4. Basic inventory control models

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Inventory management in a nutshell

Recall main questions:

- When to review?
- When should an order be placed?
- How much should be ordered?

Inventory control

- Physical inventory level
- Net inventory = physical inventory – backorders
- Inventory position = net inventory + outstanding orders
- Numerical examples (L=2)
  - 5 units stock, no backorders, outstanding order: 10
    - Inventory position = 15
  - Zero inventory, 5 backorders, outstanding order 15
    - Inventory position = 10

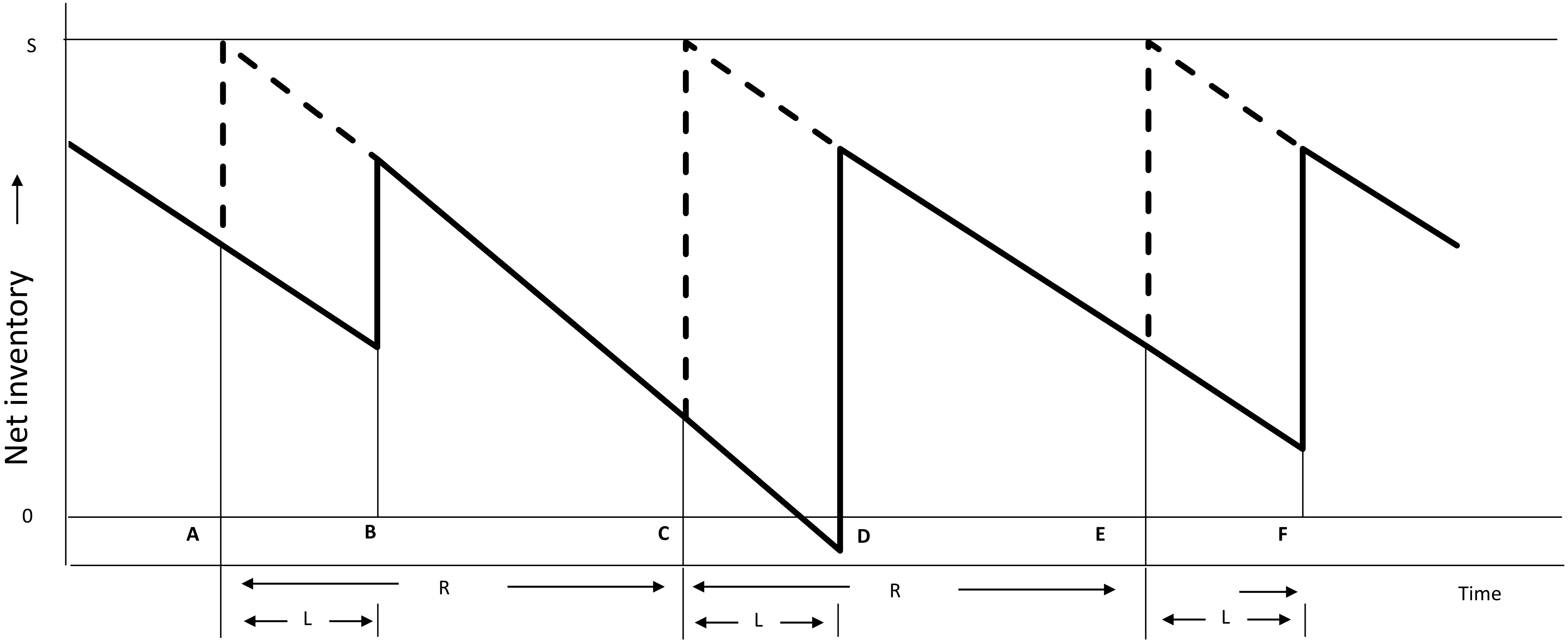
Types of inventory control rules

- (R,S) – Order-up-to policy (periodic review)
  - Order every R periods
  - Order the difference between the order-up-to-level S and the inventory **position**
- (s,Q) - Reorder-point-order-quantity policy (continuous review)
  - Order if the inventory **position** falls to or below the reorder point s
  - Order Q units (or nQ)
- (s,S) – Reorder-point-order-up-to-policy (continuous review)
  - Order if the inventory **position** falls to or below the reorder point s
  - Order the difference between the order-up-to-level S and the inventory **position**

- Question: how to set policy parameters?

