„Inventory Management“
Official Formula Sheet
Summer 2025

Economic Order Quantity Model

Average cost formulation

$$C(Q) = \frac{d}{Q}A + \frac{h}{2}Q + cd$$

$$\frac{dC(Q)}{dQ} = -\frac{dA}{Q^2} + \frac{h}{2} = 0 \rightarrow Q^* = \sqrt{\frac{2dA}{h}}$$

$$C(Q^*) = \sqrt{2dAh} + cd$$

Percentage cost penalty

$$PCP = \frac{TRC(Q') - TRC(Q^*)}{TRC(Q^*)} \times 100$$

All-Unit Quantity Discount

Unit Cost (under $h = I \cdot C_i$)

$$c(Q) = \begin{cases} C_1 & \text{for} \quad Q < q_2 \\ C_2 & \text{for} \quad q_2 \leq Q \end{cases}$$

$$EOQ_i = \sqrt{\frac{2dA}{I \cdot C_i}}$$

$$C(EOQ_i) = \sqrt{2dA(I \cdot C_i)} + dC_i$$

Algorithm

- Determine $EOQ_2, C(EOQ_2)$
- EOQ_2 feasible $(EOQ_2 \geq q_2)$?
  - Yes: $Q^* = EOQ_2$
  - No: $Q^* = \text{argmin}\{C(EOQ_1); C(q_2)\}$
  - $C(q_2) = A\frac{d}{q_2} + (I \cdot C_2)\frac{q_2}{2} + dC_2$

Incremental Quantity Discount

Purchasing cost of Q items

$$C_1(Q-q_1) \text{ for } q_1 \leq Q < q_2$$
$$C_1(q_2-q_1)+C_2(Q-q_2) \text{ for } q_2 \leq Q < q_3$$
$$C_1(q_2-q_1)+C_2(q_3-q_2)+C_3(Q-q_3) \text{ for } q_3 \leq Q$$

Algorithm

1. Compute Q_j^*

$$Q_j^* = \sqrt{\frac{2(R_j - C_j q_j + A)d}{IC_j}} \quad \forall j$$

where R_j denotes the sum of the terms that are independent of Q, if $q_j \leq Q < q_{j+1}$

2. Check if $q_{j+1} > Q_j^* \geq q_j$ and disregard the ones that do not satisfy this inequality

3. $Q^* = argmin\left\{C(Q_j^*)\right\}$, for each remaining Q_j^*,

where, $C(Q_j) = C_j \cdot d + (R_j - C_j q_j + A)\,\frac{d}{Q_j} + \frac{IC_jQ_j}{2} + \frac{I(R_j - C_j q_j)}{2}$

Statistics

Estimator for mean:

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} d_t$$

Estimator for standard deviation:

$$\hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(d_t - \hat{\mu})^2}$$

Gamma distribution:
Mean: $\alpha\beta$, variance: $\alpha\beta^2$, estimator for α: $\hat{\mu}^2/\hat{\sigma}^2$, estimator for β: $\hat{\sigma}^2/\hat{\mu}$

Compound Poisson distribution: estimator for arrival rate and individual demand size mean:

$$\hat{\lambda} = -\ln(n_0/T) \text{ and } \hat{\mu}_c = \hat{\mu}/\hat{\lambda}$$

Newsvendor

Revenue function (for price p, salvage value g, and procurement cost c)
discrete

$$\Pi(y) = -c \cdot y + \sum_{d=0}^{y-1}(p \cdot d + g \cdot (y-d)) \cdot \mathrm{P}(D=d) + \sum_{d=y}^{\infty} p \cdot y \cdot \mathrm{P}(D=d)$$

continuous

$$\Pi(y) = -c \cdot y + \int_0^y (p \cdot d + g \cdot (y-d)) \cdot f(d)\mathrm{d}d + \int_y^{\infty} p \cdot y \cdot f(d)\mathrm{d}d$$

