EITF75: Systems and Signals - Reference Sheet

Karl Hallsby

Last Edited: October 4, 2019

Contents

1 Sinusoids

There are several ways to characterize Sinusoids. The first is by dimension:

1. Multidimensional/Multichannel Signals
2. Monodimensional/Monochannel Signals

You can also classify sinusoids by their independent variable (usually time) and the values they take.

1. Continuous-Time Signals or Analog Signals
2. Discrete-Time Signals
3. There is a third way to classify sinusoids and their signals: Digital Signals

Defn 1 (Continuous-Time Signals)**.** *Continuous-time signals* or *Analog signals* are defined for every value of time and they take on values in the continuous interval (a, b), where a can be $-\infty$ and b can be ∞. Mathematically, these signals can be described by functions of a continuous variable.

For example,

$$x_1(t) = \cos \pi t,\ x_2(t) = e^{-|t|},\ -\infty < t < \infty$$

Defn 2 (Discrete-Time Signals)**.** *Discrete-time signals* are defined only at certain specifed values of time. These time instants ***need not*** be equidistant, but in practice, they are usually taken at equally speced intervals for computation convenience and mathematical tractability.

For example,

$$x(t_n) = e^{-|t_n|},\ n = 0, \pm 1, \pm 2, \ldots$$

A Discrete-Time Signals can be represented mathematically by a sequence of real or complex numbers.

Remark 2.1. To emphasize the discrete-time nature of the signal, we shall denote the signal as $x(n)$, rather than $x(t)$.

Remark 2.2. If the time instants t_n are equally spaced (i.e., $t_n = nT$), the notation $x(nT)$ is also used.

1.1 Continuous-Time Signals

1.1.1 Frequency in Continuous-Time Signals

A simple harmonic oscillation is mathematically described by Equation (1.1).

$$x_a(t) = A \cos\left(\Omega t + \theta\right),\ -\infty < t < \infty \tag{1.1}$$

Remark. The subscript a is used with $x(t)$ to denote an analog signal.

This signal is completely characterized by three parameters:

1. A, the *amplitude* of the sinusoid
2. Ω, the *frequency* in radians per second (rad/s)
3. θ, the *phase* in radians.

Instead of Ω, the frequency F in cycles per second or hertz (Hz) is used.

$$\Omega = 2\pi F \tag{1.2}$$

Plugging (1.2) into (1.1), yields

$$x_a(t) = A \cos\left(2\pi F t + \theta\right),\ -\infty < t < \infty \tag{1.3}$$

1.1.2 Properties of Continuous-Time Sinusoidal Signals

The analog sinusoidal signal in equation (1.3) is characterized by the following properties:

(i) For every fixed value of the frequency F, $x_a(t)$ is periodic.

$$x_a(t + T_p) = x_a(t)$$

where $T_p = \frac{1}{F}$ is the fundamental period.

(ii) Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct.

(iii) Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in the given time interval.

1.2 Discrete-Time Signals

1.2.1 Frequency in Discrete-Time Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta), \; n \in \mathbb{Z}, \; -\infty < n < \infty \tag{1.4}$$

The signal is characterized by these parameters:

1. n, the sample number. MUST be an integer.
2. A, the *amplitude* of the sinusoid
3. ω, the *angular frequency* in radians per sample
4. θ, is the *phase*, in radians.

Instead of ω, we use the frequency variable f defined by

$$\omega \equiv 2\pi f \tag{1.5}$$

Using (1.4) and (1.5) yields

$$x(n) = A\cos(2\pi f n + \theta), \; n \in \mathbb{Z}, \; -\infty < n < \infty \tag{1.6}$$

1.2.2 Properties of Discrete-Time Sinusoidal Signals

(i) A discrete-time sinusoid is periodic ***ONLY*** if its frequency is a rational number.
(ii) Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical. This leads us to the idea of a Frequency Alias.
(iii) The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pm\pi$ or, equivalently, $f = \pm\frac{1}{2}$.

1.2.3 Frequency Aliases

The concept of a Frequency Alias is drawn from the idea that discrete-time sinusoids whose frequencies are separated by an integer mutliple of 2π are identical and that frequencies $|f| > \frac{1}{2}$ are identical. (Properties **(ii)** and **(iii)**)

Defn 3 (Frequency Alias)**.** A *frequency alias* is a sinusoid having a frequency $|\omega| > \pi$ or $|f| > \frac{1}{2}$. This is because this sinusoid is *indistinguishable* (*identical*) to one with frequency $|\omega| < \pi$ or $|f| < \frac{1}{2}$.

A *frequency alias* is a sequence resulting from the following assertion based on the sinusoid $\cos(\omega_0 n + \theta)$.
It follows that

$$\cos[(\omega_0 + 2\pi)\,n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta$$

As a result, all sinusoidal sequences

$$x_k(n) = A\cos(\omega_k n + \theta), \; k = 0, 1, 2, \ldots$$

where

$$\omega k = \omega_0 + 2k\pi, \; -\pi \le \omega_0 \le \pi$$

are *indistinguishable* (i.e., *identical*).

