A VERY SHORT INTRODUCTION TO

CGAL

The Computational Geometry Algorithms Library

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(Based on work by Pierre Alliez, Andreas Fabri, Efi Fogel, Lutz Kettner, Sylvain Pion, Monique Teillaud, Mariette Yvinec, and probably many others.)

CGAL

I: The CGAL Project
II: Exact Geometric Computing
III: Basic Programming using a CGAL Kernel
IV: Practical Information

CGAL

PART II:

Exact Geometric Computing

GOALS

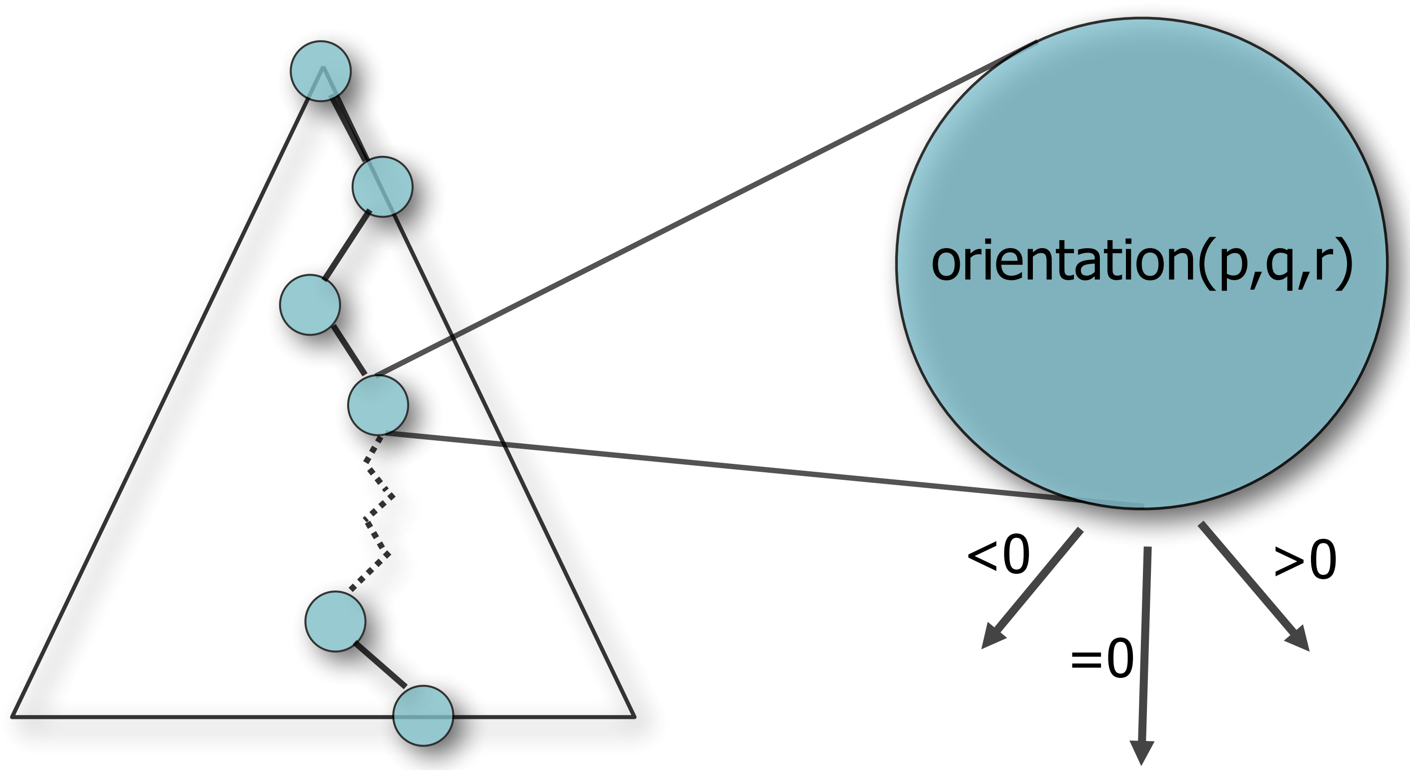
Awareness of challenges for implementing geometric algorithms.

- Consequences of using limited precision arithmetic for discrete decisions.
- Exact geometric computing: benefits and limitations

Basic knowledge of limited precision arithmetic (in C++).

- How large is `int`, `long`, `double`, …?
- How to bound results of a computation in terms of the input numbers.

LAYERS OF GEOMETRIC ALGORITHMS



$$\begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix}$$

Combinatorial Algorithm	Geometric Predicate	Algebraic Computation

Control flow depends on non-trivial algebraic computations.

How to do these efficiently and consistently?

(Tough, no universally applicable solution…)

ARITHMETIC

All operations beyond + and - are computed using limited precision floating point arithmetic.

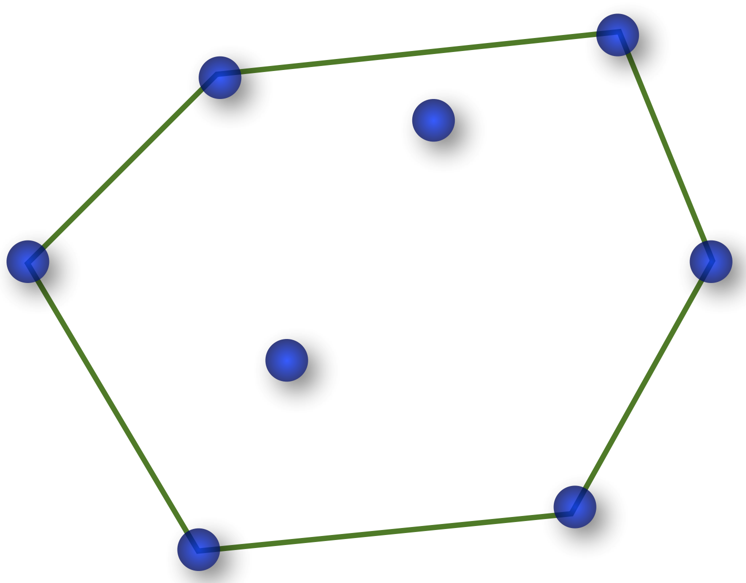
Integer multiplication and division are usually slower, often considerably.
And the precision is limited regardless...

→ Results may be **incorrect** due to roundoff.

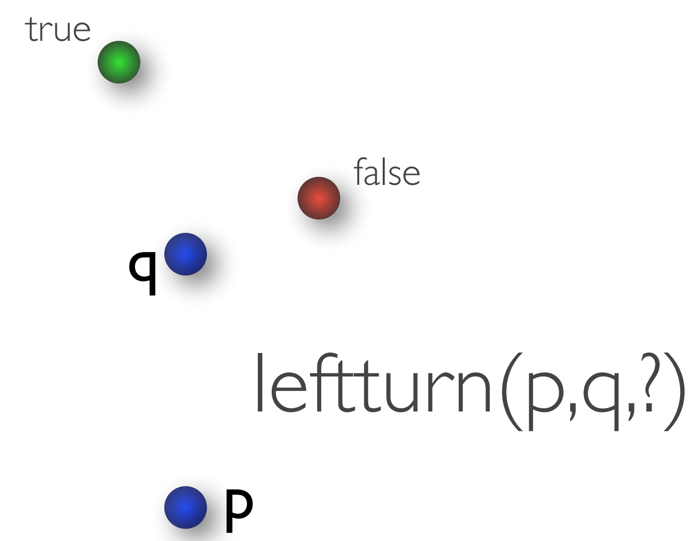
Difference to numeric computing:
Results are interpreted combinatorially: yes or no.

Incorrect results often lead to a **complete failure** rather than to a reasonable approximation.