Solution

discrete

$$F(y^* - 1) \leq \frac{p-c}{p-g} \leq F(y^*)$$

continuous

$$F(y^*) = \frac{p-c}{p-g}$$

Expected lost sales

$$ELS(y) = \int_y^\infty (d-y) f(d) \mathrm{d}d$$

Special case: Normal distribution

$$z = \frac{y-\mu}{\sigma}, ELS(y) = \sigma \cdot G(z), G(z) = f_{0,1}(z) - z(1 - F_{0,1}(z))$$

Expected sales

$$ES(y) = \int_0^y d f(d) \mathrm{d}d + \int_y^\infty y \cdot f(d) \mathrm{d}d = \mu - ELS(y)$$

Expected left-over inventory

$$ELO(y) = \int_0^y (y-d) f(d) \mathrm{d}d = y - ES(y)$$

Fill rate

$$\beta = \frac{ES(y)}{\mu} = 1 - \frac{ELS(y)}{\mu}$$

Non-stockout probability

$$\alpha = F\left(\frac{y-\mu}{\sigma}\right) = F_{0,1}(z)$$

Expected profit

$$EP(y) = -cy + pES(y) + gELO(y)$$

Newsvendor solution under normally distributed demand with estimated parameters

$$y^* = \hat{\mu} + t_{n-1}^{-1}\left(\frac{p-c}{p-g}\right)\hat{\sigma}\sqrt{1+1/n}.$$

Stochastic inventory control

Parameters for the (R,S) Policy

Service Levels
Non-stockout probability

$$\mathrm{P}(y[t+R+L]\geq 0)=\mathrm{P}(S-D(R+L)\geq 0)=\mathrm{P}(D(R+L)\leq S)=\alpha$$

Fill rate

$$\beta = 1-\frac{\int_S^\infty (d-S)f_{L+R}(d)\mathrm{d}d-\int_S^\infty (d-S)f_L(d)\mathrm{d}d}{\mu}$$

Adjusted fill rate (γ-Service-level)

$$\gamma = 1-\frac{\int_S^\infty (d-S)f_{L+R}(d)\mathrm{d}d}{\mu}$$

For normally distributed demand

$$\int_S^\infty (d-S)f_{L+R}(d)\mathrm{d}d=\sigma\sqrt{L+R}G(z)=(1-\gamma)\,\mu,\text{ where } z=\frac{S-\mu(L+R)}{\sigma\sqrt{L+R}}$$

Base stock level (normal distribution)

Stochastic demand, deterministic lead time

$$S=\mu(L+R)+z\cdot\sigma\cdot\sqrt{(L+R)}$$

Stochastic demand, stochastic lead time

$$S=\mu(L+R)+z\cdot\sqrt{(L+R)\cdot\sigma_D^2+\mu_D^2\cdot\sigma_L^2}$$

Parameters for the (s,Q) Policy (normally distributed demand)

Average cost (penalty cost per stockout occasion)

$$C(z)=\frac{\mu}{Q}A+h\left(\frac{Q}{2}+z\sigma\sqrt{L}\right)+\frac{\mu}{Q}p(1-F_{0,1}(z))$$

Average cost (penalty cost per unit short)

$$C(z)=\frac{\mu}{Q}A+h\left(\frac{Q}{2}+z\sigma\sqrt{L}\right)+\frac{\mu}{Q}p\sigma\sqrt{L}G(z)$$

Optimal z (penalty cost per stockout occasion)

$$z=\sqrt{2\ln\left(\frac{\mu p}{\sqrt{2\pi}Qh\sigma\sqrt{L}}\right)}\quad\text{if}\quad\frac{\mu p}{\sqrt{2\pi}Qh\sigma\sqrt{L}}\geq 1$$

