6. Multi-item inventory control

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Overview

- Deterministic models
  - Warehouse scheduling problem
  - Economic lot-scheduling problem
  - Joint replenishment problem

- Stochastic models
  - Multi-product-newsvendor
  - Can-order-policies

(1) Warehouse scheduling problem

- Assumptions
  - i=1,2,…,N products
  - EOQ assumptions for each product i
  - Warehouse capacity *W*
  - Capacity requirements for each unit of product *i*: a_i
- Replenishment and storage strategies
  - Dedicated space
  - Average utilization
  - Common-cycle model

Optimization problems

- Dedicated capacity
  - Convex cost function
  - Linear constraints

$$\min \sum_{i=1}^{N} \left[\frac{d_i}{Q_i} \cdot A_i + \frac{h_i}{2} \cdot Q_i \right]$$

$$s.t. \sum_{i=1}^{N} a_i Q_i \leq W$$

$$Q_i \geq 0 \quad i = 1,2,\ldots,N$$

- Average utilization

$$\min \sum_{i=1}^{N} \left[\frac{d_i}{Q_i} \cdot A_i + \frac{h_i}{2} \cdot Q_i \right]$$

$$s.t. \sum_{i=1}^{N} 0.5 a_i Q_i \leq W$$

$$Q_i \geq 0 \quad i = 1,2,\ldots,N$$

Solution

- Lagrange function $$L = \sum_{i=1}^{N}\left[\frac{d_i}{Q_i}A_i + \frac{h_i}{2}Q_i\right] + \lambda\left[\sum_{i=1}^{N} a_i Q_i - W\right]$$
- Optimality conditions
$$\frac{\partial L}{\partial Q_i} = \frac{-d_i}{Q_i^2}A_i + \frac{h_i}{2} + \lambda a_i = 0$$
$$\Rightarrow Q_i^*(\lambda) = \sqrt{\frac{2 d_i A_i}{h_i + 2\lambda a_i}} \qquad i = 1,2,\ldots,N$$
- Solution
  - Case 1: Unconstrained solution $\lambda = 0$
  - Case 2: Constrained solution, choose λ such that

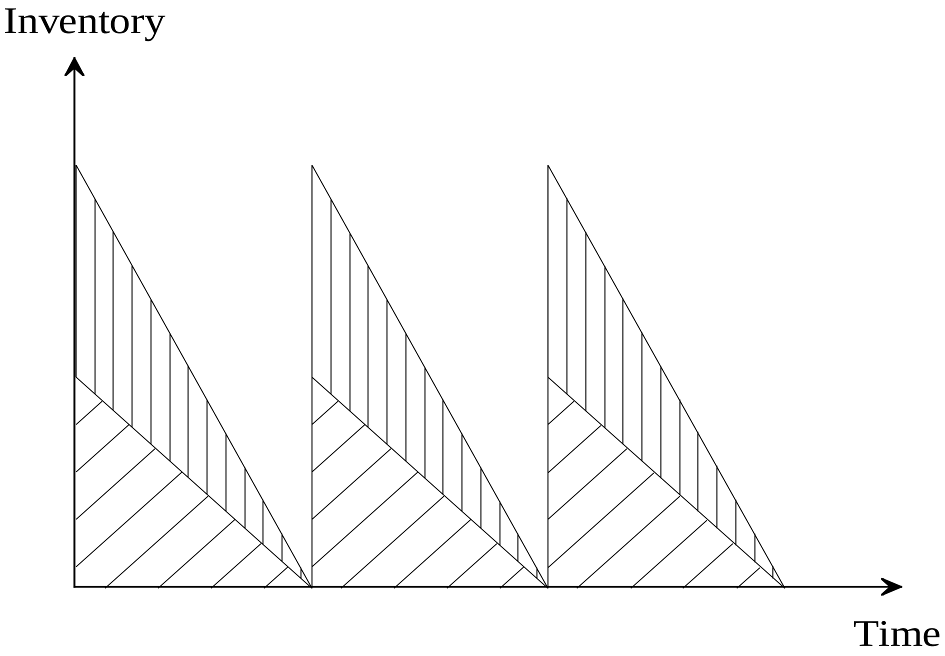
$$\sum_{i=1}^{N} a_i Q_i^*(\lambda) = W$$