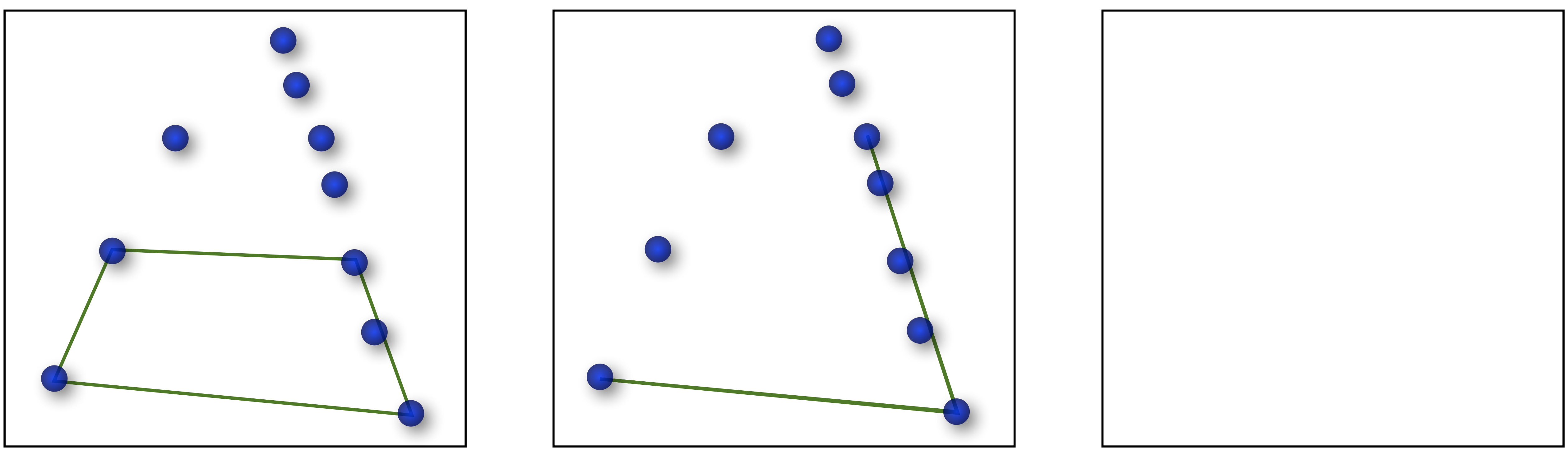
CONVEX HULL



Based on orientation test.



Possible results with an unreliable orientation test:



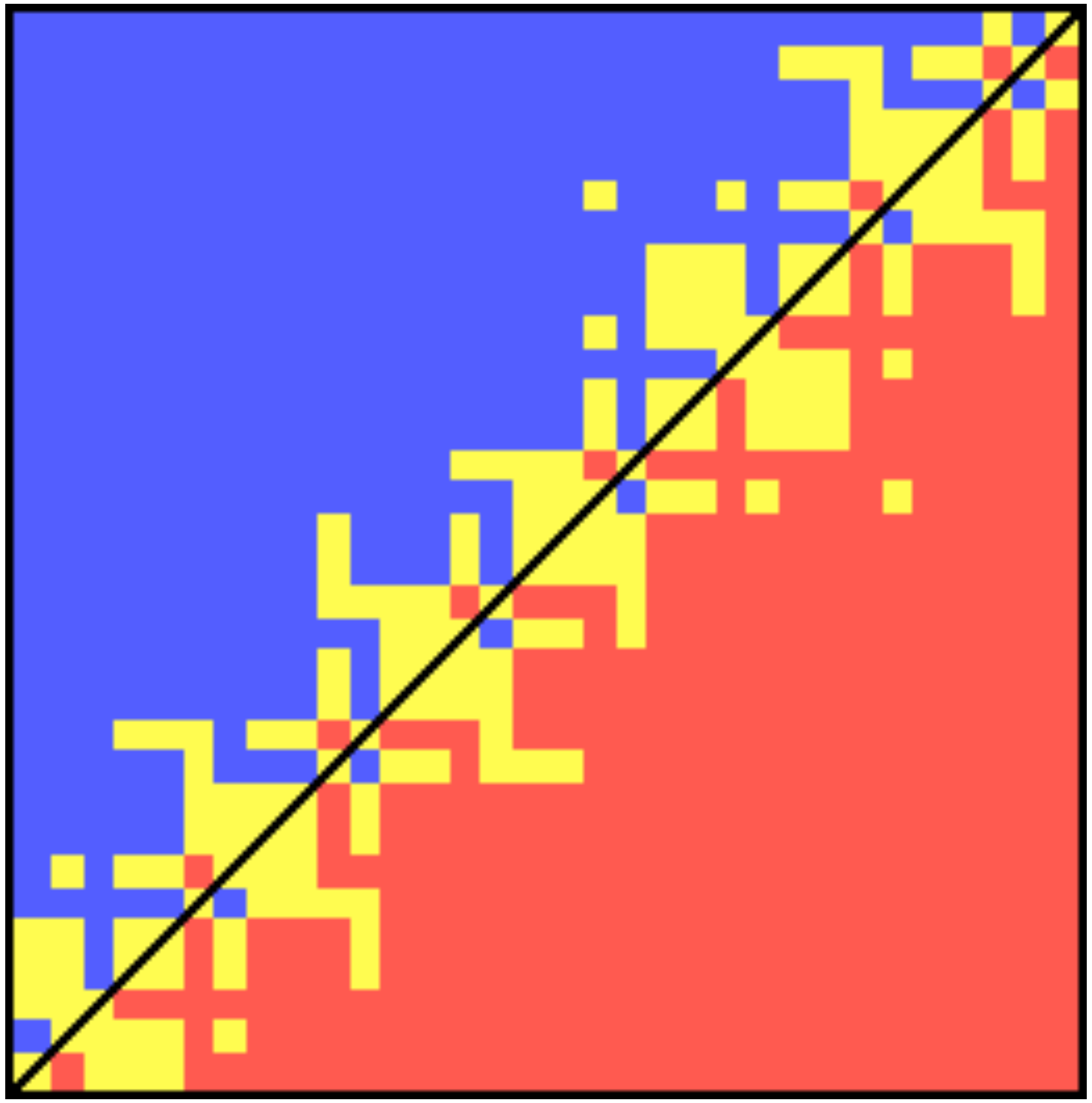
STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)$$