Optimal z (penalty cost per unit short)

$$F_{0,1}(z)=1-\frac{hQ}{p\mu}\quad\text{if}\quad\frac{hQ}{p\mu}<1$$

Optimal z (penalty cost per unit short per period)

$$G(z) = \frac{Q}{\sigma\sqrt{L}} \frac{h}{p+h}$$

Service Levels
Cycle non-stockout probability

$$\mathrm{P}(D(L) \leq s) = \alpha$$

Cycle-based fill-rate definition

$$\beta = 1 - \frac{\int_s^\infty (d_L - s) f(d_L) \mathrm{d} d_L}{Q}$$

For normally distributed demand

$$G(z) = \frac{Q}{\sigma\sqrt{L}}(1-\beta)$$

Joint optimization: Algorithm

- Start with Q=EOQ
- Repeat
  - Determine z(Q)

$$z = \sqrt{2 \ln\left(\frac{\mu p}{\sqrt{2\pi} Q h \sigma \sqrt{L}}\right)}$$

  - Determine Q(z)

$$Q = \sqrt{\frac{2\mu(A + p(1 - F_{0,1}(z)))}{h}}$$

- Until stopping criterion
- $s = \mu L + z\sigma\sqrt{L}$.

Demand Modelling

χ^2 Test

Measure

$$X = \sum_{j=1}^{m} \frac{(n_j - s_j)^2}{s_j}$$

Critical value

$$\chi^2_{1-\alpha,k-s-1}$$

Constant model

$$y_t = a + \varepsilon_t$$

Moving average

$$\widehat{a}_t = \frac{1}{n} \sum_{k=t-n+1}^{t} y_k$$

Exponential smoothing

$$\widehat{a}_t = \alpha \cdot y_t + (1-\alpha) \cdot \widehat{a}_{t-1}$$

Forecast

$$p_{t+i} = \widehat{a}_t \quad i = 1, 2...$$

Forecast error

$$e_t = y_t - p_t$$

Moving averages

$$MAD_t = \frac{1}{n} \sum_{i=t-n+1}^{t} |e_i|$$

$$MSE_t = \frac{1}{n-1} \sum_{i=t-n+1}^{t} e_i^2$$

Exponential smoothing

$$MAD_t = \gamma \cdot |e_t| + (1-\gamma) \cdot MAD_{t-1}$$

$$ERR_t = \delta \cdot e_t + (1-\delta) \cdot ERR_{t-1}$$

Error signal

$$SIG_t = \frac{ERR_t}{MAD_t}$$

Adaptive change

$$\alpha_t = |SIG_t|$$

(Linear) trend model *Holt's method*

$$\widehat{a}_t = \alpha \cdot y_t + (1-\alpha)(\widehat{a}_{t-1} + \widehat{b}_{t-1})$$

$$\widehat{b}_t = \beta(\widehat{a}_t - \widehat{a}_{t-1}) + (1-\beta)(\widehat{b}_{t-1})$$

Forecast

$$p_{t+i} = \widehat{a}_t + \widehat{b}_t \cdot i \quad i = 1, 2...$$

Seasonality (with Trend)

$$y_t = (a + b * t) \cdot s_t + \varepsilon_t$$

Winters method

$$\widehat{a}_t = \alpha \frac{y_t}{\hat{s}_{t-z}} + (1-\alpha)(\widehat{a}_{t-1} + \widehat{b}_{t-1})$$

$$\widehat{b}_t = \beta(\widehat{a}_t - \widehat{a}_{t-1}) + (1-\beta)(\widehat{b}_{t-1})$$

$$\widehat{s}_t^u = \gamma\frac{y_t}{\widehat{a}_t} + (1-\gamma)(\widehat{s}_{t-z})$$

Normalize weights

$$\widehat{s}_t = \frac{z \cdot \widehat{s}_t^u}{\sum_{j=t-z+1}^{t} \widehat{s}_j^u}$$

Forecast

$$p_{t+i} = (\widehat{a}_t + \widehat{b}_t \cdot i)\widehat{s}_{t+i-kz} \quad i = 1,2...; k = \left\lceil \frac{i}{z} \right\rceil$$

