3. Inventory analytics: Demand modelling

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Demand uncertainty

- Certainty
  - Demand is known
- Uncertainty
  - Demand is unknown
    - Statistical distribution and parameters are known
    - Unknown parameters: Estimation

Demand modelling

- Empirical distribution
- Theoretical distribution
  - Discrete distributions
    - Poisson distribution
    - Compound Poisson distribution
  - Continuous distributions
    - Normal distribution
    - Gamma distribution

Parameter estimation

- Theoretical distribution
- Selection of probability distribution
- Goodness of fit test
  - Chi-square test (χ^2 Test)
  - Kolmogorov-Smirnov test
  - Exact multinomial test
- Moment fitting procedure
  - Estimate moments from data
  - Fit distribution parameters on these moments
- Maximum Likelihood

χ^2 Test

- Hypothesis: D is distributed according to F
  - Estimate distribution parameters
- Split demand observation into k categories

  - O_j : observed frequency in the jth class interval
  - E_j : expected frequency in the jth class interval ($E_j = np_j$)
  - p_j : theoretical, hypothesized probability associated with the jth class interval.

  - Measure $\chi_0^2 = \sum_{j=1}^{k} \frac{(O_j - E_j)^2}{E_j}$

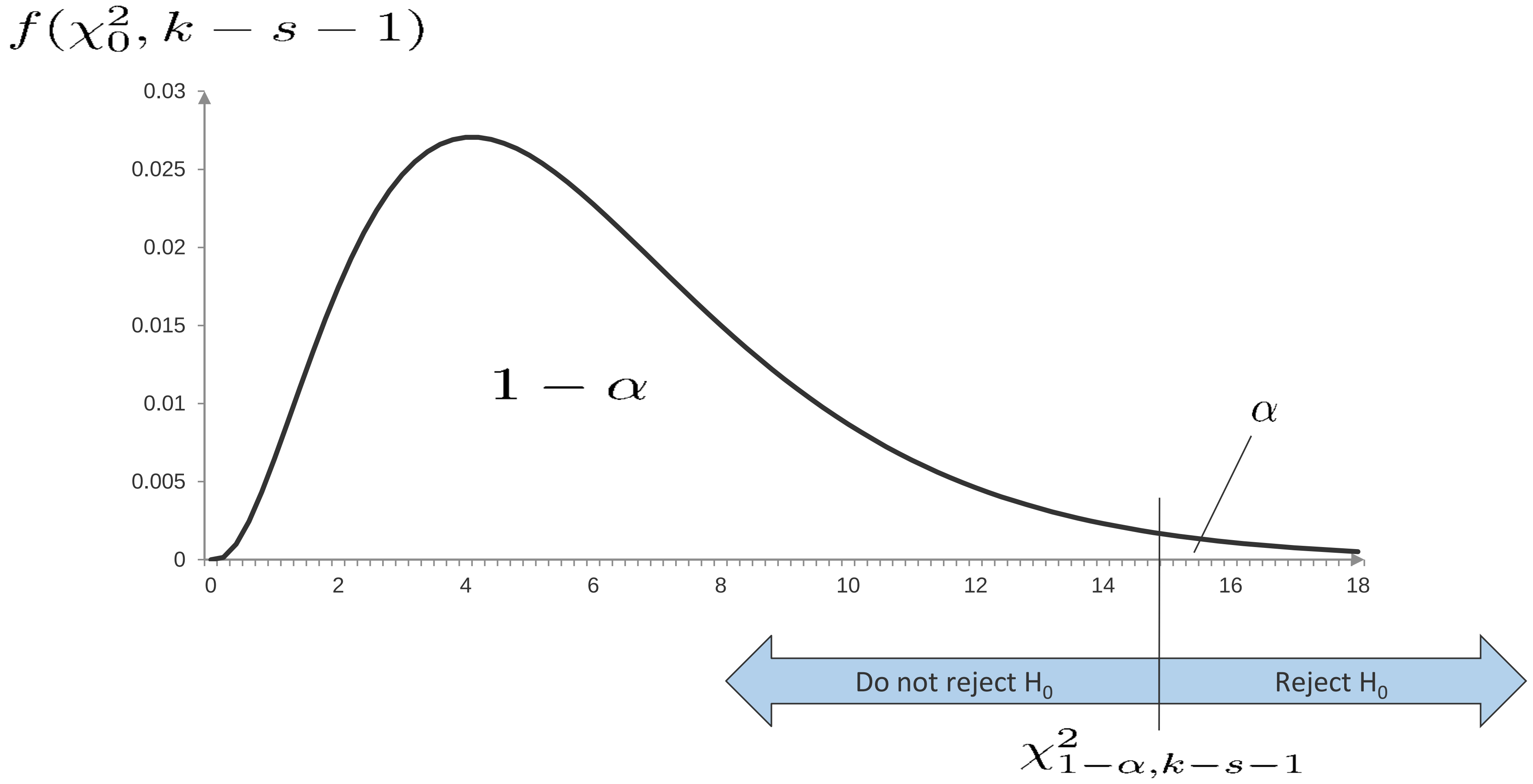
χ^2 Test

- It can be shown that χ_0^2 approximately follows the chi-square distribution with k-s-1 degrees of freedom.

- s : number of parameters of the hypothesized distribution estimated by sample statistics.
- Level of significance: α (The maximum probability for type I error)

- Critical value: $\chi^2_{1-\alpha,k-s-1}$
- H_0 is rejected if: $\chi_0^2 > \chi^2_{1-\alpha,k-s-1}$

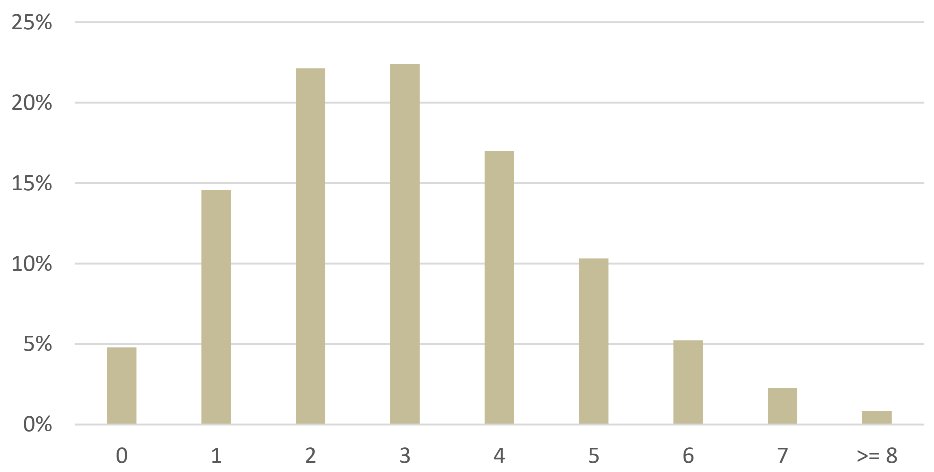
χ^2 Test


$f(\chi_0^2, k-s-1)$
0.03
0.025
0.02
0.015
0.01
0.005
0
0
2
4
6
8
10
12
14
16
18
$1-\alpha$
α
Do not reject H_0
Reject H_0
$\chi^2_{1-\alpha,k-s-1}$

Example χ^2 Test

Given the previous months data, a manager expects that the number of return shipments within a set period follows a Poisson distribution.

