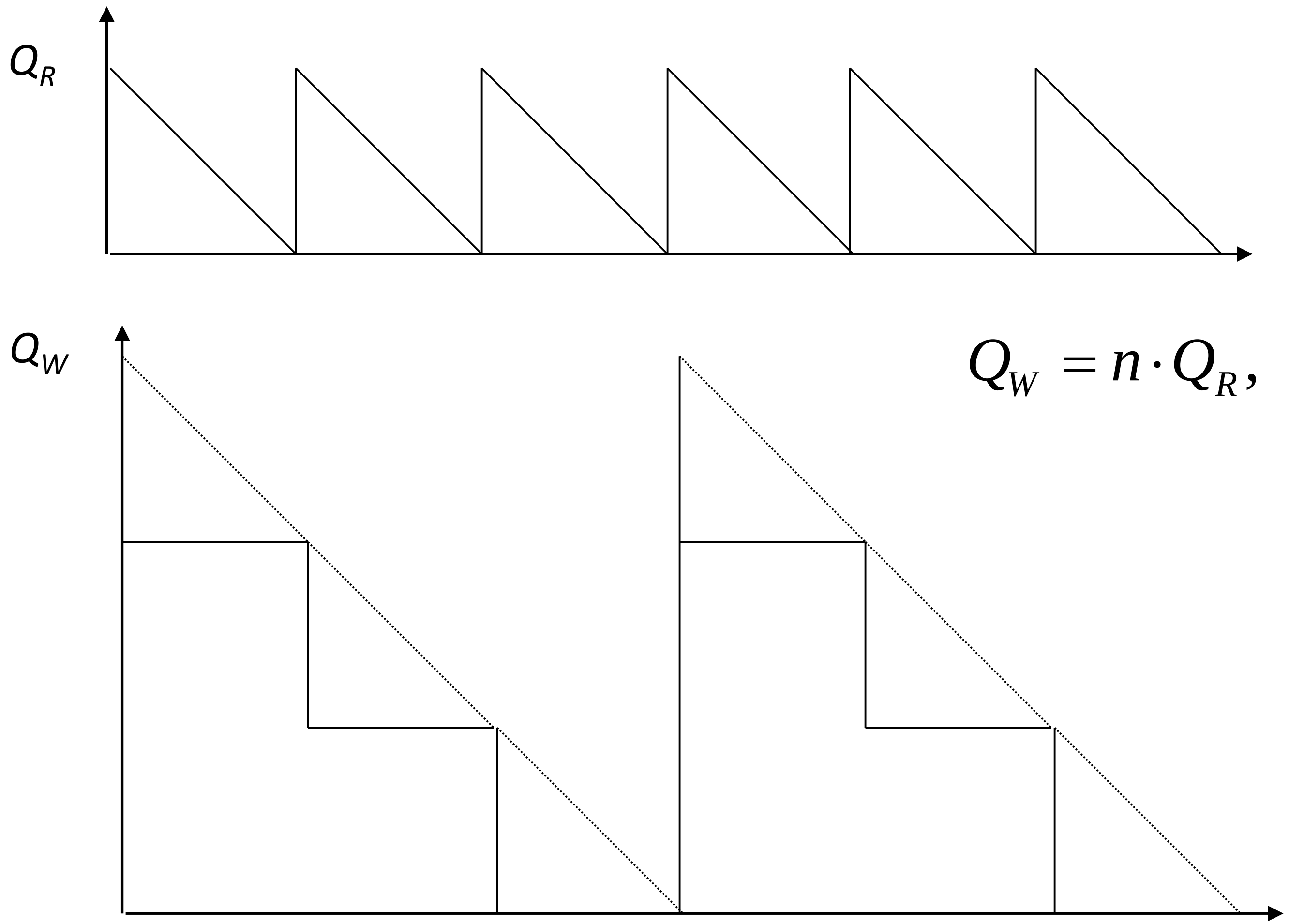
5.2 Multi-level lot-sizing

Two-stage lot-sizing

Axsäter, 9.2.1
Silver et al., 11.3



$$Q_W = n \cdot Q_R, \quad n = 3$$

Approaches

- Sequential planning
  - Determine Q_R
  - Determine Q_W=nQ_R for given Q_R

$$C_R(Q_R) = \frac{d}{Q_R} A_R + \frac{h_R}{2} Q_R \quad Q_R^* = \sqrt{\frac{2dA_R}{h_R}}$$