Intermittent demand *Croston's method*
Case 1: $y_t = 0$:

$$\widehat{a}_t = \widehat{a}_{t-1}$$
$$\widehat{x}_t = \widehat{x}_{t-1}$$

Case 2: $y_t > 0$:

$$\widehat{a}_t = \alpha y_t + (1-\alpha)\widehat{a}_{t-1}$$
$$\widehat{x}_t = \beta(t-\tau) + (1-\beta)\widehat{x}_{t-1}$$

Forecast

$$p_{t+i} = \begin{cases} \widehat{a}_t & \text{if } i = k \cdot \lfloor \widehat{x}_t \rfloor, k = 1,2,... \\ 0 & \text{otherwise} \end{cases}$$

Forecast demand per period

$$\frac{\widehat{a}_t}{\widehat{x}_t}$$

Syntetos-Boylan approximation

$$(1-\beta/2)\frac{\widehat{a}_t}{\widehat{x}_t}$$

Regression Analysis

$$\hat{b} = \frac{\sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$$

$$\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t$$

Forecast

$$p_{t+i} = \hat{a} + \hat{b} \cdot x_{t+i}$$

$$R^2 = \frac{\sum_{t=1}^{n}(p_t - \bar{y})^2}{\sum_{t=1}^{n}(y_t - \bar{y})^2}$$

Safety stocks

$$\sigma \approx 1.25 \cdot MAD$$

Adaptive Planning

Order-up-to level

$$S = (R+L)\mu + z\sigma\sqrt{R+L}$$

Update

$$\hat{\mu}_t = (1-\alpha)\hat{\mu}_{t-1} + \alpha d_t$$

Portfolio-Effect

$$\mu(n) = \sum_{i=1}^{n} \mu_i \qquad \sigma(n) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \cdot \sigma_i \sigma_j}; \quad \rho_{ii} = 1$$

$$\begin{aligned} &\sigma(2) = \sqrt{\sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2} \\ &\rho_{12} = 1: \qquad \sigma(2) = \sigma_1 + \sigma_2 \\ &\rho_{12} = -1: \quad \sigma(2) = |\sigma_1 - \sigma_2| \end{aligned}$$

Special case: identical demands, zero correlation $\sigma(n) = \sqrt{n} \cdot \sigma$

Depot-Effect

Demand consolidation

$$D = \sum_{i=1}^{n} d_i$$

Individual replenishments

$$Q_i = \sqrt{\frac{2d_iA}{h}}; \; C_i = \sqrt{2d_iAh}$$

Central replenishments

$$Q = \sqrt{\frac{2DA}{h}}; \; C = \sqrt{2DAh}$$

Cost comparison

$$\Delta = \sqrt{2AH} \cdot (\sum_{i=1}^{n} \sqrt{d_i} - \sqrt{\sum_{i=1}^{n} d_i})$$

Special case: identical demand rates (square root law) $C(D) = \sqrt{n} \cdot C(d)$

Multi-level lot-sizing

Sequential planning $Q_W = nQ_R$

$$C_R(Q_R) = \frac{d}{Q_R}A_R + \frac{h_R}{2}Q_R \quad Q_R^* = \sqrt{\frac{2dA_R}{h_R}}$$

$$C_W(n) = \frac{d}{nQ_R}A_W + \frac{h_W}{2}Q_R(n-1)$$

$$C_W(n+1) \leq C_W(n) \Leftrightarrow n(n+1) \leq \frac{A_W h_R}{A_R h_W}$$

$$n^* = \min\left\{ n | n(n+1) \geq \frac{A_W h_R}{A_R h_W} \right\}$$

Simultaneous planning

$$C(n, Q_R) = \frac{d}{nQ_R} A_W + \frac{d}{Q_R} A_R + \frac{h_W}{2} Q_R(n-1) + \frac{h_R}{2} Q_R$$

$$Q_R^*(n) = \sqrt{\frac{2d(\frac{A_W}{n} + A_R)}{nh_W + h_R - h_W}} \quad C^*(n) = \sqrt{2d(\frac{A_W}{n} + A_R)(nh_W + h_R - h_W)}$$

$$C(n+1) \leq C(n) \Leftrightarrow n(n+1) \leq \frac{A_W}{A_R} \frac{h_R - h_W}{h_W}$$

$$n^* = \min\left\{ n | n(n+1) \geq \frac{A_W}{A_R} \frac{h_R - h_W}{h_W} \right\}$$