$p = (0.5+x\cdot u, 0.5+y\cdot u)$
$q = (12, 12)$
$r = (24, 24)$

$0 \leq x, y < 256,\ \ u = 2^{-53}$

256x256 pixel image
red: <0, yellow: =0, blue: >0
evaluated with **`double`**



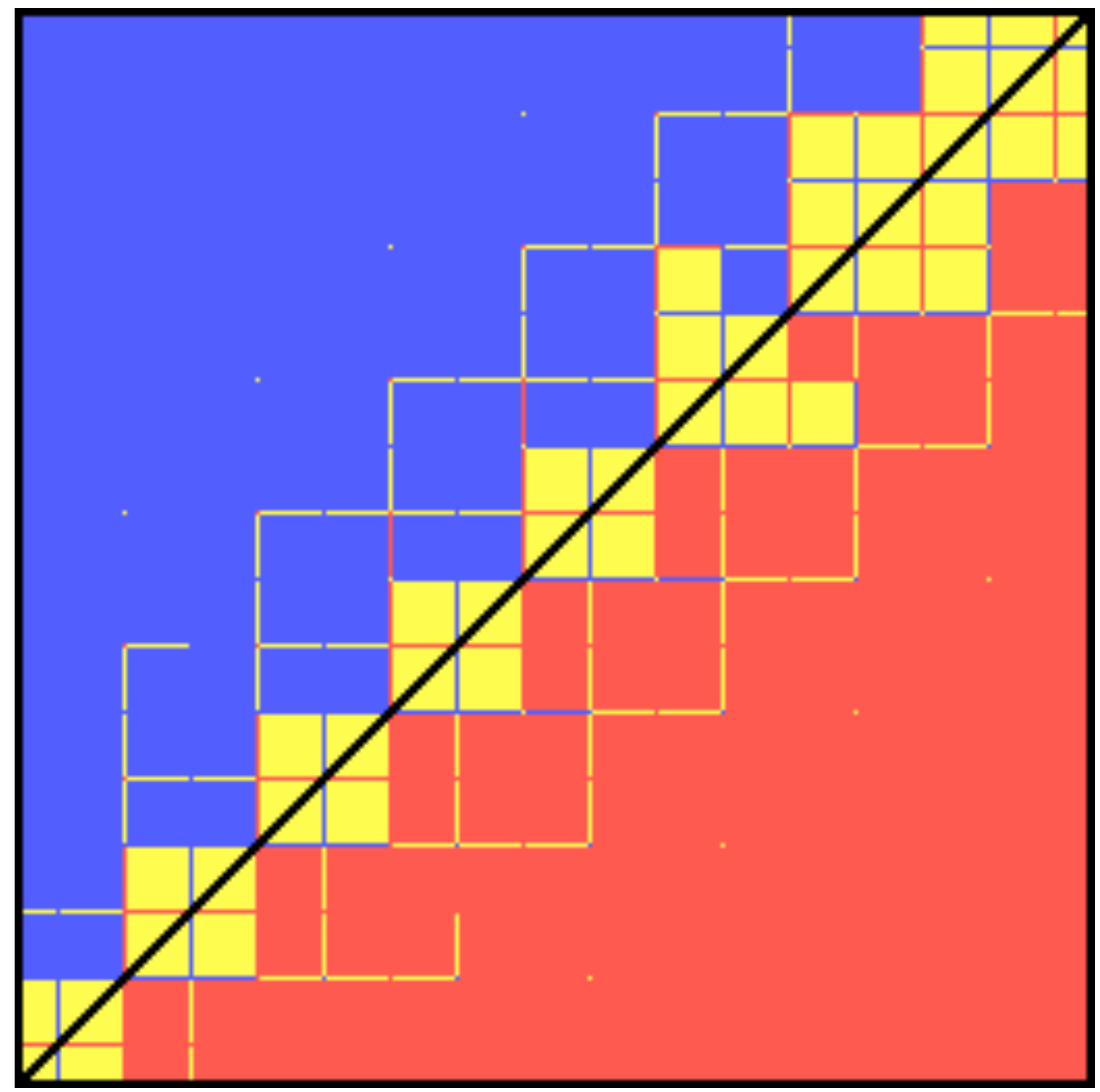
STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)$$

$p = (0.5+x \cdot u, 0.5+y \cdot u)$
$q = (8.800000000000007,$
$\quad 8.800000000000007)$
$r = (12.1, 12.1)$

$0 \leq x, y < 256, \ u = 2^{-53}$

256x256 pixel image
red: <0, yellow: =0, blue: >0
evaluated with **double**



STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)$$

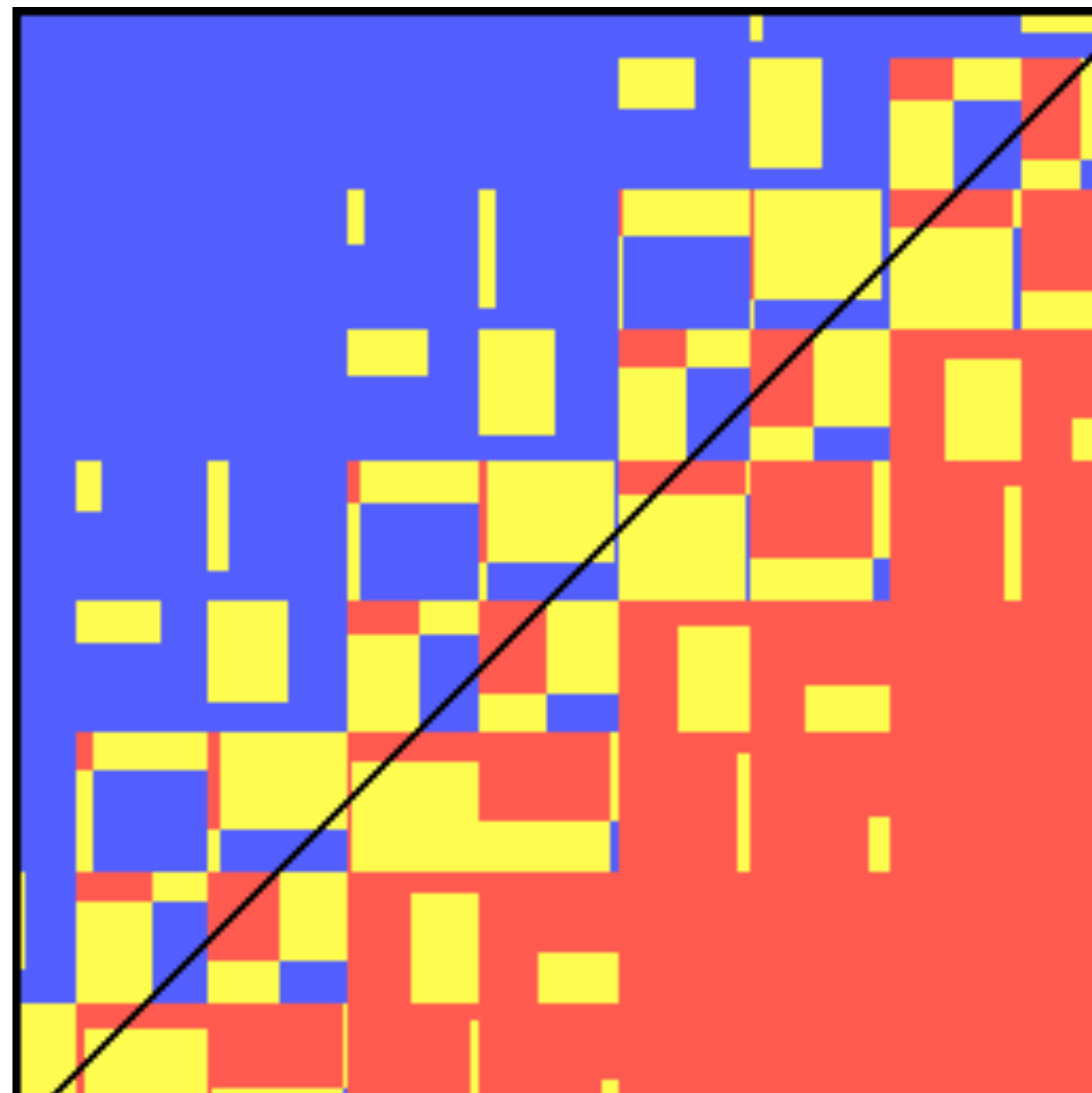
$p = (0.50000000000002531+x \cdot u,$
$\quad 0.5000000000001710+y \cdot u)$
$q = (17.300000000000001,$
$\quad 17.300000000000001)$
$r = (24.00000000000000500000,$
$\quad 24.0000000000000517765)$

$0 \leq x, y < 256, \; u = 2^{-53}$

256x256 pixel image

red: <0, yellow: =0, blue: >0

evaluated with **`ext double`**



HOW TO OBTAIN CORRECTNESS?

Several options:

- Hope things go fine ~~and~~ fiddle around if not

Sometimes possible, often hard, always messy. Very problem-specific, no general machinery.

- Adapt algorithm to cope with imprecisions

- Restrict input

Good in special cases, hard to impossible for general purpose implementations . Document and check properly!

- Use exact algebra

General approach. Easy to use. Can be very slow...

- Filtering: Check whether things go fine and use exact algebra only when needed.

General approach. Easy to use. Often quite efficient...

FLOATING POINT NUMBERS

IEEE 754 double precision

+/-	exponent	mantissa
1 bit	11 bits	53 bits

0.1 is not exactly representable

Numbers $\pm m \cdot 2^x$, $0 \leq m < 2^{53}$, $-1022 \leq x \leq 1023$.

b bits	$\pm$	b bits	$\approx$	b+1 bits
b bits	$\cdot$	b bits	$\approx$	2b bits

(q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)

→ orientation test ≈ 2b+3 bits, can be done exactly for 25-bit integer coordinates.