$$C_W(n) = \frac{d}{nQ_R} A_W + \frac{h_W}{2} Q_R(n-1)$$

$$C_W(n+1) \leq C_W(n) \Leftrightarrow n(n+1) \leq \frac{A_W h_R}{A_R h_W}$$

$$n^* = \min\left\{n \middle| n(n+1) \geq \frac{A_W h_R}{A_R h_W}\right\}$$

- Simultaneous planning

$$C(n, Q_R) = \frac{d}{nQ_R} A_W + \frac{d}{Q_R} A_R + \frac{h_W}{2} Q_R(n-1) + \frac{h_R}{2} Q_R$$

$$Q_R^*(n) = \sqrt{\frac{2d\left(\frac{A_W}{n} + A_R\right)}{nh_W + h_R - h_W}} \quad C^*(n) = \sqrt{2d\left(\frac{A_W}{n} + A_R\right)(nh_W + h_R - h_W)}$$

$$C(n+1) \leq C(n) \Leftrightarrow n(n+1) \leq \frac{A_W}{A_R} \frac{h_R - h_W}{h_W}$$

$$n^* = \min\left\{n \middle| n(n+1) \geq \frac{A_W}{A_R} \frac{h_R - h_W}{h_W}\right\}$$

Example

- Data
  - Demand per year: d=1000
  - Fixed cost: A_W=10, A_R=15
  - Inventory holding cost per unit per year: h_W =0.24; h_R=1.2

- Results
  - Independent planning
    - Q_W =288.68; Q_R =158.11; C = 69.28+189.74 = 259.02
  - Sequential planning
    - n =2; Q_W =316.22; Q_R =158.11; C=50.6+189.74=240.33
  - Simultaneous planning
    - n=2; Q_R =166.67; Q_W =333.33; C=50+190=240

5.3 Dynamic lot-sizing

Dynamic multi-level lot-sizing

- Assumptions
  - As in Wagner-Whitin model
    - Item specific setup cost A_k and holding cost h_k
  - Dynamic demands for final products d_{it}
  - k=1,2,…,K: stock keeping units at several stages
    - V(k): set of predecessor stockpoints
    - Input coefficients a_{ij}

- Planning problem
  - Decision variables
    - q_{kt} Lot-size (Production quantity) for item *k* in *t*
    - y_{kt} Inventory level for item *k* at the end of period *t*
    - γ_{kt} Setup indicator, γ_{kt}=1 if a lot is placed for item *k* in period *t*, γ_{kt}=0 otherwise
  - Cost minimization
  - Constraints

Model

- Mixed-integer Linear Programming

$$\min \sum_{t=1}^{T} \sum_{k=1}^{K} (A_k \cdot \gamma_{kt} + h_k \cdot y_{kt})$$

$$y_{kt} = y_{k,t-1} + q_{kt} - d_{kt} \quad k \in E; t = 1,2,\ldots,T$$

$$y_{kt} = y_{k,t-1} + q_{kt} - \sum_{j \in N(k)} a_{kj} q_{jt} \quad k \notin E; t = 1,2,\ldots,T$$

$$q_{kt} \leq M\gamma_{kt} \qquad k = 1,2,\ldots,K; t = 1,2,\ldots,T$$

$$y_{k0} = y_{kT} = 0$$

$$q_{kt}, y_{kt} \geq 0,\ \gamma_{kt} \in \{0,1\} \quad k = 1,2,\ldots,K; t = 1,2,\ldots,T$$