(R,S)-policy

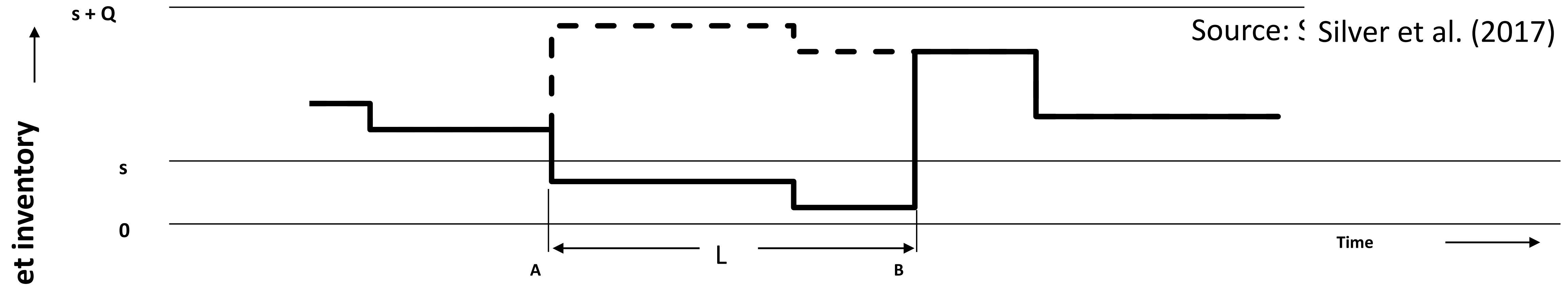
Source: Silver et al. (2017)



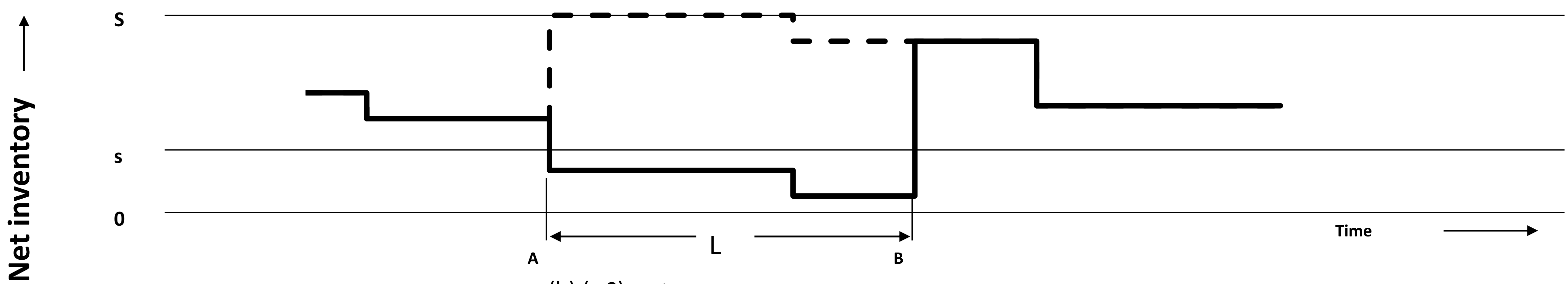
L: Lead time

Continous review (s,Q)- and (s,S)-policy

Source: Silver et al. (2017)



(a) (s,Q) system



(b) (s,S) system

Example (R,S)-Policy

- Periodic ordering (R=1)
- Positive lead time (L=1)
- S=70

t	II	IP	Q	IA	d_t	FI
1	70	70	0	70	30	40
2	40	40	30	40	20	20
3	20	50	20	50	60	-10
4	-10	10	60	10	20	-10
5	-10	50	20	50	40	10
6	10	30	40	30	40	-10
7	-10	30	40	30	30	0
					240	

Performance indicators:

α = 4/7 = 57%
β = 210/240 = 87.5%

Example

- (s,Q) - rule with s=30, Q=50, Initial Inventory (II) =50
  (IP = Inventory Position, IA = Inventory after delivery, FI=Final inventory), Demands: 30, 20, 60, 20, 40, 40, 30
- Sequence: Order, delivery, demand
- Case 1: no lead time (L=0)

t	II	IP	Q	IA	d_t	FI
1	50	50	0	50	30	20
2	20	20	50	70	20	50
3	50	50	0	50	60	-10
4	-10	-10	50	40	20	20
5	20	20	50	70	40	30
6	30	30	50	80	40	40
7	40	40	0	40	30	10
					240	