Safety stock planning

Guaranteed service

Minimize safety stock holding costs

$$\min \ \ C = h_W z_W \sigma \sqrt{L_W - ST_W} + h_R z_R \sigma \sqrt{ST_W + L_R + R} \quad \text{s.t.} \ \ 0 \leq ST_W \leq L_W$$

Optimality property: concave minimization problem

- Optimality of extreme points, $ST_W = 0$ or $ST_W = L_W$

Clark-Scarf-model

- L_i lead time for replenishment at location i
- D stochastic demand, mean μ , standard deviation σ, pdf f, cdf F
- D(L) cumulative demand over L periods, mean Lμ, variance Lσ^2
- Backorder penalty b per unit and unit of time
- Inventory holding cost h_i at location i per unit and unit of time
- Echelon holding cots $e_i = h_i - h_{i-1}, i = 2, ..., N$

Optimal solution

Stage 2

$$F_{L_2+1}(S_2) = \frac{h_1 + b}{h_2 + b}$$

Stage 1

$$\int_0^{S_2} F_{L_1}(S_1 - x) f_{L_2+1}(x) dx = \frac{b}{h_2 + b}$$

Multi-item inventory control

Warehouse scheduling problem

Dedicated capacity

$$\min \sum_{i=1}^{N} \left[\frac{d_i}{Q_i} \cdot A_i + \frac{h_i}{2} \cdot Q_i \right]$$

$$\text{s.t.} \sum_{i=1}^{N} a_i Q_i \leq W \quad Q_i \geq 0 \quad i = 1,2,...,N$$

Average utilization

$$\min \sum_{i=1}^{N} \left[\frac{d_i}{Q_i} \cdot A_i + \frac{h_i}{2} \cdot Q_i \right]$$

$$\text{s.t.} \sum_{i=1}^{N} 0.5 a_i Q_i \leq W \quad Q_i \geq 0 \quad i = 1,2,...,N$$

Lagrange function

$$L = \sum_{i=1}^{N} \left[\frac{d_i}{Q_i} \cdot A_i + \frac{h_i}{2} \cdot Q_i \right] + \lambda \left[\sum_{i=1}^{N} a_i Q_i - W \right]$$

Optimality conditions

$$\frac{\partial L}{\partial Q_i} = \frac{-d_i}{Q_i^2} A_i + \frac{h_i}{2} + \lambda a_i = 0$$

$$\rightarrow Q_i^*(\lambda) = \sqrt{\frac{2 d_i A_i}{h_i + 2\lambda a_i}} \quad i = 1,2,..N$$

Solution - Case 1: Unconstrained solution $\lambda = 0$

Solution - Case 2: Constrained solution, choose λ *such that* $\sum_{i=1}^{N} a_i Q_i^*(\lambda) = W$

Rotation (common) cycle

Capacity balance

$$(t_i - t_{i-1}) \sum_{j=1}^{N} a_j d_j = a_i d_i T \Leftrightarrow t_i = \frac{\sum_{j=1}^{i} a_j d_j}{\sum_{j=1}^{N} a_j d_j} T$$

Capacity requirement at t=0

$$W \geq \sum_{i=1}^{N} a(Q_i - d_i(T - t_i)) = \frac{\sum_{i=1}^{N} \sum_{j=1}^{i} a_i a_j d_i d_j}{\sum_{i=1}^{N} a_i d_i} T$$