Because of this, we regard frequencies in the range of $-\pi \le \omega \le \pi$ or $-\frac{1}{2} \le f \le \frac{1}{2}$ as unique, and all frequencies that fall outside of these ranges as aliases.

Remark 3.1. It should be noted that there is a difference between discrete-time sinusoids and continuous-time sinusoids here. Continuous-time sinusoids have distinct signals for Ω or F in the entire range $-\infty < \Omega < \infty$ or $-\infty < F < \infty$.

1.3 Sampling Rates and Sampling Frequency

Most signals of interest are analog. To process these signals, they must be collected and converted to a digital form, that is, to convert them to a sequence of numbers having finite precision. This is called *analog-to-digital (A/D) conversion.* Conceptually, we view this conversion as a 3-step process.

1. Sampling
2. Quantization
3. Coding

1.3.1 Nyquist Rate

1.3.2 Nyquist Frequency

1.4 Digital Signals

Defn 4 (Digital Signals)**.** *Digital signals* are a subset of Discrete-Time Signals. In this case, not only are the values being measured occuring at fixed points in time, the values themselves can only take certain, fixed values.

1.4.1 Quantization

Defn 5 (Quantization)**.** This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-value (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the Quantization Error.

1.4.1.1 Quantization Levels

1.4.1.2 Quantization Error

Defn 6 (Quantization Error)**.** The *quantization error* of something.

1.4.1.3 Bit Requirements

1.4.1.4 Bit Rate

2 Discrete-Time Systems

As discussed in Section 1.2, $x(n)$ is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is *not defined* at instants between the samples. Also, if n is not an integer, $x(n)$ is not defined.

Besides graphical representation of a discrete-time system, there are 3 ways to represent a discrete-time signal.

1. Functional Representation
2. Tabular Representation
3. Sequence Representation

2.1 Representing Discrete-Time Systems

2.1.1 Functional Representation

This representation of a discrete-time system is done as a mathematical function.

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1}$$

2.1.2 Tabular Representation

This representation of a discrete-time sysem is done as a table of corresponding values.

n	...	-2	-1	0	1	2	3	4	5	...
$x(n)$	...	0	0	0	1	4	1	0	0	...

2.1.3 Sequence Representation

There are 2 methods of representation for this. The first includes all values for $-\infty < n < \infty$. In all cases, $n = 0$ is marked in the sequence, somehow. I will do this with an underline.

$$x(n) = \{\ldots, 0, \underline{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.2}$$

The second only works if all $x(n)$ values for $n < 0$ are 0.

$$x(n) = \{\underline{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.3}$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underline{-2}, 5, 0, 4, -1\} \tag{2.4}$$

This is identified as a seven-point sequence.

A finite-duration sequence where $x(n) = 0$ for all $n < 0$ is represented as

$$x(n) = \{\underline{0}, 1, 4, 1\} \tag{2.5}$$

This is identified as a four-point sequence.

2.2 Elementary Discrete-Time Signals

The following signals are basic signals that appear often and play an important role in signal processing.

2.2.1 Unit Impulse Signal

Defn 7 (Unit Impulse Signal)**.** The *unit impulse signal* or *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.6}$$

This function is a signal that is zero everywhere, except at $n = 0$, where its value is 1.

Remark 7.1. This signal is different that the analog signal $\delta(t)$, which is also called a unit impulse, and is defined to be 0 everywhere except $t = 0$. The discrete unit impulse sequence is much less mathematically complicated.

2.2.2 Unit Step Signal

Defn 8 (Unit Step Signal)**.** The *unit step signal* is denoted as $u(n)$ or as $\mathcal{U}(n)$ and is defined as

$$\mathcal{U}(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.7}$$

2.2.3 Unit Ramp Signal

Defn 9 (Unit Ramp Signal)**.** The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.8}$$

2.2.4 Exponential Signal

Defn 10 (Exponential Signal)**.** The *exponential signal* is a sequence of the form

$$x(n) = a^n \text{ for all } n \tag{2.9}$$

If a is real, then $x(n)$ is a real signal. When a is complex valued $(a \equiv b \pm cj)$, it can be expressed as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n \left(\cos\theta n + j \sin\theta n\right) \end{aligned} \tag{2.10}$$

This can be expressed by graphing the real and imaginary parts

$$\begin{aligned} x_R(n) &\equiv r^n \cos\theta n \\ x_I(n) &\equiv r^n j \sin\theta n \end{aligned} \tag{2.11}$$

or by graphing the amplitude function and phase function.

$$\begin{aligned} |x(n)| &= A(n) \equiv r^n \\ \angle x(n) &= \phi(n) \equiv \theta n \end{aligned} \tag{2.12}$$

2.3 Classification of Discrete-Time Signals

In order to apply some mathematical methods to discrete-time signals, we must characterize these signals.

