1 The Solow Growth Model

The Solow growth model is constructed around 3 building blocks:

1. The aggregate production function:

$$Y(t) = AF(K(t), N(t)),$$

which it is assumed to satisfy a series of technical conditions:

(a) It is increasing in both arguments

$$F_K > 0, F_N > 0$$

(b) It displays decreasing marginal returns to each factor

$$F_{KK} < 0, F_{NN} < 0$$

(c) It displays constant returns to scale

$$AF(\lambda K, \lambda N) = \lambda AF(K, N)$$

(d) and it satisfies the Inada condition

$$\lim_{K \to 0} F_K = \lim_{N \to 0} F_N = +\infty.$$

For example, one production function that satisfies these properties is the Cobb-Douglas production function

$$F = AK(t)^\alpha N(t)^{1-\alpha}, \tag{1}$$

and from now on, we assume this specification. The constant A captures TFP and should be interpreted in a broad sense: it incorporates the effects of the organization of markets and production on the efficiency with which factors are utilized (e.g., in addition to the technological level, the degree of political instability, and the degree of protection of property rights would be absorbed by A).

1

2. The law of motion for the stock of capital. In discrete time we have:

$$K_{t+1} = I_t + (1 - \delta) K_t$$

To understand the equivalent formulation for the continuous time, consider a small time interval of length Δ. The dynamics of capital over the time interval can be described by

$$K(t + \Delta) = I(t)\Delta - (1 - \delta\Delta) K(t)$$

where the the flow-variables variables are the ones multiplied by the length of the interval. Dividing both sides by Δ and taking the limit as $\Delta \to 0$, we arrive at

$$\dot{K}(t) = I(t) - \delta K(t) \tag{2}$$

3. The savings/investment function. It is assumed to be of a Keynesian nature, i.e. savings (and investment in a closed economy) equals a constant fraction s of total income $Y(t)$, or

$$S(t) = I(t) = sY(t) \tag{3}$$

So, agents do not optimize in this model. The model can easily extended to include a household's problem with a dynamic consumption/saving decision (this is called the Ramsey-Cass-Koopmans model). The implications for growth and convergence remain similar.

Let's also assume that population grows at rate n, i.e. $N(t) = N_0 e^{nt}$.
Remark: *Why is n the growth rate? Take logs of both sides to obtain*

$$\ln N(t) = \ln N_0 + nt$$

and take the derivative with respect to time of both sides which yields

$$\frac{\dot{N}(t)}{N(t)} = n$$

In growth theory we are interested in the determination of income per capita (or per worker), as a measure of welfare of society. Thus, it is convenient to express all the variables of interest in per capita terms. Let's use small letters to denote per capita (or per worker) variables.

We start transforming the model from income per capita:

$$y\left(t\right) = \frac{Y\left(t\right)}{N\left(t\right)} = \frac{AK\left(t\right)^{\alpha} N\left(t\right)^{1-\alpha}}{N\left(t\right)} = A\left[\frac{K\left(t\right)}{N\left(t\right)}\right]^{\alpha} = Ak\left(t\right)^{\alpha}$$

Next, the law of motion for capital

$$\frac{\dot{K}\left(t\right)}{N\left(t\right)} = \frac{I\left(t\right)}{N\left(t\right)} - \delta\frac{K(t)}{N\left(t\right)} \Rightarrow \frac{\dot{K}\left(t\right)}{N\left(t\right)} = i\left(t\right) - \delta k\left(t\right) \tag{4}$$

What is that term on the left hand side? Consider:

$$\dot{k}\left(t\right) = \frac{\partial k\left(t\right)}{\partial t} = \frac{\partial\left(K\left(t\right)/N\left(t\right)\right)}{\partial t} = \frac{\dot{K}\left(t\right)N\left(t\right) - \dot{N}\left(t\right)K\left(t\right)}{N\left(t\right)^2} = \frac{\dot{K}\left(t\right)}{N\left(t\right)} - \frac{\dot{N}\left(t\right)}{N\left(t\right)}\frac{K\left(t\right)}{N\left(t\right)}$$

$$\dot{k}\left(t\right) = \frac{\dot{K}\left(t\right)}{N\left(t\right)} - nk\left(t\right) \Rightarrow \frac{\dot{K}\left(t\right)}{N\left(t\right)} = \dot{k}\left(t\right) + nk\left(t\right),$$

which substituted into the law of motion for capital (4) gives

$$\dot{k}\left(t\right) = i\left(t\right) - \left(\delta + n\right)k\left(t\right) \tag{5}$$

