A simple economic model with interactions[1]

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Abstract

Macroeconomic models rarely make explicit how agents actually interact. If however interaction is explicitly specified, the link between the micro and macro properties of models becomes much richer, leading in certain cases to the onset of macro-level instability. This working paper incorporates into the basic Solow model interactions among agents at a micro level to study disequilibrium behaviors and economic instability on a macro level. In particular, we investigate two limiting cases. First, we recover the classic case where the economy converges to the balanced growth path and then grows along it. In the second case, where the interactions-fueled demand dynamics become the main force driving the economy, we obtain business cycles as quasiperiodic endogenous fluctuations.

1. Introduction

This paper develops a simple dynamic model of economy based on agent interactions. This model is applicable not only around equilibrium but also away from it.

The starting point is an interactions-based stock market model developed in Gusev et al. (2015). At the core of that model is a feedback mechanism that interconnects information, expectation and price as follows. Information influences investor expectations; expectations lead to investment decisions; and investment decisions cause cash flows and move the market price. Then price changes themselves draw media attention and become an important source of information – thus creating a feedback cycle.

We wish to construct a model that would incorporate this feedback mechanism; and in addition, we assume that information and expectation are primarily driven by interactions among agents.

This model is shown in Figure 1. On a micro level, it consists of two types of agents: investors and analysts. The role of investors is to invest based on their expectations and the role of analysts is to form expectations from various news sources and channel them to investors[2]. Each individual agent has a binary expectation regarding the market return: +1 if the market is expected to go up and -1 if down.

[1] This study is a part of the project "An Interactions-based Macroeconomic Model" in the Instability Hub of the Rebuilding Macroeconomics Initiative. Principal Investigators: M. Gusev and D. Kroujiline.

[2] Analysts is a collective term that includes financial analysts, market commentators, newspaper journalists, finance bloggers and other participants who interpret relevant information, opine on how the market might react and make their views available to investors through media outlets.

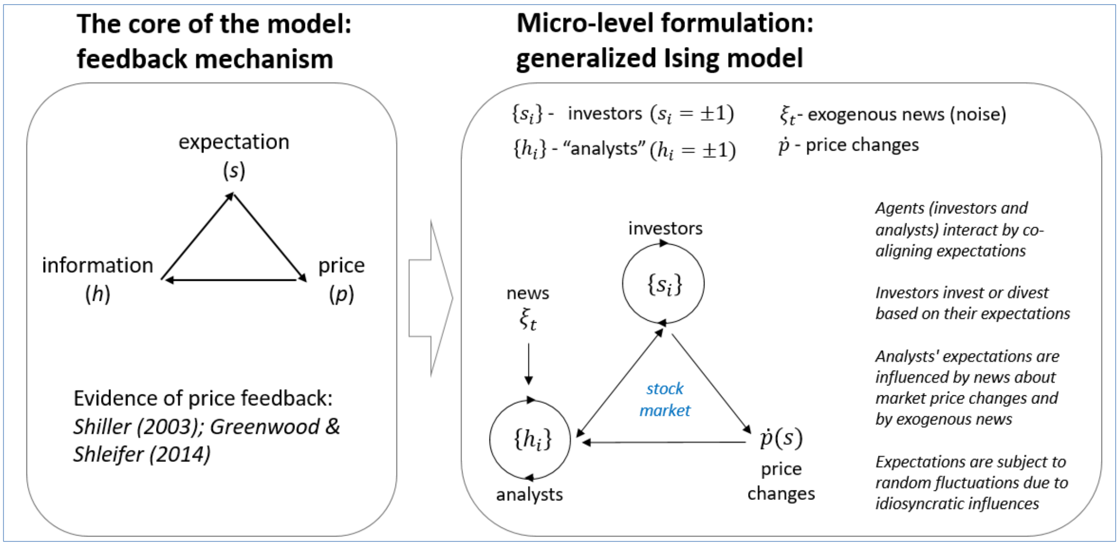

Figure 1: An interactions-based stock market model (Gusev et al., 2015) is the starting point.

