The Solow Growth Model Revisited. Introducing Keynesian Involuntary Unemployment

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Abstract

In this paper we extend the Solow growth model by introducing a simple mechanism which allows to determine involuntary unemployment explained by the weakness in aggregate demand. In our base model, we introduce a simple investment function and we find that an increase in aggregate demand (due to a reduction in the saving rate or to an increase in public expenditures) stimulates real GDP and reduces unemployment. Then, we modify the investment function in order to take into account the crowding-in/crowding-out effect on investments. This allows us to build a class of models which are between neoclassical supply-driven models and keynesian demand-driven models depending on the value of a key parameter that measures the degree of the crowding-in/crowding-out effect on investments and which lies between zero (for keynesian models) and one (for neoclassical models). Estimations on six OECD countries show that our key parameter lies between 0.6 and 0.8, implying that the fiscal multiplier is between 1 and 2, which is quite consistent with the empirical evidence.

JEL Classification: O40; E13; E12; J60.

Key-words: Neoclassical growth model; Keynesian model; Involuntary unemployment.

1 Introduction

It is quite surprising that neoclassical growth models have completely neglected a fundamental macroeconomic issue such as unemployment. Unemployment is considered as a short-term phenomenon affecting fluctuations but not as a long-term issue. In contrast, empirical data show that not only GDP growth rates but also unemployment rates fluctuate around a trend and, consequently, would deserve to be taken into account in growth models. Figure 1 shows the evolution of the unemployment rate for six OECD countries.

It is further surprising that a macroeconomic shock such as a change in public expenditures or, more generally, in one of the components of the aggregate demand, has completely different effects depending on whether one uses neoclassical supply-driven models or keynesian demand-driven models. In particular, the different vision about the functioning of the economy is reflected in the disagreement concerning the implementation of austerity policies to face the current double problem of high public debts and low economic growth.

It is well known that neoclassical models predict very low fiscal multipliers, which are not consistent with the empirical evidence.[1] This result is due to the fact that in neoclassical models an increase in public expenditures determines a strong crowding-out effect on consumption and investments, and only a small positive effect on GDP through the increase in the labor supplied by households. In contrast, DSGE models are able to produce fiscal multipliers consistent with the empirical evidence thanks to two key assumptions, namely that the markup ratio is counter-cyclical and that the labor supply elasticity is sufficiently high (Hall, 2009). The first assumption has been criticized by Hall (2009) since it is not supported by empirical analysis.[2] Concerning the second assumption, there exists a

[1]Empirical studies show that the fiscal multiplier ranges from 0.5 to 1 (see, e.g., Hall, 2009).

[2]However, Woodford (2011) states that DSGE models are able to produce high fiscal multipliers without assuming that the markup ratio is counter-cyclical. He shows that the multiplier is equal to one if the central bank is able to keep the real interest rate constant. In addition, he shows that if the monetary policy is constrained by the zero level of the nominal interest rate, than DSGE models produce much higher fiscal multipliers.

strong controversy between micro and macro labor supply elasticities.[3]

In a series of recent papers, Farmer (2010; Farmer (2012; 2013a; 2013b) and Farmer and Plotnikov (2012) use a model with search and matching frictions in the labor market in order to provide a new foundation to keynesian economics. In these works, Farmer argues that the Keynes's General Theory has nothing to do with sticky prices and unemployment is a potentially permanent feature of a market economy in the long run. In particular, the aim of Farmer is to build a model which integrates two key ideas from Keynes' General Theory: (i) there exists a continuum of labor market equilibria and a continuum of steady-state unemployment rates, and (ii) animal spirits select an equilibrium. In order to model animal spirits, Farmer introduces, instead of a traditional wage bargaining equation, a so called belief function which is a forecasting rule used by agents to predict the future value of the financial assets. In his model, Farmer assumes that firms produce as many goods as are demanded and hire the number of workers that is necessary to produce the quantity demanded. The demand, in turn, depends on beliefs of market participants about the future value of assets. The economic outcomes are then determined by self-fulfilling beliefs. Farmer shows that an exogenous and permanent drop in confidence shifts the economy from full employment to a new equilibrium characterized by high unemployment. This is coherent with the observation that during major recessions there exists a strong negative correlation between the value of the stock market and the unemployment rate. Farmer also asserts that his model provides a much better fit to data than the canonical DSGE model given its ability to explain persistent unemployment as a demand-driven phenomenon, while in DSGE models the unemployment rate has to return to its natural level.

The aim of this paper is to propose an extension of the standard Solow model (Solow, 1956) which (i) takes into account the keynesian involuntary unemployment, i.e. the unemployment that is explained by the weakness in aggregate demand and (ii) permits to generate fiscal multipliers con-

[3]Micro elasticities, computed using individual data, are much smaller than macro elasticities, based on time series data. Kean and Rogerson (2012) present an attempt to reconcile the micro and macro controversy. In particular, they show that taking into account the presence of human capital accumulation and the extensive margin allows to achieve this reconciliation.

sistent with the empirical evidence without being obliged to use a high labor supply elasticity. In our paper, we agree with some ideas proposed by Roger Farmer. First, animal spirits represent a fundamental element affecting aggregate demand, GDP and employment. Second, keynesian involuntary unemployment may prevail in the short and in the long run, even if prices and wages are assumed to be perfectly flexible. This implies that (i) unemployment has to be considered not only as a short-term phenomenon affecting fluctuations, but also as a long-term issue and (ii) in order to introduce keynesian unemployment it is not necessary to assume wage rigidity. Even if, according to the keynesian view, flexible money wages has destabilizing effects in the economy,[4] it is clearly wrong to argue that keynesian unemployment is caused by wage rigidity. In fact, if the cause of unemployment is wage rigidity, then full employment would be easily achieved by reducing the wage level. But this is exactly the contrary of the keynesian view because a reduction in the wage level reduces households' income, contracts consumption, and has a negative effect on the real activity and on employment. Of course, wage rigidity is one of the causes of unemployment but, in the keynesian view, the key element explaining unemployment is the weakness in aggregate demand and not the wage rigidity.

In our paper, the main difference with respect to the theory proposed by Farmer is that we do not model the labor market with search and matching frictions. Even if we agree that frictions in the labor market, as well as wage rigidities, play an important role in explaining involuntary unemployment, the keynesian involuntary unemployment is provoked by the lack of aggregate demand and, therefore, occurs even in the absence of frictions in the labor market. The main contribution of this paper is thus the introduction of the keynesian explanation of involuntary unemployment in a neoclassical framework, without considering wage rigidities and labor market frictions.

Our paper is organized as follows. In the next section, we discuss the characteristics of the labor market and of the instantaneous equilibrium

[4]Keynes observed that a policy of flexible money wages "would be to cause a great instability of prices, so violent perhaps as to make business calculations futile in an economic society functioning after the manner of that in which we live. To suppose that a flexible wage policy is a right and proper adjunct of a system which on the whole is one of laissez-faire, is the opposite of the truth" (Keynes, 1936, p. 269).

in the presence of keynesian involuntary unemployment. In Section 3, we present our base model which extends the Solow model to endogenize the unemployment rate. We consider the Solow model because it is a simple neoclassical growth model where the labor supply is exogenous or, equivalently, the labor supply elasticity is assumed to be equal to zero. To endogenize the unemployment rate we relax the hypothesis that investments are determined by aggregate savings to achieve full employment. The only difference with respect to the standard Solow model is that we introduce one additional equation, i.e., the investment function, and one additional variable, i.e., the unemployment rate. In our base model we use a very simple investment function in which investments are assumed to be exogenous and depend on a parameter reflecting keynesian investors' animal spirits. We show that the instantaneous equilibrium may be characterized by the presence of involuntary unemployment if the parameter that measures animal spirits is lower than a threshold value. In addition, given that in our model we assume that unemployment is entirely explained by the weakness in aggregate demand, a reduction in the level of wages, for example through the negotiation of wages between firms and potential workers, is completely useless in reducing unemployment. We also show that an under-capitalized economy converges toward its steady-state equilibrium which may be characterized by a positive value of the unemployment rate. Then, we show that an increase in the saving rate has a negative effect on employment and GDP, both in the short and the long run. This result is due to the fact that our base model, although it presents many features of neoclassical models (i.e., the production function allows for factor substitutability, the representative firm maximizes its profit, factors are remunerated at their marginal productivity, and prices are perfectly flexible), in reality it works as a keynesian model, i.e., it is demand driven. Thus, in the base model, an increase in the saving rate provokes a reduction in private consumption and in aggregate demand, and thus, increases unemployment. In Section 4, we modify the investment function in a way which allows us to take into account the fact that a change in one of the components of the aggregate demand provokes a crowding-in/crowding-out effect on investments. In particular, we introduce a parameter measuring the degree of the crowding-in/crowding-out effect and we show that (i) if this parameter is equal to zero, the model coincides with our base model, i.e. the keynesian demand-driven model; (ii) if the parameter is equal to one, the model coincides with the standard Solow model; (iii) if the parameter lies between zero and one, the model becomes an intermediate model between a keynesian demand-driven model and a neoclassical supply-driven model. In this case, a shock or a policy that increases aggregate demand (e.g., a reduction in the saving rate or the implementation of an expansionary fiscal policy) stimulates GDP and reduces unemployment (while, in neoclassical models with exogenous labor supply, the short-run effect is nil), but, at the same time, produces a (partial) crowding-out effect on investments (that is not taken into account in keynesian models with exogenous investments). Next, we analyze the effect of the introduction of an expansionary fiscal policy in our base model in Section 5 and in a model in which the investment function takes into account the crowding-in/crowding-out effect on investments in Section 6. In Section 7, we present numerical simulations which illustrate (i) the effect of an increase in the saving rate, and (ii) the effect of the introduction of public expenditures. These simulations, which are run with different values of the parameter measuring the crowding-in/crowding-out effect on investments, show that the results are highly dependent on the value of this parameter. In Section 8, we present econometric estimations of the parameter measuring the crowding-in/crowding-out effect on investments for six OECD countries. We find that the key parameter of our model lies between 0.6 and 0.8 implying that the crowding-in/crowding-out effect on investments is quite important and, as we show in Section 9, the size of the fiscal multiplier is between 1 and 2, which is quite consistent with the empirical evidence. Conclusions and possible extensions to other neoclassical growth models are discussed in Section 10.