Nr. Returns	0	1	2	3	4	5	6	7	>=8	Total
Observed Frequency	20	59	80	82	55	51	20	10	3	380
Total	0	59	160	246	220	255	120	70	24	1154
Expected Frequency	18	55	84	85	65	39	20	9	5	380



- H_0 : Return rate follows a Poisson Distribution
- $\tilde{x} = \frac{1154}{380} \approx 3$
- Significance Level (α): 5%
- Degree of Freedom = 9 – 1 – 1 = 7

$$\chi_0^2 = \frac{(20-18)^2}{18} + \frac{(59-55)^2}{55} + \dots + \frac{(3-5)^2}{5} = 6.7$$

$\chi^2_{1-\alpha,k-s-1}$ = 14.06

∴ Do not reject H_0

Table $\chi 2$ Distribution

Df \ p	0.95	0.9	0.8	0.7	0.5	0.3	0.2	0.1	0.05	0.01	0.001
1	0.0039	0.0158	0.0642	0.1485	0.4549	1.0742	1.6424	2.7055	3.8415	6.6349	10.8276
2	0.1026	0.2107	0.4463	0.7133	1.3863	2.4079	3.2189	4.6052	5.9915	9.2103	13.8155
3	0.3518	0.5844	1.0052	1.4237	2.3660	3.6649	4.6416	6.2514	7.8147	11.3449	16.2662
4	0.7107	1.0636	1.6488	2.1947	3.3567	4.8784	5.9886	7.7794	9.4877	13.2767	18.4668
5	1.1455	1.6103	2.3425	2.9999	4.3515	6.0644	7.2893	9.2364	11.0705	15.0863	20.5150
6	1.6354	2.2041	3.0701	3.8276	5.3481	7.2311	8.5581	10.6446	12.5916	16.8119	22.4577
7	2.1673	2.8331	3.8223	4.6713	6.3458	8.3834	9.8032	12.0170	14.0671	18.4753	24.3219
8	2.7326	3.4895	4.5936	5.5274	7.3441	9.5245	11.0301	13.3616	15.5073	20.0902	26.1245
9	3.3251	4.1682	5.3801	6.3933	8.3428	10.6564	12.2421	14.6837	16.9190	21.6660	27.8772
10	3.9403	4.8652	6.1791	7.2672	9.3418	11.7807	13.4420	15.9872	18.3070	23.2093	29.5883

χ^2 Test considerations

- Valid only for large sample size
- Valid for both discrete and continuous distributional assumptions
- Valid when distributional parameters are estimated
- Sensitive to the way the data are grouped
- Alternative approaches:
    - Kolmogorov-Smirnov test
    - Multinomial test

Kolmogorov-Smirnov-test

- Demand observations d_i
  - Distribution and parameters
- **Empirical distribution** (upper bound, lower bound)
  - $S(x_i)$, $S(x_{i-1})$
- **Theoretical distribution** $F(x_i)$
- Measure: $\max\left\{\left|S(x_i)-F(x_i)\right|, \left|S(x_{i-1})-F(x_i)\right|\right\}$

- Critical values: Table

Example

Test for normal distribution

Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Max
Demand	39	55	62	62	75	80	81	86	87	87	99	114	115	118	118	125	125	145	164	170	
Empirical $S(x_i)$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	
Theoretical $F(x_i)$	0.04	0.10	0.14	0.14	0.23	0.29	0.30	0.34	0.35	0.35	0.48	0.65	0.66	0.69	0.69	0.76	0.76	0.89	0.96	0.98	
$\mid S(x_i)-F(x_i) \mid$	0.01	0.00	0.01	0.06	0.02	0.01	0.05	0.06	0.10	0.15	0.07	0.05	0.01	0.01	0.06	0.04	0.09	0.01	0.01	0.02	0.15
$\mid S(x_{i-1})-F(x_i) \mid$	0.04	0.05	0.04	0.01	0.03	0.04	0.00	0.01	0.05	0.10	0.02	0.10	0.06	0.04	0.01	0.01	0.04	0.04	0.06	0.03	0.10