The agents interact by co-aligning expectations. Investors interact among themselves and with analysts. Similarly, analysts interact among themselves and with investors. In addition, the expectations of analysts are influenced by price changes (they are assumed to extrapolate current price changes to the future) and by other relevant news modeled as exogenous noise. Finally, individual expectations are subject to random fluctuations to reflect idiosyncratic influences.

Incidentally, similar models have been well studied in statistical mechanics problems. In particular, there are obvious parallels between this model and the Ising model of ferromagnetism. This model is, however, more general than the classic Ising model as it involves two classes of mutually-interacting agents (spins), an external noise forcing the system and, most importantly, the feedback effect.

As such, Gusev et al. (2015) treated this stock market model as a generalized version of the Ising model and derived equations for aggregate variables – the average expectations of investors and analysts – by considering the micro-level dynamics among the agents. We will apply similar equations for the economic model in Section 4.

We intend to adapt this model to real economy. The basic idea is that the feedback mechanism between information, expectation and price, relevant to the stock market, may also be applicable to an economic process. It is not difficult to see how this could work. Economic output depends on invested capital and the amount of invested capital in turn depends on the expected return on investment, so that it is expectations that drive investment and therefore output. Assuming analogously to the stock market model, that expectations depend on interactions across the economy and information about economic development, we again obtain a feedback cycle.

The corresponding economic model, shown in Figure 2, is very similar to the stock market model in Figure 1. Firms form expectations by interacting among themselves and with analysts. Then, based on these expectations, firms deploy capital k, which impacts output in accordance with the relevant production function. In the meantime, analysts' expectations are influenced by the output changes due to the capital flow and by other relevant news.

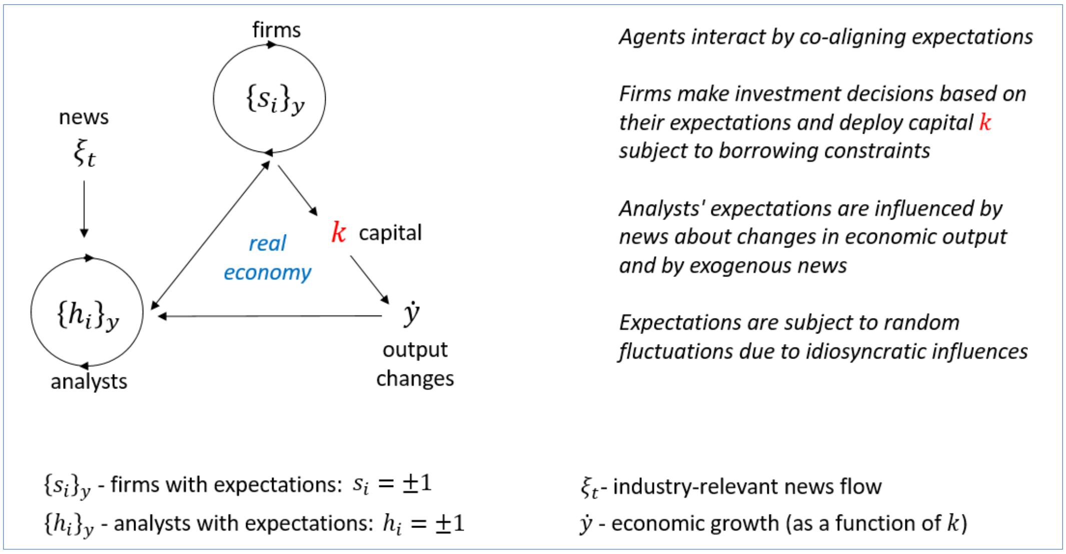

Figure 2: The expectations dynamics model relevant for real economy.

We can close the model by specifying whence the capital k comes and where the output y goes. It is easy to close it in the context of the Solow model. In the Solow model, households save a constant fraction of income and firms borrow this capital to produce output. Households consume the output and firms pay households the proceeds. Households supply capital inelastically, so that firms take into production the whole capital supplied. In other words, there is no active capital demand.