2 The instantaneous equilibrium and the labor market

In the standard Solow model, the representative firm demands the optimal quantity of labor and capital in order to maximize its profit given a technological constraint. At the optimum, the marginal productivity of each factor coincides with their real cost. Price flexibility permits to equilibrate factor demands and factor supplies. The remuneration of production factors is then determined such that the production factors available in the economy are fully employed by the representative firm. Thus, at each period, total production is fixed at the level corresponding to the full employment of the production factors. This implies that, at each period, the sum of the components of the aggregate demand is also fixed at a predetermined level. In particular, in the Solow model, which considers a closed economy without the government, consumption is determined by a fraction of the real (full employment) GDP, while investments, which are *not* determined by the optimal decision of the representative firm, are obtained residually. This implies that in the Solow model the macroeconomic equilibrium condition, which states that investments equal aggregate savings, determines the level of investments, i.e. investments are savings-driven. Consequently, the key hypothesis of the Solow model is that investments adjust in order to guarantee the full employment of the production factors. In contrast, in a keynesian model, instead, each component of the aggregate demand is determined by a specific equation, implying that the sum of the components of the aggregate demand determines real GDP. In particular, if investments are lower than a threshold level (for example, because of the investors' pessimism), then full employment cannot be achieved and unemployment, due to the weakness in aggregate demand, appears. Consequently, in a keynesian model, the macroeconomic equilibrium condition between investments and aggregate savings determines the level of real GDP. In other words, the introduction of a macroeconomic investment function, which is not directly related to the optimal behavior of the representative firm, implies that the competitive equilibrium may be characterized by the presence of unemployment.

Consider now the labor market. Patinkin (1965) asserted that "keynesian economics is the economics of unemployment disequilibrium" (pp. 337-338) because the presence of involuntary unemployment implies that the labor market is not cleared. Using a a general disequilibrium framework, Patinkin (1965) and Barro and Grossman (1971) show that a reduction in aggregate demand reduces labor demand which becomes lower than the full-employment level.[5]

Our interpretation of the functioning of the labor market, which is depicted in Figure 2, is different from that of Patinkin (1965) and Barro and Grossman (1971). In particular, our model is *not* a model of disequilibrium. Instead, our model can be defined as a model of *under-employment equilibrium*. The functioning of the labor market is depicted in Figure 2. First, the macroeconomic equilibrium condition between investments and aggregate savings determines the unemployment rate (u_B in Figure 2). In particular, this unemployment rate can be interpreted as the *equilibrium unemployment rate*[6] in the sense that it is the *only* level that guarantees the macroeconomic equilibrium between investments and aggregate savings or, equivalently, the equilibrium in the market of goods. Second, once the unemployment rate is determined and assuming that the labor supply elasticity is equal to zero as in the Solow model, it is possible to plot the (vertical) curve representing the *total* quantity of labor supplied, $\overline{L} \cdot (1 - u_B)$. Next, the profit-maximization condition determines the labor demand function, $L^d = f\left(\frac{w}{p}\right)$, as in standard neoclassical models. Next, the intersection between the labor demand curve and the vertical curve representing the total quantity of labor supplied (point B in Figure 2) determines the quantity of labor employed, $L^d_B = \overline{L} \cdot (1 - u_B)$, and the "equilibrium" wage rate $\left(\frac{w}{p}\right)_B$. Finally, the production function determines the level of production depending on the quantity of labor employed, $Y_B = F(L^d_B, \overline{K})$.[7]

[5]Barro and Grossman (1971) assumed that the reduction in aggregate demand is due to a high price level while, as have we have already said, keynesian theory states that unemployment is not caused by price rigidity. In addition, in their analysis, the quantity of labor demanded does not belong to the marginal labor productivity curve. This off-demand-curve analysis proposed by Patinkin (1965) and Barro and Grossman (1971) implies that, if labor demand is lower than the full-employment level, the real wage is lower than the marginal labor productivity, which is inconsistent with the firm's profit maximization. Interestingly, even Keynes asserted that in a competitive economy the real wage is equal to the marginal product of labor (Keynes, 1936, pp. 5 and 17).

[6]It is important to highlight that the concept of equilibrium unemployment rate used in our paper is completely different with respect to the concept used in search and matching models in which the equilibrium unemployment rate is the rate such that the number of people finding a job is equal to the number of people who lose a job.

[7]The functioning of the labor market that we have described is essentially equivalent to that discussed by Davidson (1967 and 1983). According to Davidson, the aggregate demand determines the level of production which in turn determines the level of employment, while the marginal productivity of labor determines the level of the real wage. However, we think that the fact that the wage rate is determined by the level

It is very important to note that point A in Figure 2, i.e. the intersection between the labor demand and the labor supply curves, does not represent an equilibrium in the case where the aggregate demand (and thus, the production level) is equal to $Y_B < \overline{Y}$, i.e. lower than the full-employment level. In fact, at point A, investments are lower than aggregate savings or, equivalently, the production level is greater than aggregate demand.

Thus, point B in Figure 2 represents the instantaneous equilibrium of the economy in the case in which the aggregate demand (and thus, the production level) is equal to $Y_B < \overline{Y}$. This equilibrium can be defined as an *under-employment equilibrium*, in the sense that the weakness in aggregate demand provokes involuntary unemployment. Nevertheless, it is an equilibrium: the market of goods and services is in equilibrium because the production is equal to the aggregate demand, and the labor market is in equilibrium because the demand of labor is equal to the total quantity supplied (that is equal to $(1-u)$ multiplied by the active population $\overline{L}$).

Our interpretation of the functioning of the labor market implies that, in order to take into account the keynesian involuntary unemployment, it is not necessary to introduce nominal nor real rigidities, in prices or in wages or in both. For this reason, we assume, as in the Solow model, that all the prices are perfectly flexible. Therefore, money is completely neutral and can be omitted from the analysis, and the good produced in the economy can be chosen as the *numéraire*.

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of the marginal productivity of labor is not completely satisfactory to explain the functioning of the labor market. In fact, the equality between the marginal productivity of labor and the real wage indicates that, in order to maximize profits, the quantity of labor demanded by firms must be such that the marginal productivity of labor coincides with the real wage. Thus, this equality cannot determine the real wage. In addition, if the quantity of labor demanded is already determined by the inverse of the production function (because employment represents the quantity of labor necessary to produce the quantity of goods demanded), then firms have nothing to maximize, implying that the first order condition for profit maximization is useless.

3 The base model

3.1 The instantaneous equilibrium

In this section, we present our base model which extends the standard Solow model by introducing keynesian involuntary unemployment. On the one hand, our base model is a neoclassical model in the sense that the production function allows for factor substitutability, the representative firm maximizes its profit, factors are remunerated at their marginal productivity, and all prices are perfectly flexible.[8] On the other hand, our base model works as a keynesian model. Even if the money market is not taken into account, our model is demand-driven implying that the weakness in aggregate demand provokes unemployment.

As in the Solow model, the production function is a Cobb-Douglas function with labor-augmenting productivity:

$$Y(t) = \left[K^d(t)\right]^{\alpha} \cdot \left[A(t) \cdot L^d(t)\right]^{1-\alpha} \tag{1}$$

where $K^d(t)$ and $L^d(t)$ represent respectively the demand of capital and labor, while $A(t)$ represents the productivity level assumed to grow at a constant rate g_A.