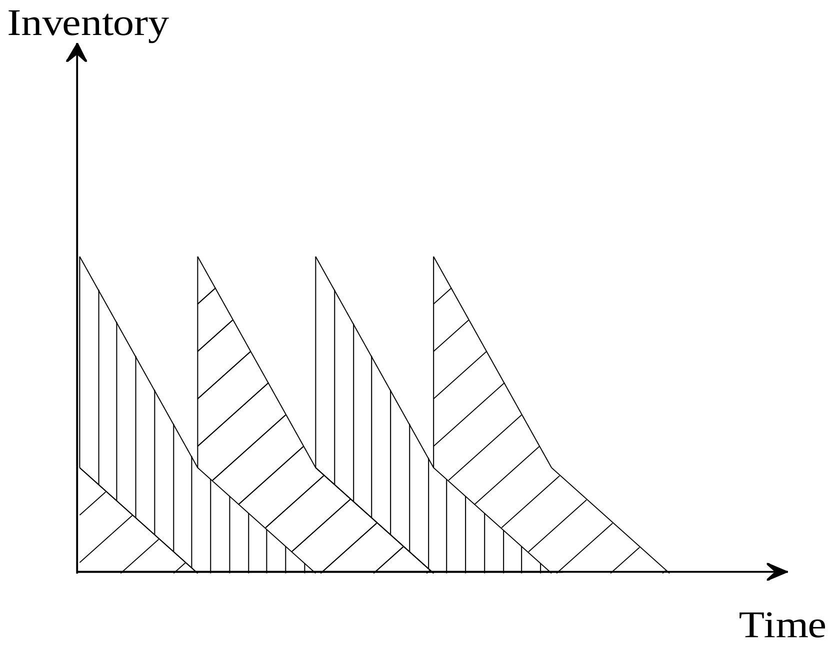
Rotation (common) cycle

- Assumption
  - Common replenishment cycle T for all products
  - Each product is ordered once in a cycle

- Question
  - Optimal timing of replenishment of each product within a cycle
  - Maximum capacity usage
  - What is the optimal cycle length?

Inventory levels





- Lot-size $Q_i = d_i T$ at time t_i
- Optimality conditions
  - Equal capacity utilization after each replenishment
  - Capacity requirement of a lot = released capacity since last order (of the other products)
  - Sequence of replenishments irrelevant

Analysis

$T = Q/d$

- Capacity balance $$(t_i - t_{i-1})\sum_{j=1}^{N} a_j d_j = a_i d_i T \Leftrightarrow t_i = \frac{\sum_{j=1}^{i} a_j d_j}{\sum_{j=1}^{N} a_j d_j} T$$

Check: write as recursion: $t_i = \ldots + t_{i-1} = \ldots + t_{i-2} = \ldots$

- Capacity requirement at t=0

$$W \geq \sum_{i=1}^{N} a_i (Q_i - d_i(T - t_i)) = \sum_{i=1}^{N} a_i d_i t_i = \frac{\sum_{i=1}^{N} \sum_{j=1}^{i} a_i a_j d_i d_j}{\sum_{i=1}^{N} a_i d_i} T$$

- Cost function $$C = \sum_{i=1}^{N} \left(\frac{A_i}{T} + \frac{h_i d_i}{2} T \right)$$

- Solution $$T^* = \min \left\{ \sqrt{\frac{2\sum_{i=1}^{N} A_i}{\sum_{i=1}^{N} h_i d_i}} ; W \frac{\sum_{i=1}^{N} a_i d_i}{\sum_{i=1}^{N} \sum_{j=1}^{i} a_i a_j d_i d_j} \right\}$$