Hence the growth rate of the labor force $N\left(t\right)$ works like an additional source of depreciation. Why? Because for a given stock of capital $K\left(t\right)$, the larger the population, the lower the stock of capital per worker.

Now, substitute the investment function (3) in per capita terms

$$i\left(t\right) = sy\left(t\right) = sAk\left(t\right)^{\alpha}$$

into (5) to obtain the **Solow growth equation**

$$\dot{k}\left(t\right) = sAk\left(t\right)^{\alpha} - \left(\delta + n\right)k\left(t\right). \tag{6}$$

This is a nonlinear first order differential equation in the variable $k\left(t\right)$ that describes the process of capital accumulation in the economy: *capital accumulation is the engine of growth*, since output per capita depends on the per-capita capital stock $k\left(t\right)$. According to Solow, to understand the growth process, we need to understand the reasons for capital accumulation.

Equation (6) lends itself to a simple interpretation: capital grows over time thanks to savings (and investments), and it declines over time due to "effective depreciation"

3

$(n + \delta)$. One key feature of that equation to notice is that the diminishing returns in capital imply that the larger is the capital stock, the smaller is the gain from additional capital, i.e. the production (and saving) technology is concave whereas the depreciation technology is linear.

To characterize the solution of (6), first we need to ask ourselves: does this dynamic system have a rest-point, i.e. a point where capital per worker is constant, i.e. $\dot{k}(t) = 0$ and $k(t) = k^*$? That would be the point where savings equal depreciation, or

$$sAk(t)^\alpha = (\delta + n) k(t)$$

Clearly, one such point is $k^* = 0$. Moreover, solving the above equation yields:

$$sAk^* = (\delta + n) k^* \Rightarrow k^* = \left(\frac{sA}{\delta + n} \right)^{\frac{1}{1-\alpha}}$$

and substituting the steady-state capital into the production function and the investment function, we obtain

$$
\begin{aligned}
y^* &= (k^*)^\alpha = \left(\frac{sA}{\delta + n} \right)^{\frac{\alpha}{1-\alpha}} \\
c^* &= (1 - s) y^* = (1 - s) \left(\frac{sA}{\delta + n} \right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

The comparative statics of the steady state capital stock with respect to the parameter vector $(\alpha, \delta, n, s, A)$ are as follows:

- k^* is increasing in A which is the level of the technology, or total factor productivity in the economy,

- k^* is increasing in the saving rate s which determines the amount of investment in the economy,

- k^* is decreasing in (δ, n) which determine how fast capital per worker depreciates in the economy,

- k^* is increasing in α, because the larger is α (closer to 1) the smaller are diminishing returns which are the key force against perpetual growth: if every unit of capital produces less and less, at some point the economy will stops accumulating capital.

4

The phase-diagram studied in class shows that the steady state $k^* = 0$ is dynamically unstable (therefore uninteresting), whereas $k^* = \left(\frac{sA}{\delta+n}\right)^{\frac{1}{1-\alpha}}$ is stable, i.e. the economy converges to this steady-state both if the initial level of k_0 is above k^* (with negative growth rates) and if k_0 is below k^*, the natural case, (with positive growth rates).

1.1 Predictions of the Solow model

- The Solow model (in this simple version) cannot generate long-run (perpetual) growth: when the system reaches the steady-state, the economy stops growing. The reason the model can't generate perpetual growth is the fact that *marginal product of capital is diminishing in capital* itself. Output per worker can grow only as long as capital per worker grows. Since as the capital stock grows larger it takes more and more investment to produce an additional unit of output, at some point the economy only invests to keep up with effective depreciation. Thus, the key to perpetual growth is the existence of non-diminishing marginal product of capital.