Example

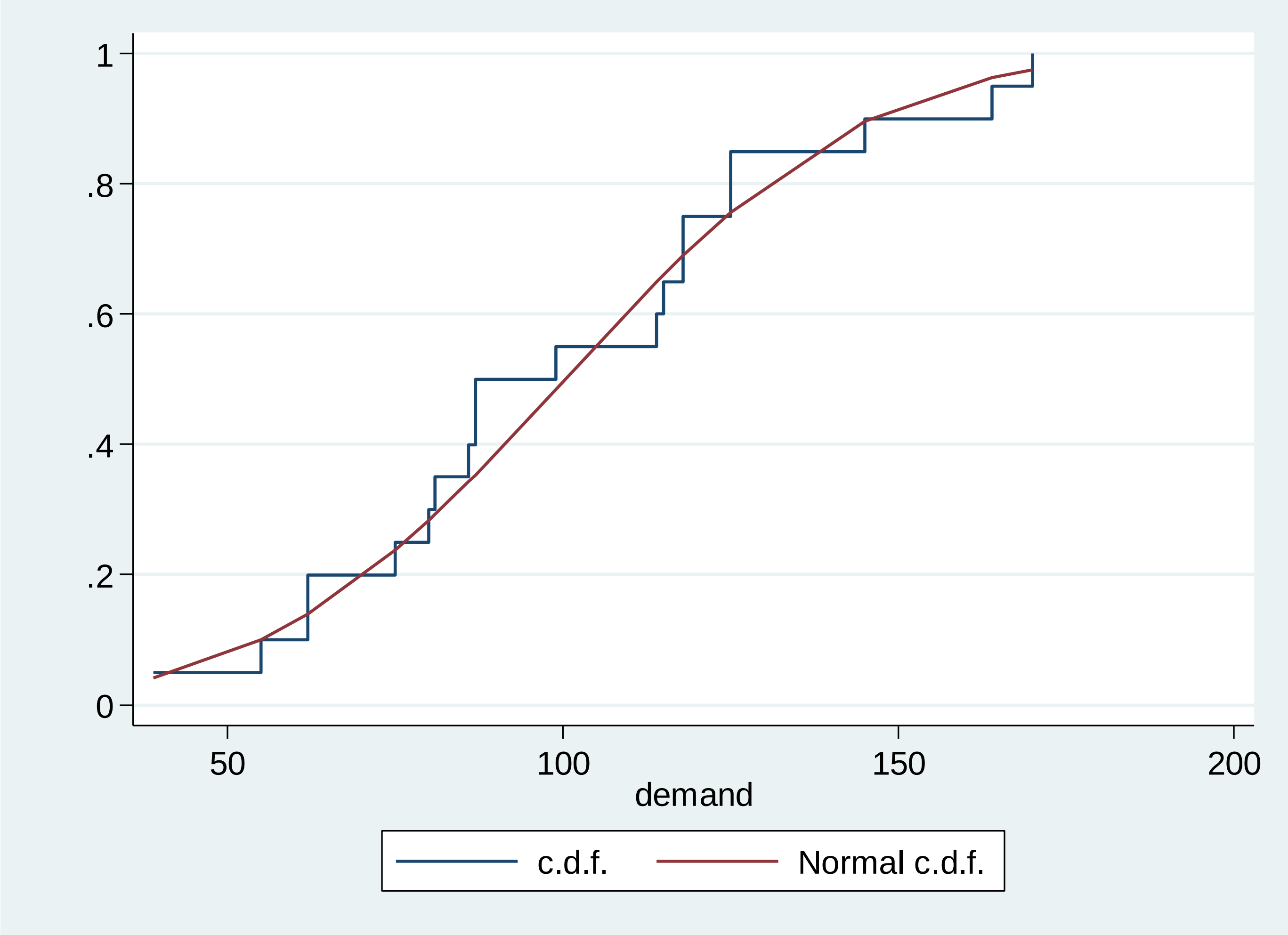


Table of critical values

SAMPLE SIZE (N)	LEVEL OF SIGNIFICANCE FOR D = MAXIMUM [$F_0(X)$ - $S_n(X)$] .20	.15	.10	.05	.01
1	.900	.925	.950	.975	.995
2	.684	.726	.776	.842	.929
3	.565	.597	.642	.708	.828
4	.494	.525	.564	.624	.733
5	.446	.474	.510	.565	.669
6	.410	.436	.470	.521	.618
7	.381	.405	.438	.486	.577
8	.358	.381	.411	.457	.543
9	.339	.360	.388	.432	.514
10	.322	.342	.368	.410	.490
11	.307	.326	.352	.391	.468
12	.295	.313	.338	.375	.450
13	.284	.302	.325	.361	.433
14	.274	.292	.314	.349	.418
15	.266	.283	.304	.338	.404
16	.258	.274	.295	.328	.392
17	.250	.266	.286	.318	.381
18	.244	.259	.278	.309	.371
19	.237	.252	.272	.301	.363
20	.231	.246	.264	.294	.356
25	.210	.220	.240	.270	.320
30	.190	.200	.220	.240	.290
35	.180	.190	.210	.230	.270
OVER 35	$\frac{1.07}{\sqrt{N}}$	$\frac{1.14}{\sqrt{N}}$	$\frac{1.22}{\sqrt{N}}$	$\frac{1.36}{\sqrt{N}}$	$\frac{1.63}{\sqrt{N}}$

Kolmogorov-Smirnov (K-S) Test vs. χ^2 Test

Chi-square test	K-S test
Valid for **large sample size** *n*	Valid for **any sample size** *n*, in particular for small *n.*
Valid for both **discrete and continuous** distributional assumptions	Valid for **continuous** distributional assumptions. If applied to discrete distributional assumptions, the critical value is too high resulting in smaller probability of type I error, with a corresponding loss of power.
Valid when distributional parameters are **estimated**.	Valid if *all* parameters of the hypothesized distribution are **known**. If distribution parameters are estimated, the critical value is too high resulting in smaller probability of type I error, with a corresponding loss of power.
A hypothesis could be **accepted** when the data are grouped in intervals one way but **rejected** when they are grouped another way.	Does **not require any grouping** of the data.

Parameter estimation

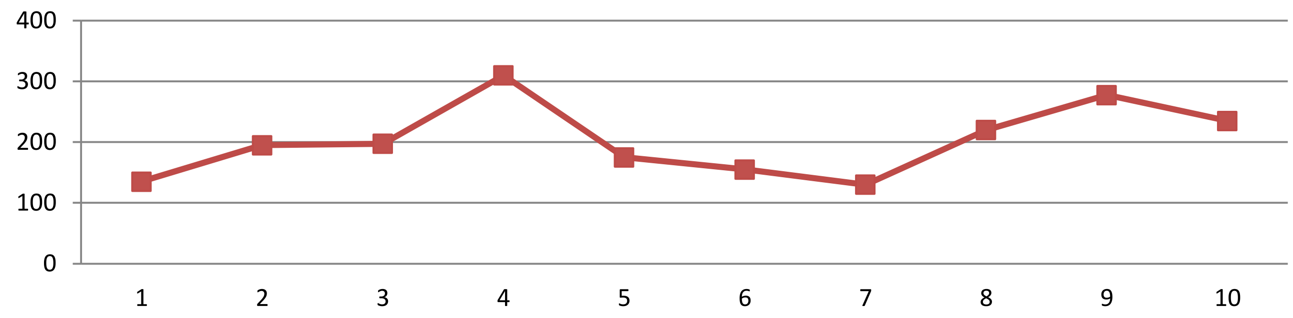
- **Theoretical distribution**
- *Selection of probability distribution—How?*
  - Shape-wise analysis (histogram) and estimation of mean and variance
  - Historical demands d_t, t=1,...,T
  - Sample mean and sample standard deviation

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} d_t \qquad \hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(d_t - \hat{\mu}\right)^2}$$

