Self-reflexive Capital Supply in the Dynamic Solow Model

1 The Dynamic Solow Model

1. **Production** (competitive markets, one good, and full technology access → Rep. firm)

$$Y_t = A_t K^\rho \tag{1}$$
$$\tau_Y \dot{Y} = -Y + AK^\rho \tag{2}$$

   with stochastic technology A_t and population $N_t = N_0 e^{nt}$.

2. **Households** - representative household investing a fraction κ of income $\Omega_t = Y_t + \max(K_{s,t-\Delta t} - K_{d,t-\Delta t}, 0)$ (excess capital returned to households), thus investment is $I_t = \kappa\Omega_t$.

3. **Capital Supply** (δ = depreciation)

$$K_{s,t+1} = (1-\delta-n)K_t + I_t \tag{3}$$
$$\dot{K}_s = \kappa\Omega - (\delta+n)K \tag{4}$$

4. **Capital Demand** via the generalised Ising model (lowercase are log variables)

$$\dot{k}_d = c_1\dot{s} + c_2 s + c_3 \tag{5a}$$
$$\tau_s\dot{s} = -s + \tanh(\beta_1 s + \beta_2 h) \tag{5b}$$
$$\tau_h\dot{h} = -h + \tanh(\gamma\dot{y} + \xi_t) \tag{5c}$$

   where s is sentiment, h is information, and ξ_t is exogenous news.

5. **Capital Markets** clearing via $K_t = \min\{K_s, K_d\}$

2 Dynamic Capital Supply

It is natural to expect that households vary their investment depending on the interactions they have. We can allow households to adjust their consumption needs as a fraction of income, $1-\kappa$, on the basis of prior incomes.

Each household $i \in \{1, \dots, M\}$ makes their savings rate decision as

$$\kappa_t^i \to F\left(\sum_{j=1, j\neq i}^{M} J_{ij}\Omega_j\right) \tag{6}$$

where $F(\cdot)$ is monotonic and increasing. Setting $J_ij = \frac{J}{M}$ and taking the $M \to \infty$ limit, we get a mean-field approximation.

We can model this directly with a shifted logistic function as

$$G(\Omega) = \kappa_{min} + \frac{\kappa_{max} - \kappa_{min}}{1 + e^{2\theta(\Omega^\star - \Omega)}} \tag{7}$$

where Ω is chosen as an input on the basis that it is the easiest for households to observe directly[1].

[1] In the Solow system, consumption, investment and income are directly proportional, so the choice is primarily economic interpretation

2.1 Candidates for κ_{max} & κ_{min}

κ_{max} In the case $K_d > K_s$, we are in the standard Solow case where $\dot{K} = \kappa Y - (\delta + n)K$ (supply-driven capital growth), which reaches a steady state at

$$K^\star = \frac{\kappa Y}{\delta + n} = \left(\frac{\kappa A_t}{\delta + n}\right)^{\frac{1}{1-\rho}} \tag{8}$$

$$Y^\star = A_t (K^\star)^\rho \tag{9}$$

which allows us to derive the consumption maximising level of κ as

$$C^\star = (1-\kappa)Y^\star = Y^\star - (\delta + n)K^\star \tag{10}$$

$$\frac{\partial C^\star}{\partial \kappa} = \left(Y^{\star\prime} - (\delta + n)\right)\frac{\partial K^\star}{\partial \kappa} \tag{11}$$

$$0 = Y^{\star\prime} - (\delta + n) = \rho A (K^\star)^{\rho - 1} - (\delta + n) \tag{12}$$

$$\kappa^\star = \rho \tag{13}$$

This is a natural target for κ in the case where the economy is perceived to be doing well. This result is specific to the Cobb-Douglas function, but the approach could be used more generally.

κ_{min} Should be a small non-zero value corresponding almost to non-saving. In this case, it is easy for any news to push $K_d > K_s$, where in the balanced growth environment the level of output and capital will increase to *catch up* with the steady state level.