The optimal level of factor demand is determined by the following conditions for profit maximization:

$$\begin{aligned} r(t) + \delta &= \frac{\partial Y(t)}{\partial K^d(t)} && (2) \\ w(t) &= \frac{\partial Y(t)}{\partial L^d(t)} && (3) \end{aligned}$$

Factor prices [$r(t) + \delta$ and $w(t)$] are determined to equilibrate the factor

[8]Given that prices are assumed to be perfectly flexible, money is completely neutral. Thus, it is useless to introduce in our model the keynesian LM curve $\frac{\overline{M}}{P} = \frac{M^d}{P}(r, Y)$. This equation would determine the price level implying that a change in money supply $\overline{M}$ provokes a proportional change in all nominal prices, and thus, no real effects because all relative prices remain unchanged.

markets:

$$K^d(t) = K(t) \tag{4}$$

$$L^d(t) = L(t) \cdot [1 - u(t)] \tag{5}$$

where $K(t)$ represents the level of capital supplied by the representative household, $L(t)$ represents the working-age population assumed to grow at a constant rate n, and $u(t)$ represents the unemployment rate. Then, $L(t) \cdot [1 - u(t)]$ represents the number of workers.

It is important to note that, regardless of the model used, the number of workers (that enters the production function) depends on the size of the working-age population $L(t)$, on the activity rate $l(t)$, and on the unemployment rate $u(t) : L(t) \cdot l(t) \cdot [1 - u(t)]$. In the standard Solow model with exogenous labor supply, the term $l(t) \cdot [1 - u(t)]$ is implicitly exogenous and constant, and thus, it does not appear in the analytical resolution. Thus, the Solow model can be interpreted as a model with exogenous and constant unemployment while, in our model, the unemployment rate is endogenous. Concerning the activity rate, both in the standard Solow model and in our model, it is exogenously fixed to one (implying that the labor supply elasticity is equal to zero) and is omitted from the analytical resolution.

Considering the equilibrium in the factor markets (Equations 4 and 5), the production function may be rewritten as follows:

$$Y(t) = K(t)^\alpha \cdot [A(t) \cdot L(t) \cdot [1 - u(t)]]^{1-\alpha} \tag{6}$$

where $A(t) \cdot L(t) \cdot [1 - u(t)])$ represents the number of units of effective labor. The initial levels of productivity and of the working-age population are normalized to 1, thus: $A(t) = e^{g_A t}$ and $L(t) = e^{nt}$. Finally, we define $A(t) \cdot L(t)$ as the number of *potential* units of effective labor, in the sense that this variable represents the number of units of effective labor in the case full employment, $u(t) = 0$.

Before proceeding to the resolution of the model, it is important to present the notation used:

- The capital per potential unit of effective labor is defined as:

$$\widehat{k}(t) = \frac{K(t)}{A(t) \cdot L(t)} \tag{7}$$

- The capital per unit of effective labor is defined as:

$$\tilde{k}(t) = \frac{K(t)}{A(t) \cdot L(t) \cdot [1 - u(t)]} = \frac{\widehat{k}(t)}{1 - u(t)}$$

- Real GDP is then given by:

$$Y(t) = A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot [1 - u(t)]^{1-\alpha} \tag{8}$$

- Real GDP per potential unit of effective labor is given by:

$$\widehat{y}(t) = \frac{Y(t)}{A(t) \cdot L(t)} = \widehat{k}(t)^{\alpha} \cdot [1 - u(t)]^{1-\alpha} \tag{9}$$

The macroeconomic equilibrium condition states that investments are equal to aggregate savings. In the case of a closed economy without government and if, as assumed in the standard Solow model, the representative agent saves an exogenous and constant fraction s of his revenue $Y(t)$, the macroeconomic equilibrium condition is:

$$I(t) = S(t) = s \cdot Y(t)$$

The key assumption of our model is that investments are not determined by the macroeconomic equilibrium condition, i.e. investments are not savings-driven, but they are determined by a specific equation as in the keynesian model. In our base model, we introduce a simple macroeconomic *investment function* as follows:[9]

$$I(t) = \gamma \cdot e^{(n+g_A)t} \tag{10}$$

[9] In Appendix 1, we present a more general model in which investments also depend on the level of the interest rate r. More precisely, we use $I(t) = \gamma \cdot e^{(n+g_A)t} \cdot (r(t) + \delta)^{-\theta}$ with $\theta > 0$. Here we use a more simple expression because in most of the models presented in our paper it is possible to find an explicit solution only by fixing $\theta = 0$.

Concerning the investment function used in our base model, it is first important to note that investments are not microfounded. However, even in the standard Solow model and in other neoclassical models where the representative firm chooses at each period the optimal demand of capital to maximize its profits (see Equation 2), investments are not microfounded. The difference between the standard Solow model and our model is that in the Solow model investments are determined by the level of aggregate savings while, in our model, investments are determined by an independent investment function. Second, Equation 10 implies, as in the Samuelson's keynesian cross diagram, that investments are exogenous. In particular, investments are assumed to depend on a positive parameter γ which may be interpreted as a parameter reflecting keynesian investors' animal spirits.

Using Equation 10, the macroeconomic equilibrium condition becomes:

$$s \cdot \widehat{k}(t)^{\alpha} \cdot A(t) \cdot L(t) \cdot [1 - u(t)]^{1-\alpha} = \gamma \cdot e^{(n+g_A)t}$$

Solving the previous equation, we get:

$$1 - u(t) = \left(\frac{\gamma}{s}\right)^{\frac{1}{1-\alpha}} \cdot \widehat{k}(t)^{-\frac{\alpha}{1-\alpha}} \tag{11}$$

Equation 11 determines the instantaneous *equilibrium unemployment rate* which represents the only value that guarantees the equilibrium between investments and aggregate savings, and thus, the equilibrium between the aggregate supply $Y(t)$ and the aggregate demand $C(t) + I(t)$.

First of all, Equation 11 shows how investors' animal spirits affect the instantaneous equilibrium unemployment rate because it negatively depends on the value of γ. In particular, if $\gamma = s \cdot \widehat{k}(t)^{\alpha}$, the instantaneous unemployment rate is equal to zero. In fact, considering Equations 10 and 7, $\gamma = s \cdot \widehat{k}(t)^{\alpha}$ implies that $I(t) = s \cdot K(t)^{\alpha} \cdot [A(t) \cdot L(t))]^{1-\alpha}$, i.e. investments are equal to aggregate savings in a full-employment economy, as assumed in the Solow model. This means that if the parameter γ is allowed to vary over time, our base model is able to exactly mimic the standard Solow model. In contrast, if $\gamma < s \cdot \widehat{k}(t)^{\alpha}$, then the unemployment rate is positive. This means that if γ is lower than the value necessary to achieve full employment, the economy is in a situation of under-employment equilibrium due to the weakness in aggregate demand and, in particular, in investments. Moreover, fluctuations in the confidence of investors affect the instantaneous equilibrium unemployment rate.

Equation 11 also implies (i) $\frac{\partial u(t)}{\partial s} > 0$, and (ii) $\frac{\partial u(t)}{\partial \widehat{k}(t)} > 0$. In particular, (i) a reduction in the saving rate induces an increase in consumption and then in aggregate demand, which permits a reduction in the equilibrium unemployment rate. (ii) An increase in the capital per potential unit of effective increases the level of savings per potential unit of effective labor and, given that investments are exogenous, the unemployment rate has to increase in order to guarantee the macroeconomic equilibrium between investments and aggregate savings.

Finally, it is worthwhile noting that a reduction in the level of wages is completely useless in order to reduce unemployment. This is because, in our model, unemployment is provoked by the weakness in aggregate demand and, in particular, in investments. In fact, if $\gamma < s \cdot \widehat{k}(t)^{\alpha}$ and if the real wage is determined in order to achieve full employment (i.e. point A in Figure 2), then private savings $s \cdot K(t)^{\alpha} \cdot [A(t) \cdot L(t))]^{1-\alpha}$ are greater than investments $\gamma \cdot e^{(n+g_A)t}$, and, consequently, production is greater than aggregate demand, implying that the economy *is not* in equilibrium.

3.2 The steady state and the transition towards the long-run equilibrium

The evolution of the capital per potential unit of effective labor is given by:

$$\dot{\widehat{k}}(t) = \frac{d\left(\frac{K(t)}{A(t)\cdot L(t)}\right)}{dt} = \frac{\dot{K}(t) \cdot A(t) \cdot L(t) - K(t) \cdot (\dot{A}(t) \cdot L(t) + A(t) \cdot \dot{L}(t))}{[A(t) \cdot L(t)]^2}$$

Given that the aggregate capital stock evolves according to $\dot{K}(t) = I(t) - \delta \cdot K(t)$, we find that :

$$\dot{\widehat{k}}(t) = s \cdot \widehat{y}(t) - (n + g_A + \delta) \cdot \widehat{k}(t)$$

Combining the previous equation with Equations 9 and 11, we find that the dynamics of the capital per potential unit of effective labor is described by:

$$\dot{\widehat{k}}(t) = \gamma - (n + g_A + \delta) \cdot \widehat{k}(t) \tag{12}$$

The steady-state condition $\dot{\widehat{k}}(t) = 0$ allows us to determine the stationary value of the capital per potential unit of effective labor:

$$\widehat{k}^* = \frac{\gamma}{n + g_A + \delta} \tag{13}$$

Combining the previous equation with Equation 11, we determine the stationary value of the unemployment rate as:

$$1 - u^* = \frac{\gamma \cdot (n + g_A + \delta)^{\frac{\alpha}{1-\alpha}}}{s^{\frac{1}{1-\alpha}}} \tag{14}$$

The previous equations imply that a permanent increase in the parameter γ which reflects investors' animal spirits determines (i) an increase in the long-run value of the capital per potential unit of effective labor and (ii) a reduction in the long-run value of the unemployment rate. These results are explained by the fact that an increase in investments permits both a greater capital accumulation and an increase in aggregate demand.