The present situation is different from the classic Solow framework. The capital demand is no longer passive; now firms determine their capital needs dynamically according to the above-discussed interaction mechanism. This modified Solow model is depicted in Figure 3.

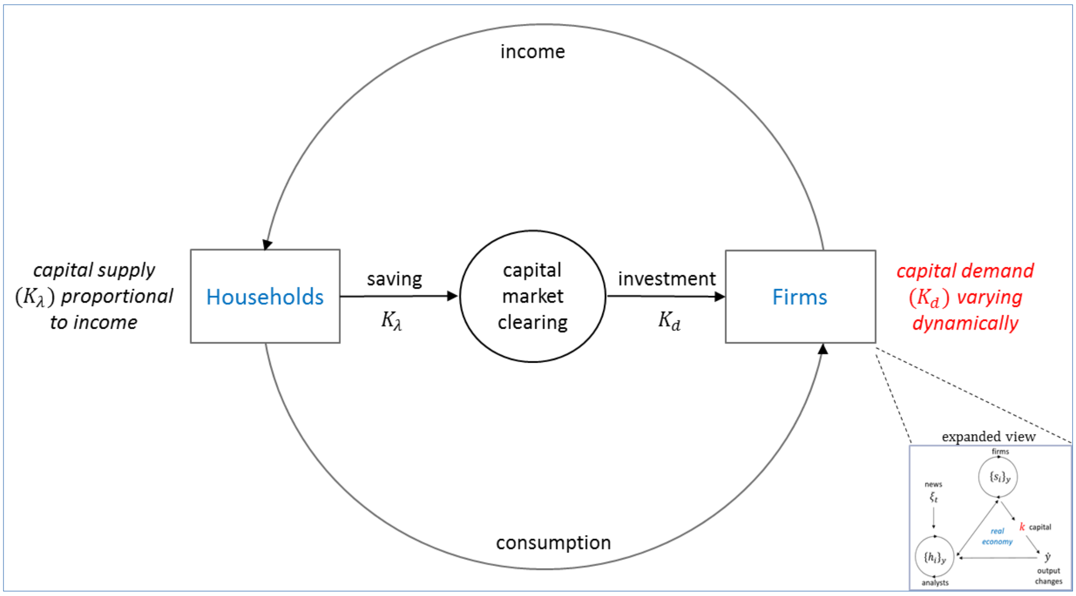

Figure 3: The modified Solow model where capital demand is determined dynamically.

We will show that this model has two limiting cases: (i) a relaxation toward and subsequent evolution along the balanced growth path if the demand exceeds the supply (classic regime) and (ii) quasiperiodic fluctuations around the balanced growth path if the supply exceeds the demand (business cycle regime). In what follows, we derive model equations and investigate their behaviors in each limiting case.

2. Model equations

The Solow model equations must be adapted to reflect the above assumptions as well as allow studying disequilibrium behaviors. Note that for simplicity we consider employment as fixed, so that all variables are per capita.

2.1. Output production

Taking the Cobb-Douglas production function with the exponential growth attributed to technological progress, the usual equation for output is

$$Y = cK^{\alpha}e^{\varepsilon t}\,, \tag{1}$$

where Y and K are output per capita and invested capital per capita, respectively; ε is the exogenous technology growth rate; and $0 < \alpha < 1$.

Equation (1) implies that output adjusts to any change in capital instantaneously. As such, it is only valid on timescales longer than the time it takes to build a factory or set up any other infrastructure relevant for production. Therefore, this equation must be adjusted to enable dynamic behaviors other than the long-term growth. We propose such a dynamic extension in the form:

$$\tau_y \dot{Y} = -Y + cK^{\alpha}e^{\varepsilon t}\,. \tag{2}$$

In accordance with equation (2), Y converges to the long-run equilibrium path over time τ_y, where $1 \ll \tau_y \ll 1/\varepsilon$. Indeed, the term $\tau_y \dot{Y}$ becomes negligibly small if $t \gg \tau_y$, thus recovering equation (1).