2.3.1 Energy Signal

Defn 11 (Energy Signal)**.** The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.13}$$

The energy of a signal can be finite or infinite. If E is finite ($0 < E < \infty$), then $x(n)$ is called an *energy signal.*

2.3.2 Power Signal

Defn 12 (Power Signal)**.** The average power of a discrete time signal $x(n)$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \tag{2.14}$$

This means that there are 2 potential outcomes:

1. If E is finite, $P = 0$
2. If E is infinite, P may be either finite or infinite

If P is finite and nonzero, the signal is called a *power signal.*

2.3.3 Periodic and Aperiodic Signals

A signal $x(n)$ is periodic with period N ($N > 0$) if and only if

$$x(n+N) = x(n) \text{for all } n \tag{2.15}$$

The smallest value of N for which (2.15) holds is called the fundamental period. If there is no value of N that satisfies (2.15), the signal is called *nonperiodic* or *aperiodic.*

2.3.4 Symmetric and Antisymmetric Signals

A real-valued signal $x(n)$ is called *symmetric* or *even* if

$$x(n) = x(-n) \tag{2.16}$$

On the other hand, a signal $x(n)$ is called *antisymmetric* or *odd* if

$$x(n) = -x(-n) \tag{2.17}$$

2.4 Discrete-Time Signal Manipulations

2.4.1 Transformation of the Independent Variable (Time)

It is important to note that Shifting in Time and Folding are not commutative. For example,

$$\mathrm{TD_k}\{\mathrm{FD}\left[x(n)\right]\} = \mathrm{TD_k}\left[x(-n)\right] = x(-n+k) \tag{2.18}$$

whereas

$$\mathrm{FD}\{\mathrm{TD_k}\left[x(n)\right]\} = \mathrm{FD}\left[x(n-k)\right] = x(-n-k) \tag{2.19}$$

2.4.1.1 Shifting in Time A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time (moves left). If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units of time (moves right).

This could be denoted by

$$\mathrm{TD_k}\left[x(n)\right] = x(n-k) \tag{2.20}$$

You cannot advance a signal that is being generated in real-time. Because that would involve signal samples that haven't been generated yet. So, you can only advance a signal that is stored on something. However, you can always introduce a delay to a signal.

2.4.1.2 Folding Another useful modification of the time base is to replace n with $-n$. The result is a *folding* or *reflection* of the original signal around $n = 0$.

This could be denoted by

$$\mathrm{FD}\left[x(n)\right] = x(-n) \tag{2.21}$$

2.4.2 Addition, Multiplication, and Scaling

Amplitude modifications include Addition, Multiplication, and Amplitude Scaling.

2.4.2.1 Addition The *sum* of 2 signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the sum of the values of these two signals at that instant.

$$y(n) = x_1(n) + x_2(n),\ -\infty < n < \infty \tag{2.22}$$

2.4.2.2 Multiplication The *product* of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$ whose value at any instant is equal to the product of the values of these two signals at that instant.

$$y(n) = x_1(n)x_2(n),\ -\infty < n < \infty \tag{2.23}$$

2.4.2.3 Amplitude Scaling *Amplitude scaling* of a signal by a constant A is accomplished by multiplying every signal smaple by A. Consequently, we obtain

$$y(n) = Ax(n),\ -\infty < n < \infty \tag{2.24}$$

3 Convolutions

Defn 13 (Convolution)**.** The *convolution* operator.

$$y(t) = \sum_{k=-\infty}^{\infty} x(k) * h(n-k) \tag{3.1}$$

4 The $\mathcal{Z}$-Transform

The $\mathcal{Z}$-Transform plays the same role in the analysis of Discrete-Time Signals and LTI systems as the Laplace Transform does in the analysis of Continuous-Time Signals and LTI systems.

4.1 The $\mathcal{Z}$-Transform

Defn 14 ($\mathcal{Z}$-Transform)**.** The *z-transform* is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Remark 14.1. For convenience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv \mathcal{Z}\{x(n)\} \tag{4.2}$$

and the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \overset{\mathrm{z}}{\longleftrightarrow} X(z) \tag{4.3}$$

4.1.1 Region of Convergence

Defn 15 (ROC)**.** The *ROC* or *region of convergence* is the region for which the infinite power series in the z-transform has a convergent solution.

Remark 15.1. ***Any time we cite a z-transform, we should also indicate its ROC***

Example 4.1: Simple $\mathcal{Z}$-Transform.