- The distance from the steady-state determines growth: the smaller is $y(t)$ and the further away we are from the steady-state, the faster is growth. Divide both sides of (6) by $k(t)$:

$$g_k(t) = \frac{sA}{k(t)^{1-\alpha}} - (\delta + n) \tag{7}$$

so the smaller is the initial capital stock $k(t)$ the faster is growth. Note that when $k(t) = k^*$ and the distance from the steady-state is zero, $g_k(t) = 0$ (just substitute $k(t) = k^*$ in the equation above).

Remark: *consider two countries with the same parameters, then growth is only determined by the distance from the common steady-state. The case of US vs Japan/Germany after WW II provides a nice illustration of this mechanism.*

- The Solow model predicts **conditional convergence**, i.e. once we control for observable differences between countries (that will affect their steady-state level) such as investment rate, human capital, population growth, and key variables that could be summarized by A such as protection of property rights, financial development, etc..., then the lower is the initial capital, the faster is growth. The growth regressions we talked about try to estimate exactly an equation like (7).

1.2   The Golden Rule of Capital Accumulation

An interesting question in the Solow model is: what is the level of capital that maximizes consumption in steady-state: is it the steady state capital of the Solow model k^* or it is a different value? In steady state

$$sf\left(k^*\right) = \left(n+\delta\right)k^* \Rightarrow c^* = f\left(k^*\right) - \left(n+\delta\right)k^*.$$

Write this equation as a function of k^* and solve

$$\max_{k^*} c^*\left(k^*\right) = \left\{\max_{k^*} f\left(k^*\right) - \left(n+\delta\right)k^*\right\}$$

which has solution

$$f_k\left(k^*\right) = n+\delta$$

Solving that equation for the Cobb-Douglas case gives

$$\alpha A\left(k_g\right)^{\alpha-1} = n+\delta \Rightarrow k_g = \left(\frac{\alpha A}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

which differs from the steady-state level insofar as $s \neq \alpha$, i.e. the saving rate differs from the capital share. This level of steady-state capital k_g that maximizes consumption is called the "golden-rule". In general, this exercise shows that certain economies might be over-saving $(s > \alpha)$ and as a consequence $k^* > k_g$, others might be under-saving $(s < \alpha)$ and $k^* < k_g$. An economy with a steady-state capital beyond k_g is said to be dynamically inefficient, because it could reduce its capital stock and free up resources to consume more; moreover, the transition to the golden rule capital stock k_g would entail higher consumption (and higher welfare) throughout.

1.3   The Solow Model with Exogenous Technical Change

One way to generate sustained growth in per capita income is allowing for exogenous technological change in the model, i.e. allowing the technological parameter A to grow exogenously over time. For example, assume that

$$A(t) = A_0 e^{\gamma t} \Leftrightarrow \frac{\dot{A}\left(t\right)}{A\left(t\right)} = \gamma$$

Suppose $A\left(t\right)$ is "labour augmenting", i.e.

$$Y\left(t\right) = K\left(t\right)^{\alpha}\left[A\left(t\right)N\left(t\right)\right]^{1-\alpha}.$$

What is the growth rate of output per worker? Dividing both sides by $N(t)$, we obtain

$$y(t) = k(t)^\alpha A(t)^{1-\alpha}$$

and differentiating with respect to time

$$\frac{\dot{y}(t)}{y(t)} = (1-\alpha)\frac{\dot{A}(t)}{A(t)} + \alpha\frac{\dot{k}(t)}{k(t)} = (1-\alpha)\gamma + \alpha\frac{\dot{k}(t)}{k(t)} \qquad (8)$$

Hence, to determine $\frac{\dot{y}(t)}{y(t)}$ we need first to determine the growth rate of capital. From the Solow growth equation:

$$\frac{\dot{k}(t)}{k(t)} = sA(t)^{1-\alpha}k(t)^{\alpha-1} - (\delta+n) = s\frac{A(t)^{1-\alpha}k(t)^\alpha}{k(t)} - (\delta+n) = s\frac{y(t)}{k(t)} - (\delta+n), \qquad (9)$$

hence the growth rate of capital is constant if capital and output per worker grow at the same rate.