2.2. Capital supply

The households own capital K_λ. The usual equation of capital motion states that capital stock K_λ increases as λY and decreases as $-\delta K_\lambda$, where λ is the constant saving rate and δ is the constant depreciation rate, i.e.

$$\dot{K}_\lambda = \lambda Y - \delta K_\lambda\,. \tag{3}$$

2.3. Market clearing

The households supply capital K_λ, while the firms demand capital K_d based on their expected return on investment. We will derive equations that determine K_d in Section 4. At this point, let us treat K_d as known in order to write down conditions for capital market clearing. In particular, we assume that both K_λ and K_d are inelastic to obtain market clearing in the form:

$$K = \begin{cases} K_\lambda\,, & K_d > K_\lambda \\ K_d\,, & K_d < K_\lambda \end{cases}\,. \tag{4}$$

Market clearing condition (4) contains two limiting cases: (i) K_d is always greater than K_λ; and (ii) K_d is always smaller than K_λ. We will study the first limit in Section 3 and the second limit in Section 4.

3. Classic regime ($\boldsymbol{K} = \boldsymbol{K}_\lambda$)

In the first limiting case, $K_d > K_\lambda$ so that $K = K_\lambda$, i.e. the invested capital is equal to the supplied capital. This regime is described by equations (2-3), which we express for convenience as a single second-order differential equation (with c absorbed in λ):

$$\tau_y \ddot{K} + \left(1 + \tau_y \delta\right)\dot{K} + \delta K = \lambda K^\alpha e^{\varepsilon t} \, . \tag{5}$$

Although it is not easy to solve this nonlinear inhomogeneous equation exactly, it is fairly straightforward to obtain its approximate solution. We use the boundary layer technique to arrive at the following expression for output, which is valid for $t \geq O\left(\tau_y\right)$:

$$Y(t) = \left(\frac{\lambda}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \left(\left(A e^{-\left(\frac{1-\alpha}{\tau_y}\right)t} + 1 \right)^{\frac{1}{1-\alpha}} + e^{\left(\frac{\varepsilon}{1-\alpha}\right)t} - 1 \right), \tag{6}$$

where A is the constant of integration.

The first exponent describes the output relaxation toward equilibrium, while the second exponent describes the output growth along the long-run equilibrium path. The approximate solution (6) and the numerical solution to equation (5) are plotted in Figure 4.