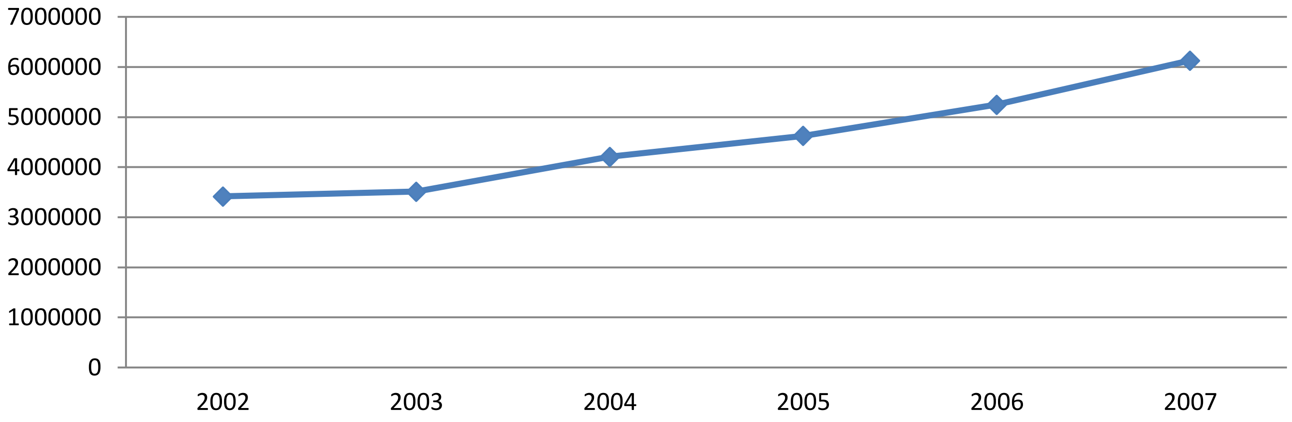
  - Example: 50, 120, 150, 80; μ = 100, σ = 44, can be used to fit a Normal distribution
- **Forecasting methods**
  - Moving average, exponential smoothing, Holt-Winters method
  - Forecast error distribution

What is the appropriate model?

Monthly electric can opener shipments for 10 months which fluctuate around a certain amount.

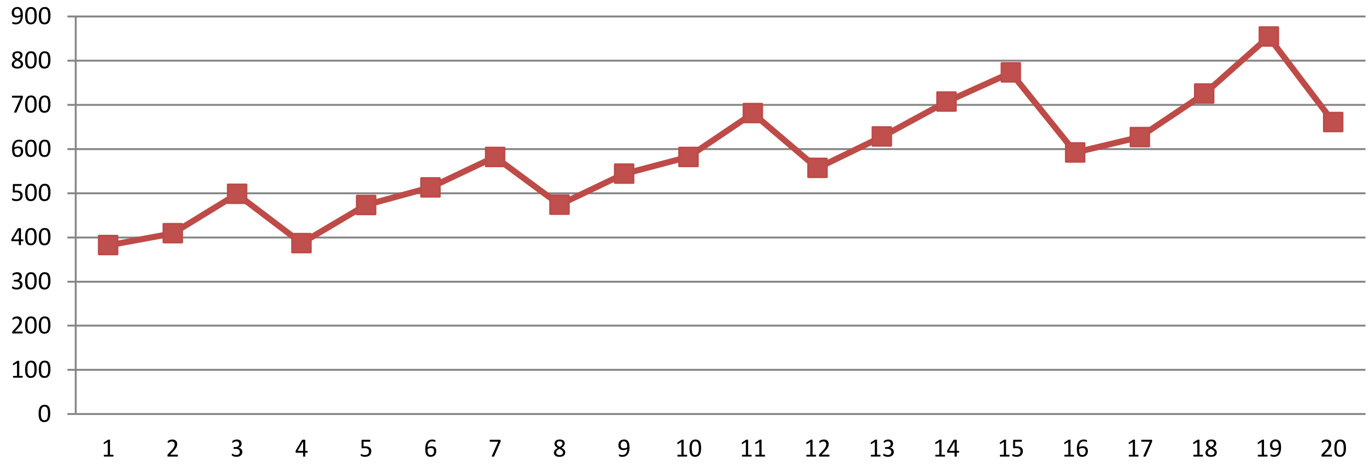


The number of visitors to Japan from other Asian Countries during the period of 2002-2007. As Japan National Tourist Organization has reported, there is a constant increase in the number of visitors.

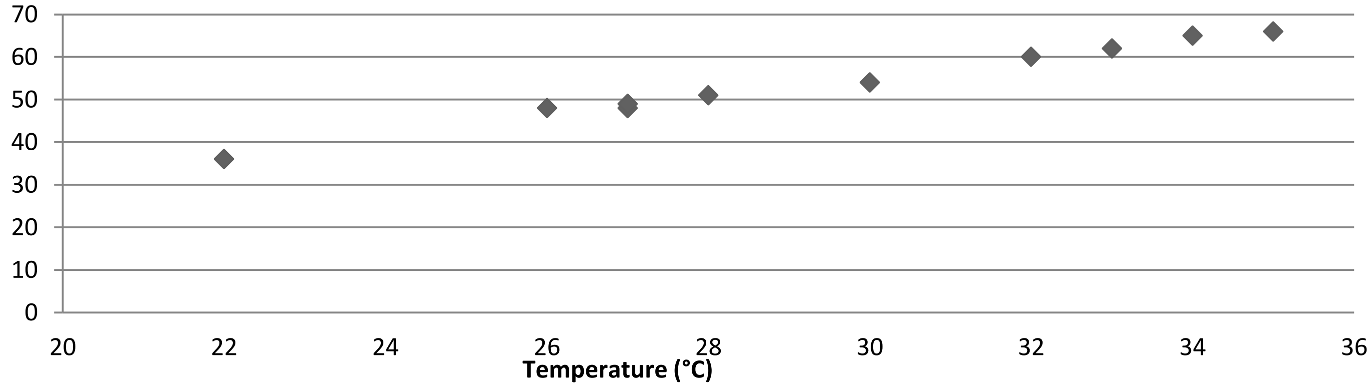


What is the appropriate model?

Quarterly sales data of a French company over a five-year period. There is a long-term increase in sales and they are influenced by quarter of the year.



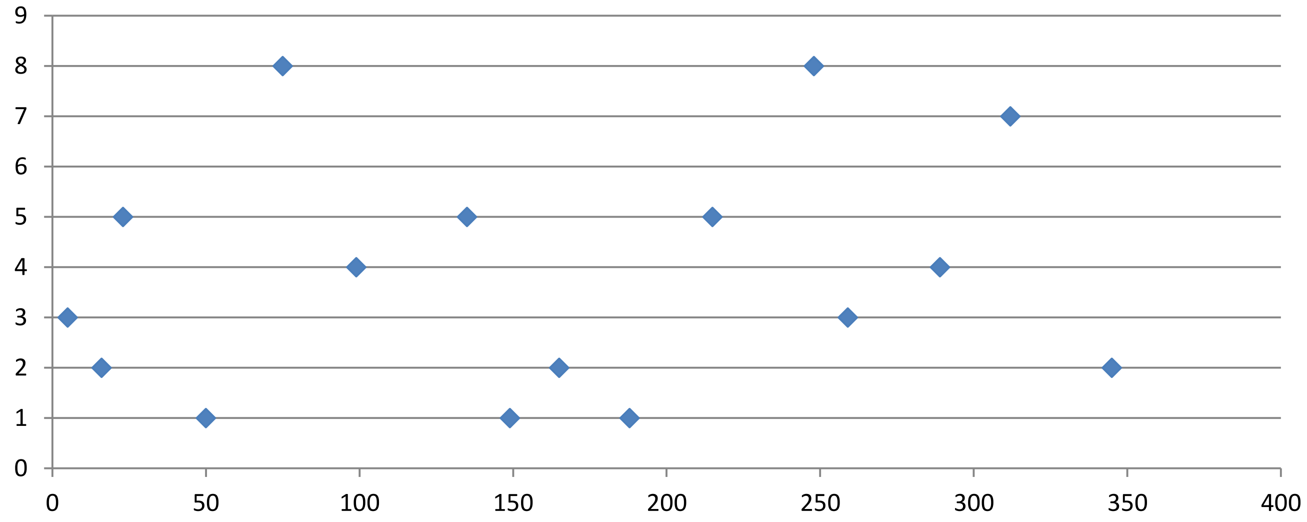
The demand for some products is strongly dependent **on other factors than time.** The graph gives an example of a product whose demand is dependent on daily temperature.