Determine the z-transform of the signal
$$x(n) = \left(\frac{1}{2}\right)^n \mathcal{U}(n)$$

The z-transform is the infinite power series

$$\begin{aligned} X(z) &= 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^{-2} + \cdots + \left(\frac{1}{2}\right)^n z^{-n} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n \end{aligned}$$

Because this is an infinite geometric series, we can solve with with our equivalency:

$$1 + A + A^2 + \cdots + A^n + \cdots = \frac{1}{1-A} \text{ if } |A| < 1$$

Thus, $X(z)$ converges to
$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC} : |z| > \frac{1}{2}$$

Signal	ROC
	Finite-Duration Signals
Causal	Entire z-plane except $z = 0$
Anticausal	Entire z-plane except $z = \infty$
Two-Sided	Entire z-plane except $z = 0$ and $z = \infty$
	Infinite-Duration Signals
Causal	$\lvert z\rvert > r_2$
Anticausal	$\lvert z\rvert < r_1$
Two-Sided	$r_2 < \lvert z\rvert < r_1$

Table 4.1: Characteristic Familes of Signals with Their Corresponding ROCs

4.1.2 The One-Sided $\mathcal{Z}$-Transform

Defn 16 (One-Sided $\mathcal{Z}$-Transform)**.** The *one-sided z-transform* is the same as the $\mathcal{Z}$-Transform, but is only defined at n values greater than or equal to 0.

$$X(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n} \tag{4.4}$$

The One-Sided $\mathcal{Z}$-Transform is generally used when there are initial conditions on a causal signal. This captures the normal causal portion of the signal, while also showing the effect of the initial condition.

4.2 The Inverse $\mathcal{Z}$-Transform

This is the formal definition of The Inverse $\mathcal{Z}$-Transform.

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz \tag{4.5}$$

where the integrals is a contour integral over a closed path C that encloses the origin and lies within the region of convergence of $X(z)$.

There are 3 methods that are often used for the evaluation of the inverse z-transform in practice:

1. Direct evaluation of (4.5).
2. Expansion into a series of terms, in the variable sz and z^{-1}.
3. Partial-fraction expansion and table lookup.

4.2.1 The Inverse $\mathcal{Z}$-Transform by Contour Integration

Defn 17 (Cauchy's Integral Theorem)**.** Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the derivative $\frac{\mathrm{d}f(z)}{\mathrm{d}z}$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \tag{4.6}$$

More generally, if the $(k+1)$-order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^k}\,\mathrm{d}z = \begin{cases} \left.\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}f(z)}{\mathrm{d}z^{k-1}}\right|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \tag{4.7}$$

4.2.2 The Inverse $\mathcal{Z}$-Transform by Power Series Expansion

4.2.3 The Inverse $\mathcal{Z}$-Transform by Partial-Fraction Expansion

4.3 Properties of the $\mathcal{Z}$-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC : $r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
$\mathcal{Z}$-Transform Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
$\mathcal{Z}$-Transform Time Shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
$\mathcal{Z}$-Domain Scaling	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
$\mathcal{Z}$-Transform Time Reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real Part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}\left[X(z) + X^*(z^*)\right]$	Includes ROC
p Imaginary Part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}j\left[X(z) - X^*(z^*)\right]$	Includes ROC
$\mathcal{Z}$-Domain Differentiation	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert r_1$
$\mathcal{Z}$-Domain Convolutions	$x_1 * x_2$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
$\mathcal{Z}$-Transform 2 Sequence Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)x_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial Value Theorem for $\mathcal{Z}$-Transform	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
$\mathcal{Z}$-Transform 2 Sequence Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least, $r_{1l}r_{2l} < \lvert a\rvert < r_{1u}r_{2u}$
Parsevals Relation for $\mathcal{Z}$-Transform	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$= \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}dv$	

Table 4.2: Properties of the $\mathcal{Z}$-Transform

4.3.1 $\mathcal{Z}$-Transform Linearity

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = a_1x_1(n) + a_2x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1X_1(z) + a_2X_2(z) \tag{4.8}$$

for any constants a_1 and a_2.

The linearity property can be generalized to an arbitrary number of signals.

Example 4.2: Simple $\mathcal{Z}$-Transform Linearity Problem. Example 3.2.1

Deermine the z-transform and the ROC of the signal

$$x(n) = [3\,(2^n) - 4\,(3^n)]\,\mathcal{U}(n)$$

Solution on Page 158.

Example 4.3: $\mathcal{Z}$-Transform Linearity on Trig Functions. Example 3.2.2

Determine the z-transform of the signals
(a) $x(n) = (\cos\omega_0 n)\,\mathcal{U}(n)$
(b) $x(n) = (\sin\omega_0 n)\,\mathcal{U}(n)$

Solution on Pages 158-159.

4.3.2 $\mathcal{Z}$-Transform Time Shifting

If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z) \tag{4.9}$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z=0$ if $k>0$ and $z=\infty$ if $k<0$.

Example 4.4: $\mathcal{Z}$-Transform Time Shifting. Example 3.2.3

By applying the time-shifting property, determine the z-transform of the signals

$$x_1(n) = \{1, 2, \underline{5}, 7, 0, 1\}$$
$$x_2(n) = \{\underline{0}, 0, 1, 2, 5, 7, 0, 1\}$$

from the z-transform of

$$x_0(n) = \{\underline{1}, 2, 5, 7, 0, 1\}$$
$$X_0(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}, \text{ROC : entire } z\text{-plane except } z=0$$

Solution on Page 160.