Performance indicators:

α=6/7=85.7% (fraction of periods without stockout)
β=230/240=95.8% (fraction of demand directly satisfied)

Example

- Case 2: positive lead time (L=1)
  - Supply based on: inventory position

t	II	IP	Q	IA	d_t	FI
1	50	50	0	50	30	20
2	20	20	50	20	20	0
3	0	50	0	50	60	-10
4	-10	-10	50	-10	20	-30
5	-30	20	50	20	40	-20
6	-20	30	50	30	40	-10
7	-10	40	0	40	30	10
					240	

Performance indicators:

α = 3/7 = 42.9%
β = 180/240 = 75%

Lead times

- Challenge: Estimate lead time demand
  - Convolution of probability distributions
  - D(L)=D+D+D….+D

- Deterministic lead time

- Stochastic lead time

Example – uniform demand

- Single period demand is uniformly distributed with demands of 0,1,2,3,4 (each with p=0.2)
- Distribution of two period demand?

D	0	1	2	3	4	5	6	7	8
P(D=d)	0.04	0.08	0.12	0.16	0.2	0.16	0.12	0.08	0.04

Lead time (L) demand modelling

- Theoretical considerations
  - Normal: D(L)~Normal(μL,σ^2L)
  - Gamma: D(L)~Gamma(Lα,β)
  - Exponential: D(L)~Erlang(L,λ)
  - Poisson: D(L)~Poisson(λL)
- Convolution
  - Distribution of a sum of (two) (independent) random variables
  - Discrete case

$$\mathrm{P}(D=x)=\sum_{d_1=0}^{x}\mathrm{P}(D_1=d_1)\mathrm{P}(D_2=x-d_1)$$

  - Continuous case

$$F(x)=\int_0^x F_1(d_1)f_2(x-d_1)\mathrm{d}d_1$$

Heuristic parameter setting

- (R,*S*)
  - *R* fixed (periodic replenishment *R*=1)
  - Base-stock level *S* = expected demand during replenishment lead time plus safety stock
- (*s*,*Q*)
  - *Q*=*EOQ*
  - Reorder point *s* = expected demand during replenishment lead time plus safety stock
- (*s*,*S*)
  - *S*-*s*=*EOQ*
  - Reorder point *s* = expected demand during replenishment lead time plus safety stock

Parameters for the (R,S) policy

Service levels

- Non-stockout probability (α-Service level)

$$\mathrm{P}(y[t+R+L] \geq 0) = \mathrm{P}(S - D(R+L) \geq 0) = \mathrm{P}(D(R+L) \leq S) = \alpha$$

- Fill rate (β-Service level)
  - Expected fraction of demand during a period that can be serviced directly
  - 1 – exp. fraction of period demand that cannot be serviced directly
  - 1 – „expected units short (EUS)" / μ
    - EUS = exp. backlog after L+R – exp. backlog after L

$$\beta = 1 - \frac{\int_S^{\infty}(d-S) f_{L+R}(d)\,\mathrm{d}d - \int_S^{\infty}(d-S) f_L(d)\,\mathrm{d}d}{\mu}$$

Service levels

- Adjusted fill rate (γ-Service-level)
  - 1 – „expected backorders as fraction of demand“

$$\gamma = 1 - \frac{\int_S^\infty (d - S) f_{L+R}(d)\mathrm{d}d}{\mu}$$

  - Transformation for normally distributed demand

$$\int_S^\infty (d - S) f_{L+R}(d)\mathrm{d}d = \sigma\sqrt{L+R}\,G(z)$$

Standard normal loss function

$$z = \frac{S - \mu(L+R)}{\sigma\sqrt{L+R}}$$

  - So: $G(z) = \frac{(1-\gamma)\mu}{\sigma\sqrt{L+R}}$. Find *z* via standard normal loss function table, transform to S