What is the appropriate model?

- W.W. Grainger is a supplier of MRO products that has 1600 stores distributed throughout the United States. Large electric motors is one of its products and they are high-value items with low demand. Demand for this type of products occurs at **infrequent** and **irregular** intervals.

- Assume that the following graph represents the demand for a specific model of the large electric motors.



Constant model

- **Model: $y_t = a + \varepsilon_t$**
  - Demand level *a* plus real-valued, zero-mean error term ε_t

- Estimation of parameter *a* at time *t*
  - **Moving average**
    - Parameter: Number of observations *n*

$$\hat{a}_t = \frac{1}{n}\sum_{k=t-n+1}^{t} y_k$$

  - **Exponential smoothing**

$$\hat{a}_t = \alpha \cdot y_t + (1-\alpha)\cdot \hat{a}_{t-1}$$

    - Forecast=weighted average of last demand and last forecast
    - Parameter: (Start value), smoothing constant α, e.g., between 0.1 and 0.3

- Forecast $p_{t+i} = \hat{a}_t \quad i = 1,2,\ldots$

Forecast accuracy

- Forecast error $e_t = y_t - p_t$
  - MAD: mean absolute deviation
  - MSE: mean squared error
  - Computation: a) moving averages, b) exponential smoothing

$$MAD_t = \frac{1}{n}\sum_{i=t-n+1}^{t}|e_i| \qquad MAD_t = \gamma \cdot |e_t| + (1-\gamma)\cdot MAD_{t-1}$$

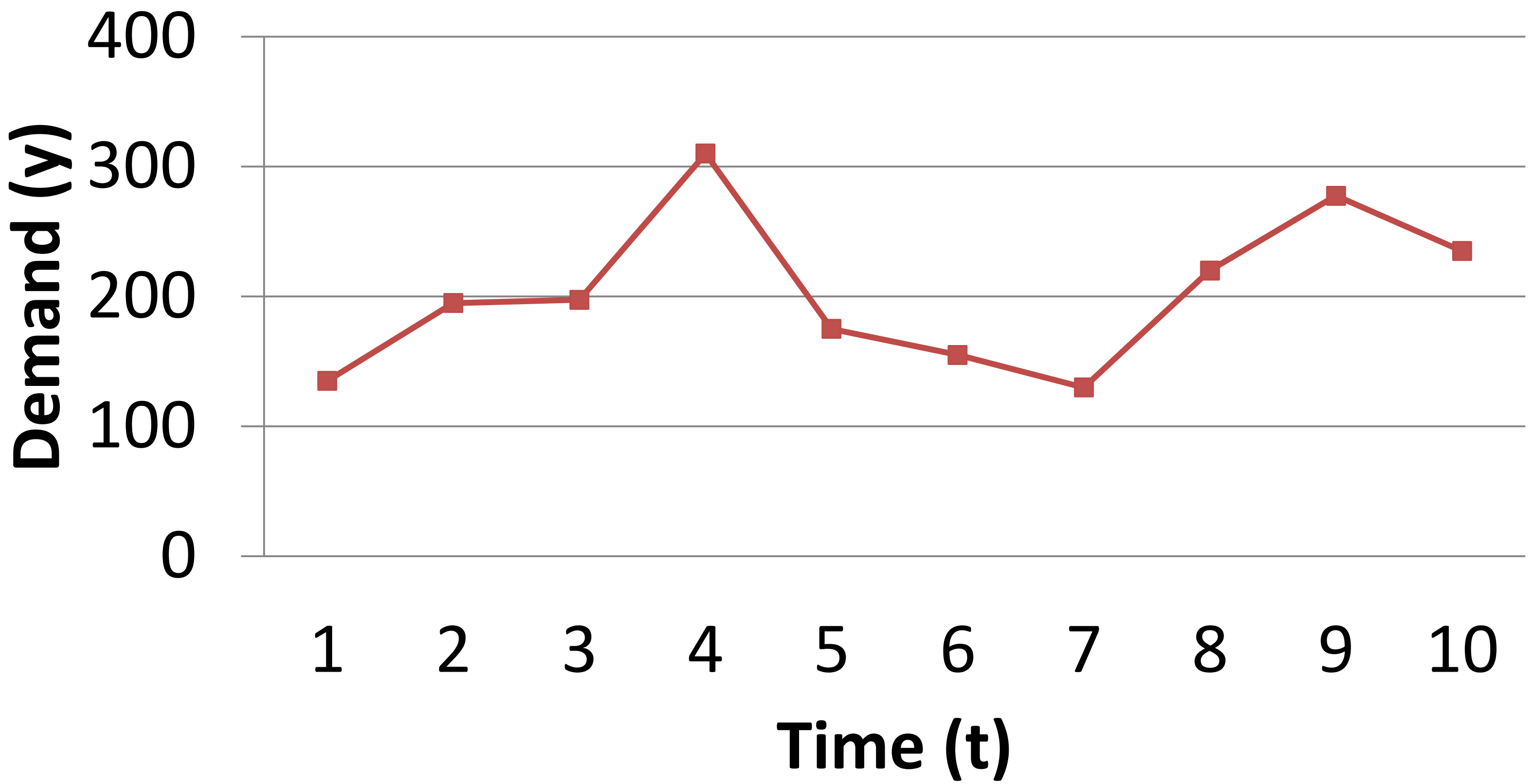
$$MSE_t = \frac{1}{n-1}\sum_{i=t-n+1}^{t} e_i^2 \qquad ERR_t = \delta \cdot e_t + (1-\delta)\cdot ERR_{t-1}$$

- Monitoring of forecast accuracy
  - Error signal $SIG_t = \dfrac{ERR_t}{MAD_t}$ $\quad -1 \le SIG_t \le 1$
  - Adaptive change of smoothing constant $\alpha_t = |SIG_t|$

Trigg, D. W., & Leach, A. G. (1967). Exponential smoothing with an adaptive response rate. *Journal of the Operational Research Society*, *18*(1), 53-59.

