta) \tag{B.19}$$

B.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{B.20}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{B.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{B.22}$$

B.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{B.23}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1^2 \tag{B.24}$$

B.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{B.25}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{B.26}$$

B.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{B.27}$$

C Calculus

C.1 L'Hopital's Rule

L'Hopital's Rule can be used to simplify and solve expressions regarding limits that yield irreconcialable results.

Lemma C.0.1 (L'Hopital's Rule)**.** *If the equation*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases}$$

then Equation (C.1) *holds.*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{C.1}$$

C.2 Fundamental Theorems of Calculus

Defn C.2.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{C.2}$$

Defn C.2.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{C.3}$$

Defn C.2.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\underset{x}{\operatorname{argmax}}$$

C.3 Rules of Calculus

C.3.1 Chain Rule

Defn C.3.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{C.4}$$

C.4 Useful Integrals

$$\int \cos(x)\ dx = \sin(x) \tag{C.5}$$

$$\int \sin(x)\ dx = -\cos(x) \tag{C.6}$$

$$\int x\cos(x)\ dx = \cos(x) + x\sin(x) \tag{C.7}$$

Equation (C.7) simplified with Integration by Parts.

$$\int x\sin(x)\,dx = \sin(x) - x\cos(x) \tag{C.8}$$

Equation (C.8) simplified with Integration by Parts.

$$\int x^2\cos(x)\,dx = 2x\cos(x) + (x^2-2)\sin(x) \tag{C.9}$$

Equation (C.9) simplified by using Integration by Parts twice.

$$\int x^2\sin(x)\,dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{C.10}$$

Equation (C.10) simplified by using Integration by Parts twice.

$$\int e^{\alpha x}\cos(\beta x)\,dx = \frac{e^{\alpha x}\big(\alpha\cos(\beta x) + \beta\sin(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.11}$$

$$\int e^{\alpha x}\sin(\beta x)\,dx = \frac{e^{\alpha x}\big(\alpha\sin(\beta x) - \beta\cos(\beta x)\big)}{\alpha^2+\beta^2} + C \tag{C.12}$$

$$\int e^{\alpha x}\,dx = \frac{e^{\alpha x}}{\alpha} \tag{C.13}$$

$$\int xe^{\alpha x}\,dx = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) \tag{C.14}$$

Equation (C.14) simplified with Integration by Parts.

$$\int \frac{dx}{\alpha+\beta x} = \int \frac{1}{\alpha+\beta x}\,dx = \frac{1}{\beta}\ln(\alpha+\beta x) \tag{C.15}$$

$$\int \frac{dx}{\alpha^2+\beta^2x^2} = \int \frac{1}{\alpha^2+\beta^2x^2}\,dx = \frac{1}{\alpha\beta}\arctan\left(\frac{\beta x}{\alpha}\right) \tag{C.16}$$

$$\int \alpha^x\,dx = \frac{\alpha^x}{\ln(\alpha)} \tag{C.17}$$

$$\frac{d}{dx}\alpha^x = \frac{d\alpha^x}{dx} = \alpha^x\ln(x) \tag{C.18}$$

C.5 Leibnitz's Rule

Lemma C.0.2 (Leibnitz's Rule)**.** *Given*

$$g(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

with $a(t)$ and $b(t)$ differentiable in t and $\frac{\partial f(x,t)}{\partial t}$ continuous in both t and x, then

$$\frac{d}{dt}g(t) = \frac{dg(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f\big[b(t),t\big]\,\frac{db(t)}{dt} - f\big[a(t),t\big]\,\frac{da(t)}{dt} \tag{C.19}$$

D Laplace Transform

D.1 Laplace Transform

Defn D.1.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int\limits_{-\infty}^{\infty} x(t)e^{-st}dt \tag{D.1}$$

D.2 Inverse Laplace Transform

Defn D.2.1 (Inverse Laplace Transform)**.** The *inverse Laplace transformation* operation is denoted as $\mathcal{L}^{-1}\{X(s)\}$ and is defined as

$$x(t) = \frac{1}{2j\pi} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}\,ds \tag{D.2}$$

D.3 Properties of the Laplace Transform

D.3.1 Linearity

The Laplace Transform is a linear operation, meaning it obeys the laws of linearity. This means Equation (D.3) must hold.

