

Image registration

Maureen van Eijnatten

Today:

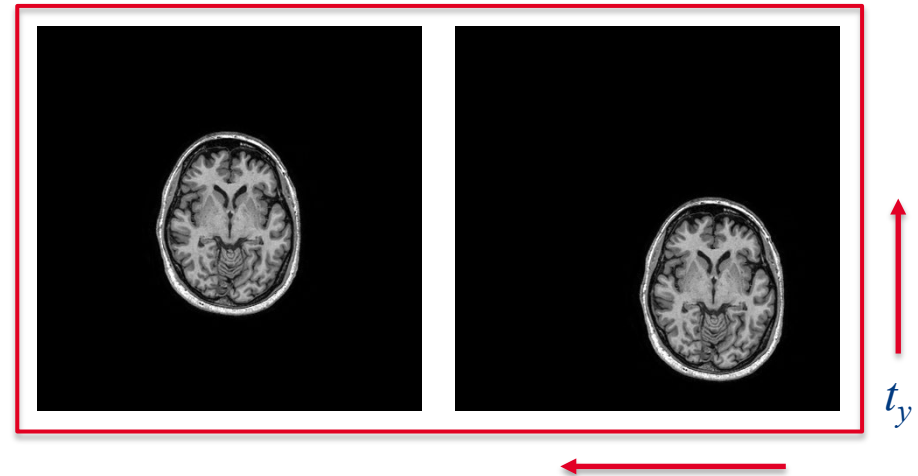
- Introduction of medical image registration
- Recap of linear algebra
- Geometrical transformations

Learning outcomes

The student can:

- name possible causes of misalignment in medical images
- name different applications of medical image registration
- classify medical image registration methods using eight different criteria
- apply the basic principles of linear algebra (i.e., matrix-vector, vector-matrix products, transpose, norms, orthogonality, determinant) to image registration tasks
- use the determinant of a transformation matrix T to predict the orientation and magnification of an object transformed with T
- compose and combine rigid and affine transformations in 2D and 3D (and rewrite them using homogeneous coordinates)
- explain the difference between affine and non-linear registrations

Image registration: Introduction

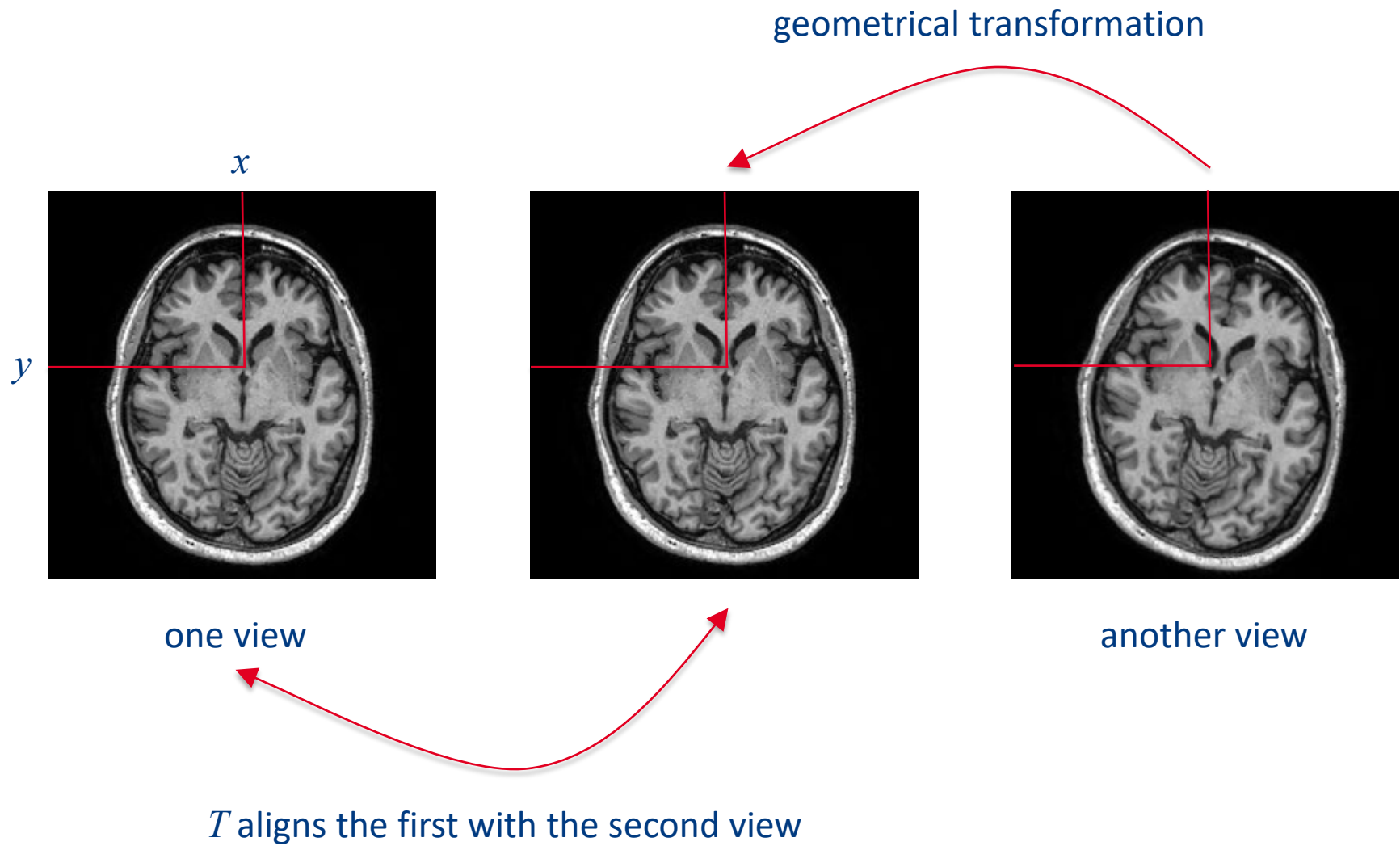


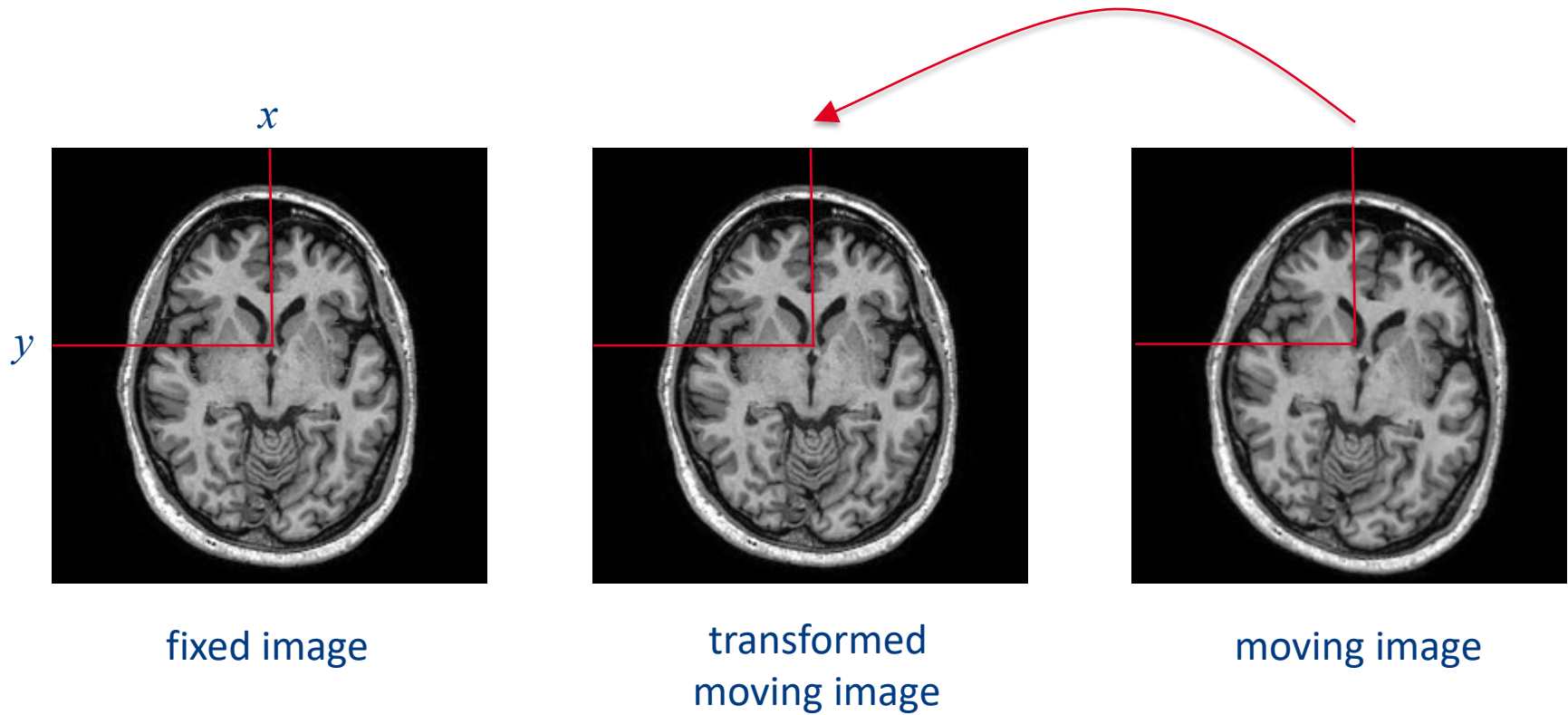
r/pics post: “I went to Milan to create a frame for this photo. Live frame.”