4.3.3 $\mathcal{Z}$-Domain Scaling

If

$$x(n) \overset{z}{\longleftrightarrow} X(z),\ \text{ROC} : r_1 < |z| < r_2$$

then

$$a^n x(n) \overset{z}{\longleftrightarrow} X\left(a^{-1}z\right),\ \text{ROC} : |a|r_1 < |z| < |a|r_2 \tag{4.10}$$

4.3.4 $\mathcal{Z}$-Transform Time Reversal

If

$$x(n) \overset{z}{\longleftrightarrow} X(z),\ \text{ROC} : r_1 < |z| < r_2$$

then

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}), \text{ROC} : \frac{1}{r_2} < |z| < \frac{1}{r_1} \tag{4.11}$$

Example 4.5: $\mathcal{Z}$-Transform Time Reversal. Example 3.2.6

Determine the z-transform of the signal

$$x(n) = \mathcal{U}(-n)$$

The transform for $\mathcal{U}(n)$ is given in Table 4.3.

$$\mathcal{U}(n) \overset{z}{\longleftrightarrow} \frac{1}{1-x^{-1}}, \text{ ROC} : |z| > 1$$

By using (4.11), we obtain

$$\mathcal{U}(-n) \overset{z}{\longleftrightarrow} \frac{1}{1-z}, \text{ROC} : |z| < 1$$

4.3.5 $\mathcal{Z}$-Domain Differentiation

If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} \tag{4.12}$$

Example 4.6: $\mathcal{Z}$-Domain Differentiation. Example 3.2.7

Determine the z-transform of the signal

$$x(n) = na^n\,\mathcal{U}(n)$$

The signal $x(n)$ can be expressed as $nx_1(n)$, where $x_1(n) = a^n\,\mathcal{U}(n)$. By passing this through the z-transform, we have

$$x_1(n) = a^n\,\mathcal{U}(n) \overset{z}{\longleftrightarrow} X_1(z) = \frac{1}{1-az^{-1}}, \text{ ROC} : |z| > |a|$$

Then by using (4.12), we obtain

$$na^n\,\mathcal{U}(n) \overset{z}{\longleftrightarrow} X(z) = -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2}$$

4.3.6 $\mathcal{Z}$-Domain Convolutions

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z) \tag{4.13}$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

Example 4.7: $\mathcal{Z}$-Domain Convolutions. Example 3.2.9

Compute the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$
$$x_2(n) = \begin{cases} 1, & 0 \le n \le 6 \\ 0, & \text{elsewhere} \end{cases}$$

When

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

According to (4.13) we carry out the multiplication of $X_1(z)$ and $X_2(z)$. Thus

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underline{1}, -1, 0, 0, 0, 0, -1, 1\}$$

4.3.7 $\mathcal{Z}$-Transform 2 Sequence Correlation

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \overset{z}{\longleftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) \tag{4.14}$$

Example 4.8: $\mathcal{Z}$-Transform 2 Sequence Correlation. Example 3.2.10

Determine the autocorrelation of the signal

$$x(n) = a^n\,\mathcal{U}(n),\ -1 < a < 1$$

Solution on Page 166.

4.3.8 $\mathcal{Z}$-Transform 2 Sequence Multiplication

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

then

$$x(n) = x_1(n)x_2(n) \overset{z}{\longleftrightarrow} X_z = \frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv \tag{4.15}$$

where C is a closed contour that encloses the origin and lies within th eregion of convergence common to both $X_1(v)$ and $X_2(\frac{1}{v})$.

4.3.9 Parsevals Relation for $\mathcal{Z}$-Transform

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv \tag{4.16}$$

4.3.10 Initial Value Theorem for $\mathcal{Z}$-Transform

If $x(n)$ is *causal* [i.e., $x(n) = 0$ for $n < 0$], then

$$x(0) = \lim_{z\to\infty} X(z) \tag{4.17}$$

4.4 Properties of the One-Sided $\mathcal{Z}$-Transform

(i)

4.5 Rational $\mathcal{Z}$-Transforms

An important family of z-transforms are those for which $X(z)$ is a rational function, a ratio of two polynomials in z^{-1} (or z).

4.5.1 Poles and Zeros of a $\mathcal{Z}$-Transform

Defn 18 (Zeros)**.** The *zeros* of a z-transform $X(z)$ are the values of z for which $X(z) = 0$.
This is analogous to "setting the numerator equal to zero."

Defn 19 (Poles)**.** The *poles* of a z transform $X(z)$ are the values of z for which $X(z) = \infty$.
This is analogous to "setting the denominator equal to zero."

If $X(z)$ is a rational function, then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid negative powers of z by factoring out the terms z^{-M} and z^{-N}.