COMPUTING WITH FLOATING POINT NUMBERS

Guideline #1: Avoid (square)roots!

$$\text{For } x, y \geqslant 0: \ \sqrt{x} < \sqrt{y} \iff x < y.$$

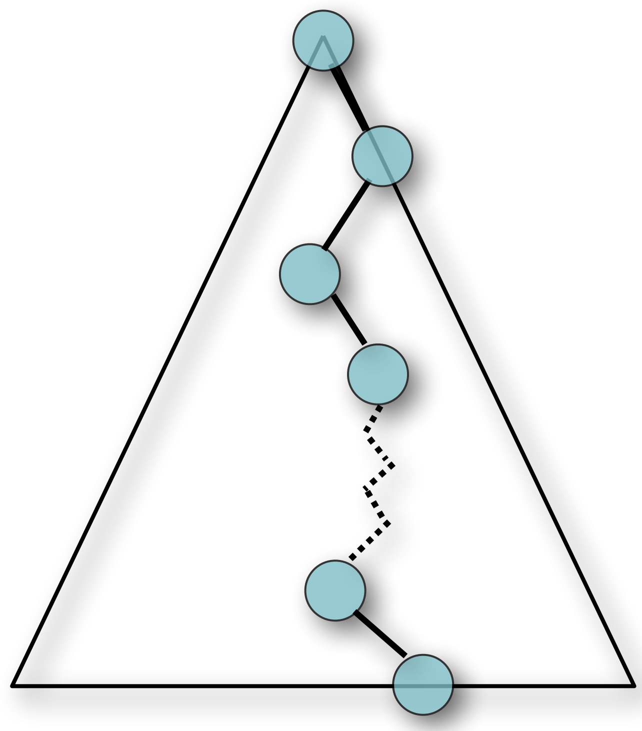
Guideline #2: Avoid divisions!

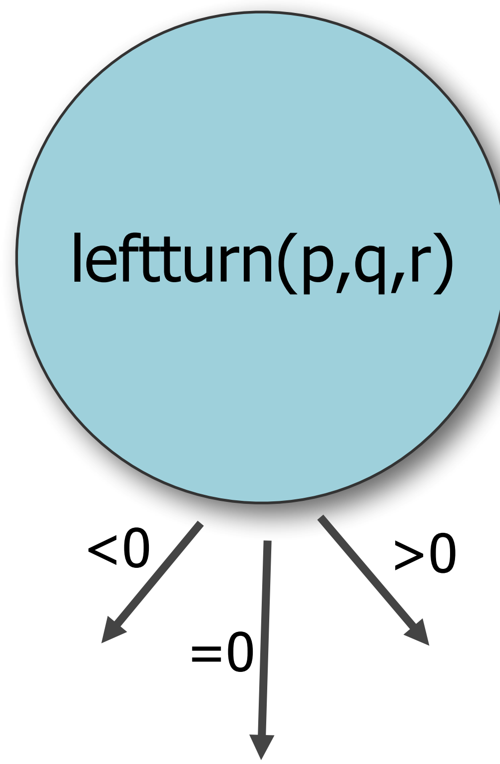
$$\text{For } b, d \neq 0: \ \frac{a}{b} < \frac{c}{d} \iff ad < bc.$$

These are just general guidelines, not hard rules. For instance, integer division can be useful to get rid of common factors.

Guideline #3: Estimate to check if loss of precision may occur! (See previous slide…)

EXACT COMPUTATION





$$\begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix}$$

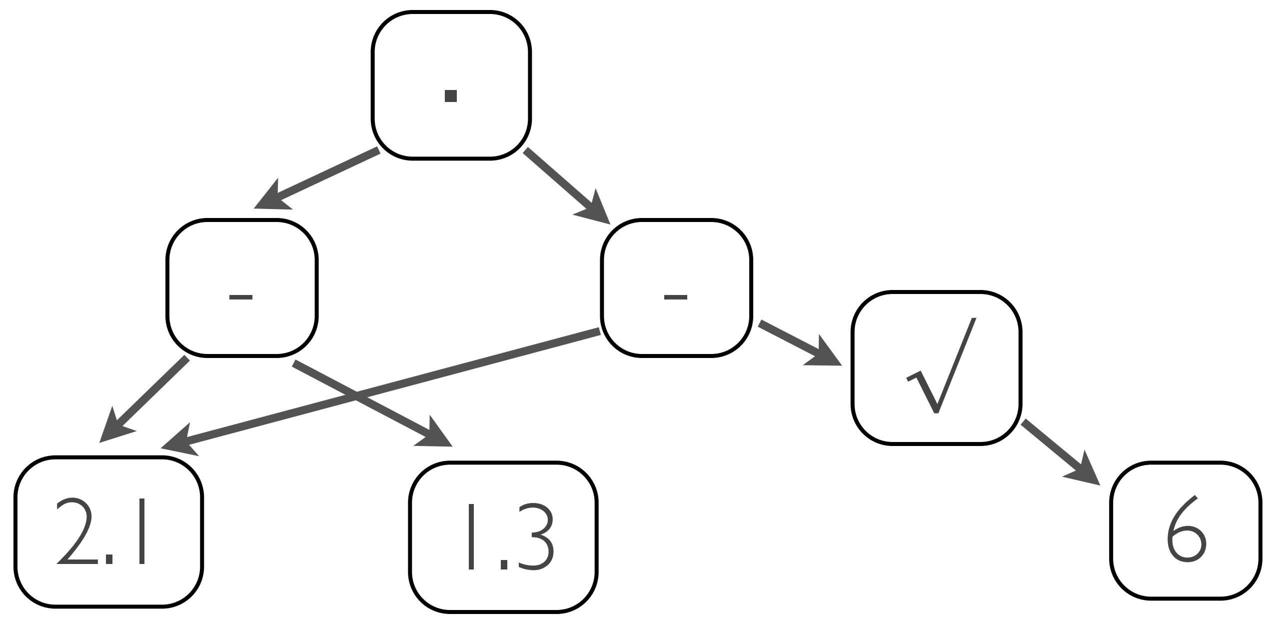
Combinatorial Algorithm	Geometric Predicate	Algebraic Computation

Ensure that the control flow in the algorithm is the same as if all algebraic computations were made exactly.

→ **Correctness**

EXACT ALGEBRAIC COMPUTATION

$(2.1-1.3)(2.1-\sqrt{6})$



- numbers represented as expression-dags
- arbitrary precision floating point data types (array of digits) to compute approximations
- sign(x): compute finer and finer approximations for x, until it becomes clear that $x>0$ or $x<0$;
- for any algebraic expression there is a *separation bound* that tells where to stop and conclude $x=0$.

FLOATING POINT FILTERS

Exact algebraic computation is expensive.

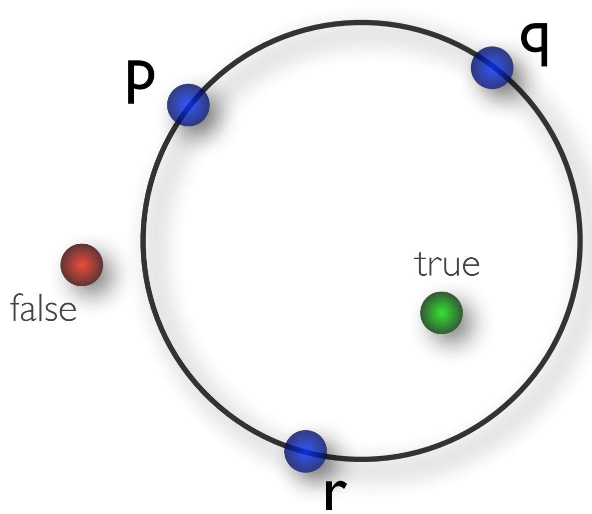
→ use when absolutely necessary only.

- maintain double approximation [l,h] using interval arithmetic (hardware support => fast)
- if $0 \notin [l,h]$, this is good enough to decide about sign.
- use exact machinery only if $0 \in [l,h]$.