Example

See Excel file

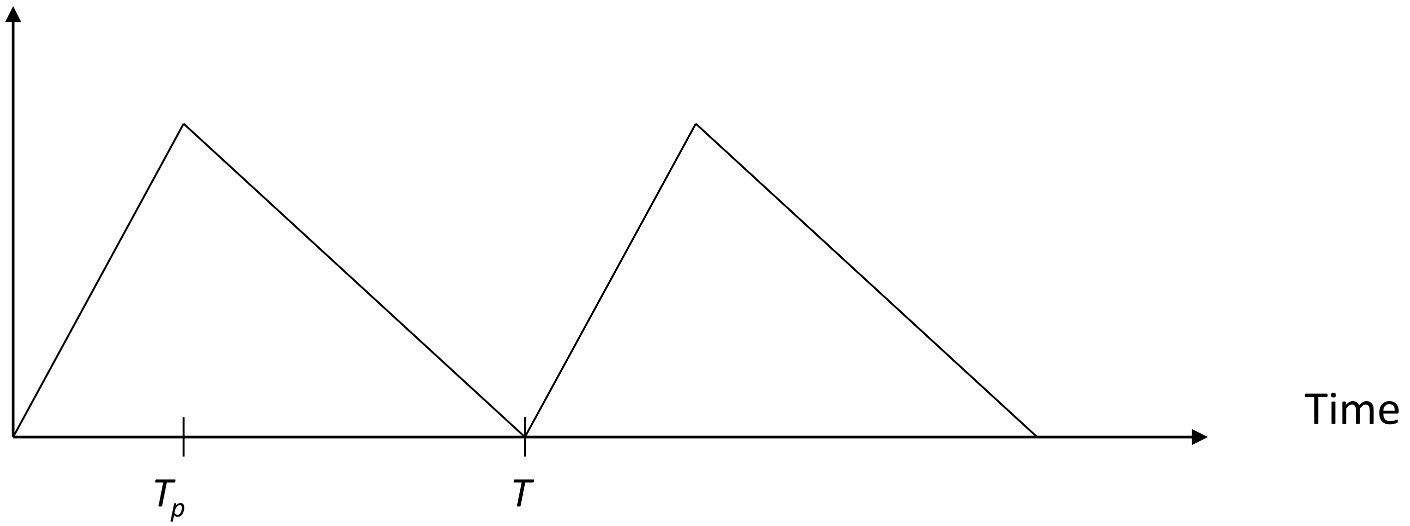
- Data

	1	2	3
d	50	200	125
A	500	1000	700
h	3	5	10
a	1	2	1

  – Case 1: W=900; Case 2: W=500
- Solution Dedicated Capacity
  – Case 1: Q_1=129.1; Q_2=282.84; Q_3=132.29; C=3124.39
  – Case 2: λ=2.735; Q_1=76.83; Q_2=158.41; Q_3=106.36; C=3453.72
- Solution Common Cycle
  – Case 1: Q_1=67.7; Q_2=270.8; Q_3=169.3; C=3249.62
  – Case 2: Q_1=56.5; Q_2=226.0; Q_3=141.3; C=3302.79

Economic production quantity (EPQ)

- Additional assumption: finite production rate p (units per unit of time), $p>d$
- Model
  - Lot-size: Q
  - Cycle length: $T=Q/d$
  - Manufacturing time: $T_p=Q/p$
- Inventory dynamics