Consider now an under-capitalized economy, i.e. an economy in which the initial value of the capital per potential unit of effective labor is lower than its stationary value, i.e. $\widehat{k}(0) < \widehat{k}^*$. Equations 12 and 11 imply that, during the transition phase, both the capital per potential unit of effective labor and the unemployment rate increase over time, until the economy reaches its steady state. In particular, the long-run unemployment rate is equal to zero, i.e. the economy converges toward the long-run full-employment equilibrium, only if $\gamma = \frac{s^{\frac{1}{1-\alpha}}}{(n+g_A+\delta)^{\frac{\alpha}{1-\alpha}}}$. In contrast, if γ is lower than this value, then the economy displays unemployment even in the long run. Clearly, this result is related to the fact that the parameter γ, which measures investors' animal spirits is assumed to be completely exogenous. Thus, the parameter γ does not converge over time to the value that guarantees the full employment in the long run.

One interesting aspect is the relationship between the growth rate of real wages and the unemployment rate during the transition phase of the economy towards its steady state equilibrium. Considering that $w(t) = (1-\alpha)\cdot A(t)\cdot\left[\frac{\widehat{k}(t)}{1-u(t)}\right]^{\alpha}$ (from Equation 3), $\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)} = \frac{\gamma}{\widehat{k}(t)} - (n+g_A+\delta)$ (from Equation 12) and $\frac{1-\dot{u}(t)}{1-u(t)} = -\frac{\alpha}{1-\alpha}\cdot\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)}$ (from Equation 11), it is possible to write the growth rate of the real wage as:

$$\frac{\dot{w}(t)}{w(t)} = g_A + \frac{\alpha}{1-\alpha}\cdot\left[\frac{\gamma}{\widehat{k}(t)} - (n+g_A+\delta)\right]$$

Considering again Equation 11 which implies that $\widehat{k}(t) = [1-u(t)]^{-\frac{\alpha}{1-\alpha}}\cdot\left(\frac{\gamma}{s}\right)^{\frac{1}{\alpha}}$, the growth rate of the real wage can be written as:

$$\frac{\dot{w}(t)}{w(t)} = g_A + \frac{\alpha}{1-\alpha}\cdot\left[\frac{[1-u(t)]^{\frac{\alpha}{1-\alpha}}\cdot\gamma}{\left(\frac{\gamma}{s}\right)^{\frac{1}{\alpha}}} - (n+g_A+\delta)\right] \tag{15}$$

Interestingly, Equation 15 may be interpreted as the Phillips curve. In fact, it shows that in our model there exists a negative relationship between the growth rate of the real wage and the unemployment rate. In fact, during the transition phase towards the long-run equilibrium, the growth rate of the real wage decreases over time while the unemployment rate increases over time.

4 Introduction of a crowding-in/crowding-out effect on investments

Our base model discussed in the previous section implies that an increase in the saving rate has a negative effect on employment and on real GDP, both in short and the long run. This result, which is not consistent with the empirical evidence, is related to the fact that an increase in the saving rate reduces private consumption and aggregate demand, while investments are assumed to be unaffected. This assumption is relaxed in this section.

In this section we consider an increase in the saving rate from the initial value s_{old} to the value s_{new} and we assume, for simplicity, that before the shock the economy is at the steady state. With respect to what supposed in the previous section, we assume here that a change in private consumption and savings could affect investments. In particular, we modify the macroeconomic investment function by adding a term that allows to consider the crowding-in/crowding-out effect on investments:

$$I(t) = \gamma \cdot e^{(n+g_A)t} + \beta \cdot \Delta S_H(t) \tag{16}$$

where β is a parameter lying between 0 and 1 that measures the degree of the crowding-in/crowding-out effect on investments, and $\Delta S_H(t)$ represents the change in private savings with respect to the pre-shock situation.

To analyze Equation 16, it is important to note that the change in private savings can be decomposed in two effects: (i) the effect provoked by the increase in the saving rate and computed at a given level of the unemployment rate; (ii) the effect provoked by the change in the unemployment rate and computed using the new value of the saving rate. Thus:

$$\begin{aligned} s_{new} \cdot Y_{new}(t) - s_{old} \cdot Y_{old}(t) &= \left[s_{new} \cdot \overline{Y}(t) - s_{old} \cdot Y_{old}(t)\right] \\ &+ \left[s_{new} \cdot Y_{new(t)} - s_{new} \cdot \overline{Y}(t)\right] \end{aligned}$$

where $\overline{Y}(t)$ represents the post-shock value of GDP computed at a given level of the unemployment rate, as follows:

$$\overline{Y}(t) = A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot (1 - u^*)^{1-\alpha} \tag{17}$$

In the investment function (Equation 16), the change in private savings has to be computed by considering only the first effect, i.e., by neutralizing the effect provoked by the change in the unemployment rate. Otherwise, the investment function becomes an identity (if $\beta = 1$) or it is never verified (if $\beta \neq 1$). In both cases, the parameter β cannot be identified in the econometric analysis. Thus, the change in private savings has to be computed as follows:

$$\Delta S_H(t) = s_{new} \cdot \overline{Y}(t) - s_{old} \cdot Y_{old}(t) \tag{18}$$

Concerning the effect of an increase in the saving rate, it is interesting to consider three cases:

1. If β is equal to 1, the increase in private savings computed at a given level of the unemployment rate produces an identical increase in investments, implying that the crowding-in effect on investments is *complete*. The increase in investments coincides with the first positive effect on private savings, whereas the second effect is nil which implies that the unemployment rate is not affected. In fact, the increase in the saving rate produces a reduction in consumption which is perfectly compensated by an increase in investments. Thus, the unemployment rate and real GDP are not affected in the first period. This is exactly what happens in the standard Solow model or in a neoclassical model where the elasticity of labor supply is equal to zero, implying that real GDP is a predetermined variable. Consequently, our model with $\beta = 1$ reproduces the standard Solow model.

2. If β is equal to 0, investments remain unchanged. Thus, the first positive effect is perfectly compensated by the second effect, i.e., by the reduction in private savings due to the increase in the unemployment rate. The crowding-in effect on investments is *nil*. This is exactly what happens in a keynesian model where investments are exogenous. In this case, an increase in the saving rate produces a reduction in consumption and in real GDP, and an increase in the unemployment rate. Consequently, our model with $\beta = 0$ reproduces a keynesian model with exogenous investments.

3. If $0 < \beta < 1$, the crowding-in effect on investments is *partial*. As we will see later, according to the value of β, an increase in the saving rate provokes (i) an increase in the level of investments (which is lower with respect to the case of a neoclassical model where the labor supply elasticity is equal to zero, but higher with respect to the case of a keynesian model with exogenous investments); and (ii) an increase in the unemployment rate (which is lower with respect to the case of a keynesian model with exogenous investments, but higher with respect to the case of a neoclassical model where the unemployment rate is exogenous and constant). This allows to build a class of models that are between the keynesian and the neoclassical models.

Introducing Equations 17 and 18 in Equation 16, the investment function becomes:

$$\begin{aligned} I(t) &= \gamma \cdot e^{(n+g_A)t} \\ &+ \beta \cdot s_{new} \cdot A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\ &- \beta \cdot s_{old} \cdot A(t) \cdot L(t) \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \end{aligned} \tag{19}$$

The investment function can also be written as follows:[10]

$$I(t) = \gamma \cdot e^{(n+g_A)t} + \beta \cdot \left[S(t) \cdot \left(\frac{1-u^*}{1-u(t)} \right)^{1-\alpha} - S^*(t) \right] \tag{20}$$

Equation 20 will be used in Section 8 in order to empirically investigate the value of the coefficient β.