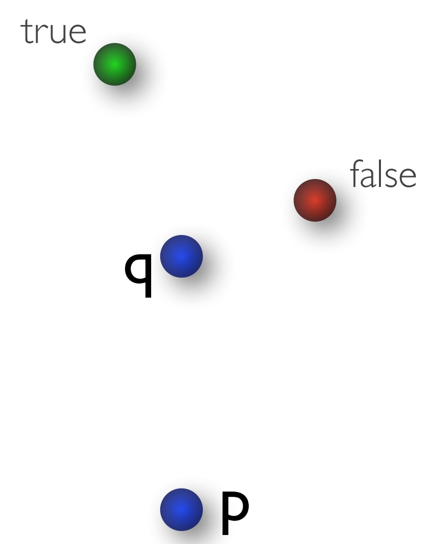
Minimal overhead as long as filter works.

In particular, if only predicates are used and no constructions.

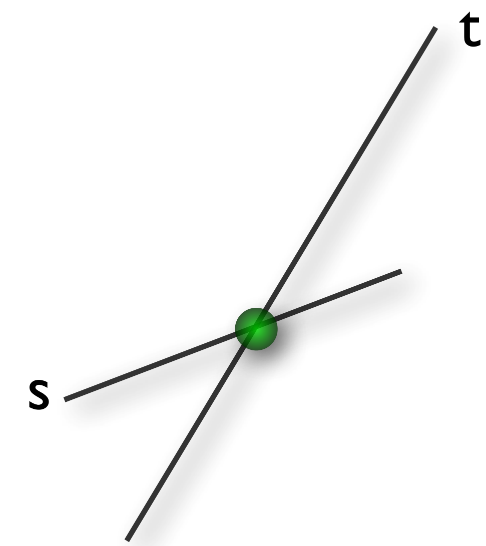
GEOMETRIC OPERATIONS



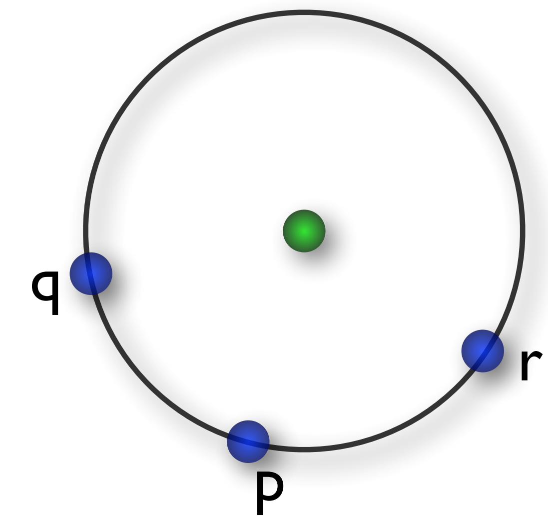
incircle(p,q,r,?)



leftturn(p,q,?)



intersection(s,t)



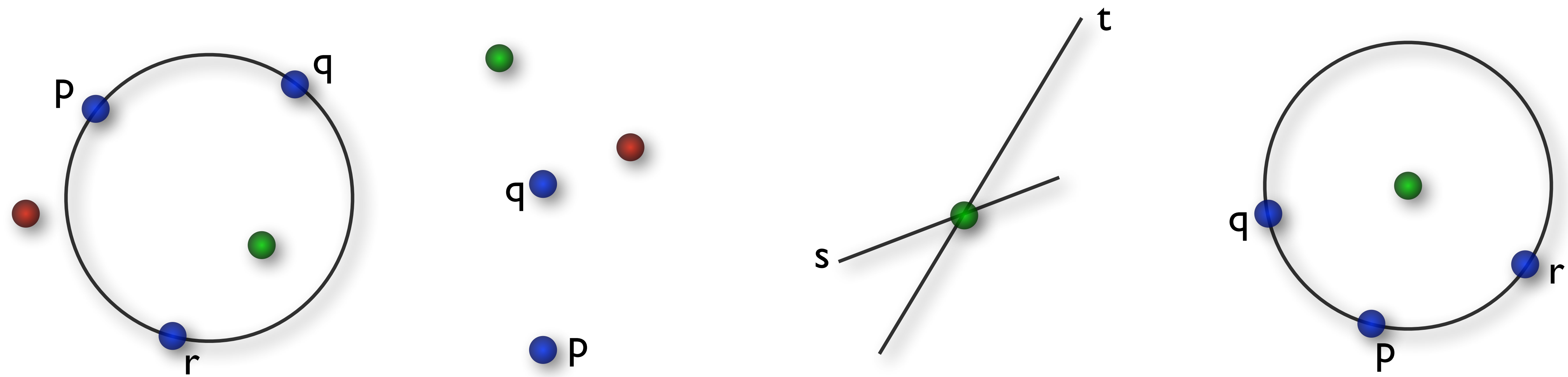
circumcircle(p,q,r)

Predicates

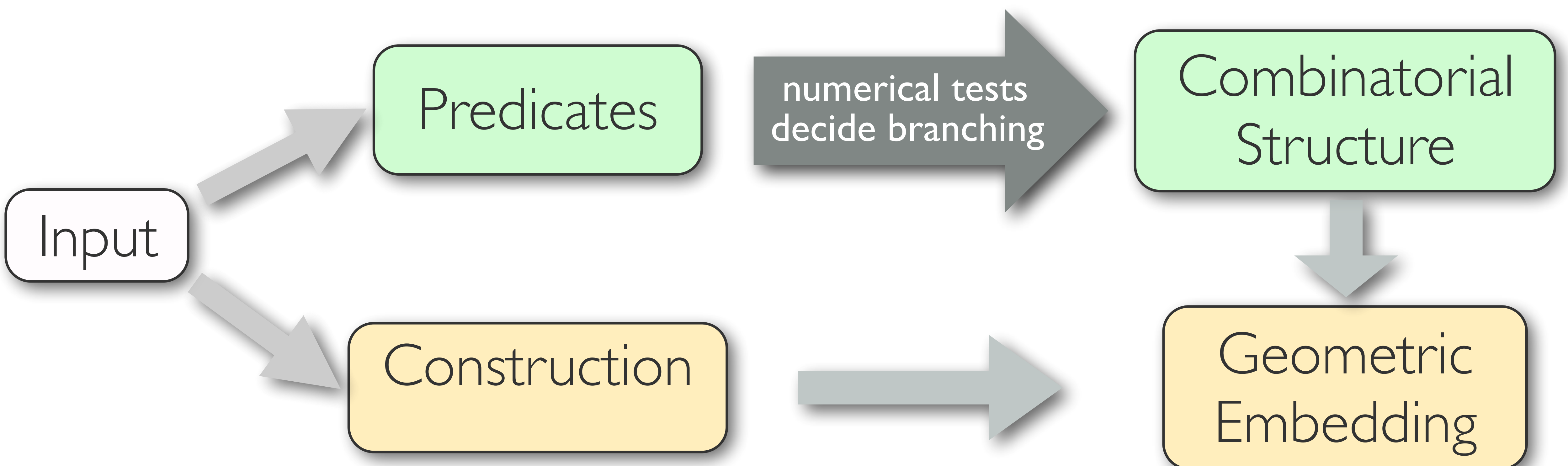
Construction

→ Do you need (exact) constructions?

GEOMETRIC OPERATIONS



incircle(p,q,r,?) leftturn(p,q,?) intersection(s,t) circumcircle(p,q,r)



FLEXIBILITY

There is no single true way to do geometric computing.

CGAL offers different kernels to serve various needs

Collection of geometric data types and operations.

You have to choose the right one for your particular case.

Predefined defaults:

All three compute predicates exactly using filters for efficiency.

- `CGAL::Exact_predicates_inexact_constructions_kernel`
  Constructions use `double`.
- `CGAL::Exact_predicates_exact_constructions_kernel`
  Constructions use an exact number type supporting +,-,*,/.
- `CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt`
  Constructions use an exact number type supporting +,-,*,/, and roots.