Optimal EPQ

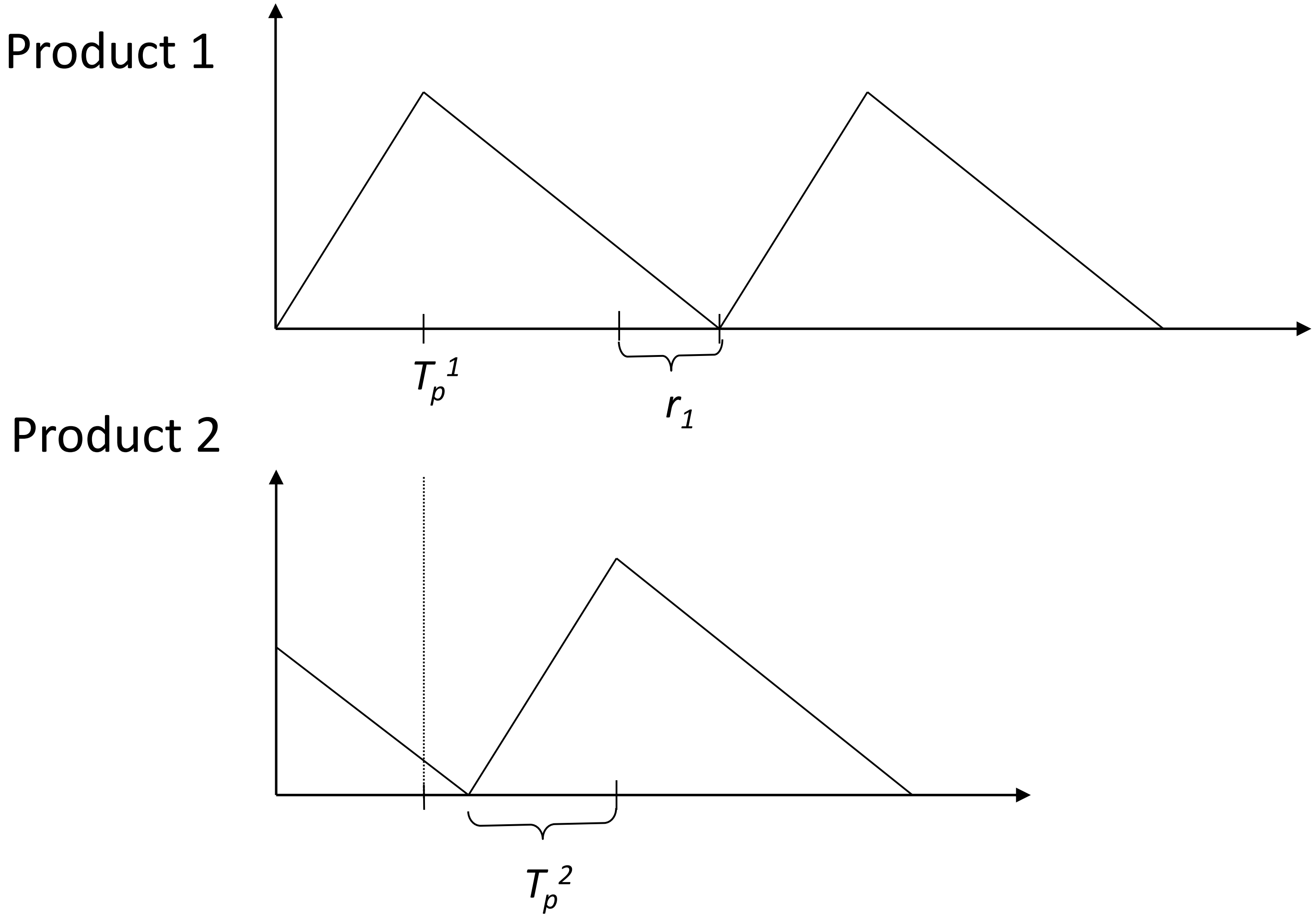
- Cost function
  - Setup cost per cycle: *A*
  - Inventory cost per cycle $$\frac{h}{2}(p-d)T_p^2 + \frac{h}{2}(p-d)T_p(T-T_p) = \frac{h}{2}(p-d)T_pT$$
  - Total cost per unit of time $$C(Q) = \frac{d}{Q}\cdot A + \frac{h}{2}(p-d)\cdot\frac{Q}{p}$$
- Optimal lot-size

$$Q^* = \sqrt{\frac{2dA}{h\cdot\left(1-\frac{d}{p}\right)}} = \sqrt{\frac{p}{p-d}}\cdot EOQ > EOQ$$

(2) Economic lot scheduling problem

- Assumptions
  - Products j=1,2,...,n; one resource/machine
  - Setup cost A_j, setup time r_j, production rate p_j
  - Inventory holding costs per unit and unit of time h_j
  - Demand rates d_j
- Planning problem
  - Determination of lot-sizes
  - Determination of manufacturing sequences
  - Demand satisfaction
- Common cycle model
  - Each product is produced once in a cycle
  - Optimal cycle length

Scheduling problem



Relationship T_p vs T

- $T = T_p$ + "depletion time"
- Depletion time = time to deplete $(p-d)T_p$, at rate d
- So, $T = T_p + (p-d)T_p/d = pT_p/d$
- Or, $T_p = dT/p$

Optimization problem (Common cycle approach)

- Objective function
  - Minimize setup and inventory holding cost

$$\min C(T) = \sum_{j=1}^{n} \left(\frac{A_j}{T} + \frac{h_j}{2}(p_j - d_j)\frac{d_j}{p_j}T \right)$$

- Constraints
  - Sum of setup and manufacturing times cannot exceed cycle length

$$\sum_{j=1}^{n} \left(r_j + \frac{d_j T}{p_j} \right) \leq T \Rightarrow T \geq \frac{\sum_{j=1}^{n} r_j}{1 - \sum_{j=1}^{n} \frac{d_j}{p_j}}$$

Solution

- Lagrange-Multiplier
- Unconstrained solution

$$T^* = \sqrt{\frac{2\sum_{j=1}^{n} A_j}{\sum_{j=1}^{n} h_j d_j \left(1 - \frac{d_j}{p_j}\right)}} \qquad Q_j^* = d_j T^* \quad j = 1,2,\ldots,n$$

- Lower bound

$$T^* = \frac{\sum_{j=1}^{n} r_j}{1 - \sum_{j=1}^{n} \frac{d_j}{p_j}}$$

Basic period – approach

- Assumptions
  - Basic period of length *W*
  - Individual cycle times T_i are an integer multiple of basic period, $T_i = n_i W$, and $\mathbf{n_i = 2^m}$.
- Cost function

$$\min C(W, n_i) = \sum_{j=1}^{n} \left(\frac{A_j}{n_j W} + \frac{h_j}{2}(p_j - d_j)\frac{d_j}{p_j} n_j W \right)$$

