§13 January 7, 2021

AMC-12 results were supposed to come out today but I highly doubt it because its sooo hard to grade a 25 question multiple choice test.

§13.1 Wedge Product / Determinants

Definition 13.1. The 2-wedge product is (mostly) the same as the tensor product. Here are the formal definitions:

$$(v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w$$
$$v \wedge (w_1 + w_2) = v \wedge w_1 + v \wedge w_2$$
$$(c \cdot v) \wedge w = v \wedge (c \cdot w)$$

And more notably,

$$v \wedge v = 0.$$

When we take $(v + w) \wedge (v + w) = 0$, we also find that

$$v \wedge w = -(w \wedge v).$$

When you expand to n-wedge product, the same is pretty much true, but

- if any 2 adjacent wedges switch, the the whole thing is negated.
- If any 2 adjacent wedges are equal the whole thing is zero.
- Everything expands distributively still.

If we have a linear map $T : V \to V$, and we take $\Lambda^n(T)$, the constant that the original wedge product of bases is multiplied by is the determinant.

$$(a_{11}e_1 + a_{21}e_2 + \dots) \wedge \dots \wedge (a_{1n}e_1 + a_{2n}e_2 + \dots + a_{nn}e_n)$$

you will find that it turns into a "neat" formula.

Theorem 13.2 (Leibniz formula)

Recall that S_n is the symmetric group. $\operatorname{sgn}(\delta)$ returns $+1$ when it is an even permutation, and -1 when it is an odd permutation.

$$\det A = \sum_{\delta \in S_n} \operatorname{sgn}(\delta) a_{1,\delta(1)} a_{2,\delta(2)} \dots a_{n,\delta(n)}$$

Definition 13.3. If we have a map with eigen alues $\lambda_1, \lambda_2, \dots, \lambda_n$ (with repetition), then the characteristic polynomial is defined as

$$p_T(X) = (X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_n).$$

Theorem 13.4 (Cayley-Hamilton)

Let $T : V \to V$ be a map of finite-dimensional vector spaces over an algebraically closed field. Then for any $T : V \to V$, the map $p_T(T)$ is the zero map.

Problem 13.5 (Determinant is product of eigenvalues)

Let V be an n-dimensional vector space over an algebraically closed field k. Let $T : V \to V$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with algebraic multiplicity). Show that $\det T = \lambda_1 \cdots \lambda_n$.

Solution. First step is to convert T to Jordan form. Then the determinant is $\Lambda^n(T)$. We take the n-wedge product. Since it is in jordan form, let c_i be either 0 or 1, representing the entry next to the eigenvalues. So the wedge product can be written as:

$$\det(T) = (\lambda_1 e_1 + c_1 e_2) \wedge (\lambda_2 e_2 + c_2 e_3) \wedge \cdots \wedge (\lambda_{n-1} e_{n-1} + c_{n-1} e_n) \wedge (\lambda_n e_n).$$

If we ever wedge the same bases, then the whole thing becomes zero! So the only way to wedge without 0 is by wedging all the eigenvalues:

$$\det(T) = (\lambda_1 \cdots \lambda_n)(\bigwedge_{i=1}^{n} e_i)$$

Thus the determinant is the product of eigenvalues as desired. □

Problem 13.6 (Exponential matrix)

X is a $n \times n$ matrix with complex coefficients. Define the exponential map by

$$\exp(X) = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots.$$

Prove that

$$\det(\exp(X)) = e^{\operatorname{Tr} X}.$$

Solution. Turn it into an upper triangular matrix T. The property of triangular matrices is that the diagonal has each of its terms to the power n when we take T^n. Therefore, a term in the diagonal d_i becomes

$$1 + d_i + \frac{d_i}{2!} + \frac{d_i}{3!} + \cdots = e^{d_i}$$

Determinant is just product of all of these in a triangular matrix, so

$$\det(\exp(X)) = e^{d_1} e^{d_2} \cdots e^{d_n} = e^{d_1 + d_2 + \cdots + d_n} = e^{\operatorname{Tr} X}$$

□

Problem 13.7

Let $T : V \to V$ be a map of finite-dimensional vector spaces. Prove that T is an isomorphism iff $\det T \neq 0$.

Solution. (1. $\det T \neq 0$ implies T isomorphism) We can replace isomorphic with injective as long as the dimensions of the vector spaces are equal, which they are. If $T(v) = 0$ for any nonzero v, then we can take a basis for which $e_1 = v$.

$$\Lambda^n(T) : (e_1 \wedge e_2 \wedge \dots) \mapsto (0 \wedge T(e_2) \wedge \dots) = 0.$$

Hence we have zero map (??) so $\det(T) = 0$.

(2. T isomorphism implies $\det T \neq 0$) Assume that $\det(T) = 0$ and T is an isomorphism. Then we construct inverse matrix $S = T^{-1}$.

$$\det(T \circ S) = \det(1) = 1$$

$\det(T) \neq 0$ as desired.

$\square$