4.1 Instantaneous equilibrium

Using Equation 19, the macroeconomic equilibrium condition, $s_{new} \cdot Y(t) = I(t)$, becomes:

$$\begin{aligned} s_{new} \cdot \widehat{k}(t)^{\alpha} \cdot e^{(n+g_A)t} \cdot [1-u(t)]^{1-\alpha} &= \gamma \cdot e^{(n+g_A)t} \\ &+ \beta \cdot s_{new} \cdot A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\ &- \beta \cdot s_{old} \cdot A(t) \cdot L(t) \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \end{aligned}$$

Equations 13 and 14 imply that $(\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} = \gamma / s_{old}$. Thus, the instantaneous equilibrium unemployment rate is given by:

$$1 - u(t) = \left[\frac{\gamma \cdot (1-\beta) + \beta \cdot s_{new} \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha}}{s_{new}} \right]^{\frac{1}{1-\alpha}} \cdot \widehat{k}(t)^{-\frac{\alpha}{1-\alpha}} \tag{21}$$

Two extreme cases are interesting: the case $\beta = 0$ implying that the crowding-in effect on investments is nil, and the case $\beta = 1$ implying that

[10] Computation details are reported in Appendix 2.

the crowding-in effect on investments is complete:

$$1 - u(t) = \begin{cases} \left(\frac{\gamma}{s_{new}}\right)^{\frac{1}{1-\alpha}} \cdot \widehat{k}(t)^{-\frac{\alpha}{1-\alpha}} & \text{if} \quad \beta = 0 \\ 1 - u^{*} & \text{if} \quad \beta = 1 \end{cases} \tag{22}$$

Note that the two polar cases reproduce, respectively, the keynesian model presented in the previous section and the Solow model where the unemployment rate is exogenous. Consequently, an increase in the saving rate increases the unemployment rate (except for the case of a complete crowding-in effect, i.e. $\beta = 1$) and the size of the negative effect is a decreasing function of β.

4.2 The steady state

The evolution of the capital per potential unit of effective labor is given by $\dot{\widehat{k}}(t) = s_{new} \cdot \widehat{y}(t) - (n + g_A + \delta) \cdot \widehat{k}(t)$. Considering Equations 9 and 21, we find:

$$\dot{\widehat{k}}(t) = \gamma \cdot (1 - \beta) + \beta \cdot s_{new} \cdot \widehat{k}(t)^{\alpha} \cdot (1 - u^{*})^{1-\alpha} - (n + g_A + \delta) \cdot \widehat{k}(t)$$

The steady-state condition $\dot{\widehat{k}}(t) = 0$ allows us to determine the new stationary value of the capital per potential unit of effective labor. In particular, the long-run value of the capital per potential unit of effective labor in the two polar cases is:

$$\widehat{k}^{**} = \begin{cases} \frac{\gamma}{n+g_A+\delta} & \text{if} \quad \beta = 0 \\ \left(\frac{s_{new}}{s_{old}}\right)^{\frac{1}{1-\alpha}} \cdot \frac{\gamma}{n+g_A+\delta} & \text{if} \quad \beta = 1 \end{cases} \tag{23}$$

This implies that the capital per potential unit of effective labor is not affected by an increase in the saving rate when the crowding-in effect is nil, as in our base model. However, the effect is positive when the crowding-in effect is complete (as in the Solow model), but also when the crowding-in effect is partial (i.e. when $0 < \beta < 1$).

By combining Equations 21 and 23, we can determine the new stationary value of the unemployment rate for the two polar cases:

$$1 - u^{**} = \begin{cases} (1-u^*) \cdot \left(\frac{s_{old}}{s_{new}}\right)^{\frac{1}{1-\alpha}} & \text{if} \quad \beta = 0 \\ 1 - u^* & \text{if} \quad \beta = 1 \end{cases} \tag{24}$$

An increase in the saving rate increases the steady state unemployment rate, $u^{**} > u^*$, except for the case in which the crowding-in effect is complete, i.e. $\beta = 1$.

Concerning the short-run effect on real GDP, for the two polar cases and assuming that the saving rate increases at $t = 0$, real GDP is:

$$Y(0) = \begin{cases} \frac{\gamma}{s_{new}} \cdot A(0) \cdot L(0) & \text{if} \quad \beta = 0 \\ \widehat{k}(0)^{\alpha} \cdot (1-u^*)^{1-\alpha} \cdot A(0) \cdot L(0) & \text{if} \quad \beta = 1 \end{cases}$$

This result implies that, with the exception of the case $\beta = 1$, i.e. the case in which the crowding-in effect on investments is complete, the short-run effect is negative because $\widehat{k}(t)$ is a predetermined variable and unemployment increases. In contrast, if $\beta = 1$, there is no effect on real GDP in the short-run, because the unemployment rate is not affected.

In the long run, the GDP level is given by:

$$Y(t) = \begin{cases} \frac{\gamma}{s_{new}} \cdot A(t) \cdot L(t) & \text{if} \quad \beta = 0 \\ \left(\frac{s_{new}}{n+g_A+\delta}\right)^{\frac{\alpha}{1-\alpha}} \cdot (1-u^*) \cdot A(t) \cdot L(t) & \text{if} \quad \beta = 1 \end{cases}$$

The long-run effect on GDP of an increase in the saving rate is negative if $\beta = 0$ and positive if $\beta = 1$. This implies that there exists a threshold value $\widetilde{\beta}$ such that if $\beta > \widetilde{\beta}$ the long-run effect on GDP is positive, while if $\beta < \widetilde{\beta}$ the long-run effect is negative.

5 Introduction of public expenditures and lump-sum taxes

Now we consider again our base model (i.e. the model with the investment function defined by Equation 10) and we assume that, starting from a situation of steady state, the government introduces expenditures $G(t)$. Public expenditures are assumed to be equal to an exogenous and constant fraction g of real GDP. This shock is assumed to be permanent and unanticipated. Of course, the government has to introduce taxes such that the present value of all the taxes equals the present value of all the public expenditures. The easiest way to introduce in our model the taxes in order to respect the intertemporal budget constraint of the government is to assume that the government introduces a lump-sum tax such that, at each instant, $T(t) = G(t) = g \cdot Y(t)$.

5.1 The instantaneous equilibrium

Assuming that private savings are equal to an exogenous fraction s of the disposable income $Y(t) - T(t)$, and given that public savings are equal to zero, the macroeconomic equilibrium condition becomes:

$$I(t) = s \cdot (1-g) \cdot Y(t)$$

Using the investment function defined in our base model (Equation 10), we find:

$$s \cdot (1-g) \cdot \widehat{k}(t)^{\alpha} \cdot [1-u(t)]^{1-\alpha} \cdot A(t) \cdot L(t) = \gamma \cdot e^{(n+g_A)t}$$

Then, the instantaneous equilibrium unemployment rate is given by:

$$1 - u(t) = \left[\frac{\gamma}{s \cdot (1-g)}\right]^{\frac{1}{1-\alpha}} \cdot \widehat{k}(t)^{-\frac{\alpha}{1-\alpha}} \tag{25}$$

The previous expression implies that the equilibrium unemployment rate depends negatively on the value of g. Thus, given that $\widehat{k}(t)$ is a predetermined variable, the implementation of an expansionary fiscal policy, represented by the simultaneous introduction of public expenditures and lump-sum taxes, allows to reduce the unemployment rate and to stimulate real GDP in the short term, through the increase in aggregate demand.

5.2 The steady state

In the presence of public expenditures and lump-sum taxes as previously described, the evolution of the capital per potential unit of effective labor is given by $\dot{\widehat{k}}(t) = s \cdot \widehat{y}(t) \cdot (1-g) - (n+g_A+\delta) \cdot \widehat{k}(t)$. Considering Equations 9 and 25, we find:

$$\dot{\widehat{k}}(t) = s \cdot \widehat{k}(t)^{\alpha} \cdot \frac{\gamma}{s \cdot (1-g)} \cdot \widehat{k}(t)^{-\alpha} \cdot (1-g) - (n+g_A+\delta) \cdot \widehat{k}(t)$$

Thus, the dynamics of the capital per potential unit of effective labor is described by:

$$\dot{\widehat{k}}(t) = \gamma - (n+g_A+\delta) \cdot \widehat{k}(t) \tag{26}$$

The steady-state condition $\dot{\widehat{k}}(t) = 0$ allows us to determine the stationary value of the capital per potential unit of effective labor:

$$\widehat{k}^* = \frac{\gamma}{n+g_A+\delta} \tag{27}$$

Considering Equation 25 and the stationary value of the capital per potential unit of effective labor (Equation 27), we can determine the stationary value of the unemployment rate:

$$1 - u^* = \frac{\gamma \cdot (n+g_A+\delta)^{\frac{\alpha}{1-\alpha}}}{[s \cdot (1-g)]^{\frac{1}{1-\alpha}}} \tag{28}$$

The two previous expressions imply that an increase in public expenditures (i) does not affect the steady state value of the capital per potential unit of effective labor and (ii) allows to reduce the long-term level of the unemployment rate. This implies that an expansionary fiscal policy can be adopted in order to restore full employment. In fact, $u^* = 0$ if $g = 1 - \frac{\gamma^{1-\alpha}}{s} \cdot (n+g_A+\delta)^{\alpha}$. This implies that the lowest is the value of the parameter γ reflecting investors' animal spirits, the higher will be the value of public expenditures (and lump-sum taxes) necessary to restore full employment.

The long-run effect on real GDP, as the short-run effect previously presented, is positive. Real GDP is then stimulated when an expansionary fiscal policy is introduced, both in the short and the long run. This result is explained by the fact that the model is demand-driven and by the fact that the specification of the investment function implies that an increase in public expenditures produces no crowding-out effect on investments. Of course, in neoclassical models, the effect is completely different because an expansionary fiscal policy reduces aggregate savings and investments which produces a negative effect on capital accumulation and on GDP. The hypothesis that an increase in public expenditures produces no crowding-out effect on investments is relaxed in the next section.