Example

- Data
  - Normally distributed demand with mean 100, standard deviation 30
  - Deterministic lead time L=1, periodic review R=1
  - Note: D(L+R) is normal with mean μ(L+R) and s.d. $\sigma\sqrt{L+R}$
  - Service level: 95% (each type)
- Results
  - Non-stockout probability:

$$S = \mu(L+R) + z\sigma\sqrt{L+R} = 200 + 1.645 \cdot 30 \cdot \sqrt{2} \approx 270$$

  - Adjusted fill rate:

$$G(z) = \frac{(1-\gamma)\mu}{\sigma\sqrt{L+R}} = 0.1179 \Rightarrow z = 0.81$$

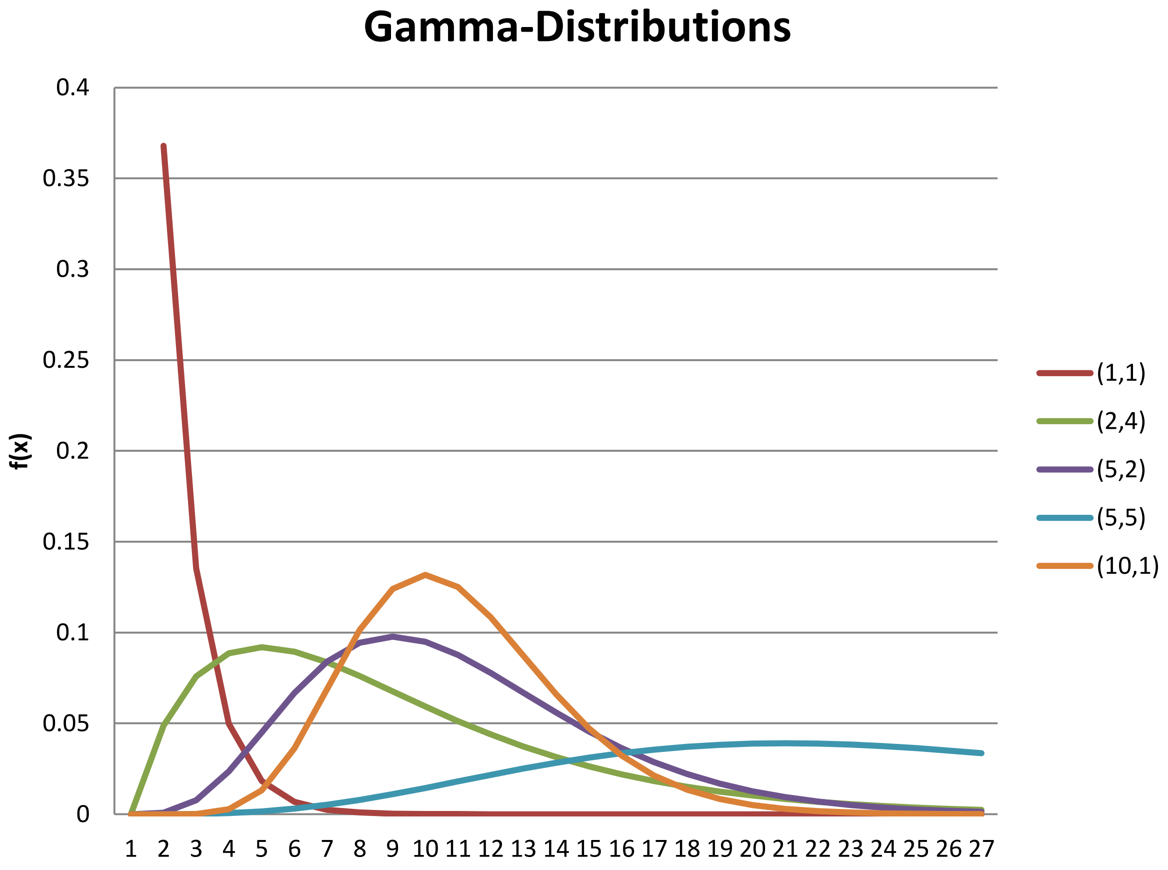
$$S = 200 + 0.81 \cdot 30 \cdot \sqrt{2} \approx 234$$