PART III:

Basic Programming using a CGAL Kernel

GOALS

For a geometric algorithm, you are able to pick an adequate CGAL kernel.

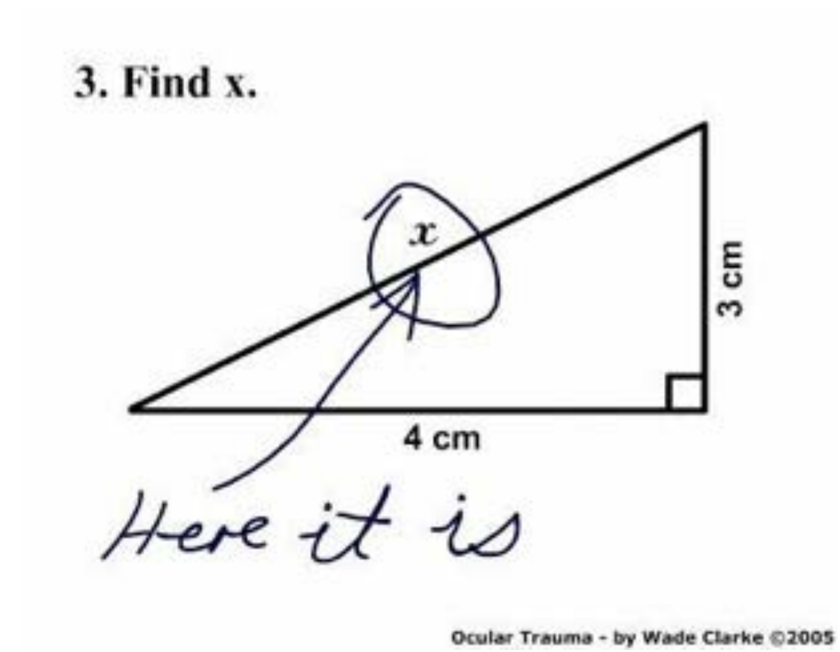
- Are non-trivial geometric constructions needed?
- Are exact roots needed?

You are able to do some basic geometric computations using CGAL.

- 2D kernel objects
- Intersections
- Bounding Volumes

PREREQUISITES

You know basic Euclidean geometry (e.g., distance/area/volume, angles, Pythagoras, ...) and can apply this knowledge to describe and analyze problems, to design models and algorithms.



You know basic algorithmic techniques (e.g., D.P., binary search, sorting, line sweep...). → You skillfully combine them with the geometric techniques discussed here.

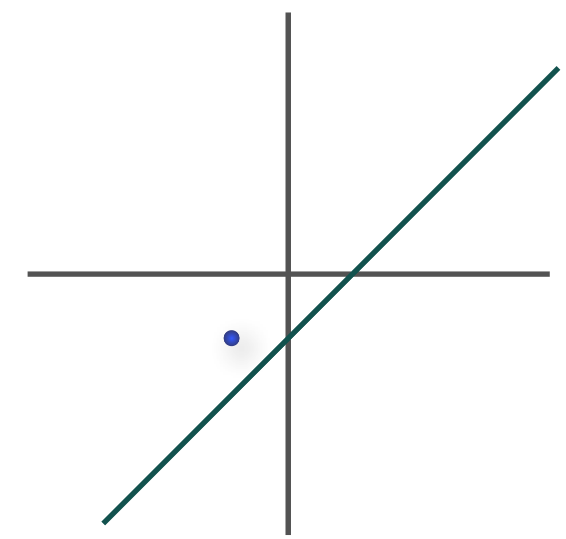
HELLO POINT

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <iostream>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = CGAL::squared_distance(r,l);
  std::cout << d << std::endl;
}
```

There is a bunch of hyperlinks here.
Click me to get to the CGAL manual.

Does this code use constructions?

YES!



Output: 0.5

avoids square root computation

FT = field type

The number type used for the underlying algebra. Supports all field operations, i.e., +-*/.

Some (few) field types also support exact roots.

To obtain an approximation of the real distance, use

```
std::sqrt(CGAL::to_double(CGAL::squared_distance(r,l)))
```

This function must be defined for any field type.

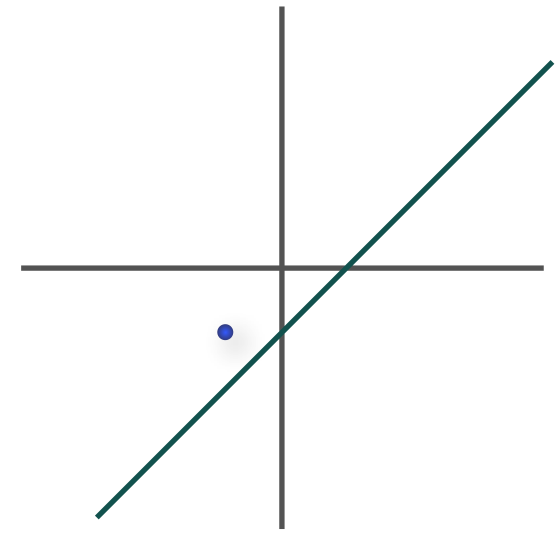
Even if the field type supports exact square roots, in order to output it numerically you have to resort to an approximation...

HELLO POINT

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <iostream>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = CGAL::squared_distance(r,l);
  std::cout << d << std::endl;
}
```

For the small coordinates used here, things are probably fine. But in general...

this code is not safe!



Constructing a point from Cartesian `double` coordinates. All default kernels can do this exactly, by just storing the coordinates.
=> trivial construction, no problem

Constructing a line from two points.
Trivial?

Depends on representation of lines... equation => non-trivial construction

Also a non-trivial construction.
(Squared distance may be considerably larger than input coordinates, which may lead to overflow.)

CGAL::Line_2<Kernel>

Class

Definition

An object *l* of the data type *Line_2*<*Kernel*> is a directed straight line in the two-dimensional Euclidean plane $\mathbb{E}^2$. It is defined by the set of points with Cartesian coordinates (x,y) that satisfy the equation l : ax + by + c = 0

The line splits $\mathbb{E}^2$ in a *positive* and a *negative* side. A point p with Cartesian coordinates (px, py) is on the positive side of *l*, iff $a\,px + b\,py + c > 0$, it is on the negative side of *l*, iff $a\,px + b\,py + c < 0$. The positive side is to the left of *l*.

HELLO POINT (EXACTLY)

```
#include <CGAL/Exact_predicates_exact_constructions_kernel_with_sqrt.h>
#include <iostream>
#include <iomanip>

typedef CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt K;

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = sqrt(CGAL::squared_distance(r,l));
  std::cout << CGAL::to_double(d) << std::endl;
  std::cout << std::setiosflags(std::ios::fixed) << std::setprecision(2)
            << CGAL::to_double(d) << std::endl;
}
```

Set precision (number of digits after the decimal point) for floating point number output. Round to nearest, but tie-breaking is not well defined!

Output:

```
0.707107
0.71
```

Compute squareroot (here: exactly).

Round to some `double` nearby. (There is no easy way to output the exact internal representation.)

Problem: No guarantee on precision and rounding.

Output floating point numbers in fixed point notation from now on.

`std::resetiosflags(std::ios::fixed)` switches back to default behaviour.

HELLO POINT (EVEN MORE EXACTLY)

```
#include <CGAL/Exact_predicates_exact_constructions_kernel_with_sqrt.h>
#include <iostream>
#include <cmath>

typedef CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt K;

double floor_to_double(const K::FT& x)
{
  double a = std::floor(CGAL::to_double(x));
  while (a > x) a -= 1;
  while (a+1 <= x) a += 1;
  return a;
}

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = sqrt(CGAL::squared_distance(r,l));
  std::cout << floor_to_double(d) << std::endl;
}
```

for `std::floor(...)`

Compute approximation of the closest integer ≤ x. (Usually, this is pretty good. But we cannot be sure that it is always...)

Compare to the exact value to be sure.

(This assumes that x is somewhere within the range of `double`, which will be the case in all our problems.)

Compute squareroot exactly.