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M}}{z^{-N}} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

Since $B(z)$ and $A(z)$ are polynomials in z, they can be expressed in factored form as

$$X(z) = \frac{B(z)}{A(z)} = \frac{z^{-M}}{z^{-N}} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \tag{4.18}$$

Thus, $X(z)$ has M finite Zeros at $z = z_1, z_2, \ldots, z_M$ (the roots of the numerator polynomial), N finite Poles at $z = p_1, p_2, \ldots, p_N$ (the roots of the denominator polynomial, and $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$. Poles and zeroes may occur at $z = \infty$. A zero exists at $z = \infty$ if $X(\infty) = 0$ and a pole exists at $z = \infty$ if $X(\infty) = \infty$.

Defn 20 (Pole-Zero Plot)**.** Poles and Zeros of a z-transform can be shown graphically by a *pole-zero plot* in the complex plane, which shows the location of poles by crosses ($\times$) and the location of zeros by circles. Multiplicity is shown by a number close to the corresponding cross or circle. The ROC of a z-transform should not contain any poles, by definition.

4.5.2 Decomposition of Rational $\mathcal{Z}$-Transforms

4.6 Analysis of LTI Systems in the $\mathcal{Z}$-Domain

4.7 Common $\mathcal{Z}$-Transforms

5 The Fourier Transform and Fourier Series

When a signal is decomposed with either the Fourier Transform or the Fourier Series, you receive either sinusoids or complex-valued exponentials. This decomposition is said to be represented in the *frequency domain.*

Defn 21 (Fourier Transform)**.** When decomposing the class of signals with finite energy, you perform a *Fourier transform.* This is generally shown as the function

$$c_k = \mathrm{F}\{x(t)\}$$

There are 2 possible equations for the Fourier Transform, depending of the function is continuous-time or discrete-time.

1. Continuous-Time: Equation (5.1)
2. Discrete-Time: Equation (5.2)

Signal, $x(n)$	z-Transform, $X(z)$	ROC
$\delta(n)$	1	All z
$\mathcal{U}(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n\,\mathcal{U}(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n\,\mathcal{U}(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n\,\mathcal{U}(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$-na^n\,\mathcal{U}(-n-1)$	$\frac{az^{-z}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$(\cos\omega_0 n)\,\mathcal{U}(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(\sin\omega_0 n)\,\mathcal{U}(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(a^n\cos\omega_0 n)\,\mathcal{U}(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$(a^n\sin\omega_0 n)\,\mathcal{U}(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 4.3: Common $\mathcal{Z}$-Transforms

The Fourier Transform is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt \tag{5.1}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \tag{5.2}$$

Remark 21.1. Sometimes $X(F)$ and $X(f)$ will be denoted with Ω and ω ($X(\Omega)$ and $X(\omega)$) respectively. In both cases, Ω and ω mean something similar.

$$\Omega = 2\pi F$$
$$\omega = 2\pi f$$

This means that we can rewrite Equations (5.1) to (5.2) as

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{5.3}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.4}$$

Remark 21.2. Generally, when people say the Fourier Transform, they are referring to the transform on Continuous-Time Signals. There is a distinction that occurs with the *DTFT* or *Discrete-Time Fourier Transform*.

This document explains them side-by-side, but will primarily focus on the Discrete-Time Fourier Transform.

Defn 22 (Fourier Series)**.** When decomposing the class of periodic signals, you are returned a *Fourier series*. This is generally shown as the function

$$X(F) = \mathrm{F}\{x(t)\}$$

Defn 23 (Discrete-Time Fourier Transform)**.** The *Discrete-Time Fourier Transform, DTFT* is a special case of the Fourier Transform that occurs when the input function $x(n)$ is a case of Discrete-Time Signals.

The transformation (analysis) equations are:

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \tag{5.5a}$$

$$\omega = 2\pi f$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.5b}$$

The reverse (synthesis) equations are:

$$x(n) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f n} df \tag{5.6a}$$

$$x(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega n} d\omega \tag{5.6b}$$

These equations are expanded more upon in Section 5.2, The Inverse Fourier Transform.

5.1 Fourier Transform Relations

Each of these relations is just a side-note, the only relation of real importance is Equation (5.7). The Fourier Transform is just a special case in each of these scenarios. The Fourier Transform is evaluated around the unit circle on the real-imaginary plane.

5.1.1 Laplace Transform Fourier Transform Relation

There is a correlation between the Laplace Transform and the Fourier Transform. The Fourier Transform is a more specific case of the Laplace Transform, when

$$e^{-st} = e^{-j2\pi f t}$$

5.1.2 $\mathcal{Z}$-Transform Discrete-Time Fourier Transform Relation

There is a relationship between the $\mathcal{Z}$-Transform and the Discrete-Time Fourier Transform.

$$\begin{aligned} z &= e^{j2\pi f} \\ z &= e^{j2\pi n} \end{aligned} \tag{5.7}$$

The Discrete-Time Fourier Transform can be viewed as the $\mathcal{Z}$-transform at of the sequence evaluated at the unit circle. If $X(z)$ does not converge in the region $|z| = 1$ (i.e., if the unit circle is not contained within the ROC of $X(z)$), the Fourier Transform $X(f)$ does not exist.