Cost function

$$C = \sum_{i=1}^{N} \left(\frac{A_i}{T} + \frac{h_i d_i}{2} T \right)$$

Solution

$$T^* = \min \left\{ \sqrt{\frac{2 \sum_{i=1}^{N} A_i}{\sum_{i=1}^{N} h_i d_i}} ; W \frac{\sum_{i=1}^{N} a_i d_i}{\sum_{i=1}^{N} \sum_{j=1}^{i} a_i a_j d_i d_j} \right\}$$

Economic production quantity (EPQ)

Cycle length

$$T = Q/d$$

Manufacturing time

$$T_P = Q/p$$

Inventory cost per cycle

$$\frac{h}{2}(p-d){T^2}_p + \frac{h}{2}(p-d)T_p(T-T_p) = \frac{h}{2}(p-d)T_pT$$

Total cost per unit of time

$$C(Q) = \frac{d}{Q} \cdot A + \frac{h}{2}(p-d) \cdot \frac{Q}{p}$$

Optimal lot-size

$$Q^* = \sqrt{\frac{2dA}{h \cdot \left(1 - \frac{d}{p}\right)}} = \sqrt{\frac{p}{p-d}} \cdot EOQ > EOQ$$

Economic lot scheduling problem

Unconstrained solution

$$T^* = \sqrt{\frac{2\sum_{j=1}^{n} A_j}{\sum_{j=1}^{n} h_j d_j \left(1 - \frac{d_j}{p_j}\right)}} \qquad Q_j^* = d_j T^* \qquad j = 1,2,\ldots,n$$

Lower bound

$$T^* = \frac{\sum_{j=1}^{n} r_j}{1 - \sum_{j=1}^{n} \frac{d_j}{p_j}}$$

Basic period - approach

- Cost function

$$\min\ C(W, n_i) = \sum_{j=1}^{n} \left(\frac{A_j}{n_j W} + \frac{h_j}{2}(p_j - d_j)\frac{d_j}{p_j} n_j W \right)$$

- Determine W (minimum cycle length from independent optimization)
- Determine n_i, given W

$$n_i = 2^m \quad T_i = n_i W$$

- Determine

$$W = \sqrt{\frac{2\sum_{i=1}^{N} \frac{A_i}{n_i}}{\sum_{i=1}^{N} h_i (p_i - d_i)\frac{d_i}{p_i} n_i}}$$

- If multipliers n_i converged, then check feasibility. If not feasible, adjust n_i and determine W again.
- Feasibility:

$$\sum_{j=1}^{n} \left(\frac{r_j}{n_j} + \frac{Q_j}{n_j p_j} \right) \leq W.$$

Joint replenishment problem

Solution approach

- Let T_1 the smallest cycle time and assume that all cycle times $T_i = n_i T_1$ for i=1,...,N are integer multiples of T_1, and that $n_i = 1$
- Objective

$$C = \frac{A_0 + \sum_{i=1}^{N} \frac{A_i}{n_i}}{T_1} + \frac{T_1 \sum_{i=1}^{N} h_i d_i n_i}{2}$$

- Optimal initial cycle time

$$T_1^* = \sqrt{\frac{2(A_0 + \sum_{i=1}^{N} \frac{A_i}{n_i})}{\sum_{i=1}^{N} h_i d_i n_i}}$$

Optimal cost

$$C^* = \sqrt{2\left(A_0 + \sum_{i=1}^{N} \frac{A_i}{n_i}\right) \sum_{i=1}^{N} h_i d_i n_i}$$

Optimal (non-integer) multipliers

$$n_i = \sqrt{\frac{A_i h_1 d_1}{h_i d_i (A_0 + A_1)}}$$

Lower bound

$$\underline{C} = \sqrt{2(A_0 + A_1) h_1 d_1} + \sum_{i=2}^{N} \sqrt{2 A_i h_i d_i}$$

Iterative solution approach

- Determine initial multipliers by rounding non-integer values
- Determine new T_1
- Determine new integers from cost function
  - Minimum integer that satisfies

$$n_i (n_i + 1) \geq \frac{2A_i}{h_i d_i T_1^2}$$

- Repeat if any integer changed