Dynamic multi-echelon lot-sizing heuristics

- Algorithm
  - Cost adjustment on each echelon level
  - Treat the upper echelon level as a single level planning problem (Wagner/Whitin)
  - Resulting replenishments represent demand for the next echelon
  - Consider the next echelon again as a single level planning problem

Approach

- Cost adjustment heuristic:

Total cost: $$C(Q_W, Q_R) = \frac{D}{Q_R}\left(A_R + \frac{A_W}{n}\right) + \frac{Q_R}{2}\left((n-1)h_W + h_R\right)$$

Adjusted ordering cost: $$\hat{A}_R = A_R + \frac{A_W}{n}$$

Adjusted holding cost: $$\hat{h}_R = (n-1)h_W + h_R$$

Good estimate: $$n = \max\left[\sqrt{\frac{A_W(h_R - h_W)}{A_R h_W}}, 1\right]$$

Example

- Demands over the next 6 month : 750, 100, 50, 100, 400, 1000
- Serial two-stage process: one warehouse, one retailer
- Setup cost: A_W=700, A_R= 500
- Inventory holding cost per unit and month: h_W=2, h_R=3

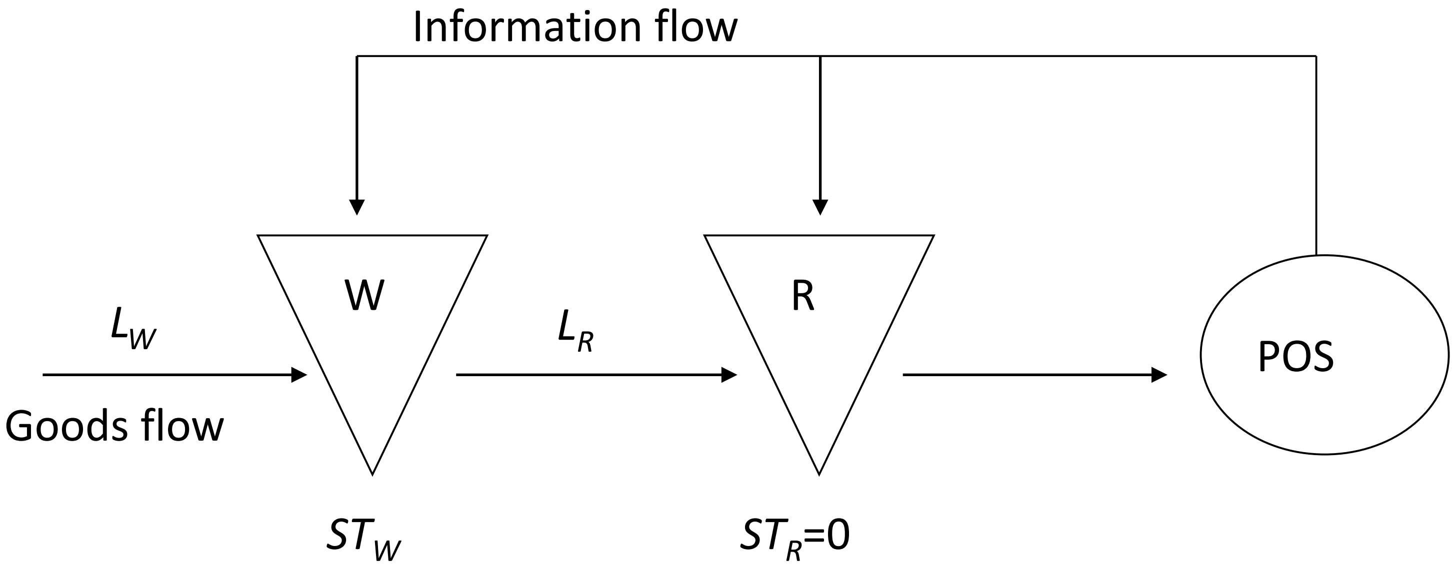
Results:

		1	2	3	4	5	6	Cost	Total Cost
Level by level Wagner Whitin Algorithm	Retailer	900	0	0	100	400	1000	2600	5300
	Warehouse	1000	0	0	0	400	1000	2700	
Cost adjustment Heuristic + WW-Algorithm	Retailer	1000	0	0	0	400	1000	3000	5100
	Warehouse	1000	0	0	0	400	1000	2100	
Cost adjustment Heuristic + Silver-Meal Heuristic	Retailer	900	0	0	100	400	1000	2600	5400
	Warehouse	900	0	0	100	400	1000	2800	

Assumptions

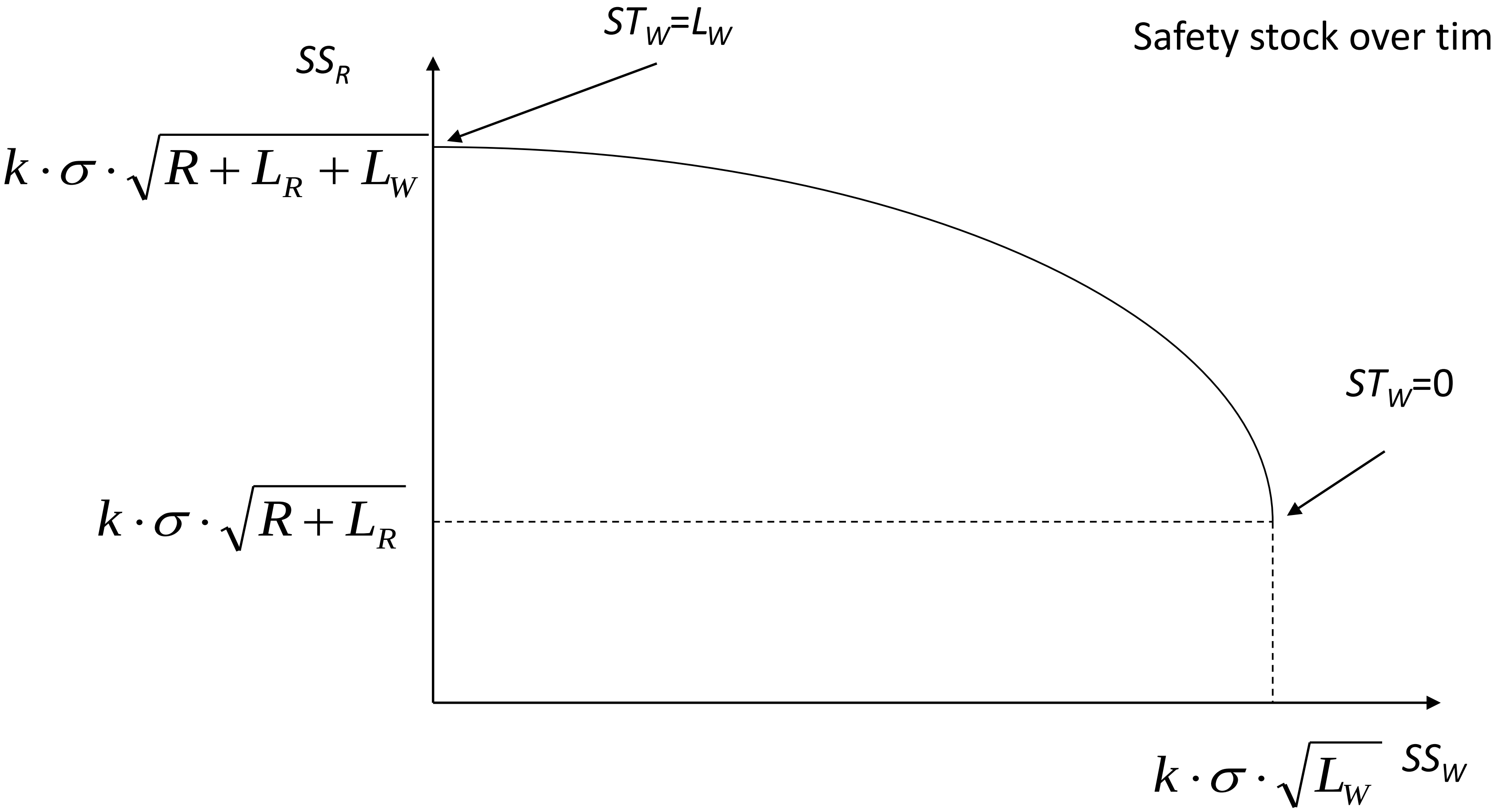
- Retailer and warehouse, in series
- L_i lead time for replenishment at location i
- D stochastic demand, mean μ, standard deviation σ, pdf f, cdf F
- D(L) cumulative demand over L periods, mean $L\mu$, variance $L\sigma^2$
- Inventory holding cost h_i at location i per unit and unit of time
- Inventory at the retailer is reviewed every R periods, decisions at earlier stages are coordinated directly
- Safety stocks are only meant to cover „maximum reasonable demand“ at a location, depending on predetermined internal service levels α_i
- Normally distributed demand:
  - maximum reasonable demand at stage i in a period of length t: $t\mu + k_i\sigma\sqrt{t}$
  - k_i is the safety factor that corresponds to the service level α_i