Gamma distribution

- Moment fitting
  - Mean: αβ, variance: $\alpha\beta^2$

$$\alpha = \frac{\mu^2}{\sigma^2}, \quad \beta = \frac{\sigma^2}{\mu}$$

- Example:
  - $\mu = 100, \sigma = 50, L = 1, R = 1$
  - $\mu(L+R) = 200, \sigma\sqrt{L+R} = 70.71 \rightarrow \alpha = 8, \beta = 25$
  - Non-stockout probability = 90%: S=294.27
  - Adjusted fill-rate = 90%: S=257.27
  - Excel: GAMMADIST(x,α,β,1), GAMMAINV(x,α,β),
  - ELS=αβ(1-GAMMADIST(s,α+1,β,1))-s(1-GAMMADIST(s,α,β,1))



Discrete distribution

- Cumulative distribution function

$$P(D(R+L) \leq x) = \sum_{d \leq x} P(D_{L+R} = d)$$

- Cost optimal policy

$$P(D(R+L) \leq S) \geq \frac{p-c}{p+h}$$

- Non-stockout probability:

$$\alpha = P(D(R+L) \leq S) = \sum_{d \leq S} P(D_{L+R} = d)$$

- Adjusted Fill rate

$$\gamma = 1 - \frac{\sum_{d=S+1}^{d_{\max}} (d-S)\mathrm{P}(D_{L+R} = d)}{\mu}$$

Stochastic lead time

- Approach 1: Normal distribution
- Approach 2: Discrete distribution for lead time

Safety stock formulas (Normal Distribution)

- Repeated replenishment (every *R* periods)
  - Lead time *L*

- Approach 1: Normally distributed demand and lead time
  - Demand stochastic, lead time deterministic

$$SS = z \cdot \sigma_D \cdot \sqrt{L+R}$$

  - Case 2: Demand and lead time uncertainty

$$SS = z \cdot \sqrt{(L+R) \cdot \sigma_D^2 + \mu_D^2 \cdot \sigma_L^2}$$

$\sigma_D = Standard\ Deviation\ of\ Demand$

$\sigma_L = Standard\ Deviation\ of\ Lead\text{-}Time$

$\mu_D = Mean\ Demand$

Example

- Normally distributed demand

  Mean 100, Variance 300
- Periodic review R=1
- Non-stockout probability: 95%
  - Case 1: No uncertainty in the Lead time, L=3

    $$SS = z\sigma\sqrt{L+R} = 1.645\sqrt{300\,(3+1)} = 56.98$$
  - Case 2: Stochastic lead time, lead time demand normally distributed

    Mean 3, Variance 2

$$SS = z \cdot \sqrt{(L+R)\cdot\sigma_D^2 + \mu_D^2 \cdot \sigma_L^2}$$

$$SS = 1.645 \cdot \sqrt{(3+1)\cdot 300 + 100^2 \cdot 2} = 239.52$$

Stochastic lead time

- Discretely distributed Lead time *L:* $l_{min} \leq l \leq l_{max}$

- Density of lead time demand

$$f_{L+R}(d) = \sum_{l=l_{\min}}^{l_{\max}} f_{L+R}(d|L = l) \cdot \mathrm{P}\{L = l\}$$

Distribution function

$$F_{L+R}(d) = \sum_{l=l_{\min}}^{l_{\max}} \left[\int_0^d f_{L+R}(x|L = l) \cdot \mathrm{d}x\right] \cdot \mathrm{P}\{L = l\}$$