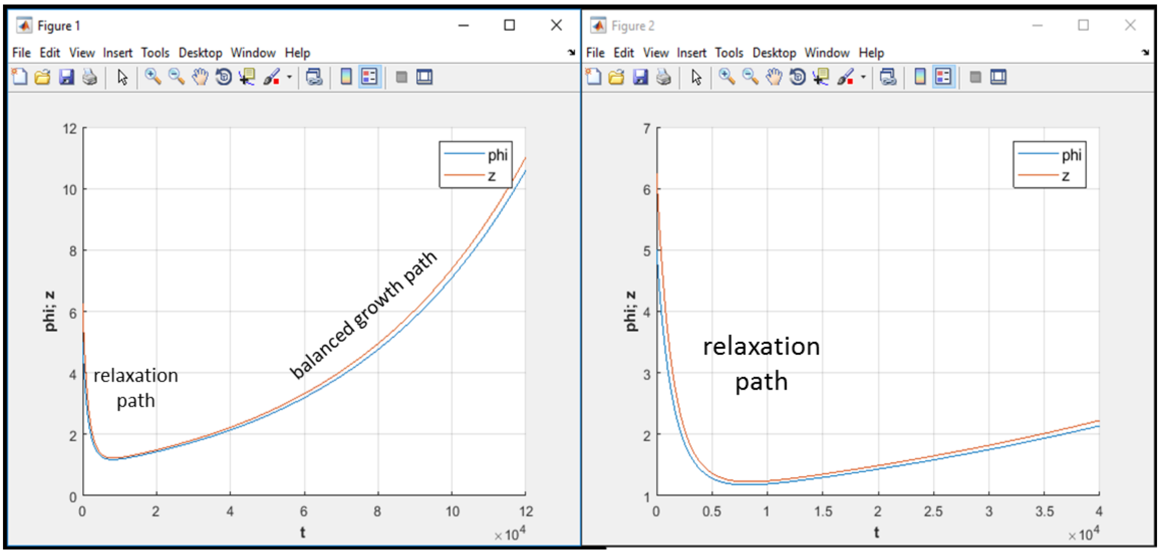

Figure 4: Output $Y(t)$ in the classic regime: the orange line is the approximate solution and the blue line is the numerical solution ($\alpha = 0.5$, $\tau_y = 1'000$ days, $1/\varepsilon = 100'000$ days). (a) The relaxation to the long-run equilibrium growth path. (b) An expanded view of relaxation to equilibrium.

4. Business cycle regime ($\boldsymbol{K} = \boldsymbol{K}_d$)

In the second limiting case, $K_d < K_\lambda$ so that $K = K_d$, i.e. the invested capital is equal to the demanded capital.

Modeling expectations is central to determining capital demand K_d. As explained in Section 1, we treat these expectations as dependent on the exchange of opinions across the industry (i.e. interactions) and news about economic development[3], which implies a feedback mechanism interconnecting information, expectation, capital and output (see Figure 2 in Section 1).

We consider these interactions-based dynamics on a micro level to obtain a closed-form dynamical system on a macro level[4]:

$$\dot{k} = c_1\dot{s} + c_2(s - s_*), \tag{7a}$$

$$\tau_s\dot{s} = -s + \tanh(\beta_1 s + \beta_2 h), \tag{7b}$$

$$\tau_h\dot{h} = -h + \tanh(\gamma\dot{y} + \xi_t), \tag{7c}$$

where y is log output ($y = \ln Y$), k is log capital ($k = \ln K$), s is the average firms' expectation (henceforth called sentiment), h is the average analysts' expectation (henceforth called information) and ξ_t is the exogenous news noise. Note that the characteristic timescales differ by at least one order of magnitude: $\tau_h \ll \tau_s \ll \tau_y \ll 1/\varepsilon$.

To close the model, equations (7) must be considered in conjunction with equation (2) rewritten for log variables,

$$\tau_y\dot{y} = ce^{\alpha k + \varepsilon t - y} - 1\,. \tag{8}$$

Let us introduce a new variable z that makes system (7-8) bounded, three-dimensional and autonomous in the absence of noise:[5]

$$z = k + \varepsilon t - y\,. \tag{9}$$

Thus, finally, we obtain a self-contained dynamical system in the (s, h, z)-space:

$$\dot{z} = c_1\dot{s} + c_2(s - s_*) - \omega_y(ce^z - 1) + \varepsilon, \tag{10a}$$

$$\tau_s\dot{s} = -s + \tanh(\beta_1 s + \beta_2 h), \tag{10b}$$

$$\tau_h\dot{h} = -h + \tanh\big(\gamma\omega_y(ce^z - 1) + \xi_t\big), \tag{10c}$$

where $\omega_y = 1/\tau_y$ for convenience.

System (10) has three equilibria for the relevant range of parameters[6]. Two equilibrium points are stable foci, located in the region where sentiment is negative ($s < 0$) and where sentiment is positive ($s > 0$). The former corresponds to receding economy and the latter to growing economy. The third equilibrium is an unstable saddle located between the two stable points. In the absence of external news ($\xi_t = 0$), the economy converges to stable equilibrium where, depending on the

[3] Normally, it is consumption, not output, that firms take as a key input when planning budget. However, these are equivalent in the Solow model, where consumption is proportional to output.

[4] This derivation follows Gusev et al. (2015) and Kroujiline et al. (2016, 2019), which obtained using statistical mechanics methods equations for agent-based models with feedback depicted in Figure 2.

[5] Note that α can be absorbed into k, c_1 and c_2, so that we set $\alpha = 1$ without loss of generality.

[6] The dynamical system is tuned up in a subcritical regime close to the formation of a limit cycle.

initial conditions, it either contracts or expands. The nonzero ξ_t does not let the economy to settle at equilibrium, making it evolve dynamically in accordance with equations (10). The equilibrium points and phase portrait, projected on the (s, h)-plane, are shown in Figure 5.