Base-stock-system



Base-stock-system

Safety stock over time t: $k\sigma\sqrt{t}$



Optimization problem

- Decision variables
  - Service times, safety stock coverage times
- Minimize safety stock holding costs

$$\min C = h_W k_W \sigma \sqrt{L_W - ST_W} + h_R k_R \sigma \sqrt{ST_W + L_R + R}$$

$$s.t. \quad 0 \leq ST_W \leq L_W$$

- Optimality property: concave minimization problem
  - Optimality of bang-bang solutions (extreme points)
  - $ST_W=0$ or $ST_W=L_W$

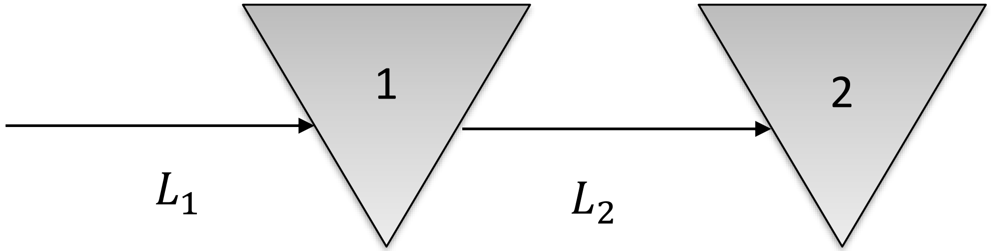
Example

- Data
  - μ=20, σ=8
  - L_R=1; L_W=5;
  - α_i=95%, k=1.645
  - Holding cost h_R=0.024; h_W=0.0048
- Result
  - $C(ST_W=L_W)=0.024*34.82=0.836$
  - $C(ST_W=0)=0.0048*29.43+0.024*18.61=0.588$