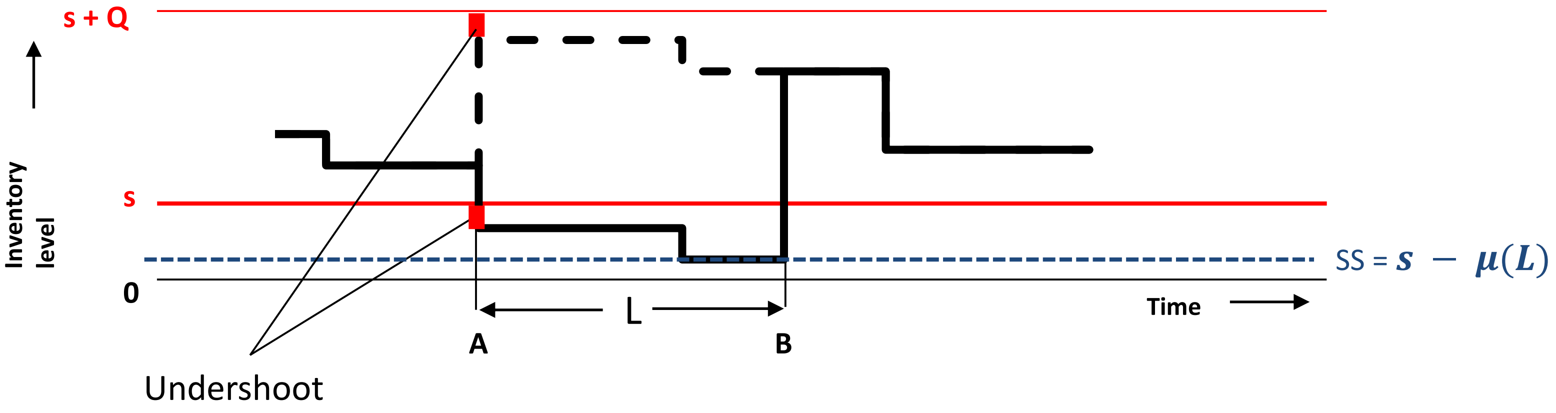
- Non-stockout probability (α-Service level)

$$\mathrm{P}(D(L + R) \leq d) = \sum_{l=l_{\min}}^{l_{\max}} \mathrm{P}\{D(L + R) \leq d|L = l\} \cdot \mathrm{P}\{L = l\} = \alpha$$

Parameters for the (s,Q) policy

Preliminaries

- **Unit demand**
  - – Uniform distribution of inventory position [s+1,s+Q]
- **Demand in batches (several at the same time)**
  - – Undershoot (time delay: inventory < s → order)



Service-level considerations

- Cycle non-stockout probability

$$P(D(L) \leq s) = \alpha$$

- Cycle-based fill-rate definition $ELS \leq (1-\beta) \cdot Q$

$$\beta = 1 - \frac{\int_s^{\infty} (d_L - s) f(d_L)\, \mathrm{d}d_L}{Q}$$

$$G(z) = \frac{Q}{\sigma\sqrt{L}}(1-\beta) \text{ (normal distr.)}$$

(continuous distribution)

$$\beta = 1 - \frac{\sum_{d_L = s+1}^{d_{L\,\max}} (d_L - s) \cdot \mathrm{P}\{D_L = d_L\}}{Q}$$

(discrete distribution)

Example (continuous)

- Data:
  - Normally distributed demand with mean 100, standard deviation 30
  - Order quantity Q = 500, Deterministic lead time L=3,
  - Service level: 95% (each type)

- Cycle non-stockout probability:

$$s_\alpha = \mu(L) + z\sigma\sqrt{L} = 300 + 1.645 \cdot 30 \cdot \sqrt{3} = 386$$

- Cycle-based fill-rate definition:

$$G(z) = \frac{Q}{\sigma\sqrt{L}}(1-\beta) = 0.4811 \Rightarrow z = -0.15$$

$$s_\beta = 300 - 0.15 \cdot 30 \cdot \sqrt{3} = 293$$

Example (discrete)

- Data:
  - Demand in Lead time

d	0	1	2	3	4	5	6
P(D=d)	0.0046	0.0392	0.1418	0.2704	0.2890	0.1800	0.0750
P(D≤d)	0.0046	0.0438	0.1856	0.4560	0.7450	0.9250	1.000

  - Order quantity Q = 20, Deterministic lead time L=2, SL = 95%

- Cycle non-stockout probability:

$$s_\alpha = \min[s \mid P(D(L) \leq s) \geq \alpha] = 6$$

- Cycle-based fill-rate definition:

$$ELS \leq (1-\beta) \cdot Q = (1-0.95) \cdot 20 = 1.0$$

$$\Rightarrow s_\beta = 3$$

s	*ELS(s)*
0	3.640
1	2.645
2	1.688
3	**0.874**
4	0.330
5	0.075
6	0.000

Cost optimization for given lot-size

- Expected cost per period
  - Expected ordering cost
  - Inventory holding cost h
    - Cycle stock
    - Safety stock
  - Shortage penalty p
    - Per stockout occasion
    - Per unit short
    - Per unit short per unit of time

Cost considerations (normal distribution)