Image registration: determination of a **geometrical transformation** that aligns **one view** of an object with **another view** of that object or another object.









Causes of misalignment:

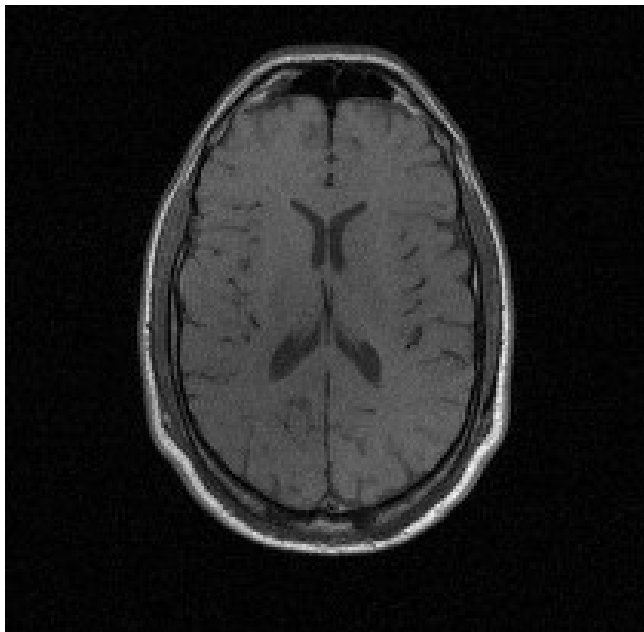
- Different positioning of patient in scanner
- Movements of organs due to physiology
- Movements of patient
- Distortions caused by imaging system
- Changes caused by interventions (e.g. surgery, chemotherapy) in between acquisition of the images

Applications of image registration:

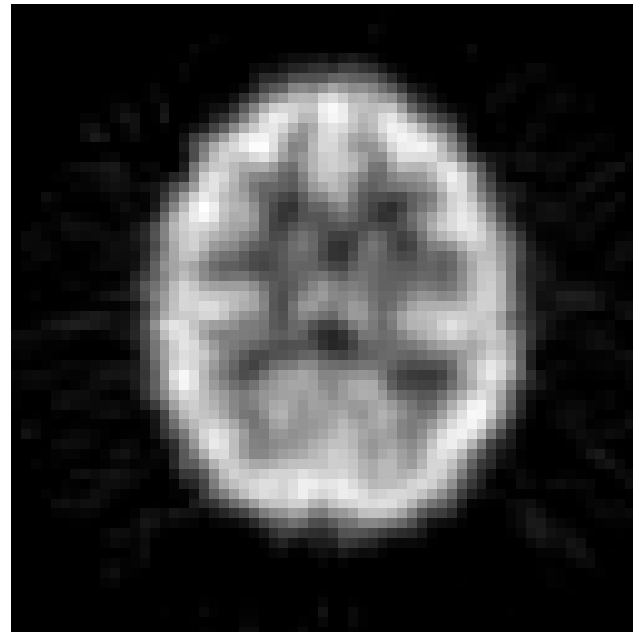
- Combining information from different sources
- Comparison: differences in (groups of) subjects
- Comparison: monitoring changes in a single subject
- Segmentation
- Motion correction
- Image-guided treatment
- Atlas, model of average anatomy

Applications of image registration:

- **Combining information from different sources**
- Comparison: differences in (groups of) subjects
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- Segmentation
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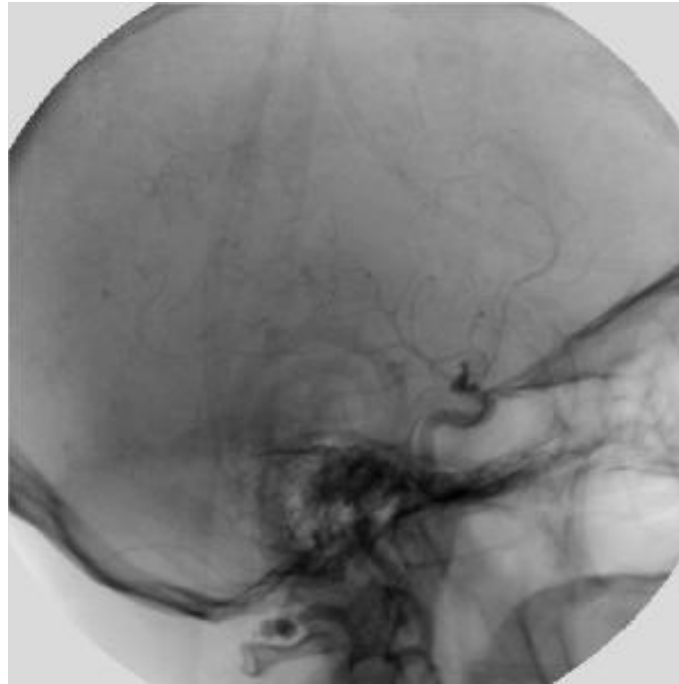
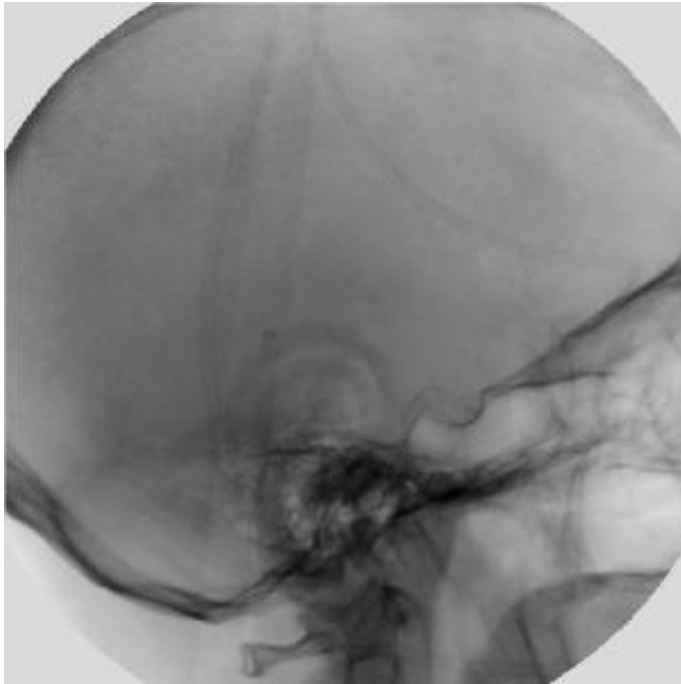
MRI, information about form



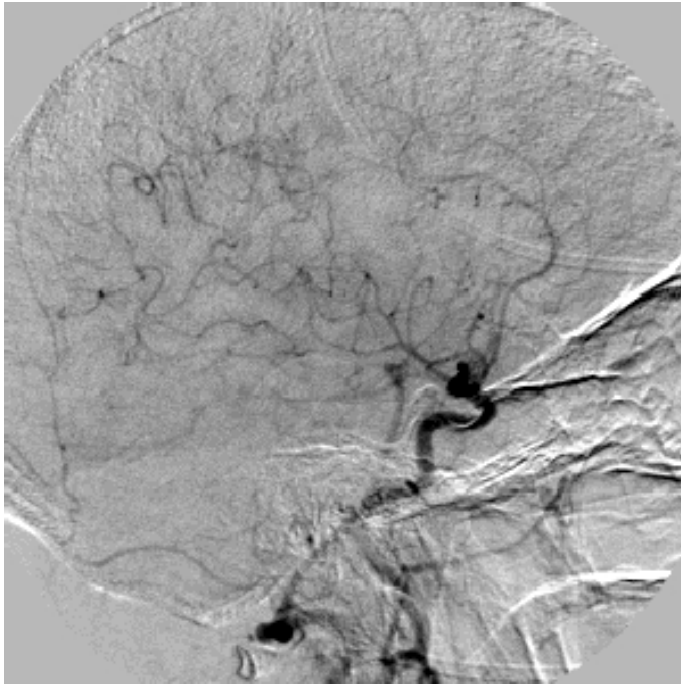
PET, information about function

Applications of image registration:

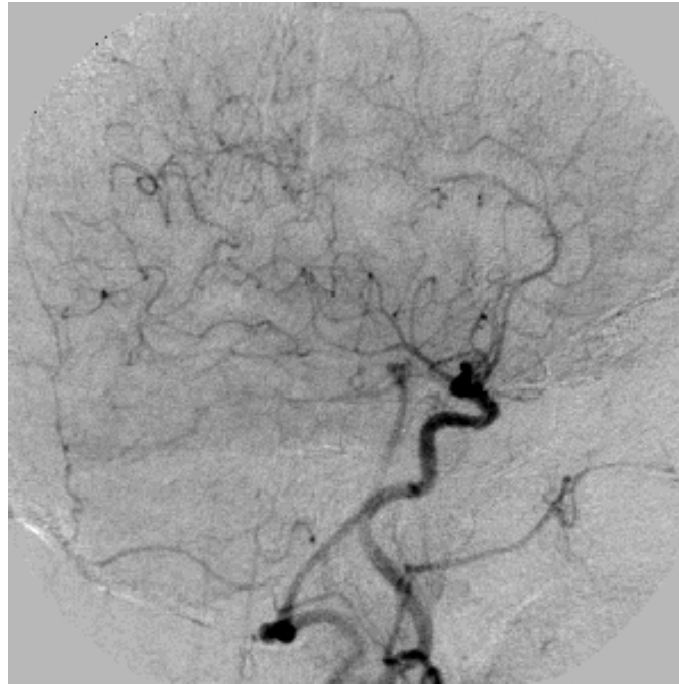
- Combining information from different sources
- Comparison: differences in (groups of) subjects
- **Comparison: monitoring changes in a single subject**
- Segmentation
- Motion correction
- Image-guided treatment
- Atlas, model of average anatomy



Digital subtraction angiography



Without registration



With registration

Digital subtraction angiography

Classification of image registration:

- Image dimensionality: 2D, 3D, 3D + time...
- Registration basis: point sets, intensity...
- Geometrical transformations: rigid, affine, nonlinear...
- Degree of interaction: automatic, semi-automatic
- Optimization procedure: closed-form solution, iterative
- Modalities: multi-modal, intra-modal
- Subject: inter-patient, intra-patient, atlas
- Object: head, vertebra, liver...

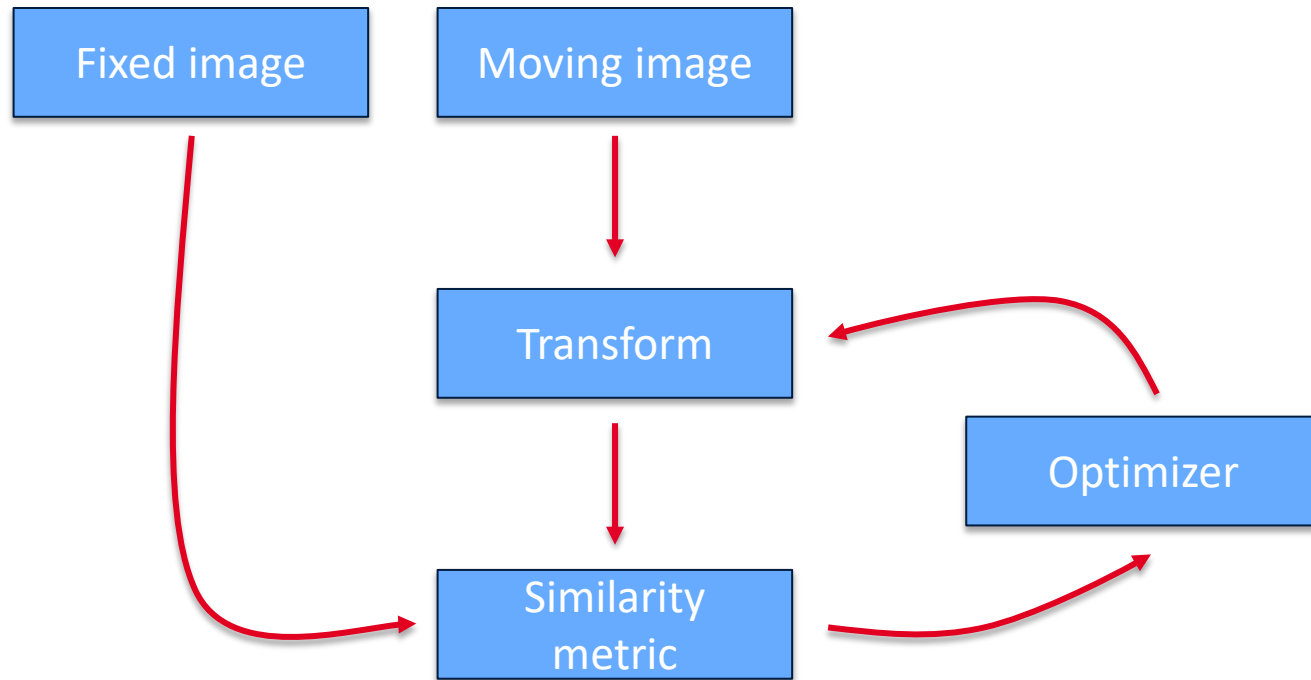
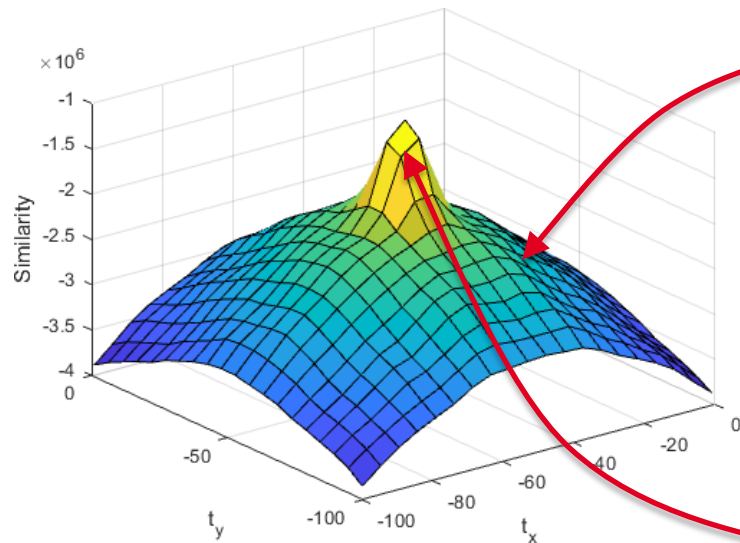
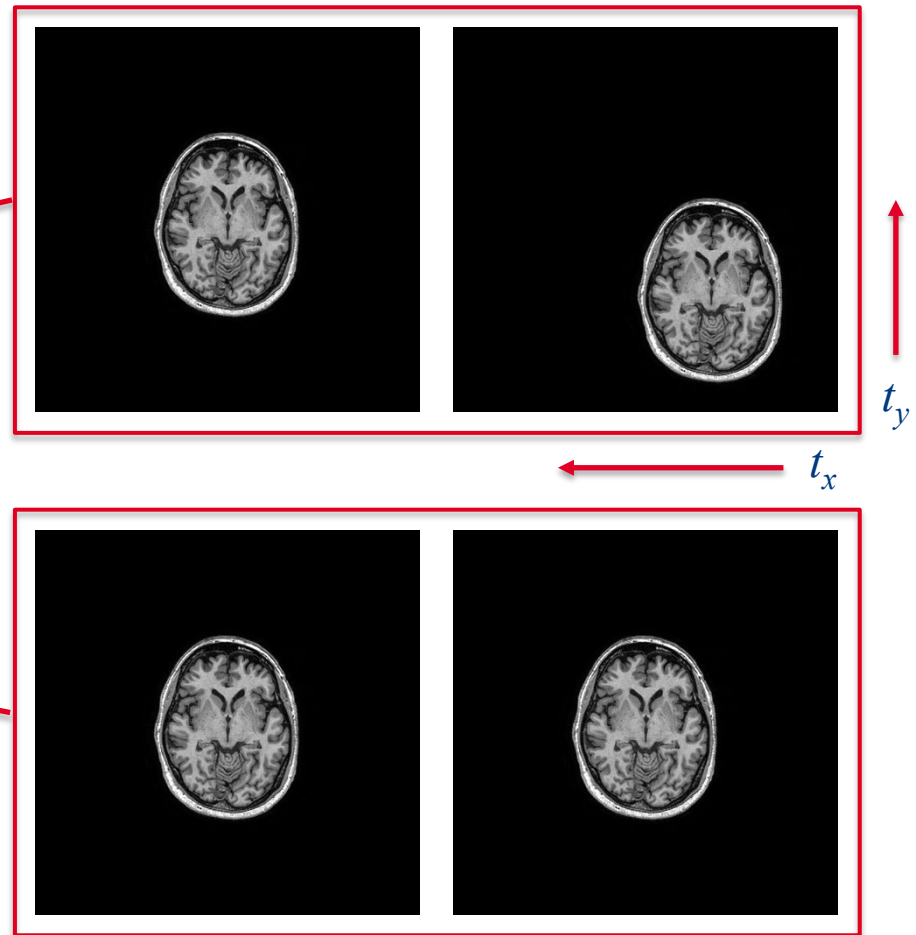


