Derivation of the Generalised Ising Framework

1 Dynamical System

The complete dynamical system for the stock-market is described by

$$\tau_s \dot{s} = -s + \tanh(\beta_1 s + \beta_2 h) \tag{1a}$$

$$\tau_h \dot{h} = -h + \tanh(k_1 \dot{p} + \xi_t) \tag{1b}$$

$$\dot{p} = c_1 \dot{s} + c_2 (s - s_\star) \tag{1c}$$

with $\dot{x}$ denoting the derivative of x with respect to time, and the variable definitions are given in Table **??**.

Table 1: Variables of the dynamic model

Symbol	Name	Description
s	sentiment	
h	information	
p	price	
τ_i	Characteristic timeframe	$tau_s >> tau_h$
β_1	Sensitivity of $\dot{s}$ to avg. s	Mean field approximation s
β_2	Sensitivity of $\dot{s}$ to avg. h	Mean field approximation h
k_1	Sensitivity $\dot{h}$ to $\dot{p}$	Feedback cycle for price changes to sentiment
ξ_t	Exogenous news flow	
c_1	Sensitivity of p to $\dot{s}$	short-term allocations only occur when sentiment changes
c_2	Sensitivity of p to sentiment divergence	long-term allocations with reference to long-term growth expectations

2 Derivation of the sentiment dynamics

There is a large group of $i \in \{1, \dots, N\}$ investors who each have a sentiment $s_i \in \{-1, 1\}$. The fraction of investors that are optimists ($s_i = 1$) is given by $n_+ = \frac{N_+}{N}$ (pessimists with $s_i = -1$ have fraction $n_- = \frac{N_-}{N}$). The average sentiment is thus $s = n_+ - n_-$ with $n_- + n_+ = 1$, which leads to the definitions

$$n_- = \frac{1-s}{2} \tag{2a}$$

$$n_+ = \frac{1+s}{2} \tag{2b}$$

Conceptually agent sentiment is affected by

1. The **sentiment of the other agents** (s) they are around, who influence the sentiment of i to co-align with their own. We take a mean-field approximation.

2. The **opinion of analysts** (h), which also aim to co-align investor sentiment with their own sentiment. Again, a mean-field approximation is taken.

The *Force* acting on investor sentiment is the sum of interaction (s) and dissemination (h), yielding

$$F(s,h) = \beta_1 s + \beta_2 h \tag{3}$$

However, this is deterministic. Investor sentiment might change due to idiosyncrasies. Thus, we define the *transition probabilities* for agents to switch from negative (positive) to positive (negative) sentiment as p^{-+} (p^{+-}). The relation between these probabilities is proportional to the force F acting on sentiment. Specifically:

$$p^{+-} = p^{-+} \quad \text{if } F = 0 \tag{4a}$$
$$p^{+-} > p^{-+} \quad \text{if } F < 0 \tag{4b}$$
$$p^{+-} < p^{-+} \quad \text{if } F > 0 \tag{4c}$$

The discrete development of n_- and n_+ is then given by

$$n_+(t+\Delta t) = n_+(t) + \Delta t \left(n_-(t)p^{-+}(t) - n_+(t)p^{+-}(t)\right) \tag{5a}$$
$$n_-(t+\Delta t) = n_-(t) + \Delta t \left(n_+(t)p^{+-}(t) - n_-(t)p^{-+}(t)\right) \tag{5b}$$

Using the relationship between n and s from Eqs. **??** the relations of Eq (**??**) become

$$\frac{s(t+\Delta t) - s(t)}{\Delta t} = (1 - s(t))p^{-+}(t) + (1 + s(t))p^{+-}(t) \tag{6}$$

Taking the limit $\Delta t \to 0$ yields

$$\dot{s} = (1-s)p^{-+} - (1+s)p^{+-} \tag{7}$$

To relate the transition probabilities to the proportion of investors who are optimists / pessimists, the equilibrium can be used.

$$n_\pm(t+\Delta t) - n_\pm(t) = 0 \tag{8}$$
$$n_-(t)p^{-+}(t) - n_+(t)p^{+-}(t) = 0 \tag{9}$$
$$\frac{p^{-+}}{p^{+-}} = \frac{n_+}{n_-} \tag{10}$$

Assumption: the transition rates are the same inside and outside of equilibrium.