Solution algorithm

1. Determine W
   - E.g. choose minimum cycle length from independent optimization
2. Determine n_i, given W
   $$C_i(n_i) = \frac{A_i}{n_i W} + h_i(p_i - d_i)\frac{d_i}{p_i}\frac{n_i W}{2}$$
3. Determine
   $$W = \sqrt{\frac{2\sum_{i=1}^{N}\frac{A_i}{n_i}}{\sum_{i=1}^{N} h_i(p_i - d_i)\frac{d_i}{p_i} n_i}}$$
4. Back to Step 2 unless the procedure has converged (multipliers do not change anymore). If converged, check feasibility. If not feasible, adjust n_i and determine W from 3.

Feasibility: $\sum_{j=1}^{n}\left(\frac{r_j}{n_j} + \frac{Q_j}{n_j p_j}\right) \leq W$

Example

d	50	60	150	100	200	40
p	500	500	1000	1000	2000	500
r	0.5	1	2	1	2	0.5

A_j=10, h_j=0.0002

See Excel file

Solutions

- Independent solution
  - Lot-sizes
  - Provides lower bound
- Common cycle
  - Provides upper bound
- Basic period
  - Heuristic solution

(3) Joint replenishment problem

Axsäter, 7.3.1.1

- Different ordering cost structure
  - Major setup cost A_0
    - For each order (regardless of product and quantity)
  - Minor setup cost A_i
    - For each order of a product *i* (regardless of order quantity)

Solution approach

- Without major setup costs: classical EOQ formula: $T_i = \sqrt{\dfrac{2A_i}{h_i d_i}}$

- Let T_1 the smallest cycle time and assume that all cycle times $T_i = n_i T_1$ for i=2,…,N are integer multiples of T_1 ($n_1 = 1$)

- Objective $$C = \frac{A_0 + \sum_{i=1}^{N} \frac{A_i}{n_i}}{T_1} + \frac{T_1 \sum_{i=1}^{N} h_i d_i n_i}{2}$$

- Optimal initial cycle time (note: NOT according to the classical EOQ formula above):

$$T_1^* = \sqrt{\frac{2(A_0 + \sum_{i=1}^{N} \frac{A_i}{n_i})}{\sum_{i=1}^{N} h_i d_i n_i}}$$

Bounding

- Optimal cost $$C^* = \sqrt{2\left(A_0 + \sum_{i=1}^{N} \frac{A_i}{n_i}\right)\sum_{i=1}^{N} h_i d_i n_i}$$
- Optimal (non-integer) multipliers $$n_i = \sqrt{\frac{A_i h_1 d_1}{h_i d_i (A_0 + A_1)}}$$
- Lower bound $$\underline{C} = \sqrt{2(A_0 + A_1) h_1 d_1} + \sum_{i=2}^{N} \sqrt{2 A_i h_i d_i}$$

Iterative solution approach

1. Determine initial multipliers by rounding non-integer values

2. Determine T_1

3. Determine new integers from cost function

   – Minimum integer that satisfies $n_i(n_i+1) \geq \frac{2A_i}{h_i d_i T_1^2}$

4. Back to step 2 if any integer changed

Numerical example

- N=4 products
- Major setup cost A_0=300, minor setup costs A_i=50, holding cost h_i=10
- Demand rates d_1=5000, d_2=1000, d_3=700, d_4=100

Solution

- Lower bound cost: 8069
- Initial integers: $n_1=n_2=n_3=1, n_4=3$
- T_1=0.1155
- New integers: $n_1=n_2=n_3=1, n_4=3$
- Cost: 8082.90

See Excel file

Dynamic joint replenishment

- Assumptions
  - Discrete time periods $t = 1,2,...,T$
  - Dynamic demands d_{kt} for product k
  - Other assumptions as in Joint Replenishment Problem
- Decision variables
  - q_{kt} Lot-size (Order quantity) for product k in t
  - y_{kt} Inventory level for product k at the end of period t
  - γ_{kt} Setup indicator, $\gamma_{kt} = 1$ if a lot is placed for product k in period t, $\gamma_{kt} = 0$ otherwise