Example

- The electric can opener



Example

Estimation of standard deviation
(assumption: normal distribution)

$$\sigma \approx 1.25 \cdot MAD$$

			n=4		alpha				gamma, delta		
					0.1				0.1		
t	y	a	p		a	p	e	|e|	ERR	MAD	SIG
0					200				0	0	
1	135				193.50	200.00	-65.00	65.00	-6.50	6.50	-1.00
2	195				193.65	193.50	1.50	1.50	-5.70	6.00	-0.95
3	197.5				194.04	193.65	3.85	3.85	-4.75	5.79	-0.82
4	310	209.38			205.63	194.04	115.97	115.97	7.33	16.80	0.44
5	175	219.38	209.38		202.57	205.63	-30.63	30.63	3.53	18.19	0.19
6	155	209.38	219.38		197.81	202.57	-47.57	47.57	-1.58	21.12	-0.07
7	130	192.50	209.38		191.03	197.81	-67.81	67.81	-8.20	25.79	-0.32
8	220	170.00	192.5		193.93	191.03	28.97	28.97	-4.49	26.11	-0.17
9	277.5	195.63	170		202.28	193.93	83.57	83.57	4.32	31.86	0.14
10	235	215.63	195.63		205.56	202.28	32.72	32.72	7.16	31.94	0.22

(Linear) Trend Model

- Demand model $y_t = a + b \cdot t + \varepsilon_t$
  - Demand level a at time t=0
  - Trend factor b, increase of demand per period
  - Error term ε_t

- Estimation of parameters: Holt‘s method:

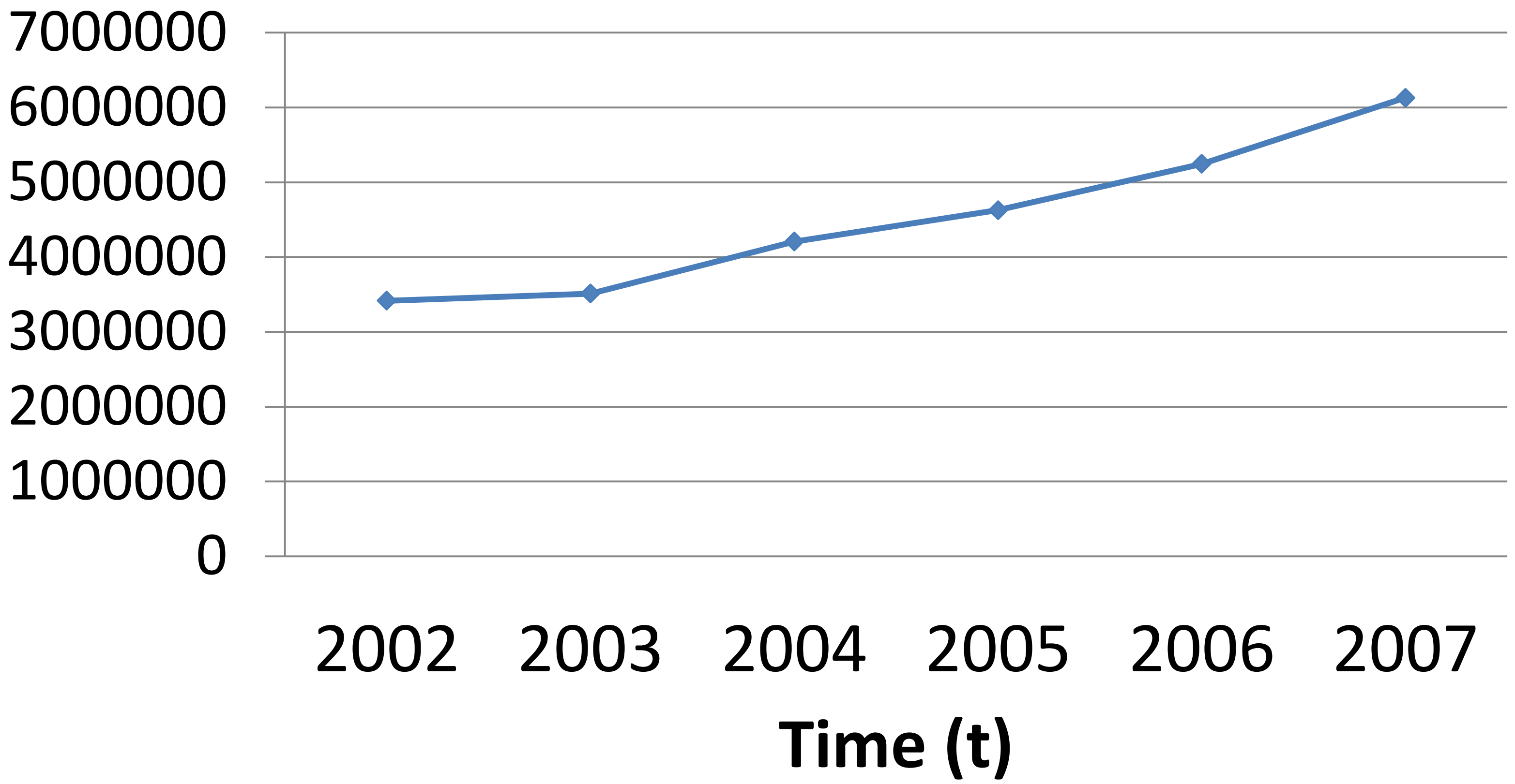
$$\hat{a}_t = \alpha \cdot y_t + (1-\alpha)(\hat{a}_{t-1} + \hat{b}_{t-1})$$

$$\hat{b}_t = \beta(\hat{a}_t - \hat{a}_{t-1}) + (1-\beta)\hat{b}_{t-1}$$

- Forecast $p_{t+i} = \hat{a}_t + \hat{b}_t \cdot i \quad i = 1,2,\ldots$

Example

- Visitors to Japan



Example

α=0.1 β=0.2

t	y	a	b	p	e	|e|	ERR	MAD	SIG
2001		2604842	548247						
2002	3417774	3179557.50	553540.70	3153089	264685.00	264685.00	26468.50	26468.50	1.00
2003	3511513	3710939.68	549109.00	3733099	-221586.00	221586.00	1663.05	45980.25	0.04
2004	4208095	4254853.31	548069.92	4260049	-51954.00	51954.00	-3698.66	46577.63	-0.08
2005	4627478	4785378.71	544561.02	4802924	-175446.00	175446.00	-20873.39	59464.46	-0.35
2006	5247125	5321658.25	542904.72	5329940	-82815.00	82815.00	-27067.55	61799.52	-0.44
2007	6130262	5891132.88	548218.70	5864563	265699.00	265699.00	2209.10	82189.46	0.03