6 Introduction of public expenditures with (partial) crowding-out effect on investments

6.1 The instantaneous equilibrium

As in the previous section, we assume that, starting from the steady state, the government introduces expenditures and a lump-sum tax such that $T(t) = G(t) = g \cdot Y(t)$. Now, we modify the investment function as follows:

$$I(t) = \gamma \cdot e^{(n+g_A)t} + \beta \cdot [\Delta S_H(t) + \Delta S_G(t)] \tag{29}$$

where β is again a parameter between 0 and 1 that measures the degree of the crowding-in/crowding-out effect on investments, $\Delta S_H(t)$ represents the change in private savings (with respect to the situation before a shock) computed at a given level of the unemployment rate, and $\Delta S_G(t)$ represents the change in public savings with respect to the situation before a shock. Thus, the investment function defined in Equation 29 allows to take into account the crowding-out effect provoked by an increase in public expenditures.

Starting from a situation of steady state, the introduction of public expenditures (accompanied by the introduction of a lump-sum tax), has no effect on public savings ($\Delta S_G(t) = 0$) and produces the following change in private savings:

$$
\begin{aligned}
\Delta S_H(t) &= s \cdot (1-g) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \cdot A(t) \cdot L(t) \\
&- s \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \cdot A(t) \cdot L(t)
\end{aligned}
$$

As shown in Appendix 3, the investment function can also be written as follows:

$$
I(t) = \gamma \cdot e^{(n+g_A)t} + \beta \cdot \left[S(t) \cdot \left(\frac{1-u^*}{1-u(t)} \right)^{1-\alpha} - S^*(t) \right] \tag{30}
$$

This equation is identical to Equation 20 and will be used in the econometric analysis.

The macroeconomic equilibrium condition can be written as:

$$
\begin{aligned}
s \cdot (1-g) \cdot \widehat{k}(t)^{\alpha} \cdot [1-u(t)]^{1-\alpha} &= \gamma \\
&+ \beta \cdot s \cdot (1-g) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
&- \beta \cdot s \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha}
\end{aligned}
$$

Considering that Equations 13 and 14 imply that $(\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} = \gamma/s$, the instantaneous equilibrium unemployment rate is given by:

$$
1-u(t) = \left[\frac{\gamma \cdot (1-\beta) + \beta \cdot s \cdot (1-g) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha}}{s \cdot (1-g)} \right]^{\frac{1}{1-\alpha}} \cdot \widehat{k}(t)^{-\frac{\alpha}{1-\alpha}} \tag{31}
$$

The previous expression implies that the introduction of public expenditures, accompanied by a simultaneous introduction of a lump-sum tax, permits a reduction in the level of unemployment, except for the case $\beta = 1$. In particular, the unemployment rate in the two polar cases is:

$$
1-u(t) = \begin{cases} \left[\frac{\gamma}{s \cdot (1-g)} \right]^{\frac{1}{1-\alpha}} \cdot \widehat{k}(t)^{-\frac{\alpha}{1-\alpha}} & \text{if} \quad \beta = 0 \\ 1-u^* & \text{if} \quad \beta = 1 \end{cases} \tag{32}
$$

6.2 The steady state

The introduction of public expenditures and lump-sum taxes as previously described, implies that the evolution of the capital per potential unit of effective labor is given by $\dot{\widehat{k}}(t) = s \cdot (1-g) \cdot \widehat{y}(t) - (n + g_A + \delta) \cdot \widehat{k}(t)$. Considering Equations 9 and 31, the dynamics of the capital per potential unit of effective labor is described by:

$$\dot{\widehat{k}}(t) = \gamma \cdot (1-\beta) + \beta \cdot s \cdot (1-g) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} - (n+g_A+\delta) \cdot \widehat{k}(t) \quad (33)$$

The steady-state condition $\dot{\widehat{k}}(t) = 0$ allows us to determine the new stationary value of the capital per potential unit of effective labor in the two polar cases:

$$\widehat{k}^{**} = \begin{cases} \frac{\gamma}{n+g_A+\delta} & \text{if} \quad \beta = 0 \\ \left[\frac{s \cdot (1-g)}{n+g_A+\delta}\right]^{\frac{1}{1-\alpha}} \cdot (1-u^*) & \text{if} \quad \beta = 1 \end{cases} \quad (34)$$

This implies that the capital per potential unit of effective labor is not affected by an increase in public expenditures when the crowding-in effect is nil, as in our base model while, with $\beta > 0$, an increase in public expenditures reduces capital accumulation.

Considering again Equation 31 and the stationary value of the capital per potential unit of effective labor (Equation 34), we can determine the new stationary value of the unemployment rate for the two polar cases:

$$1 - u^{**} = \begin{cases} \left[\frac{\gamma}{s \cdot (1-g)}\right]^{\frac{1}{1-\alpha}} \cdot \left(\frac{\gamma}{n+g_A+\delta}\right)^{-\frac{\alpha}{1-\alpha}} & \text{if} \quad \beta = 0 \\ 1 - u^* & \text{if} \quad \beta = 1 \end{cases} \quad (35)$$

This implies that the expansionary fiscal policy permits a reduction in the long-term unemployment rate, except for the case $\beta = 1$.

7 Numerical simulations

In this section we present numerical simulations in order to analyze the evolution of (i) of an economy in which the saving rate increases and (ii) of an economy in which public expenditures and lump-sum taxes are introduced.

We first calibrate our model at the steady state without public expenditures and taxes. Our economy is characterized by a population growth rate of 0.5%, a productivity growth rate of 1.5%, a saving rate of 20%, and a depreciation rate of 4%. Moreover, α in the Cobb-Douglas production function is fixed at 1/3 and γ in the investment equation has been calibrated in order to obtain a stationary unemployment rate equal to 10%.

7.1 Increase in the saving rate

In the first simulation we assume that the economy is at the steady state and the private saving rate increases from 20% to 21%.

We first solve the model using the Solow model, i.e. by assuming that investments are determined by aggregate savings instead of by the investment function defined in Equation 10 and by fixing the unemployment rate at 10% or, equivalently, by assuming that the number of workers is equal, at each period, to 90% of the active population. Then, we solve the model by introducing Equation 10 and by endogenizing the unemployment rate. Finally, we solve the model by considering different values of β, i.e. different degrees of the crowding-in/crowding-out effect on investments.

The economic effects are reported in Figure 3. First, Figure 3a shows the effect on the unemployment rate. In the case in which $\beta = 0$ (which corresponds to our base model and to the keynesian model with exogenous investments), i.e. with , the increase in the saving rate determines a strong increase in the unemployment rate because this shock induces a reduction in private consumption and in aggregate demand. In particular, the unemployment rate increases to 16.4%. The negative effect on unemployment is less important if a crowding-in effect on investments is produced. For example, in the case in which β is equal to 0.2, the unemployment rate becomes equal to 15.2% in the short run and to 15.5% in the long run. In addition, a more important value of β implies a lower negative impact on the unemployment rate. In the case in which $\beta = 1$ (which corresponds to the Solow model), the reduction in private consumption is perfectly compensated by the increase in investments, implying that aggregate demand is unaffected and the unemployment rate remains equal to 10%, as before the shock.

Figure 3b shows the effect on GDP, measured as the percentage deviations with respect to the situation before the shock. The most negative effect is obtained with $\beta = 0$ where the value of GDP is 4.8% lower than before the shock. The negative effect is less important if β is positive. Interestingly, if β is equal to 0.8 and to 0.9, the effect on GDP is negative in the short run (-1%) and becomes positive after some periods. In the case in which β is equal to 1, as in the Solow model, there is no effect on GDP in the short run (because unemployment remains unchanged), while the long-run effect is positive (+2.5%).

7.2 Introduction of public expenditures and lump-sum taxes

In the second simulation, we assume that the economy is at the steady state and the government permanently introduces public expenditures and lump-sum taxes which represent 2% of GDP in each period.

Figures 4a and 4b show the effect on the unemployment rate and on GDP, respectively. With $\beta = 0$, the introduction of the expansionary fiscal policy reduces the unemployment rate from 10% to 7.2% and stimulates GDP (+2%), both in the short and in the long run. In the Solow model and in our model with $\beta = 1$, the unemployment remains unchanged, while the GDP is negatively affected in the long run (-1%). Interestingly, if β is equal to 0.8 and to 0.9, the effect on GDP is positive in the short run (thanks to the reduction in the unemployment rate), but becomes negative after some periods due to the unfavorable evolution of capital accumulation.

8 Econometric analysis

Both the theoretical analysis and the numerical simulations have shown that the key element of our model is the parameter β which measures the degree of the crowding-in/crowding-out effect on investments. In this section we present a first attempt to estimate this parameter. In particular, using OECD yearly data from 1955 to 2012, we estimate the investment function defined in Equation 20 for six OECD countries: France, Germany, Italy, Japan, the UK, and the United States. For each of the six countries we separately estimate the following equation:

$$I(t) = \gamma \cdot e^{(n+g_A)t} + \beta \cdot \left[S(t) \cdot \left(\frac{1-u^*}{1-u(t)} \right)^{1-\alpha} - S^*(t) \right] + \epsilon(t) \tag{36}$$

where γ and β are the parameters to be estimated.