The existence of the $\mathcal{Z}$-transform requires that the sequence $\{x(n)r^{-n}\}$ be absolutely summable for some value of r, that is,

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \tag{5.8}$$

Therefore, if Equation (5.8) converges only for values of $r < r_0 < 1$, the ***$\mathcal{Z}$-transform exists***, but the ***Discrete-Time Fourier Transform DOES NOT EXIST***. This is the case for causal sequences of the form $x(n) = a^n \mathcal{U}(n)$, where $|a| > 1$.

There are sequences that do not satisfy Equation (5.8), for example

$$x(n) = \frac{\sin \omega_c n}{\pi n}, \; -\infty < n < \infty$$

This sequences does not have a $\mathcal{Z}$-transform. However, since it is a finite Energy Signal, it has a Discrete-Time Fourier Transform that converges to

$$X(f) = \begin{cases} 1, & |f| < f_c \\ 0, & f_c < |f| \leq \frac{1}{2} \end{cases}$$

> The existece of the $\mathcal{Z}$-transform requires that Equation (5.8) be satisfied for some region in the z-plane. If this region contains the unit circle, the Discrete-Time Fourier Transform, $X(f)$ exists. However, the existence of the Discrete-Time Fourier Transform, which is defined for finite Energy Signals, does not necessarily ensure the existence of the $\mathcal{Z}$-transform.

5.2 The Inverse Fourier Transform

Defn 24 (Inverse Fourier Transform)**.** Since the Fourier Transform is a "lossless" function (the definition of a transformation), the *inverse fourier transform* is just the opposite setup of Equations (5.1) to (5.2).

In both cases, a Continuous-Time signal and a Discrete-Time signal, you use the below synthesis equations (Equations (5.9) to (5.10)).

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}dF \\ x(n) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi fn}df \end{aligned} \tag{5.9}$$

If you're calculating with Ω or ω instead of F or f, then use these synthesis equations.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \\ x(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega n}d\omega \end{aligned} \tag{5.10}$$

5.3 Properties of the Discrete-Time Fourier Transform

One thing to keep in mind with all of these properties is that $\omega = 2\pi f$.

Property	Time Domain $x(n)$	Frequency Domain $X(f)$ or $X(\omega)$
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(\omega)$ $X_1(\omega)$ $X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time Shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time Reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1,x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1,x_2}(\omega) = X_1(\omega)X_2(\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener-Khintchine Theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency Shifting	$e^{j\omega_0 n}x(n)$	$X(\omega-\omega_0)$
Modulation	$x(n)\cos(\omega_0 n)$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication in Time Domain	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in Frequency Domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

Table 5.1: Properties of the Fourier Transform for Discrete-Time Signals

5.3.1 Linearity

If

$$x_1(n) \overset{\text{F}}{\longleftrightarrow} X_1(f)$$

$$x_2(n) \overset{\text{F}}{\longleftrightarrow} X_2(f)$$

then

$$a_1x_1(n) + a_2x_2(n) \overset{\text{F}}{\longleftrightarrow} a_1X_1(f) + a_2X_2(f) \tag{5.11}$$

5.3.2 Time Shifting

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$x(n-k) \overset{\mathrm{F}}{\longleftrightarrow} e^{-j\omega k} X(f) \tag{5.12}$$

5.3.3 Time Reversal

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$x(-n) \overset{\mathrm{F}}{\longleftrightarrow} X(-f) \tag{5.13}$$

5.3.4 Convolution

If

$$x_1(n) \overset{\mathrm{F}}{\longleftrightarrow} X_1(f)$$
$$x_2(n) \overset{\mathrm{F}}{\longleftrightarrow} X_2(f)$$

then

$$x(n) = x_1(n) * x_2(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f) = X_1(f)X_2(f) \tag{5.14}$$

Remark. There is one thing to note here. Both $x_1(n)$ and $x_2(n)$ must be reasonably well-behaved and have be BIBO-stable for this relation to hold.

5.3.5 Correlation

If

$$x_1(n) \overset{\mathrm{F}}{\longleftrightarrow} X_1(f)$$
$$x_2(n) \overset{\mathrm{F}}{\longleftrightarrow} X_2(f)$$

then

$$r_{x_1x_2}(m) \overset{\mathrm{F}}{\longleftrightarrow} S_{x_1x_2}(f) = X_1(f)X_2(-f) \tag{5.15}$$

5.3.6 Wiener-Khintchine Theorem

Let $x(n)$ be a real signal. Then

$$r_{xx}(l) \overset{\mathrm{F}}{\longleftrightarrow} S_{xx}(f) \tag{5.16}$$

That is, the energy spectral density of an energy signal is the Fourier Transform of its autocorrelation sequence. This is a special case of Equation (5.15).