Forecast for the first period:

$$p_1 = \hat{a}_0 + \hat{b}_0 = 3153089$$

Forecast for the second period:

$$\hat{a}_1 = (0.1)3417774 + (0.9)3153089 = 3179557.50$$

$$\hat{b}_1 = 0.2(3179557.50 - 2604842) + (0.8)548247 = 553540.7$$

$$p_2 = \hat{a}_1 + \hat{b}_1 = 3733099$$

Seasonality (with Trend)

- Demand model $y_t = (a + b \cdot t) \cdot s_t + \varepsilon_t$
  - (multiplicative) season weight s_t, t=1,2,...,z
  - Cycle z (e.g. 4 quarters, 12 month)
  - Sum of weights equals z
- Parameter estimation: Winters method

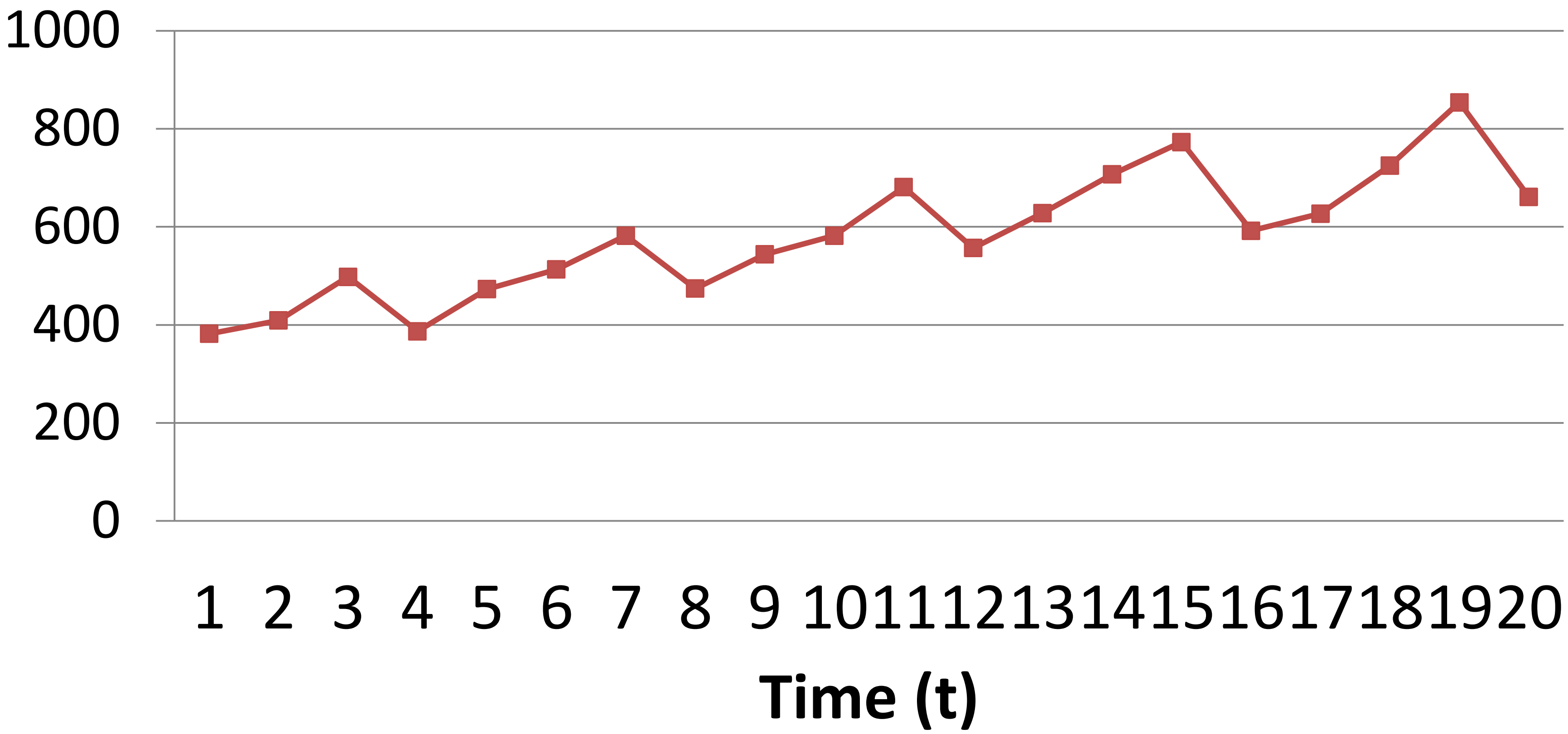
$$\hat{a}_t = \alpha \frac{y_t}{\hat{s}_{t-z}} + (1-\alpha)(\hat{a}_{t-1} + \hat{b}_{t-1})$$
$$\hat{b}_t = \beta(\hat{a}_t - \hat{a}_{t-1}) + (1-\beta)\hat{b}_{t-1}$$
$$\hat{s}_t^u = \gamma \frac{y_t}{\hat{a}_t} + (1-\gamma)\hat{s}_{t-z}$$

Normalize weights after every cycle $\quad \hat{s}_t = \frac{z \cdot \hat{s}_t^u}{\sum_{j=t-z+1}^{t} \hat{s}_j^u}$

- Forecast $\quad p_{t+i} = (\hat{a}_t + \hat{b}_t \cdot i) \cdot \hat{s}_{t+i-kz} \quad i = 1,2,\ldots;\ k = \left\lceil \frac{i}{z} \right\rceil$

Example

- Sales of a French company



Example

		alpha	beta	gamma		
		0.2	0.1	0.3		
T	**y**	**a**	**b**	**su**	**s**	**p**
-3	362			0.953	0.953	
-2	385			1.013	1.013	
-1	432			1.137	1.137	
0	341	380	9.75	0.897	0.897	
1	382	391.97	9.97	0.959	0.953	371.43
2	409	402.30	10.01	1.014	1.008	407.16
3	498	417.45	10.52	1.154	1.146	468.80
4	387	428.66	10.59	0.899	0.893	383.89
5	473	450.64	11.73	0.982	0.963	418.72
6	513	471.73	12.67	1.032	1.012	465.85
7	582	489.06	13.13	1.159	1.137	555.27
8	474	507.92	13.71	0.905	0.888	448.42
9	544	530.24	14.57	0.982	0.967	502.51
10	582	550.90	15.18	1.025	1.010	551.21
11	681	572.62	15.83	1.153	1.135	643.75
12	557	596.26	16.61	0.902	0.888	522.35
13	628	620.16	17.34	0.981	0.972	592.77
14	707	650.05	18.60	1.033	1.024	643.60
15	773	671.09	18.84	1.140	1.130	759.12
16	592	685.29	18.38	0.881	0.873	612.60
17	627	691.90	17.20	0.953	0.954	684.23
18	725	708.87	17.18	1.024	1.026	726.17
19	854	731.93	17.76	1.141	1.143	820.75
20	661	751.17	17.91	0.875	0.877	654.56