We then use two variables to explain the level of investments. The first one may be interpreted as a trend component. In particular, for each country, we approximate the term $n + g_A$ by the average growth rate of investments during the period. The second variable may be interpreted as a cyclical component which is related to the crowding-in/crowding-out effect produced by a change in aggregate savings with respect to the pre-shock situation. In order to construct this second variable it is necessary to define the initial steady state value of the unemployment rate and the evolution of aggregate savings in the pre-shock situation. In our regressions, we consider three values for u^* (5%, 6% and 7%) and we compute the value of aggregate savings in the pre-shock situation using a HP filter with the smoothing parameter fixed at 6.25 (see Ravn and Uhlig, 2002).

The econometric results, reported in Table 1, provide a strong evidence that the parameter β is positive and lower than one, implying the existence of a partial crowding-in/crowding-out mechanism on investments. In particular, for Germany, Italy and the UK the estimated parameter is close to 0.7 and it is robust to changes in the steady state value of the unemployment rate used. The estimated parameter is higher for the USA (around 0.9) and lower for France and Japan, even if the parameter is quite sensitive to changes in the steady state value of the unemployment rate used.

9 Fiscal multiplier

In this section, we compute the value of the fiscal multiplier using the econometric results presented in the previous section. We assume that the government introduces public expenditures at time $t = 0$ for just one period, without introducing taxes.[11]

[11] Note that at time $t = 0$, we have $A(t) \cdot L(t) = 1$ and $e^{(n+g_A)t} = 1$.

The investment function is given by Equation 29 where the change in public savings at time $t = 0$ is $\Delta S_G(0) = -G(0)$, while the change in private savings (computed at a given level of the unemployment rate) is:

$$\Delta S_H(0) = s \cdot \widehat{k}(0)^{\alpha} \cdot (1 - u^*)^{1-\alpha} - s \cdot (\widehat{k}^*)^{\alpha} \cdot (1 - u^*)^{1-\alpha} = 0$$

because the capital stock is a predetermined variable.

Then, the macroeconomic equilibrium condition at time $t = 0$ can be written as:

$$s \cdot \widehat{k}(0)^{\alpha} \cdot (1 - u(0))^{1-\alpha} - G(0) = \gamma - \beta \cdot G(0)$$

Thus, the instantaneous equilibrium unemployment rate at time $t = 0$ is given by:

$$1 - u(0) = \left[\frac{\gamma + (1-\beta) \cdot G(0)}{s}\right]^{\frac{1}{1-\alpha}} \cdot \widehat{k}(0)^{-\frac{\alpha}{1-\alpha}}$$

Hence, an increase in public expenditures at time $t = 0$ reduces the short-run level of the unemployment rate, except for the case $\beta = 1$.

Real GDP at time $t = 0$ is given by:

$$Y(0) = \frac{\gamma}{s} + \frac{1-\beta}{s} \cdot G(0) \tag{37}$$

Equation 37 implies that the fiscal multiplier is equal to $\frac{1-\beta}{s}$ and lies between 0 (if $\beta = 1$ as in neoclassical models with elasticity of labor supply equal to zero) and $1/s$ (if $\beta = 0$ as in keynesian models with exogenous investments). Considering a saving rate s equal to 20% and a value of β ranging between 0.6 and 0.8 according to the econometric estimations presented in the previous section, our model predicts that the fiscal multiplier lies between 1 and 2. In particular, in the case of $\beta = 0.7$, which is the case of Germany, Italy and the UK, the implied fiscal multiplier is equal to 1.5.

10 Conclusions

The aim of this paper is to extend the standard Solow model in a way that allows to endogenize unemployment provoked by the weakness in aggregate demand. The introduction of keynesian unemployment in the Solow model is made it possible by relaxing the hypothesis, used in the classical and neoclassical theory, of full-utilization of the production factors.

With respect to the standard Solow model, our base model presents one additional equation (a simple investment function in which investments are driven by investors' animal spirits) and one additional variable (the unemployment rate). We show that both the instantaneous and the steady-state equilibria may be *under-employment equilibria*, implying that involuntary unemployment occurs because of the weakness in aggregate demand provoked by the low level of investors' confidence.

We analyze the effects of a change in the saving rate and in the value of public expenditures. Using our base model, that works as a keynesian demand-driven model, we find that an increase in aggregate demand (due to a reduction in the saving rate or to an increase in public expenditures), reduces unemployment and stimulates real GDP. Then, we modify the investment function in a way that allows us to take into account the crowding-in/crowding-out effect on investments. In particular, we introduce a parameter β that measures the degree of the crowding-in/crowding-out effect. We show that if β is equal to zero, the model coincides with our base model, i.e. the keynesian demand-driven model with exogenous investments; if β is equal to one, the model coincides with the Solow model and the unemployment rate remains unchanged; if β is between zero and one, the model is an intermediate model. In this case, a shock that increases the aggregate demand stimulates real GDP and reduces unemployment (while in neoclassical models where the elasticity of labor supply is equal to zero the real effect is nil), but also produces a (partial) crowding-out effect on investments (that is not taken into account in keynesian models with exogenous investments). Simulation results show that the effect of a policy or a shock on real GDP may be positive or negative according to the value of β which indicates how much a change in private and public savings affects investments. Estimations on six OECD countries reveal that β lies between 0.6 and 0.8, implying that the crowding-out effect on investments is quite important, but not complete as assumed in neoclassical models. These estimation results imply that the fiscal multiplier is between 1 and 2, which is quite consistent with the empirical evidence.

Finally, in the present paper we introduced keynesian involuntary unemployment in the Solow model which is the simplest neoclassical model. However, involuntary unemployment can be introduced in other neoclassical models. Possible extensions of the model presented in this paper are the introduction of keynesian unemployment in models with infinitely-lived households (as the Ramsey-Cass-Koopmans model) and in models where households have a finite horizon (as the Diamond model), i.e. models where households have to decide the optimal path of consumption. The interesting point is that the optimal level of consumption is chosen by households without considering that, at the aggregate level, consumption affects the aggregate demand. This implies that these possible extensions would permit to take into account that keynesian involuntary unemployment may appear not only because of the weakness in the level of investments, but also because of the weakness in the level of consumption if, for instance, the real interest rate is sufficiently high or the rate of time preference is sufficiently low.

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Figure 1: Unemployment rate in six OECD countries

Figure 2: Involuntary unemployment in the labor market

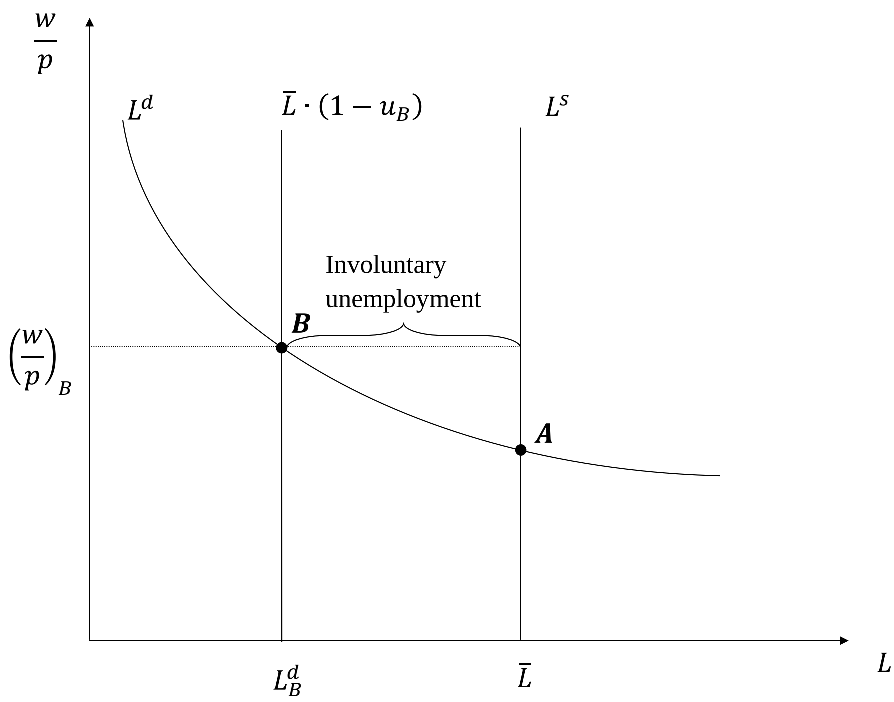
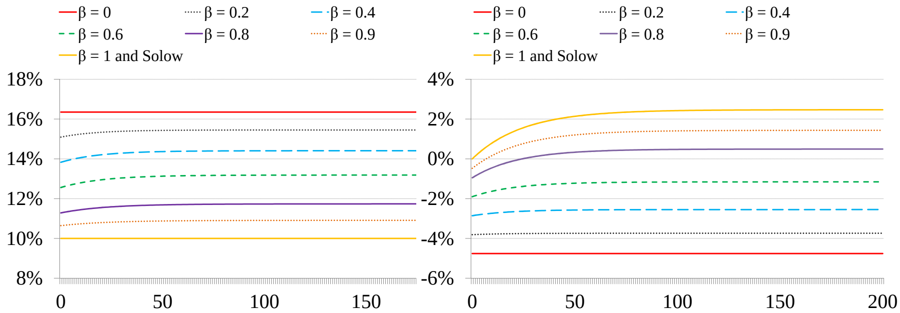