Clark-Scarf-model

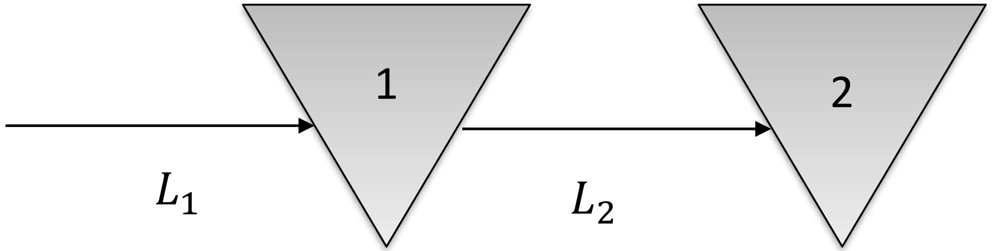
Axsäter, 10.1

- Serial system
- Discrete time periods, stock reviewed every period
- L_i lead time for replenishment at location i
- D stochastic demand, mean μ, standard deviation σ, pdf f, cdf F
- D(L) cumulative demand over L periods, mean $L\mu$, variance $L\sigma^2$
- Backorder penalty b per unit and unit of time
- Inventory holding cost h_i at location i per unit and unit of time



Sequence of events

- Location 1 orders from an external supplier with infinite capacity
- Outstanding order from supplier arrives at location 1
- Location 2 orders from location 1
- Delivery from location 1 arrives at location 2
- Demand realization at location 2
- Evaluation of holding and shortage (backorder) costs



Optimization problem

- Decision: Echelon-order-up-to levels S_i
- Objective: minimize expected average backorder and inventory holding costs
- Approach, consider:
  - at location 1: order at time t, cost at time $t+L_1$
  - at location 2: order at time $t+L_1$, cost at time $t+L_1+L_2$
- Note that a demand occuring in period t, affects location 2 still in period t, but location 1 in period t+1

Analysis

- Say we are at time t
- Echelon 1 raises echelon inventory position to S_1

  (external supplier can always deliver)
  - Echelon stock at time $t+L_1$, **before** the demand arrival: $IL_1^e = S_1 - D(L_1)$
  - Costs at echelon 1 are incurred at time $t+L_1$

- Echelon 2 raises inventory position to S_2, if echelon 1 provides sufficient supply
  - Also happens **before** the demand arrival
  - Installation stock level at location 1 at time $t+L_1$: $IL_1^i = IL_1^e - S_2 = S_1 - D(L_1) - S_2$
  - **Realized** inventory position S at location 2 after ordering: $S = \min(S_2, S_1 - D(L_1))$
  - Costs at echelon 2 are incurred at time $t+L_1+L_2$
  - Stock level then: $S - D(L_2+1)$

Average cost analysis

- Stage 1

$$C_1(S_1, S_2) = h_1 E(S_1 - D(L_1) - S_2)^+$$

$$= h_1 \int_0^{S_1 - S_2} (S_1 - S_2 - d) f_{L_1}(d) \mathrm{d}d$$

- Stage 2

$$C_2(S_1, S_2) = h_2 E((S - D(L_2 + 1))^+) + bE((S - D(L_2 + 1))^-)$$

$$= h_2 \int_0^S (S - d) f_{L_2+1}(d) \mathrm{d}d + \mathrm{b} \int_S^\infty (d - S) f_{L_2+1}(d) \mathrm{d}d$$

Optimal solution

- Stage 2 $$F_{L_2+1}(S_2) = \frac{h_1 + b}{h_2 + b}$$

- Stage 1 $$\int_0^{S_2} F_{L_1}(S_1 - x) f_{L_2+1}(x) \mathrm{d}x = \frac{b}{h_2 + b}$$

Example

- L_1=L_2=5, μ=10, σ=5, b=10, h_1=1, h_2=1.5
- Computation
  - Normal distribution, numerical integration
  - S_2*=81, S_1*=129.7
  - Use of Excel

METRIC

Axsäter, 10.2

- METRIC: A multi-echelon technique for recoverable item control (Sherbrooke, 1968)
- Assumptions
  - One warehouse, multiple retailers
  - Independent Poisson demand processes at the retailers with rate λ_i
  - Complete backordering
  - Continuous review base-stock policies
  - Retailer orders are filled by the warehouse on a FCFS basis