Image registration overview



Similarity as a function of translation



Study materials:

- Primary: lecture slides, exercises, virtual reader
- Recommended reading – relevant sections from:
[Fitzpatrick, J.M., Hill, D.L. and Maurer Jr, C.R., Image registration.](#)

Review of linear algebra

Kolter, Z. Do, C., Linear Algebra
Review and Reference

([http://cs229.stanford.edu/section/
cs229-linalg.pdf](http://cs229.stanford.edu/section/cs229-linalg.pdf))

Topics to review:

- Matrix-vector, vector-matrix products
- Transpose
- Norms
- Orthogonality
- Determinant

Scalars

- A scalar is a single number
- Integers, real numbers, rational numbers, etc.
- We denote it with italic font:

a, n, x

Vectors

- A vector is a 1-D array of numbers:

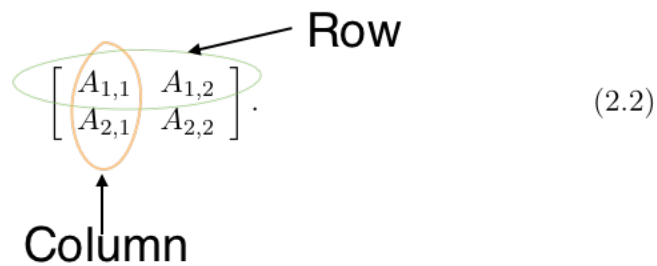
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \quad (2.1)$$

- Can be real, binary, integer, etc.
- Example notation for type and size:

$$\mathbb{R}^n$$

Matrices

- A matrix is a 2-D array of numbers:



The diagram shows a 2x2 matrix $\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$. A green oval encircles the top row, with an arrow pointing to it from the label "Row". An orange oval encircles the first column, with an arrow pointing to it from the label "Column". The matrix is labeled with the equation number (2.2) to its right.

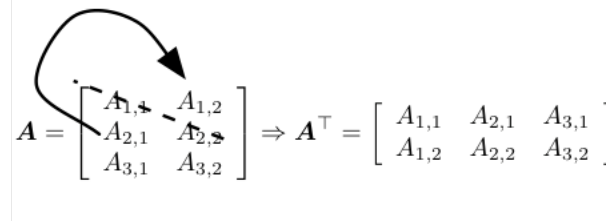
$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \quad (2.2)$$

- Example notation for type and shape:

$$\mathbf{A} \in \mathbb{R}^{m \times n}$$

Matrix Transpose

$$(\mathbf{A}^\top)_{i,j} = A_{j,i}. \quad (2.3)$$



$$\mathbf{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix} \Rightarrow \mathbf{A}^\top = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

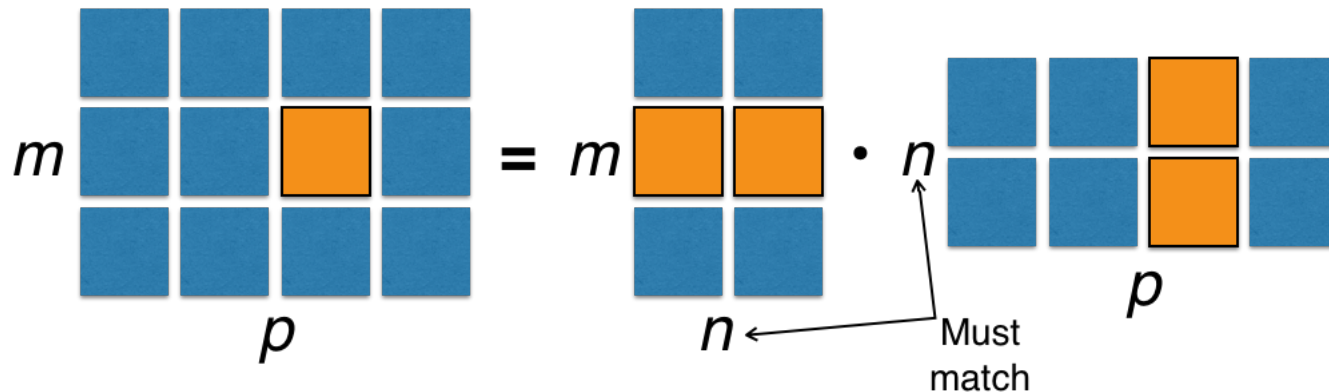
Figure 2.1: The transpose of the matrix can be thought of as a mirror image across the main diagonal.

$$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top. \quad (2.9)$$

Matrix (Dot) Product

$$C = AB. \quad (2.4)$$

$$C_{i,j} = \sum_k A_{i,k} B_{k,j}. \quad (2.5)$$



Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 2.2: *Example identity matrix*: This is \mathbf{I}_3 .

$$\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{I}_n \mathbf{x} = \mathbf{x}. \quad (2.20)$$

Systems of Equations

$$Ax = b \quad (2.11)$$

expands to

$$A_{1,:}x = b_1 \quad (2.12)$$

$$A_{2,:}x = b_2 \quad (2.13)$$

$$\dots \quad (2.14)$$

$$A_{m,:}x = b_m \quad (2.15)$$

System of Linear Equation

$$2.0x + 4.0y + 6.0z = 18$$

$$4.0x + 5.0y + 6.0z = 24$$

$$3.0x + 1y - 2.0z = 4$$

Matrix representation

$$A = \begin{bmatrix} 2.0 & 4.0 & 6.0 \\ 4.0 & 5.0 & 6.0 \\ 3.0 & 1.0 & -2.0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 18.0 \\ 24.0 \\ 4.0 \end{bmatrix}$$

Matrix Inversion

- Matrix inverse:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n. \quad (2.21)$$

- Solving a system using an inverse:

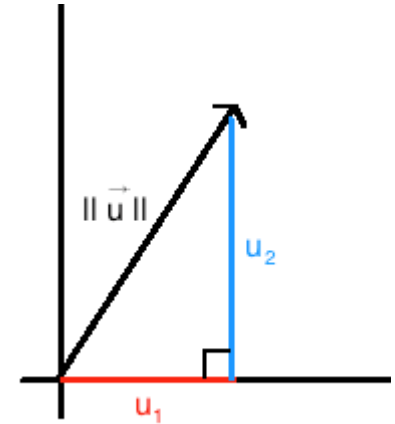
$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad (2.22)$$

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad (2.23)$$

$$\mathbf{I}_n\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad (2.24)$$

- Numerically unstable, but useful for abstract analysis

Norms



- Functions that measure how “large” a vector is
- Similar to a distance between zero and the point represented by the vector
 - $f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} = \mathbf{0}$
 - $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ (the *triangle inequality*)
 - $\forall \alpha \in \mathbb{R}, f(\alpha \mathbf{x}) = |\alpha| f(\mathbf{x})$

Special Matrices and Vectors

- Unit vector:

$$||\boldsymbol{x}||_2 = 1. \quad (2.36)$$

- Symmetric Matrix:


$$\boldsymbol{A} = \boldsymbol{A}^\top. \quad (2.35)$$

- Orthogonal matrix:

$$\begin{aligned} \boldsymbol{A}^\top \boldsymbol{A} &= \boldsymbol{A} \boldsymbol{A}^\top = \boldsymbol{I}. \\ \boldsymbol{A}^{-1} &= \boldsymbol{A}^\top \end{aligned} \quad (2.37)$$