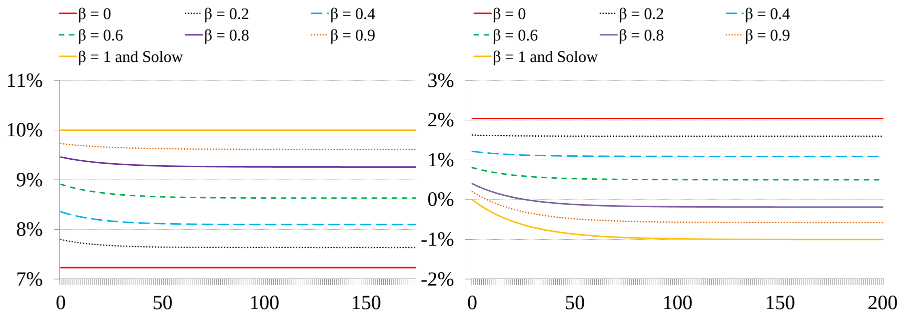
Figure 3: Economic impacts of an increase in the saving rate



(a) Evolution of the unemployment rate

(b) Evolution of GDP (% deviations to the baseline)

Figure 4: Economic impacts of the introduction of public expenditures



(a) Evolution of the unemployment rate

(b) Evolution of GDP (% deviations to the baseline)

Table 1: Estimation of the investment function for six OECD countries

		5%	***6%***	***7%***
	γ	51 361	51 646	51 859
France	β	0.594	0.492	0.386
	R^2	0.78	0.78	0.78
	γ	223 330	224 642	225 948
Germany	β	0.744	0.740	0.736
	R^2	0.82	0.82	0.82
	γ	128 562	129 291	129 996
Italy	β	0.691	0.677	0.661
	R^2	0.79	0.79	0.79
	γ	52 672 731	52 821 434	52 883 964
Japan	β	0.685	0.574	0.458
	R^2	0.49	0.48	0.48
	γ	63 606	63 913	64 214
UK	β	0.726	0.718	0.710
	R^2	0.82	0.82	0.82
	γ	503 740	506 660	509 324
USA	β	0.962	0.918	0.873
	R^2	0.90	0.90	0.90

Appendix 1. Base model with a general investment function

Here we solve our base model by considering a more general investment function where investments negatively depend on the gross rate of remuneration of capital as follows:

$$I(t) = \gamma \cdot e^{(n+g_A)t} \cdot (r(t) + \delta)^{-\theta} \tag{38}$$

where γ and θ are positive parameters.

Considering that $r(t)+\delta = \alpha \cdot [1-u(t)]^{1-\alpha} \cdot \widehat{k}(t)^{\alpha-1}$, the macroeconomic equilibrium between investments and aggregate savings is given by:

$$s \cdot \widehat{k}(t)^{\alpha} \cdot A(t) \cdot L(t) \cdot [1-u(t)]^{1-\alpha} = \gamma \cdot e^{(n+g_A)t} \cdot \left[\alpha \cdot [1-u(t)]^{1-\alpha} \cdot \widehat{k}(t)^{\alpha-1}\right]^{-\theta}$$

Then, the instantaneous *equilibrium unemployment rate* is given by:

$$1 - u(t) = \left(\frac{\gamma}{s \cdot \alpha^{\theta}}\right)^{\frac{1}{(1+\theta)(1-\alpha)}} \cdot \widehat{k}(t)^{\frac{\theta(1-\alpha)-\alpha}{(1+\theta)(1-\alpha)}} \tag{39}$$

Equation 39 implies that $\frac{\partial u(t)}{\partial \gamma} < 0$, $\frac{\partial u(t)}{\partial s} > 0$, and $\frac{\partial u(t)}{\partial \widehat{k}(t)} > 0$ with $\theta < \frac{\alpha}{1-\alpha}$.

Given that $\dot{\widehat{k}}(t) = s \cdot \widehat{y}(t) - (n + g_A + \delta) \cdot \widehat{k}(t)$, $\widehat{y}(t) = \frac{Y(t)}{A(t) \cdot L(t)} = \widehat{k}(t)^{\alpha} \cdot [1-u(t)]^{1-\alpha}$ and considering Equation 39, we find that the dynamics of the capital per potential unit of effective labor is described by:

$$\dot{\widehat{k}}(t) = \frac{s^{\frac{\theta}{1+\theta}} \cdot \gamma^{\frac{1}{1+\theta}}}{\alpha^{\frac{\theta}{1+\theta}}} \cdot \widehat{k}(t)^{\frac{\theta}{1+\theta}} - (n + g_A + \delta) \cdot \widehat{k}(t)$$

The steady-state condition $\dot{\widehat{k}}(t) = 0$ allows us to determine the stationary value of the capital per potential unit of effective labor:

$$\widehat{k}^{*} = \frac{s^{\theta} \cdot \gamma}{\alpha^{\theta} \cdot (n + g_A + \delta)^{1+\theta}} \tag{40}$$

Considering again Equation 39 and the stationary value of the capital per potential unit of effective labor, we can determine the stationary value of the unemployment rate u^{*}:

$$1 - u^{*} = \frac{\gamma}{s \cdot \alpha^{\theta}} \cdot \left(\frac{s}{n + g_A + \delta}\right)^{\theta - \frac{\alpha}{1-\alpha}} \tag{41}$$

Appendix 2. Demonstration of Equation 20

$$
\begin{aligned}
I(t) &= \gamma \cdot e^{(n+g_A)t} \\
&+ \beta \cdot s_{new} \cdot A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
&- \beta \cdot s_{old} \cdot A(t) \cdot L(t) \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
\\
&= \gamma \cdot e^{(n+g_A)t} \\
&+ \beta \cdot s_{new} \cdot A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \cdot \frac{[1-u(t)]^{1-\alpha}}{[1-u(t)]^{1-\alpha}} \\
&- \beta \cdot s_{old} \cdot A(t) \cdot L(t) \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
\\
&= \gamma \cdot e^{(n+g_A)t} \\
&+ \beta \cdot s_{new} \cdot A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot [1-u(t)]^{1-\alpha} \cdot \frac{(1-u^*)^{1-\alpha}}{[1-u(t)]^{1-\alpha}} \\
&- \beta \cdot s_{old} \cdot A(t) \cdot L(t) \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
\\
&= \gamma \cdot e^{(n+g_A)t} + \beta \cdot s_{new} \cdot Y(t) \cdot \left(\frac{1-u^*}{1-u(t)}\right)^{1-\alpha} - \beta \cdot s_{old} \cdot Y_{old} \\
\\
&= \gamma \cdot e^{(n+g_A)t} + \beta \cdot \left[s_{new} \cdot Y(t) \cdot \left(\frac{1-u^*}{1-u(t)}\right)^{1-\alpha} - s_{old} \cdot Y_{old}\right] \\
\\
&= \gamma \cdot e^{(n+g_A)t} + \beta \cdot \left[S(t) \cdot \left(\frac{1-u^*}{1-u(t)}\right)^{1-\alpha} - S^*(t)\right]
\end{aligned}
$$

Appendix 3. Demonstration of Equation 30

$$
\begin{aligned}
I(t) &= \gamma \cdot e^{(n+g_A)t} \\
&+ \beta \cdot s \cdot (1-g) \cdot A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
&- \beta \cdot s \cdot A(t) \cdot L(t) \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
\\
&= \gamma \cdot e^{(n+g_A)t} \\
&+ \beta \cdot s \cdot (1-g) \cdot A(t) \cdot L(t) \cdot \widehat{k}(t)^{\alpha} \cdot (1-u^*)^{1-\alpha} \cdot \frac{[1-u(t)]^{1-\alpha}}{[1-u(t)]^{1-\alpha}} \\
&- \beta \cdot s_{old} \cdot A(t) \cdot L(t) \cdot (\widehat{k}^*)^{\alpha} \cdot (1-u^*)^{1-\alpha} \\
\\
&= \gamma \cdot e^{(n+g_A)t} + \beta \cdot s \cdot (1-g) \cdot Y(t) \cdot \left(\frac{1-u^*}{1-u(t)}\right)^{1-\alpha} - \beta \cdot s \cdot Y_{old} \\
\\
&= \gamma \cdot e^{(n+g_A)t} + \beta \cdot \left[S(t) \cdot \left(\frac{1-u^*}{1-u(t)}\right)^{1-\alpha} - S^*(t)\right]
\end{aligned}
$$