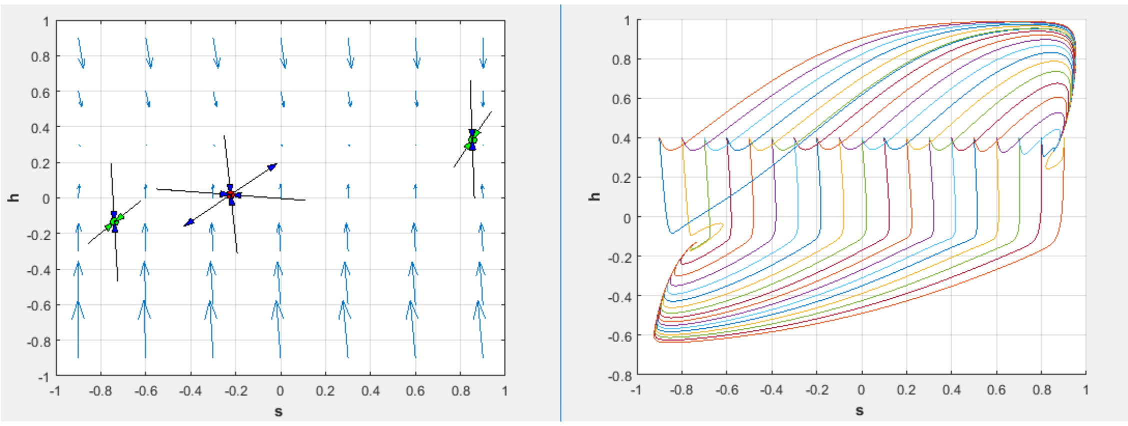

Figure 5: (a) The equilibrium points and velocity field projected on the (s, h)-plane. From left to right: stable focus (recession attractor), unstable saddle and stable focus (expansion attractor). (b) Phase portrait projected on the (s, h)-plane.

The economy's trajectory spends most of the time near the stable equilibrium points that act as attractors. The transition between the attractors, i.e. between recession and expansion, is triggered by random news events (Figure 6a). These news shocks throw the economy across the border between the attracting regions in the phase space once its trajectory passes close by. As a result, the model acquires both regular and stochastic features and thus generates endogenous quasiperiodic fluctuations characterized by a peak in length distribution at 40-60yrs (Figure 6b). This phenomenon, whereby noise facilitates quasiperiodic "limit cycles", is called coherence resonance in dynamical systems.

Simulated trajectory in 3D phase space

Histogram of business cycle lengths

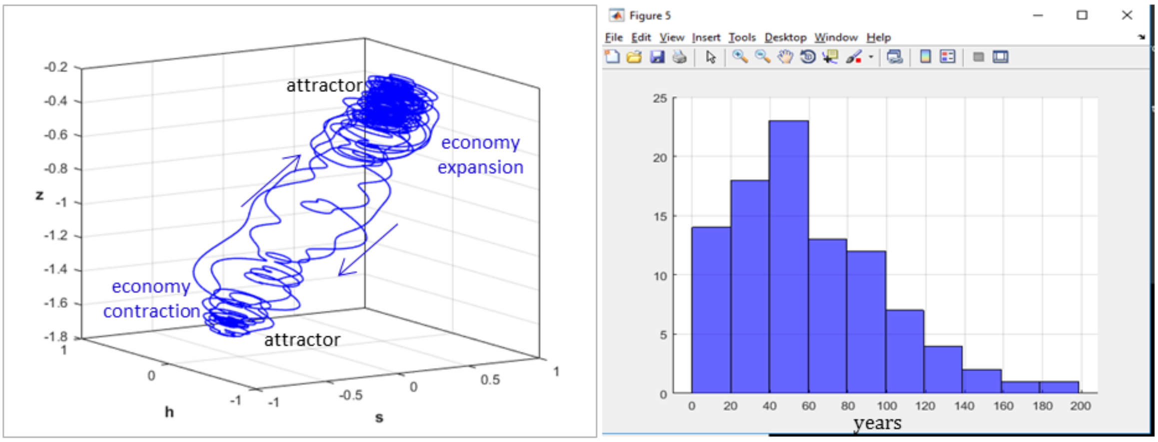

Figure 6: (a) A simulated trajectory in the phase space (from Kroujiline et al (2018) for illustration purposes). (b) Histogram of the lengths of business cycles (the horizontal axis denominated in years).