The transition probabilities are related to the force, F, acting on the sentiment, i.e. $g = \frac{n_+}{n_-} \sim F$. Specifically, we assume that the relationship is of the form

$$\frac{\partial g}{g} = \alpha \partial F \qquad \alpha > 0 \tag{11}$$

which implies

$$\frac{n_+}{n_-} = \frac{p^{-+}}{p^{+-}} = e^{\alpha F} \tag{12}$$

Assumption: the relation between the relative transition rates and the force F is of the form presented in Eq. (**??**)

Let τ_s be the characteristic time over which random disturbances lead to a change of s_i. This is equivalent to the total time over which the probability that s_i flips is equal to unity, which translates to

$$\tau_s(p^{+-} + p^{-+}) = 1 \tag{13}$$

The relationship of Eq. (**??**) assumes that the agents state after an interaction is independent of the initial state. Then the Markov chain properties lead to this relationship.
Combining the relations of Eq. (**??**) and Eq. (**??**) yields

$$p^{-+} = \frac{1}{\tau_s\left(1 + e^{-\alpha F}\right)} \tag{14}$$

$$p^{+-} = \frac{1}{\tau_s\left(1 + e^{\alpha F}\right)} \tag{15}$$

Using the derivation of Gusev Et. Al (2015, appendix A), this relationship together with the definition of $\dot{s}$ and F yields the relationship

$$\tau_s \dot{s} = -s + \tanh(\beta_1 s + \beta_2 h) \tag{16}$$

where $\frac{\alpha}{2}$ is absorbed into β_1 and β_2.
The relationship for h can be derived in a similar manner. However, for h the force acting on the analyst sentiment is defined as

$$F_h = k_1 \dot{p} + \xi_t \tag{17}$$

on the basis of the following observations:

1. The *time-to-market* of news (from discovery to publication) is very short, and journalism operates on a non-collaborative basis, which implies that the interactions term for the analysts is omitted.

2. Thus, the newsflow can be split into two key components: the newsflow modulation from changes in the stock price $\dot{p}$, which represents a feedback effect, and the arrival of random news ξ_t.

3 Price-Sentiment relation

The price of a security changes as a function of the trades that are made by investors. In this framework trades are made on the basis of investor sentiment. The price sentiment relation (Eq. (**??**)) is established on the basis of two observations

1. in the **short-term** an agent has a pre-existing allocation, and would only change their allocation if their sentiment changes in reference to the prior level of sentiment (i.e. they recall sentiment). This implies that for $t << \tau_s$ we have $\dot{p} \sim \dot{s}$.

2. In the **long-run** an agent makes their allocation decision based on the sentiment level itself. In contrast to the short-run, the prior sentiment wont be in the agent's memory. Hence, at timescales $t >> \tau_s$ we have $\dot{p} \sim s$.

   This leads to the formulation

$$\dot{p} = c_1 \dot{s} + c_2 s + c_3 \tag{18}$$

   that can be re-written as

$$\dot{p} = c_1 \dot{s} + c_2 (s - s_\star) \tag{19}$$

with $s_\star = -\frac{c_3}{c_2} \neq 0$, such that $s_\star$ can be seen as "an implied reference sentiment level that investors are accustomed to and consider normal." (Gusev Et Al, 2015). Gusev Et Al (2015) also find that "in the leading order $s_\star$ coincides with the equilibrium value of sentiment", which is determined by β_1 (equilibrium with $\dot{s} = h = 0$.