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{D.3a}$$

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) \tag{D.3b}$$

D.3.2 Time Scaling

Scaling in the time domain (expanding or contracting) yields a slightly different transform. However, this only makes sense for $\alpha > 0$ in this case. This is seen in Equation (D.4).

$$\mathcal{L}\big\{x(\alpha t)\big\} = \frac{1}{\alpha} X\left(\frac{s}{\alpha}\right) \tag{D.4}$$

D.3.3 Time Shift

Shifting in the time domain means to change the point at which we consider $t = 0$. Equation (D.5) below holds for shifting both forward in time and backward.

$$\mathcal{L}\big\{x(t-a)\big\} = X(s)e^{-as} \tag{D.5}$$

D.3.4 Frequency Shift

Shifting in the frequency domain means to change the complex exponential in the time domain.

$$\mathcal{L}^{-1}\big\{X(s-a)\big\} = x(t)e^{at} \tag{D.6}$$

D.3.5 Integration in Time

Integrating in time is equivalent to scaling in the frequency domain.

$$\mathcal{L}\left\{\int_0^t x(\lambda)\,d\lambda\right\} = \frac{1}{s}X(s) \tag{D.7}$$

D.3.6 Frequency Multiplication

Multiplication of two signals in the frequency domain is equivalent to a convolution of the signals in the time domain.

$$\mathcal{L}\big\{x(t) * v(t)\big\} = X(s)V(s) \tag{D.8}$$

D.3.7 Relation to Fourier Transform

The Fourier transform looks and behaves very similarly to the Laplace transform. In fact, if $X(\omega)$ exists, then Equation (D.9) holds.

$$X(s) = X(\omega)|_{\omega=\frac{s}{j}} \tag{D.9}$$

D.4 Theorems

There are 2 theorems that are most useful here:

1. Intial Value Theorem
2. Final Value Theorem

Theorem D.1 (Intial Value Theorem)**.** *The Initial Value Theorem states that when the signal is treated at its starting time, i.e. $t = 0^+$, it is the same as taking the limit of the signal in the frequency domain.*

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Theorem D.2 (Final Value Theorem)**.** *The Final Value Theorem states that when taking a signal in time to infinity, it is equivalent to taking the signal in frequency to zero.*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

D.5 Laplace Transform Pairs

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$\delta(t)$	1
$\delta(t-T_0)$	e^{-sT_0}
$\mathcal{U}(t)$	$\frac{1}{s}$
$t^n\,\mathcal{U}(t)$	$\frac{n!}{s^{n+1}}$
$\mathcal{U}(t-T_0)$	$\frac{e^{-sT_0}}{s}$
$e^{at}\,\mathcal{U}(t)$	$\frac{1}{s-a}$
$t^n e^{at}\,\mathcal{U}(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)\,\mathcal{U}(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)\,\mathcal{U}(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)\,\mathcal{U}(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)\,\mathcal{U}(t)$	$\begin{cases} a: & \frac{sr\cos(\theta)+ar\cos(\theta)-br\sin(\theta)}{s^2+2as+(a^2+b^2)} \\ b: & \frac{1}{2}\left(\frac{re^{j\theta}}{s+a-jb}+\frac{re^{-j\theta}}{s+a+jb}\right) \\ c: & \frac{As+B}{s^2+2as+c}\ \begin{cases} r & = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} \\ \theta & = \arctan\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) \end{cases} \end{cases}$
$e^{-at}\left(A\cos(\sqrt{c-a^2}t)+\frac{B-Aa}{\sqrt{c-a^2}}\sin(\sqrt{c-a^2}t)\right)\mathcal{U}(t)$	$\frac{As+B}{s^2+2as+c}$

D.6 Higher-Order Transforms

Time Domain	Frequency Domain
$x(t)$	$X(s)$
$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left(X(s+j\omega_0)-X(s-j\omega_0)\right)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left(X(s+j\omega_0)+X(s-j\omega_0)\right)$
$t^n x(t)$	$(-1)^n\frac{d^n}{ds^n}X(s)\ n\in\mathbb{N}$
$\frac{d^n}{dt^n}x(t)$	$s^nX(s)-\sum\limits_{0}^{n-1}s^{n-1-i}\frac{d^i}{dt^i}x(t)\|_{t=0^-}\ n\in\mathbb{N}$