Analysis - warehouse

- Demand process at the warehouse
  - Poisson process with rate equal to sum of retailer rates: $\lambda_0 = \sum_{i=1}^{N} \lambda_i$

- Distribution of the warehouse inventory level: $\mathrm{P}(IL_0 = j) = \mathrm{P}(D_0(L_0) = S_0 - j) = \dfrac{(\lambda_0 L_0)^{S_0 - j}}{(S_0 - j)!} e^{-\lambda_0 L_0}, \ j \leq S_0$

- Average on-hand inventory level at the warehouse: $\mathrm{E}(IL_0^+) = \sum_{j=1}^{S_0} j\mathrm{P}(IL_0 = j)$

$E(IL_0) = E(IL_0^+) - E(IL_0^-)$

- Average backorder at the warehouse: $E(IL_0^-) = \sum_{j=-\infty}^{-1} (-j)\mathrm{P}(IL_0 = j) = \mathrm{E}(IL_0^+) - (S_0 - \lambda_0 L_0)$

Analysis - retailer

- Retailer lead time: stochastic due to delays from the warehouse
  - Approximation: replace stochastic lead time by its mean: $\bar{L}_i = L_i + \frac{E(IL_0^-)}{\lambda_0}$

Little's law: Avg. queue length = arrival rate * avg. waiting time

- Distribution of the inventory level: $P(IL_i = j) = P(D_i(\bar{L}_i) = S_i - j) = \frac{(\lambda_i \bar{L}_i)^{S_i - j}}{(S_i - j)!} e^{-\lambda_i \bar{L}_i}, \; j \le S_i$

- Average inventory and backorder: $E(IL_i^+) = \sum_{j=1}^{S_i} jP(IL_i = j), \quad E(IL_i^-) = E(IL_i^+) - (S_i - \lambda_i \bar{L}_i)$

Optimization

Axsäter, 10.4

- Notation
  - h_i inventory holding cost at location i=0,1,…,N
  - b_i backorder cost per unit and unit of time at location i=0,1,…,N (typically, b_0=0)
  - $C_0(S_0)$ average holding cost at the warehouse
  - $C_i(S_i)$ average holding and backorder cost at retailer i
  - C average system costs per unit time

- Expression

$$C = C_0(S_0) + \sum_{i=1}^{N} \left(h_i \mathrm{E}(IL_i^+) + b_i \mathrm{E}(IL_i^-)\right)$$

Properties and algorithm

$$\overline{L}_i = L_i + \frac{E(IL_0^-)}{\lambda_0}$$

- Convexity of objective function for given S_0
- Bounds on inventory level at retailer
  - Lower: find S_i for lowest possible lead time L_i
  - Upper: find S_i for largest possible lead time L_0+L_i
  - Independent optimization for each retailer by successively increasing S_i

- Warehouse parameter S_0
  - Not necessarily convex
  - Bounds in inventory level at warehouse
    - Lower: Optimize w.r.t. to upper bounds on S_i
    - Upper: Optimize w.r.t. to lower bounds on S_i
  - Enumeration of values

Exact analysis

- Distribution of backorders at the warehouse

$$P(IL_0 = -k) = P(D_0(L_0) = S_0 + k) = \frac{(\lambda_0 L_0)^{S_0+k}}{(S_0+k)!} e^{-\lambda_0 L_0}$$

- Distribution of backorders from retailer i at the warehouse

$$P(B_i = j) = \sum_{k=j}^{\infty} P(IL_0 = -k) \binom{k}{j} \left(\frac{\lambda_i}{\lambda_0}\right)^j \left(\frac{\lambda_0 - \lambda_i}{\lambda_0}\right)^{k-j} \quad , j > 0$$

$$P(B_i = 0) = 1 - \sum_{j=1}^{\infty} P(B_i = j)$$

- Exact distribution of the inventory level at a retailer

$$P(IL_i = j) = P(B_i + D_i(L_i) = S_i - j) = \sum_{k=0}^{S_i - j} P(B_i = k) \frac{(\lambda_i L_i)^{S_i - j - k}}{(S_i - j - k)!} e^{-\lambda_i L_i}, \quad j \le S_i$$