Dynamic joint replenishment

$$\min \sum_{t=1}^{T} (A_0 \gamma_{0t} + \sum_{k=1}^{K} \left(A_k \cdot \gamma_{kt} + h_k \cdot y_{kt} \right))$$

$$y_{kt} = y_{k,t-1} + q_{kt} - d_{kt} \qquad t = 1,2,\ldots,T; k = 1,2,\ldots,K$$

$$q_{kt} \le M \gamma_{kt} \qquad t = 1,2,\ldots,T; k = 1,2,\ldots,K$$

$$\gamma_{kt} \le \gamma_{0t} \qquad t = 1,2,\ldots,T; k = 1,2,\ldots,K$$

$$y_{k0} = y_{kT} = 0 \qquad k = 1,2,\ldots,K$$

$$q_{kt}, y_{kt} \ge 0, \gamma_{kt} \in \{0,1\} \qquad t = 1,2,\ldots,T; k = 1,2,\ldots,K$$

Capacitated lot-sizing problem (CLSP)

- Multi-products $k = 1,2,\ldots,K$
- Discrete time periods $t = 1,2,\ldots,T$
- One machine
- Finite production rate a_k
- Dynamic demand d_{kt}
- Objective: Cost minimization

CLSP cont'd

- Assumptions
  - Product specific setup cost A_k and holding cost h_k
  - M_t available production time in period t
- Decision variables
  - q_{kt} Lot-size (Production quantity) for product *k* in *t*
  - y_{kt} Inventory level for product *k* at the end of period *t*
  - γ_{kt} Setup indicator, γ_{kt}=1 if a lot is placed for product *k* in period *t*, γ_{kt}=0 otherwise

Model

$$\min \sum_{k=1}^{K}\sum_{t=1}^{T} h_k y_{kt} + \sum_{k=1}^{K}\sum_{t=1}^{T} A_k \gamma_{kt}$$

$$y_{kt} = y_{k,t-1} + q_{kt} - d_{kt} \quad k = 1,2,\ldots,K; t = 1,2,\ldots,T$$

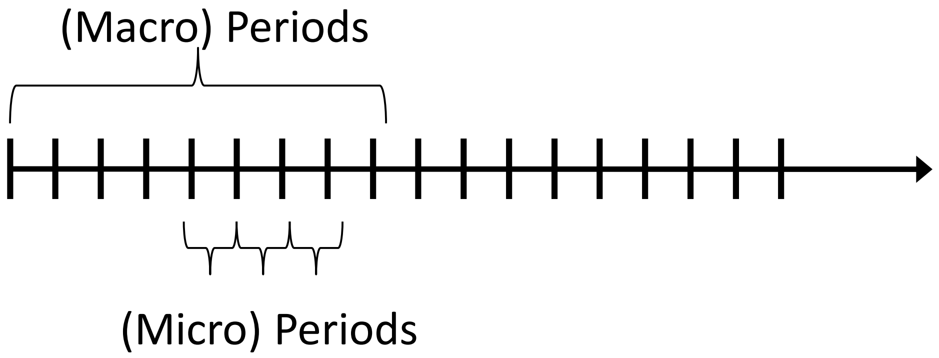
$$\sum_{k=1}^{K} a_k q_{kt} \le M_t \quad t = 1,2,\ldots,T$$

$$q_{kt} \le \frac{M_t}{a_k}\gamma_{kt} \quad k = 1,2,\ldots,K; t = 1,2,\ldots,T$$

$$q_{kt}, y_{kt} \ge 0,\ \gamma_{kt} \in \{0,1\} \quad t = 1,2,\ldots,T$$

Discrete lot-sizing and scheduling problem (DLSP)

- Decision about the sequence



- Additional notations and variables to (DLSP)
  - s = 1,2,…,S : (Micro) Periods
  - $\bar{\gamma}_{ks}$ Setup indicator, $\bar{\gamma}_{ks}$ =1 if a lot is placed for product *k* in period *s* = 0 otherwise $\bar{\gamma}_{ks}$
  - γ_{ks} Production indicator, γ_{ks} =1 if a lot is placed for product *k* in period *s* = 0 otherwise γ_{ks}