Forecast for the first period: $p_1 = (380 + 9.75) \cdot 0.953 = 371.43$

Forecast for the second period:

$$\hat{a}_1 = 0.2 \cdot \frac{382}{0.953} + 0.8 \cdot (380 + 9.75) = 391.97$$

$$\hat{b}_1 = 0.1 \cdot (391.97 - 380) + 0.9 \cdot 9.75 = 9.97$$

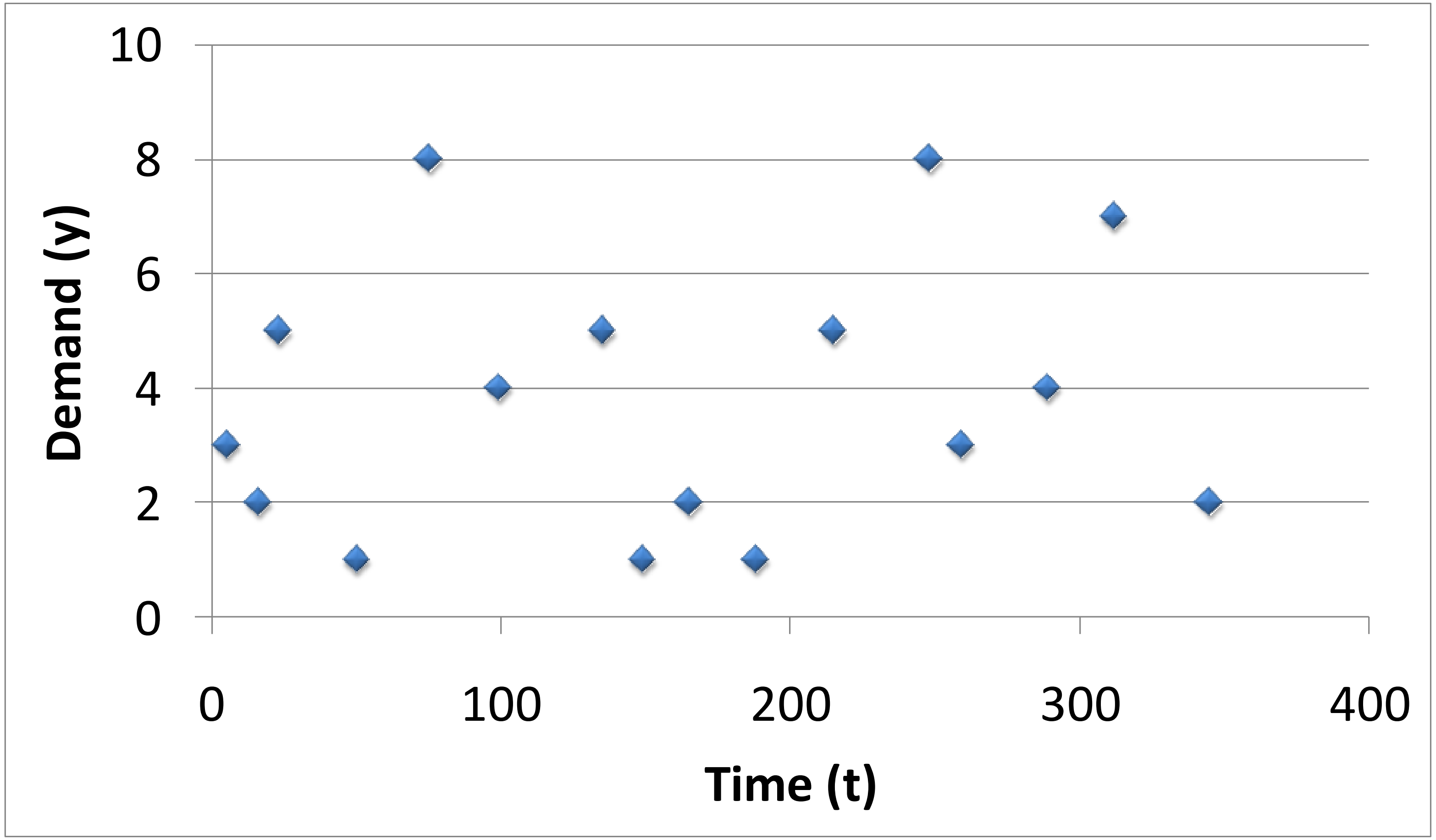
$$\hat{s}_1^u = 0.3 \cdot \frac{382}{391.97} + 0.7 \cdot 0.953 = 0.959$$

$$p_2 = (391.97 + 9.97) \cdot 1.013 = 407.16$$

Intermittent demand

- Several periods without demand
- Idea: model separately the demand size (if positive) (a_t) and the demand interval (x_t)
- Both by exponential smoothing
- Period of last demand: τ
- Parameter estimation (Croston method)
  - Case 1: y_t=0
    $$\hat{a}_t = \hat{a}_{t-1}$$
    $$\hat{x}_t = \hat{x}_{t-1}$$
  - Case 2: y_t>0
    $$\hat{a}_t = \alpha y_t + (1-\alpha)\hat{a}_{t-1}$$
    $$\hat{x}_t = \beta(t-\tau) + (1-\beta)\hat{x}_{t-1}$$
- Forecast
  $$p_{t+i} = \begin{cases} \hat{a}_t & i = k \cdot \lfloor \hat{x}_t \rfloor, k = 1,2,\ldots \\ 0 & otherwise \end{cases}$$

Example



Example

$\alpha=\beta=0.3$

Day	Demand	x	a	Day	Quantity
0		20.00	5.00	20	5
5	3	15.50	4.40	20	5
16	2	14.15	3.68	30	4
23	5	12.01	4.08	35	5
50	1	16.50	3.15	66	4
75	8	19.05	4.61	94	5
99	4	20.54	4.43	119	5
135	5	25.18	4.60	160	5
149	1	21.82	3.52	170	4
165	2	20.08	3.06	185	4
188	1	20.95	2.44	208	3
215	5	22.77	3.21	237	4
248	8	25.84	4.65	273	5
259	3	21.39	4.15	280	5
289	4	23.97	4.11	312	5
312	7	23.68	4.98	335	5
345	2	26.48	4.08	371	5