Example METRIC

- Serial supply chain
- (S-1,S) base stock policy
- Poisson customer demand with $\lambda=5$
- Cost parameters
  - $h_0=1$, $h_1=2$, $b_0=0$, $b_1=10$
  - $L_0=L_1=1$

Evaluation

- $S_0=5$, $S_1=5$
- Warehouse analysis
  - E(IL+)=0.88, E(IL-)=0.88
  - E(L)=1.175
- Retailer analysis
  - E(IL+)=0.55, E(IL-)=1.43
- Cost (C) = 16.30

h0	1
h1	2
b	10
L0	1
L1	1
Lambda	5
S0	5
S1	5
L1_bar	1.17546737

WAREHOUSE						RETAILER						SUPPLY CHAIN
d	p{D=d}	S-d	d-S	E[IL_0+]	0.877337	d	p{D=d}	S-d	d-S	E[IL+]	0.554046	
0	0.006737947	5	0	E[IL_0-]	0.877337	0	0.002802	5	0	E[IL-]	1.431383	
1	0.033689735	4	0			1	0.01647	4	0			
2	0.084224337	3	0	C0(S0)		2	0.048399	3	0	Ci(S0,Si)		C
3	0.140373896	2	0	0.877337		3	0.094819	2	0	15.42192		16.30
4	0.17546737	1	0			4	0.139321	1	0			
5	0.17546737	0	0			5	0.163767	0	0			
6	0.146222808	0	1			6	0.160419	0	1			
7	0.104444863	0	2			7	0.134691	0	2			
8	0.065278039	0	3			8	0.098953	0	3			
9	0.036265577	0	4			9	0.06462	0	4			

Example - optimization

- Bounds for retailer
  - L=1: S_1=7
  - L=2: S_1=13

- METRIC
  - S_0=2, S_1=11, C*=8.96
  - S_0=3, S_1=10, C*=8.63
  - S_0=4, S_1=9, C*=8.42
  - S_0=5, S_1=8, C*=8.53
  - S_0=6, S_1=7, C*=9.19

h0	1
h1	2
b	10
L0	1
L1	1
Lambda	5
S0	0
S1	13
L1_bar	2

S1_lower	S1_upper
7	13
S0_upper	S0_lower
6	0

S_0	S_1	C
0	13	9.869673
1	12	9.395685
2	11	8.959367
3	10	8.611765
4	**9**	8.422805
5	8	8.526411
6	7	9.199882

WAREHOUSE

d	p{D=d}	S-d	d-S			
					E[IL_0+]	0
0	0.006737947	0	0		E[IL_0-]	5
1	0.033689735	0	1			
2	0.084224337	0	2		C0(S0)	
3	0.140373896	0	3		0	
4	0.17546737	0	4			
5	0.17546737	0	5			
6	0.146222808	0	6			
7	0.104444863	0	7			
8	0.065278039	0	8			
9	0.036265577	0	9			
10	0.018132789	0	10			
11	0.008242177	0	11			
12	0.00343424	0	12			

RETAILER

d	p{D=d}	S-d	d-S			
					E[IL+]	3.322473
0	4.53999E-05	13	0		E[IL-]	0.322473
1	0.000453999	12	0			
2	0.002269996	11	0		Ci(S0,Si)	
3	0.007566655	10	0		9.869673	
4	0.018916637	9	0			
5	0.037833275	8	0			
6	0.063055458	7	0			
7	0.090079226	6	0			
8	0.112599032	5	0			
9	0.125110036	4	0			
10	0.125110036	3	0			
11	0.113736396	2	0			
12	0.09478033	1	0			

SUPPLY CHAIN

C
9.869673