The Determinant

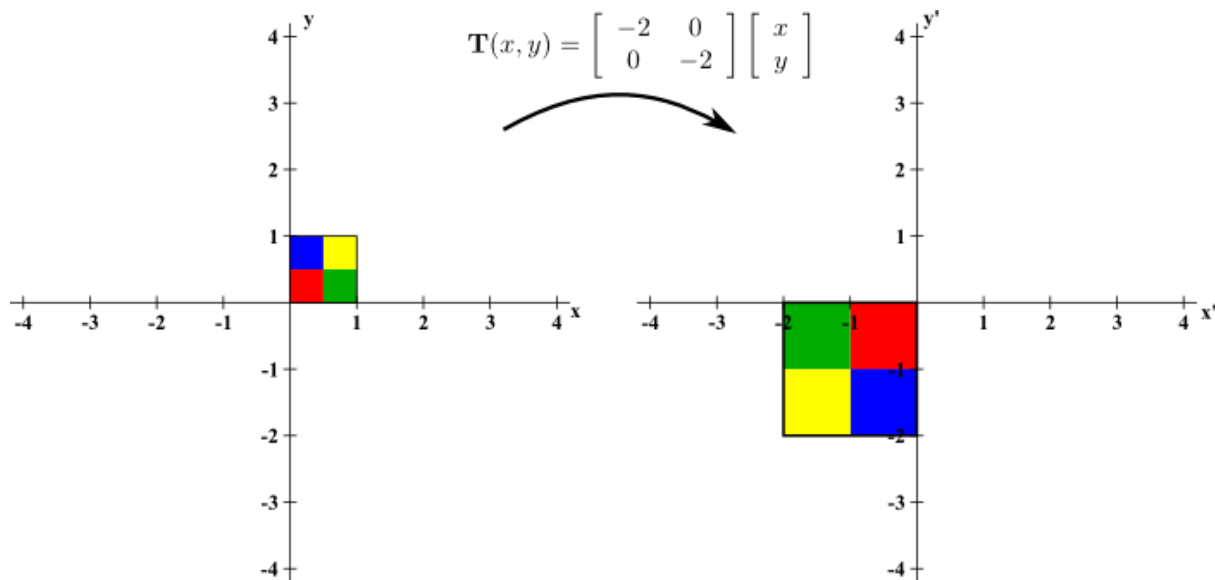
Symbol of Determinant

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$


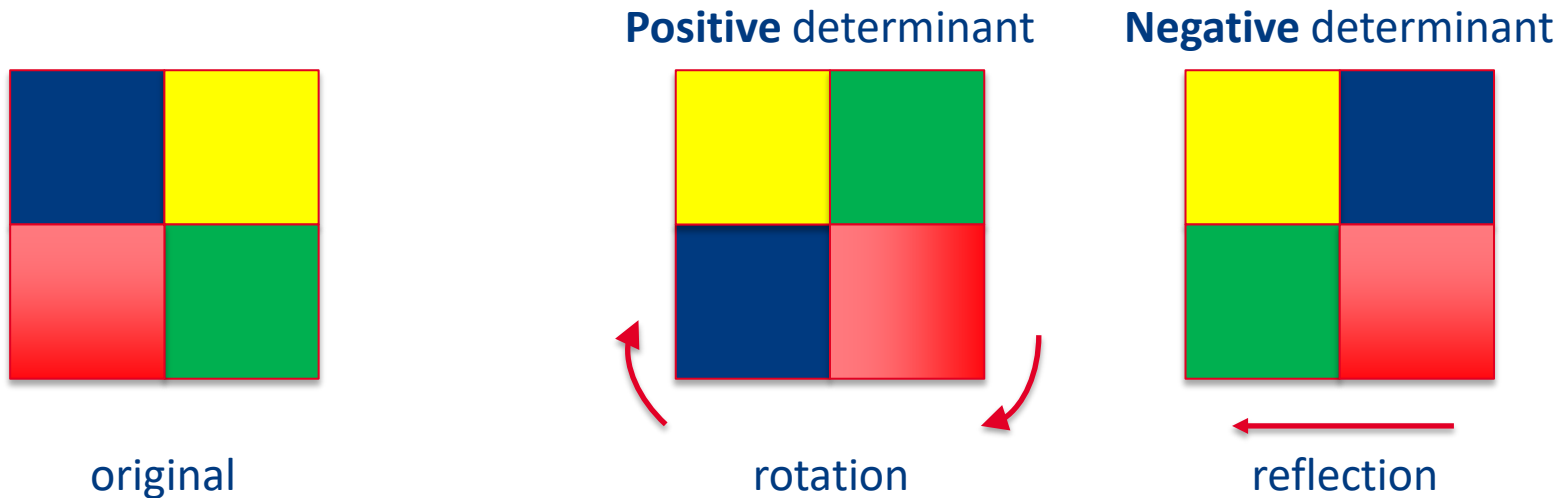
- The determinant of a square matrix maps matrices to real scalars

Question: What is the determinant of the identity matrix $\det(\mathbf{I})$?

Determinant of a transformation matrix **T**: the signed area of a unit square shape after transforming with **T**. The sign reflects whether the orientation has changed.

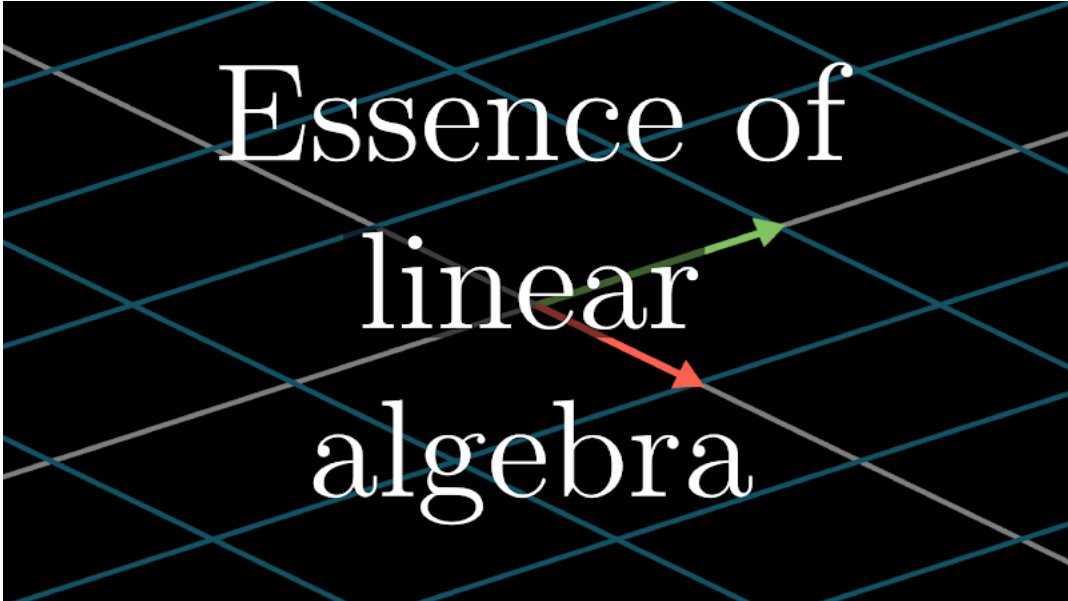


Determinant of a transformation matrix **T**: the signed **area** of a unit square shape after transforming with **T**. The **sign** reflects whether the orientation has changed.



$|T|$

- = 1 → no magnification
- > 1 → the matrix has magnification property
- < 1 → the matrix has shrinking property
- = 0 → shrink any object to a dot / matrix is not invertable