The Kaldor facts suggest that economic growth in the long-run is **balanced**, i.e. output and capital grow indeed at the same rate. So, for the model to replicate this important feature of the data, we impose a *balanced growth path* to the model (all variables are growing at the same constant rate g). How do we determine the exact value of g, the growth rate of capital and output?

From equation (8) and the requirement that $\frac{\dot{y}(t)}{y(t)} = \frac{\dot{k}(t)}{k(t)}$, we obtain

$$\frac{\dot{y}(t)}{y(t)} = (1-\alpha)\gamma + \alpha\frac{\dot{k}(t)}{k(t)} \Rightarrow (1-\alpha)g = (1-\alpha)\gamma \Rightarrow g = \gamma$$

As a by-product, from equation (9) one obtains the long-run capital/output ratio

$$\gamma = s\frac{y^*}{k^*} - (\delta+n) \Rightarrow \frac{k^*}{y^*} = \frac{s}{(\delta+n+\gamma)} \simeq \frac{30\%}{5\% + 2\% + 3\%} \simeq 3$$

where the values substituted are the typical ones for the U.S. economy.

Consider now the Solow growth equation (6)

$$\dot{k}(t) = sA(t)^{1-\alpha}k(t)^\alpha - (\delta+n)k(t),$$

and divide both sides by $A(t)$:

$$\frac{\dot{k}(t)}{A(t)} = s\left[\frac{k(t)}{A(t)}\right]^\alpha - (\delta+n)\frac{k(t)}{A(t)}.$$

To study the dynamics, it is useful to define the capital-technology ratio $\tilde{k}(t) \equiv \dfrac{k(t)}{A(t)}$.

What is $\dfrac{\dot{k}(t)}{A(t)}$? Consider

$$\frac{d}{dt}\left[\frac{k(t)}{A(t)}\right] = \frac{\dot{k}(t)A(t) - \dot{A}(t)k(t)}{A(t)^2} = \frac{\dot{k}(t)}{A(t)} - \gamma\tilde{k}(t) \Rightarrow \frac{\dot{k}(t)}{A(t)} = \dot{\tilde{k}}(t) + \gamma\tilde{k}(t),$$

which implies that we can write the Solow equation in the presence of technological progress, in terms of capital-technology ratio as

$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^\alpha - (\delta + n + \gamma)\tilde{k}(t) \tag{10}$$

The transformed system is extremely similar to the one we studied. The steady-state of the transformed model is

$$\tilde{k}^* = \left(\frac{s}{\delta + n + \gamma}\right)^{\frac{1}{1-\alpha}}.$$

So the model is stationary in terms of capital-technology ratio $\tilde{k}(t)$ which is convenient to study the dynamics. But it will not be stationary in terms of the original variable, capital per worker $k(t)$. In fact the steady state $\tilde{k}^*$ corresponds to a balanced-growth path where $k(t)$ (and output, consumption, investment) grows at the constant rate γ, as seen from

$$\tilde{k}^* = \frac{k^*(t)}{A(t)} \Rightarrow k^*(t) = \left(\frac{s}{\delta + n + \gamma}\right)^{\frac{1}{1-\alpha}} A(t).$$

Outside the steady-state, $k(t)$ does not grow at a constant rate. The economy will approach the balanced growth path (where it grows at rate γ) at a decreasing (increasing) rate, if started below (above, respectively) the balanced growth value $k^*(t)$.

Think now, for example, about the following experiment. Suppose the economy is on its balanced growth path and the saving rate is s_0. Now, increase the saving rate to s_1. The balanced growth path will shift upward and the economy will fins itself below the new balanced growth path, hence it will start growing faster to converge to the new balanced growth path.

So, the Solow model with exogenous technical change can generate long-run growth, albeit exogenously. However, if we accept that the frontier technology $A(t)$ grows at the same rate for all countries, it cannot explain differences in long-run growth rates across

8

countries. More in general, the Solow model does not address the question: what are the determinants of long-run productivity growth? The challenge is to develop models that can generate perpetual growth without assuming that $A(t)$ grows exogenously. This is the scope of *endogenous growth* theory.