Output:
0

We need a precise specification for all output, in order to compare on the judge.

This is the recommended way to round down to an integer.
(The symmetric function `ceil_to_double(...)` to round up should be an easy exercise...)

TWO KERNELS IN ONE PROGRAM

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <iostream>
#include <stdexcept>

typedef CGAL::Exact_predicates_inexact_constructions_kernel IK;
typedef CGAL::Exact_predicates_exact_constructions_kernel   EK;

int main()
{
  IK::Point_2 p(2,1), q(1,0), r(-1,-1);
  // do something that needs predicates only, e.g., ...
  std::cout << (CGAL::left_turn(p, q, r) ? "y" : "n") << "\n";

  // now we use non-trivial constructions...
  EK::Point_2 ep(p.x(), p.y()), eq(q.x(), q.y()), er(r.x(), r.y());
  EK::Circle_2 c(ep, eq, er);
  if (!c.has_on_boundary(ep))
    throw std::runtime_error("ep not on c");
}
```

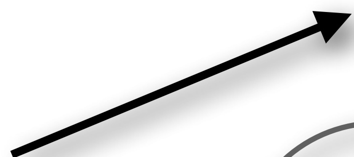
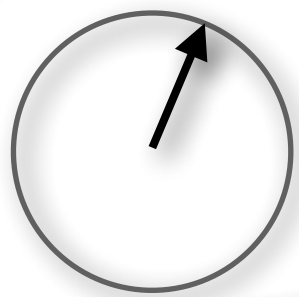
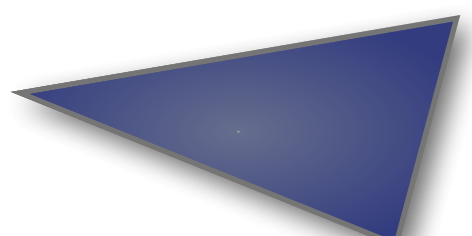
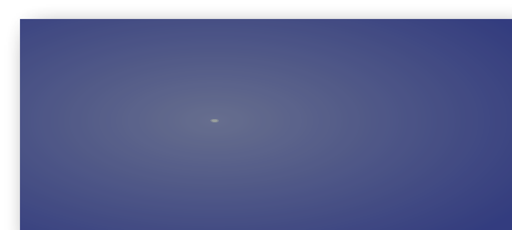
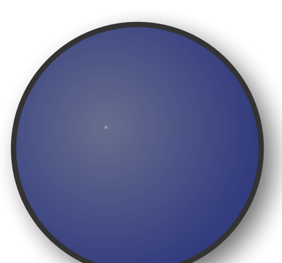
This works because the coordinates of `IK::Point_2` are actually `double`.

It would not work the other way round, because the coordinates of `EK::Point_2` are of some elaborate number type.

We cannot just write `c(p, q, r)` because these are `IK::Point_2` and there is no general conversion between points from different kernels.

Output:
n

2D (LINEAR) KERNEL

- Point_2 
- Vector_2 
- Direction_2 
- Line_2
- Ray_2
- Segment_2 
- Triangle_2 
- Iso_rectangle_2 
- Circle_2 

Follow the links to see the manual.

2D KERNEL REPRESENTATIONS

- Point_2
- Vector_2
- Direction_2

two FTs (Cartesian coordinates)

- Line_2 three FTs (coefficients of line equation)
- Ray_2
- Segment_2

two points

- Triangle_2 three points (corners)
- Iso_rectangle_2 (two points, opposite corners)
- Circle_2 point and FT (center and squared radius)

So that you can tell which constructions are trivial…

2D KERNEL FUNCTIONALITY

See the CGAL Manual: http://www.cgal.org

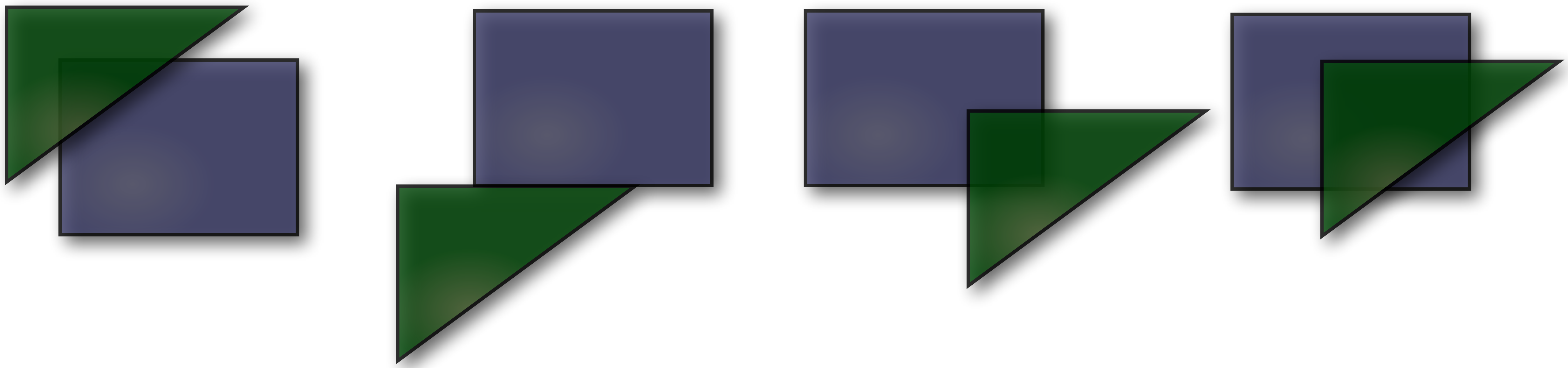
Most manual chapters have two parts:

- User Manual: general introduction and examples.
- Reference Manual: complete list of functionality.

Often one deals with several different interacting types and has to jump back and forth.
=> html is very convenient

https://judge.inf.ethz.ch/doc/cgal/doc_html/Kernel_23/group__PkgKernel23.html

INTERSECTIONS



Problem: We do not know the return type.

```
K::Iso_rectangle_2 r = … ;
K::Triangle_2 t = … ;
??? i = CGAL::intersection(r, t);
```

Solution: Use a generic wrapper class (based on `boost::variant`).

Test whether it contains an object of type `T` using `boost::get<T>`.

INTERSECTIONS

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <iostream>
#include <stdexcept>

typedef CGAL::Exact_predicates_exact_constructions_kernel K;
typedef K::Point_2 P;
typedef K::Segment_2 S;

int main()
{
  P p[] = { P(0,0), P(2,0), P(1,0), P(3,0), P(.5,1), P(.5,-1) };
  S s[] = { S(p[0],p[1]), S(p[2],p[3]), S(p[4],p[5]) };
  for (int i = 0; i < 3; ++i)
    for (int j = i+1; j < 3; ++j)
      if (CGAL::do_intersect(s[i],s[j])) {
        auto o = CGAL::intersection(s[i],s[j]);
        if (const P* op = boost::get<P>(&*o))
          std::cout << "point: " << *op << "\n";
        else if (const S* os = boost::get<S>(&*o))
          std::cout << "segment: " << os->source() << " "
                    << os->target() << "\n";
        else // how could this be? -> error
          throw std::runtime_error("strange segment intersection");
      } else
        std::cout << "no intersection\n";
}
```

The actual type is `std::result_of<K::Intersect_2(S,S)>::type`

Needs `#include <type_traits>`

Test for intersection (predicate)

Construct intersection (construction :-))