Model

$$\min \sum_{k=1}^{K}\sum_{s=1}^{S} h_k y_{ks} + \sum_{k=1}^{K}\sum_{s=1}^{S} A_k \gamma_{ks}$$

$$y_{ks} = y_{k,s-1} + \frac{M_s}{a_k}\bar{\gamma}_{ks} - d_{ks} \qquad \forall k = 1,2,\ldots,K; s = 1,2,\ldots,S$$

$$\sum_{k=1}^{K} \bar{\gamma}_{ks} \le 1 \qquad \forall s = 1,2,\ldots,S$$

$$\gamma_{ks} \ge \bar{\gamma}_{ks} - \bar{\gamma}_{k,s-1} \qquad \forall k = 1,2,\ldots,K; s = 1,2,\ldots,S$$

$$q_{ks}, y_{ks} \ge 0, \bar{\gamma}_{ks}, \gamma_{ks} \in \{0,1\} \qquad \forall s = 1,2,\ldots,S$$

Example: Single-machine multi-item DLSP

- A single machine produces 6 different products. The planning horizon is devided into 60 periods. Demand takes place in 6 equidistant macro periods (every 10th period, first demand at T=10). At the beginning of the planning horizon a certain initial inventory for each product is available. The production speed is one unit per period. Demands, initial inventory (Ini.Inv.), production speed (a) and setup-cost (A) are given below (‚dlsp.dat‘).

- What is the optimal production schedule?

Demand Product	Demand instances 10	20	30	40	50	60	Ini. Inv.	a	A
1	3	3	2	1	2	5	3	1	60
2	3	1	0	1	0	3	1	1	60
3	0	2	0	0	2	3	3	1	60
4	2	0	0	1	3	1	1	1	60
5	0	1	0	0	2	4	1	1	60
6	0	0	0	1	0	1	1	1	60

See Xpress file -> IPYNB!

Continuous setup lot-sizing problem (CSLP)

- Waiver of a strict DLSP assumption

- Additional notations and variables to (CLSP)
  - q_{ks} Lot-size (Production quantity) for product *k* in *s*
  - β_{ks} Setup indicator, β_{ks}=1 if a lot is placed for product *k* in period s, β_{ks}=0 otherwise

Model

$$\min \sum_{k=1}^{K}\sum_{s=1}^{S} h_k y_{ks} + \sum_{k=1}^{K}\sum_{s=1}^{S} A_k \gamma_{ks}$$

$$y_{ks} = y_{k,s-1} + q_{ks} - d_{ks} \qquad \forall k = 1,2,\ldots,K;\, s = 1,2,\ldots,S$$

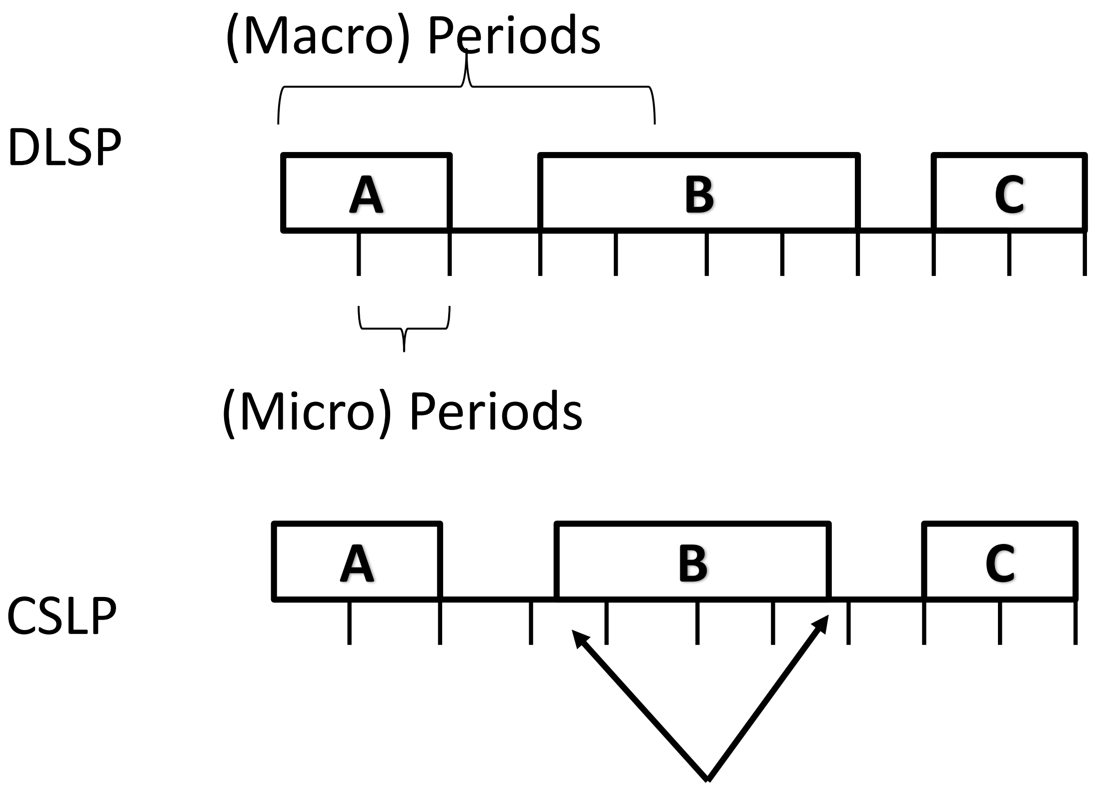
$$\sum_{k=1}^{K} \beta_{ks} \leq 1 \qquad \forall s = 1,2,\ldots,S$$