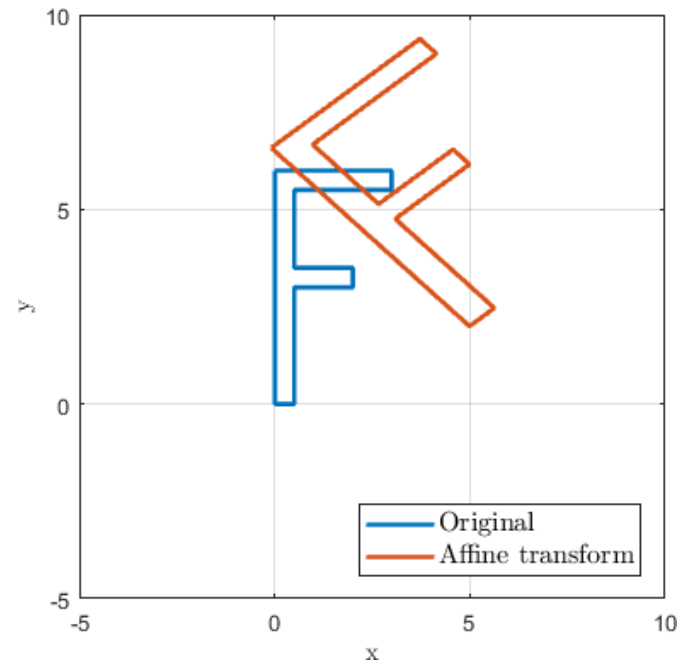


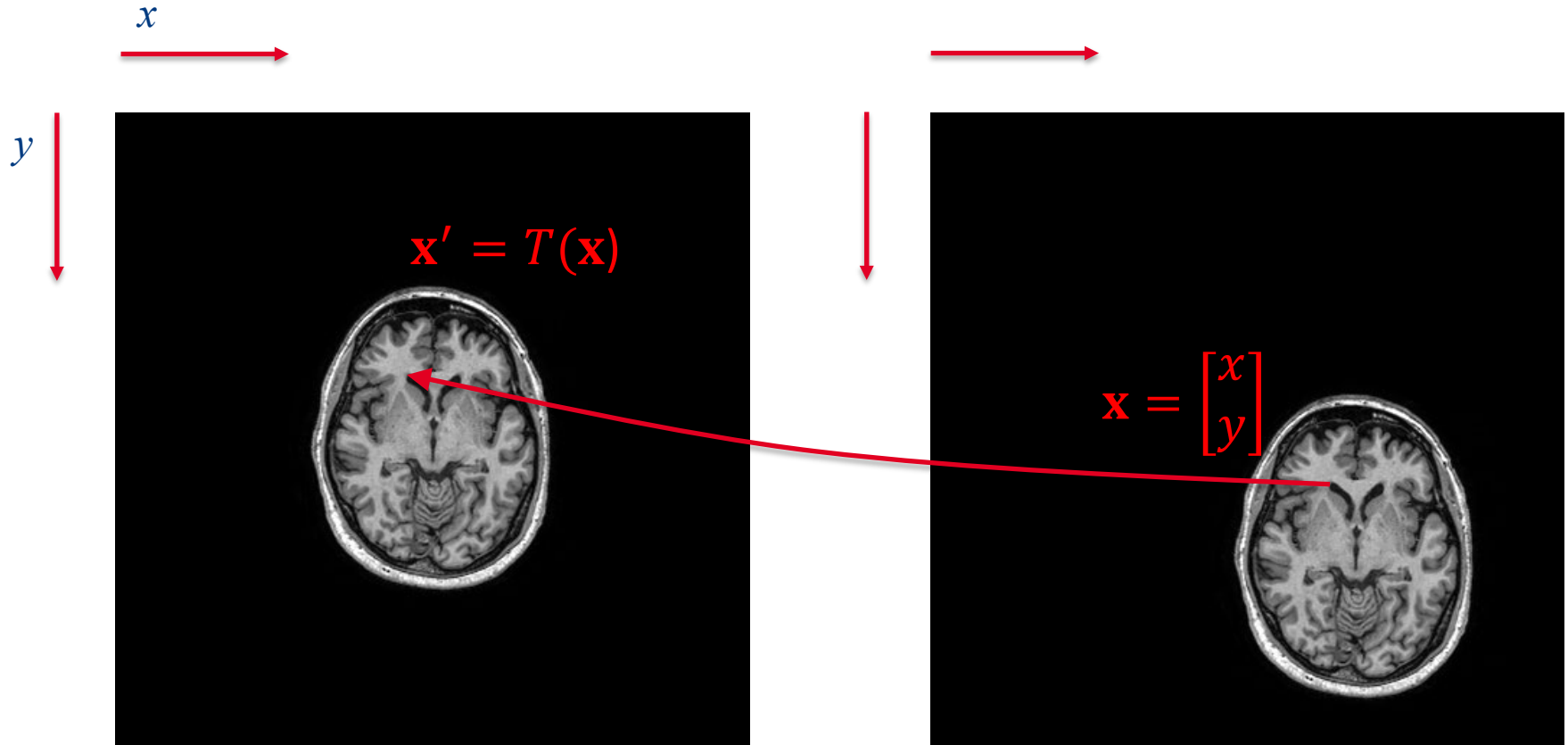
Essence of linear algebra

[Essence of linear algebra, 3Blue1Brown channel](#)

Geometrical transformations

Maureen van Eijnatten





All examples will be for 2D geometrical shapes and images, but they can be easily generalized to 3D.

Translation:

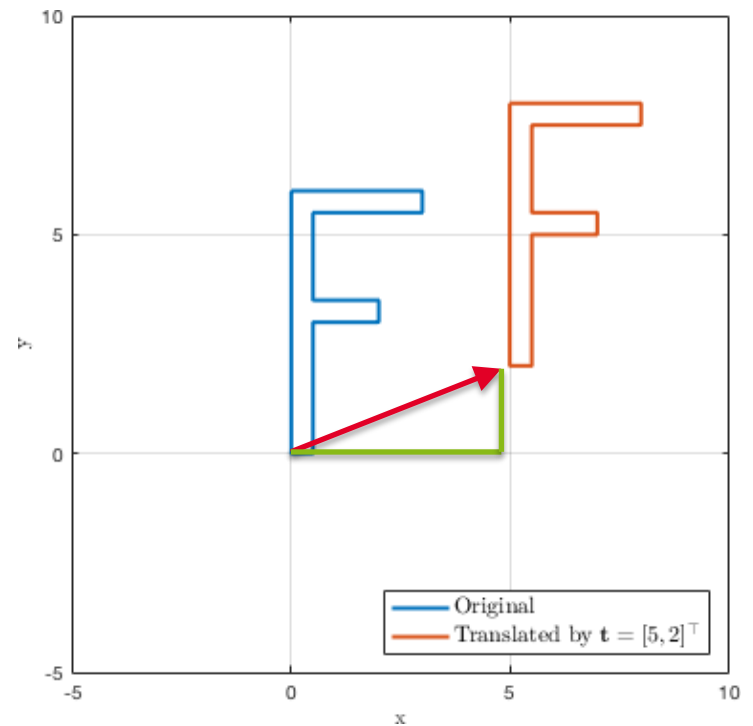
$$x = x + t_x$$

$$y = y + t_y$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



Distance between two points in 2D:

$$D(\mathbf{x}, \mathbf{x}') = \sqrt{(x - x')^2 + (y - y')^2} = \sqrt{(\mathbf{x} - \mathbf{x}')^T (\mathbf{x} - \mathbf{x}')}^{\frac{1}{2}}$$

Rotation:

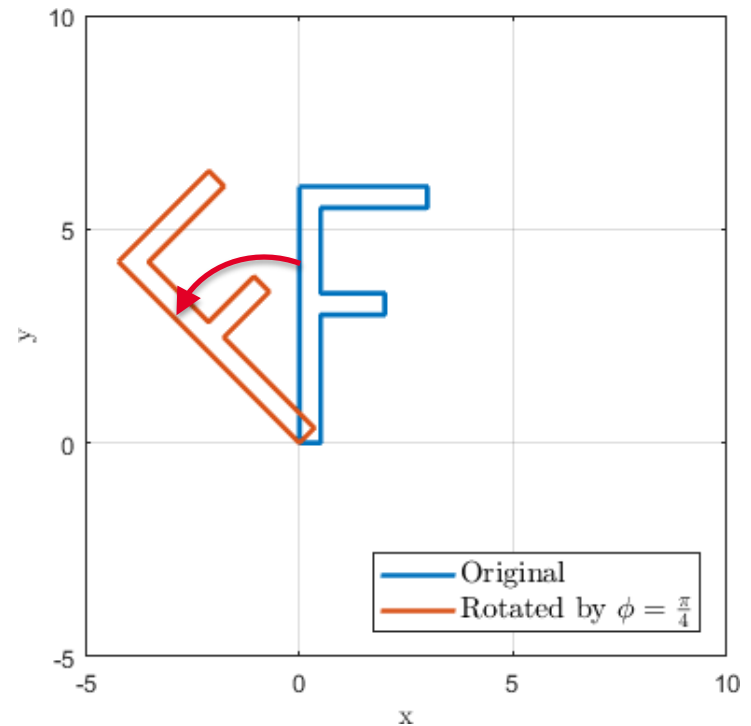
$$x' = \cos(\phi)x - \sin(\phi)y$$

$$y' = \sin(\phi)x + \cos(\phi)y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$



Not every matrix can be considered a rotation matrix.

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Rotation matrices:

- Are orthogonal:

$$\mathbf{R}\mathbf{R}^{-1} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$$

- Have determinant equal to 1:

$$\det(\mathbf{R}) = 1$$

Scaling:

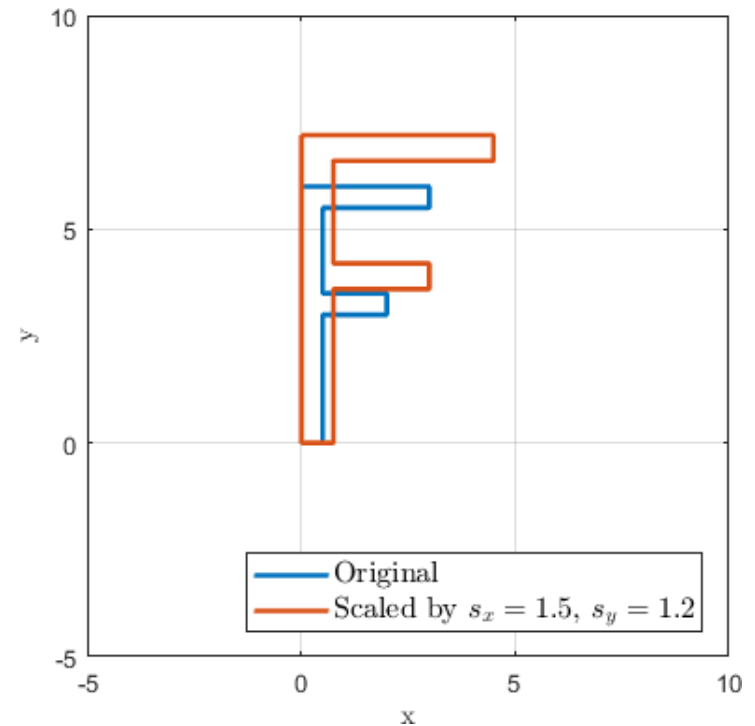
$$x' = s_x x$$

$$y' = s_y y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{S}\mathbf{x}$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



Scaling (on whiteboard)

Rotation (on whiteboard)

Shearing:

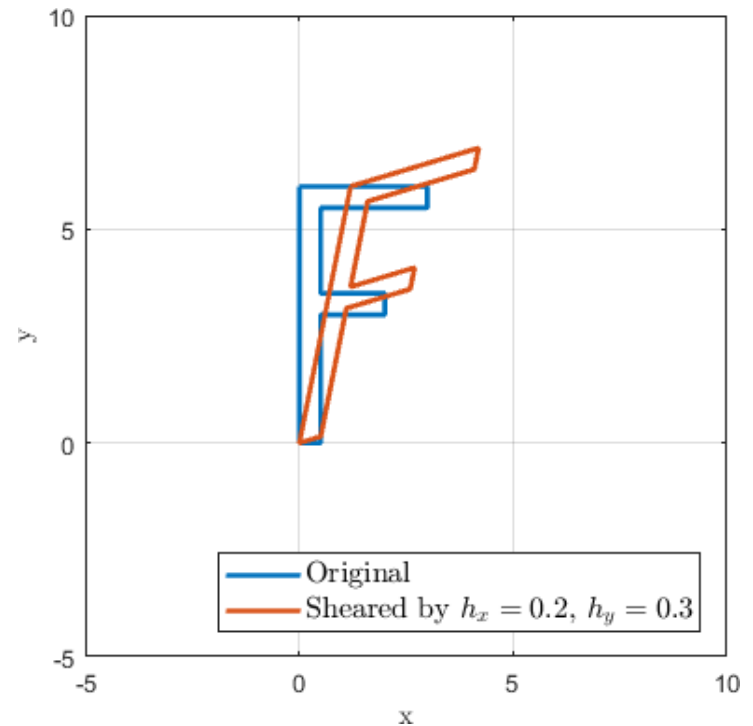
$$x' = x + h_x y$$

$$y' = h_y x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{H} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix}$$



Reflection:

Horizontal:

$$x' = -x$$

$$y' = y$$

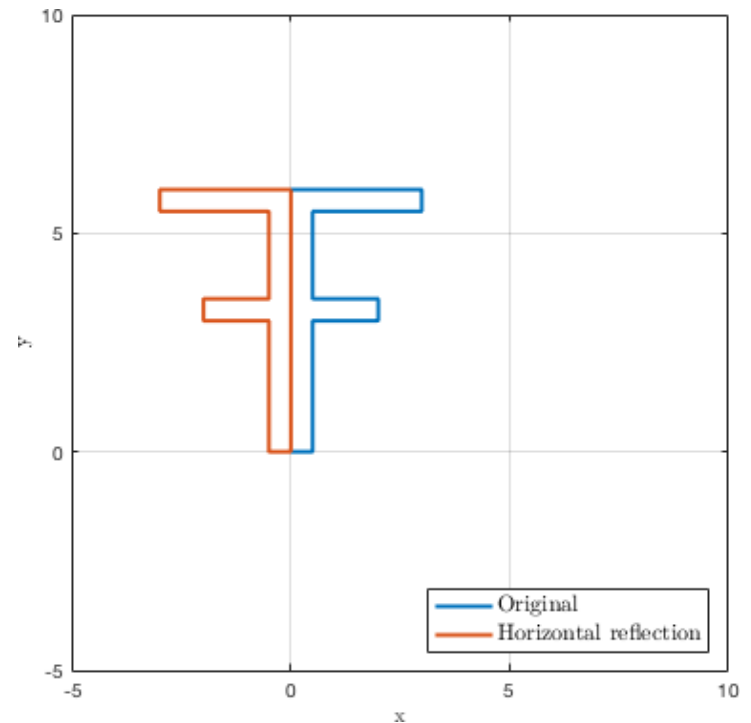
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Vertical:

$$x' = x$$

$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Composition of transformations:

Rotation + translation (rigid):

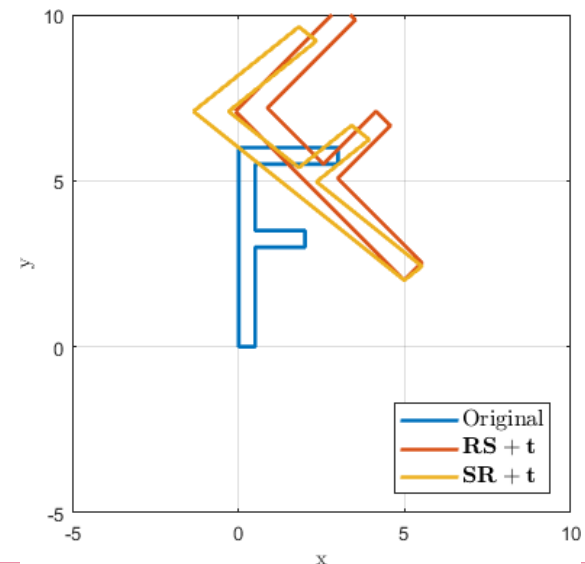
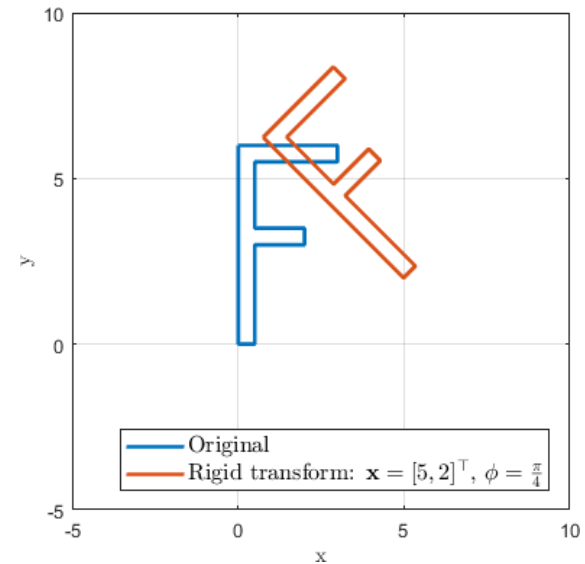
$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

Transformations can be combined by multiplying the transformation matrices.

Rotation, scaling + translation:

$$\mathbf{x}' = \mathbf{R}\mathbf{S}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \mathbf{S}\mathbf{R}\mathbf{x} + \mathbf{t}$$



Note that matrix multiplication is not commutative:

$$\mathbf{T}_1 \mathbf{T}_2 \mathbf{x} \neq \mathbf{T}_2 \mathbf{T}_1 \mathbf{x}$$

First scaling, then rotation, then translation:

$$\mathbf{x}' = \mathbf{R} \mathbf{S} \mathbf{x} + \mathbf{t}$$

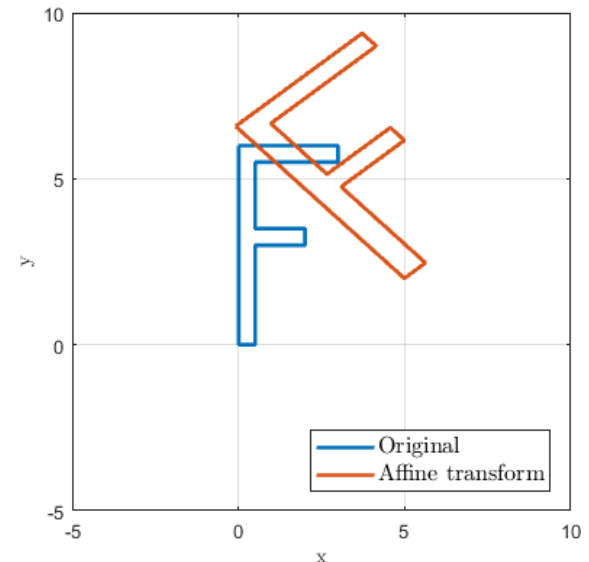
First rotation, then scaling, then translation:

$$\mathbf{x}' = \mathbf{S} \mathbf{R} \mathbf{x} + \mathbf{t}$$

Affine transformation (no restriction on the transformation parameters):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}$$



It can be considered as a composition of any combination of rotations, scalings, shearings, reflections + translations.

Note that the affine transformation has **6 parameters**: 2×2 transformation matrix and 2×1 translation vector.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

The combination of rotation, scaling, shearing, reflection + translation has **9 parameters**: 1 rotation angle, 2 scaling parameters, 2 shearing parameters, 2 reflection parameters and 2×1 translation vector.

$$[\phi \quad s_x \quad s_y \quad h_x \quad h_y \quad r_x \quad r_y \quad t_x \quad t_y]$$

However, the first 7 parameters are not independent.

The first parameterization is more compact, the second more human-readable.

Affine transformation in 2D has only **6 degrees of freedom**.

In medical image registration, **reflections do not usually occur**, and it can be very problematic if two images are incorrectly registered with a reflection (e.g. can cause a surgical procedure to be performed on the wrong side of the body).

Thus, reflections should be excluded from affine registration.

$$\begin{bmatrix} \phi & s_x & s_y & h_x & h_y & t_x & t_y \end{bmatrix}$$

When using the unrestricted transformation matrix, a check for reflection can be made by examining $\det(\mathbf{A})$. If a reflection has occurred $\det(\mathbf{A}) < 0$.

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}$$

A transformation matrix and a translation vector can be combined when using homogeneous coordinates:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This largely simplifies the notation and implementation of complex transformations.