Figure 7 depicts sentiment s and output y as the functions of time. Economic recessions and expansions correspond to the periods where sentiment is negative and positive, respectively. Sentiment mostly stays captive to the recession and expansion attractors, with abrupt and infrequent transitions between them. This sentiment dynamic produces business cycles manifested as quasiperiodic output fluctuations along the economy's growth trend.

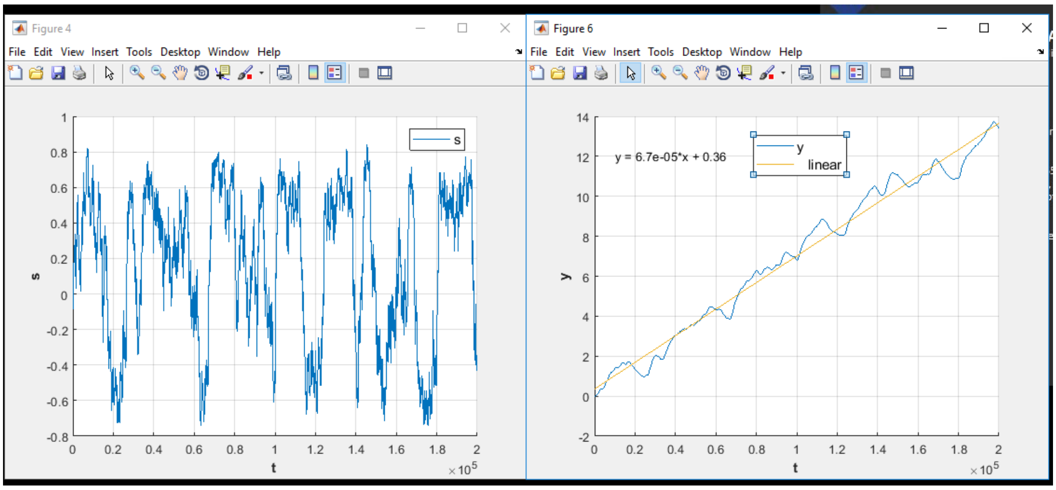

Figure 7: (a) Sentiment evolution. (b) Output evolution.

5. Conclusion

Our objective was to construct a basic economic model that would incorporate interactions among agents. To enable interactions, we assumed that the firms' demand for capital depends on their expected net return on investment. We also assumed the firms' expectations evolve dynamically, driven by interactions among them. Finally, to close the model, we assumed that households save a constant fraction of income to supply the capital from which firms borrow, as in the Solow model. In addition, we adapted the usual equation for economic output to be able to examine behaviors outside equilibrium.

Treating the demand and supply as inelastic, which leads to a binary market clearing depending on whether the demand exceeds the supply or vice versa, we obtained the model equations in closed form. We then investigated two limiting cases. In the first case, where the demand always exceeds the supply, the economy converges to the balanced growth path and then grows along it (classic regime). In the second case, where the supply always exceeds the demand, the interactions-fueled demand dynamics become the main force driving the economy. As a result, the economy undergoes quasiperiodic endogenous fluctuations (business cycle regime).

Regarding further research, this model may serve as an elementary building block for a realistic model with interactions. For example, we can try to integrate into it the interactions-based stock market model of Gusev et al. (2015) (the main steps of an integration via credit frictions were outlined in Gusev et al. (2019)). Such an integration may have a strong impact on dynamic behaviors as the feedback mechanism driving stock market cycles is operating on shorter timescales than that forcing the economy. In particular, we may expect shorter, more realistic business cycles.

6. References

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