5.3.7 Frequency Shifting

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$e^{-i2\pi f_0 n} x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f-f_0) \tag{5.17}$$

5.3.8 Modulation

If

$$x(n) \overset{\mathrm{F}}{\longleftrightarrow} X(f)$$

then

$$x(n)\cos(2\pi f_0 n) \overset{\mathrm{F}}{\longleftrightarrow} \frac{1}{2}\left[X(f+f_0) + X(f-f_0)\right] \tag{5.18}$$

5.3.9 Multiplication in Time Domain

This is also called the *Windowing Theorem*.

If

$$x_1(n) \xleftrightarrow{\text{F}} X_1(f)$$
$$x_2(n) \xleftrightarrow{\text{F}} X_2(f)$$

then

$$x_3(n) \equiv x_1(n)x_2(n) \xleftrightarrow{\text{F}} X_3(f) = \int_{\frac{1}{2}}^{\frac{1}{2}} X_1(\lambda)X_2(f-\lambda)d\lambda \tag{5.19}$$

5.3.10 Differentiation in Frequency Domain

If

$$x(n) \xleftrightarrow{\text{F}} X(f)$$

then

$$nx(n) \xleftrightarrow{\text{F}} j\frac{dX(f)}{df} \tag{5.20}$$

5.3.11 Parseval's Theorem

If

$$x_1(n) \xleftrightarrow{\text{F}} X_1(f)$$
$$x_2(n) \xleftrightarrow{\text{F}} X_2(f)$$

then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \int_{-0.5}^{0.5} X_1(f)X_2^*(f)df \tag{5.21}$$

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega \tag{5.22}$$

Both Equations (5.21) to (5.22) can be expressed in another format.

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \int_{-0.5}^{0.5} |X_1(f)|^2 df \tag{5.23}$$

$$\sum_{n=-\infty}^{\infty} |x_1(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_1(\omega)|^2 d\omega \tag{5.24}$$

A Trigonometry

A.1 Trigonometric Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{A.1}$$

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{A.2}$$

A.2 Euler Equivalents of Trigonometric Functions

$$e^{\pm j\alpha} = \cos(\alpha) \pm j\sin(\alpha) \tag{A.3}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{A.4}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{A.5}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \tag{A.6}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{A.7}$$

A.3 Angle Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{A.8}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{A.9}$$

A.4 Double-Angle Formulae

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{A.10}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{A.11}$$

A.5 Half-Angle Formulae

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos(\alpha)}{2}} \tag{A.12}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos(\alpha)}{2}} \tag{A.13}$$

A.6 Exponent Reduction Formulae

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{A.14}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{A.15}$$

A.7 Product-to-Sum Identities

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{A.16}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{A.17}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{A.18}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{A.19}$$

A.8 Sum-to-Product Identities

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{A.20}$$

$$\cos(\alpha) + \cos(\alpha) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{A.21}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{A.22}$$

A.9 Pythagorean Theorem for Trig

$$\cos^2(\alpha) + \sin^2(\alpha) = 1^2 \tag{A.23}$$

A.10 Rectangular to Polar

$$a + jb = \sqrt{a^2 + b^2}e^{j\theta} = re^{j\theta} \tag{A.24}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a > 0 \\ \pi - \arctan\left(\frac{b}{a}\right) & a < 0 \end{cases} \tag{A.25}$$

A.11 Polar to Rectangular

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \tag{A.26}$$

B Calculus

B.1 Fundamental Theorems of Calculus

Defn B.1.1 (First Fundamental Theorem of Calculus)**.** The *first fundamental theorem of calculus* states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{B.1}$$

Defn B.1.2 (Second Fundamental Theorem of Calculus)**.** The *second fundamental theorem of calculus* holds for f a continuous function on an open interval I and a any point in I, and states that if F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

then

$$\begin{aligned} \frac{d}{dx} \int_a^x f(t)\, dt &= f(x) \\ F'(x) &= f(x) \end{aligned} \tag{B.2}$$

Defn B.1.3 (argmax)**.** The arguments to the *argmax* function are to be maximized by using their derivatives. You must take the derivative of the function, find critical points, then determine if that critical point is a global maxima. This is denoted as

$$\operatorname*{argmax}_x$$

B.2 Rules of Calculus

B.2.1 Chain Rule

Defn B.2.1 (Chain Rule)**.** The *chain rule* is a way to differentiate a function that has 2 functions multiplied together.

If

$$f(x) = g(x) \cdot h(x)$$

then,

$$\begin{aligned} f'(x) &= g'(x) \cdot h(x) + g(x) \cdot h'(x) \\ \frac{df(x)}{dx} &= \frac{dg(x)}{dx} \cdot g(x) + g(x) \cdot \frac{dh(x)}{dx} \end{aligned} \tag{B.3}$$

C Laplace Transform

Defn C.0.1 (Laplace Transform)**.** The *Laplace transformation* operation is denoted as $\mathcal{L}\{x(t)\}$ and is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{C.1}$$