$$q_{ks} \leq \frac{M_s}{a_k} \beta_{ks} \qquad \forall k = 1,2,\ldots K;\, s = 1,2,\ldots,S$$

$$\gamma_{ks} \geq \beta_{ks} - \beta_{k,s-1} \qquad \forall k = 1,2,\ldots,K;\, s = 1,2,\ldots,S$$

$$q_{ks}, y_{ks} \geq 0, \beta_{ks}, \gamma_{ks} \in \{0,1\} \qquad \forall s = 1,2,\ldots,S$$

Comparison DLSP, CSLP



Batches for
Products A, B, C

No unnecessary Inventory, free scheduling of machine idle time

Proportional lot-sizing and scheduling problem (PLSP)

- Maximum two different products can be manufactured per period
- Setup condition changes maximum once a period
- Production quantities are proportional to the moment of setup
- Setup condition remains in periods with no production

Model

$$\min \sum_{k=1}^{K}\sum_{s=1}^{S} h_k y_{ks} + \sum_{k=1}^{K}\sum_{s=1}^{S} A_k \gamma_{ks}$$

$$y_{ks} = y_{k,s-1} + q_{ks} - d_{ks} \qquad \forall k = 1,2,\ldots,K;\ s = 1,2,\ldots,S$$

$$\sum_{k=1}^{K} a_k q_{ks} \le M_s \qquad \forall s = 1,2,\ldots,S$$

$$\sum_{k=1}^{K} \beta_{ks} \le 1 \qquad \forall s = 1,2,\ldots,S$$

$$q_{ks} \le \frac{M_s}{a_k}(\beta_{k,s-1} + \beta_{ks}) \qquad \forall k = 1,2,\ldots,K;\ s = 1,2,\ldots,S$$

$$\gamma_{ks} \ge \beta_{ks} - \beta_{k,s-1} \qquad \forall k = 1,2,\ldots,K;\ s = 1,2,\ldots,S$$

$$q_{ks}, y_{ks} \ge 0,\ \beta_{ks}, \gamma_{ks} \in \{0,1\} \qquad \forall k = 1,2,\ldots,S$$

Multi-product newsvendor

- Aggregate inventory constraint
  - Warehouse space
  - Budget constraint
  - Shelf-space allocation
- Aggregate service level constraint
  - Substitution in case of a stockout
  - Group service level

Aggregate inventory constraint

- Optimization problem

$$\max \Pi = \sum_{i=1}^{N} \left((p_i - c_i) S_i - p_i \int_0^{S_i} F(x) dx \right)$$

$$s.t. \quad \sum_{i=1}^{N} a_i S_i \leq W$$

- Solution (Lagrange approach)

$$F_i(S_i) = \frac{p_i - c_i - \lambda}{p} \quad i = 1, \ldots, N$$

$$\sum_{i=1}^{N} a_i S_i(\lambda) \leq W$$

Service level constraint

- Application: Group service level
- Example
  - Not every type of salad has to be availalbe at the end of the day, but there should be at least some salad available

- Products i=1,2,...,N
  - Newsvendor assumptions for each product
  - Aggregate non-stockout probability α_G

Optimization and solution

- Optimization problem

$$\max \Pi = \sum_{i=1}^{N}\left((p_i - c_i)S_i - p_i \int_0^{S_i} F(x)dx \right)$$

$$s.t. \quad P\left(\sum_{i=1}^{N} D_i \le \sum_{i=1}^{N} S_i \right) \ge \alpha$$

- Solution (Lagrange approach)

$$F_i(S_i) = \frac{p_i - c_i + \lambda}{p} \quad i = 1,\ldots,N$$

$$F_G\left(\sum_{i=1}^{N} S_i(\lambda) \right) \ge \alpha$$