- Penalty cost per stockout occasion

$$C(z)=\frac{\mu}{Q}A+h\left(\frac{Q}{2}+z\sigma\sqrt{L}\right)+\frac{\mu}{Q}p(1-F_{0,1}(z))$$

Recall!
$s = \mu L + z\sigma\sqrt{L}$

- Penalty cost per unit short

$$C(z)=\frac{\mu}{Q}A+h\left(\frac{Q}{2}+z\sigma\sqrt{L}\right)+\frac{\mu}{Q}p\sigma\sqrt{L}G(z)$$

- Penalty cost per unit short per period
  - Derivation skipped due to equivalence with fill rate model for $\beta = \frac{p}{p+h}$

Derivations

- Cost per stockout occasion

$$\frac{dC}{dz} = h\sigma\sqrt{L} - \frac{\mu p}{Q} f_{0,1}(z) = 0$$

$$z = \sqrt{2\ln\left(\frac{\mu p}{\sqrt{2\pi}Qh\sigma\sqrt{L}}\right)} \text{ if } \frac{\mu p}{\sqrt{2\pi}Qh\sigma\sqrt{L}} \geq 1$$

- Cost per unit short

$$\frac{dC}{dz} = h\sigma\sqrt{L} - \frac{\mu p\sigma\sqrt{L}}{Q}\left(F_{0,1}(z) - 1\right) = 0$$

$$F_{0,1}(z) = 1 - \frac{hQ}{p\mu} \text{ if } \frac{hQ}{p\mu} < 1$$

Recall!
$s = \mu L + z\sigma\sqrt{L}$

Derivations (cont.)

- Cost per unit short per period
  - Optimal solution under a fill-rate $\beta = \frac{p}{p+h}$

$$G(z) = \frac{Q}{\sigma\sqrt{L}} \frac{h}{p+h}$$

Joint optimization for cost per stock-out

- Necessary conditions

$$\frac{\partial C(z,Q)}{\partial Q} = \frac{h}{2} - \mu \frac{A + p\left(1 - F_{0,1}(z)\right)}{Q^2} = 0 \Leftrightarrow Q = \sqrt{\frac{2\mu(A + p(1 - F_{0,1}(z)))}{h}}$$

$$\frac{\partial C(z,Q)}{\partial z} = 0 \Leftrightarrow z = \sqrt{2\ln\left(\frac{\mu p}{\sqrt{2\pi}Qh\sigma\sqrt{L}}\right)}$$

- Algorithm
  - Start with Q=EOQ
  - Repeat
    - Determine z(Q)
    - Determine Q(z)
  - Until stopping criterion

Example

- Data
  - Normally distributed annual demand with μ=500, σ=100
  - Lead time L of 3 months
  - Fixed ordering cost 250€ per order
  - Inventory holding cost h=2 per unit per period
  - Shortage penalty p=1000 per occasion

- Solve this problem using
  - Successive approach
  - Simultaneous approach

Solution

- Successive approach
  - Using EOQ

$$Q = \sqrt{\frac{2dA}{h}} = \sqrt{\frac{2 \times \$250 \times 500\text{units/yr}}{\$2/\text{unit/yr}}} = 354$$

$$z = \sqrt{2\ln\left(\frac{\mu p}{\sqrt{2\pi}Qh\sigma\sqrt{L}}\right)} = \sqrt{2\ln\left(\frac{500 * 1000}{\sqrt{2\pi} * 353.55 * 2 * 100\sqrt{1/4}}\right)} = 1.86$$

$$s = \mu L + z\sigma\sqrt{L} = 500 * \frac{1}{4} + 1.86 * 100\sqrt{1/4} = 218$$

- The resulting cost of when Q=354 and s=218

$$C(s,Q) = \frac{500}{353.55}250 + 2\left(\frac{353.55}{2} + 1.86 * 100\sqrt{\frac{1}{4}}\right) + \frac{500}{353.55}1000(1 - F_{0,1}(1.86)) = 937.57$$

Solution

- Simultaneous solution
- By starting with Q=EOQ and computing s and Q iteratively;

	Q	z(Q)	C(s,Q)
1	353.55	1.86	937.57
2	375.12	1.83	936.17
3	376.67	1.83	936.16
4	376.79	1.83	936.16
5	376.79	1.83	936.16

- We find s=216, Q=377, C(s,Q)=936.16