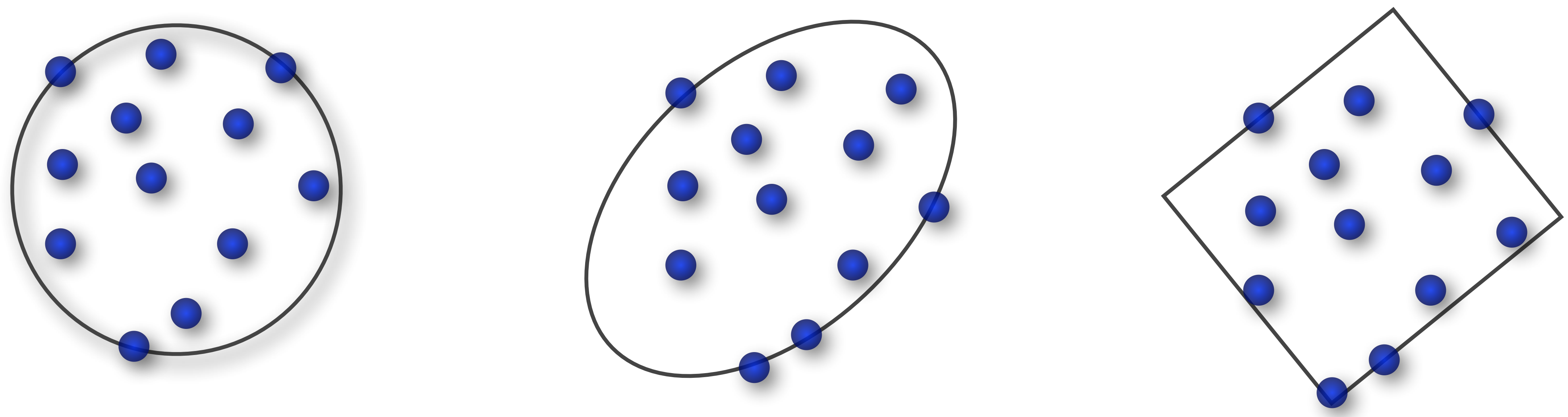
Cast fails (=0) if o is not of type P.

Use auto with caution! It disables static type checks that can be quite useful…

Output:

```
segment: 1 0 2 0
point: 0.5 0
no intersection
```

BOUNDING VOLUMES



Problem: Given n points in $\mathbb{R}^2$, what is their minimum enclosing ... ?

- Circle
- Ellipse
- (Circular) annulus

→ Can be computed in expected linear time.

- Rectangle
- Parallelogram
- Strip

→ Can be computed in linear time once the convex hull is known.

https://judge.inf.ethz.ch/doc/cgal/doc_html/Bounding_volumes/

MINIMUM ENCLOSING CIRCLE

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <CGAL/Min_circle_2.h>
#include <CGAL/Min_circle_2_traits_2.h>
#include <iostream>

// typedefs
typedef  CGAL::Exact_predicates_exact_constructions_kernel K;
typedef  CGAL::Min_circle_2_traits_2<K>  Traits;
typedef  CGAL::Min_circle_2<Traits>      Min_circle;

int main()
{
  const int n = 100;
  K::Point_2 P[n];

  for (int i = 0; i < n; ++i)
    P[i] = K::Point_2((i % 2 == 0 ? i : -i), 0);
  // (0,0), (-1,0), (2,0), (-3,0), ...

  Min_circle mc(P, P+n, true);
  Traits::Circle c = mc.circle();
  std::cout << c.center() << " " << c.squared_radius() << std::endl;
}
```

Many data structures and algorithms have their own traits concept. It defines the geometric primitives needed.

Separate: Combinatorial algorithm <=> geometry

Build from a range of points.

Attention! Constructions are used inside...

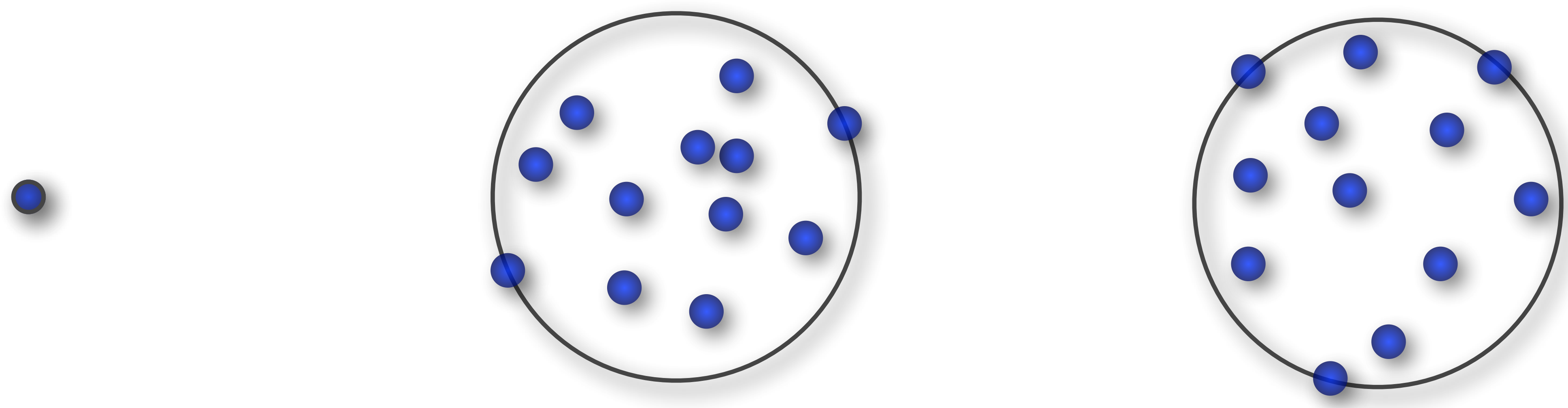
Randomize input order? Generally a good idea, unless input is known to be random, anyway.

Construct and return the circle.

Output:

```
-0.5 0 9702.25
```

MINIMUM ENCLOSING CIRCLE



The minimum enclosing circle for a set of $n \geq 1$ points in $\mathbb{R}^2$ is determined ... by at most three points on its boundary.

These so-called support points can be obtained using corresponding member functions and iterators of `CGAL::Min_circle_2`.

https://judge.inf.ethz.ch/doc/cgal/doc_html/Bounding_volumes/

PART IV:

Practical Information

INTEGER IO

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <iostream>

typedef CGAL::Exact_predicates_exact_constructions_kernel K;
...
// this is nicer ... (and usually ok for the inexact_constructions kernel)
K::Point_2 p;
std::cin >> p;
...
// this is faster ... assuming the input fits in a double
double x, y;
std::cin >> x >> y;
K::Point_2 p(x, y);
...
// this is even faster ... assuming the input fits in an int
int x, y;
std::cin >> x >> y;
K::Point_2 p(x, y);
```

IO PERFORMANCE

- If possible, read as an `int`
- else read as a `long`

32bit on the judge

64bit on the judge

Typical for 64-bit computers, but not universally true....

Sanity check

`double` literal for 2^25

```
#include <limits>

if (std::numeric_limits<int>::max() < 33554432.0)
  throw std::range_error("max(int) < 2^(25)");
```

USING CGAL

Best start in a new directory, name source file s.t. it ends with `.cpp`.

Run `cgal_create_cmake_script` in this directory.

`cmake .`

Note the dot (current directory)!

This creates a makefile with rules and targets for every `.cpp` file.

You can then build your program using `make`

If you want to use C++11 features, add the line

```
set(CMAKE_CXX_FLAGS "${CMAKE_CXX_FLAGS} -std=c++11")
```

somewhere in the `CMakeLists.txt` file.

You have to re-run `cgal_create_cmake_script` whenever you add a new application/`.cpp` file.

No need to re-run `cmake` because that's done by `make` automatically.

As a default, makefiles are created in release mode. If you want to debug, run `cmake -DCMAKE_BUILD_TYPE=Debug .`

To go back to release mode, run `cmake -DCMAKE_BUILD_TYPE=Release .`

If you want to see the actual compiler and linker calls, run `cmake -DCMAKE_VERBOSE_MAKEFILE=ON .`

That's it!

For more, see...

If you want to install CGAL on your private computer:

- Check/install prerequisites first: compiler, cmake, boost, gmp, mpfr, (qt)
- Install cgal (on the judge we run CGAL-4.6.2)
  `https://judge.inf.ethz.ch/doc/cgal/doc_html/Manual/installation.html`
- Or download CGAL packages of your distribution if they exist (don't forget cgal-devel).

HTTP://WWW.CGAL.ORG