Example:

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix}$$

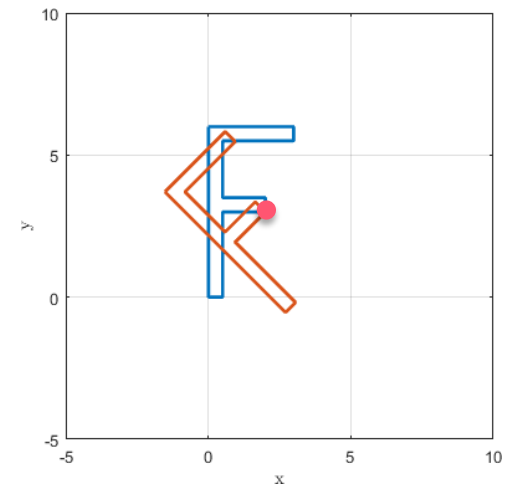
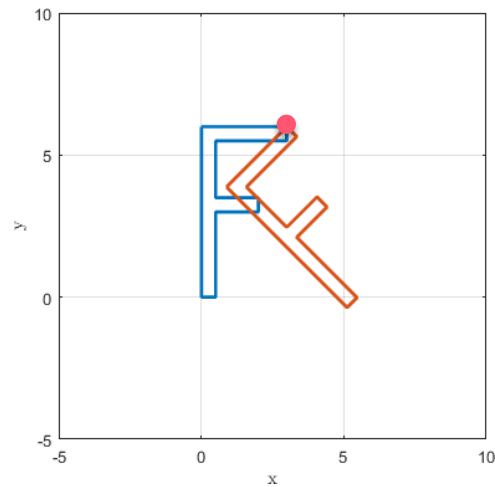
$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_1 \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x,1} + t_{x,2} \\ 0 & 1 & t_{y,1} + t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

Example – rotation around an arbitrary point $\mathbf{x}_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Example – rotation around an arbitrary point $\mathbf{x}_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$:



Inverse transformation can be achieved by taking the inverse of the transformation matrix:

$$\mathbf{T} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Affine transformation in 3D:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How many rotation angles in 3D?

How many degrees of freedom?

Non-linear transformations:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = ax + by + t_x$$

$$y' = cx + dy + t_y$$



Linear polynomial

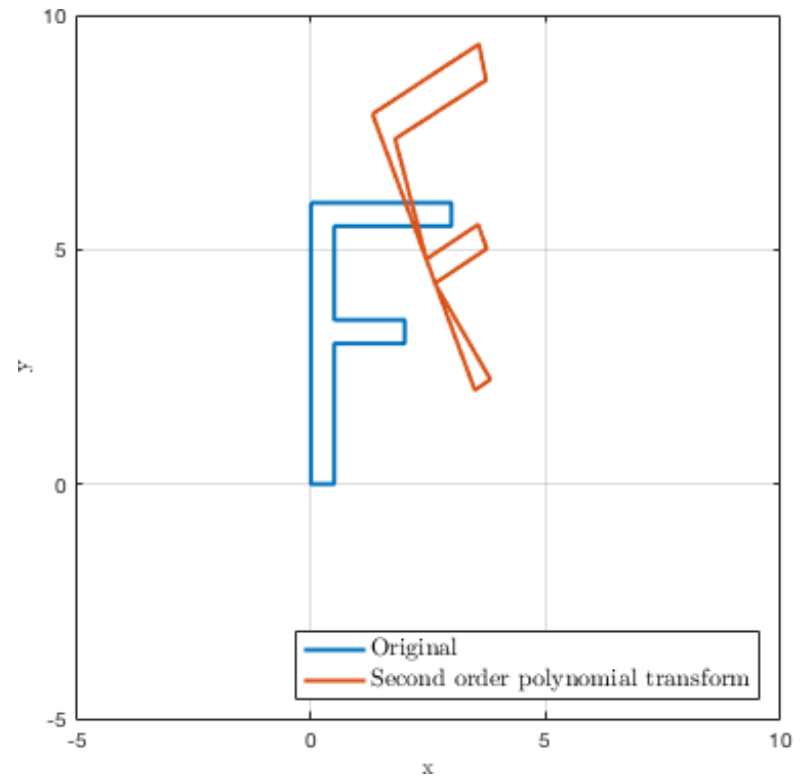
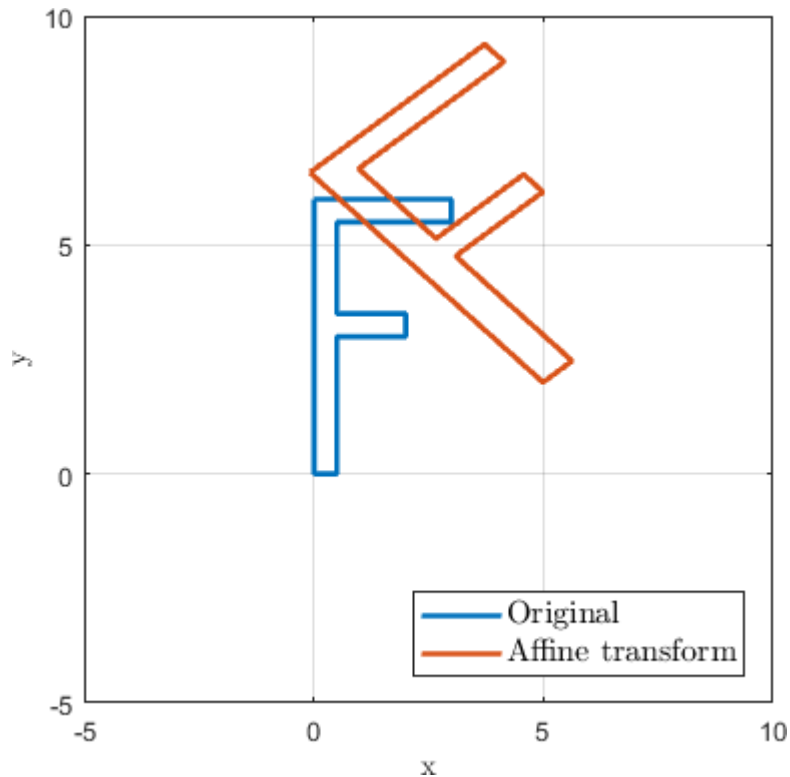
$$x' = ax + by + t_x + u_1x^2 + u_2y^2 + u_3xy \dots$$

$$y' = cx + dy + t_y + v_1x^2 + v_2y^2 + v_3xy \dots$$



Higher order polynomial

Non-linear transformations:

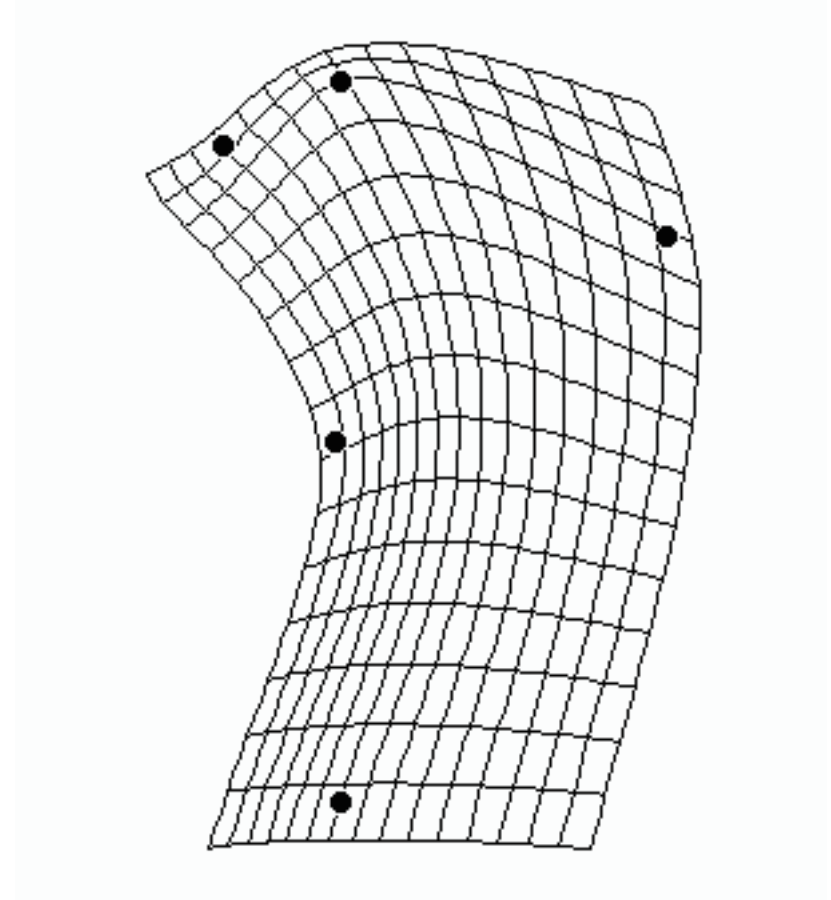


Non-linear transformations: thin-plate spline

$$x' = ax + by + t_x + \sum_{i=1}^N u_i r_i^2 \ln r_i^2$$

$$y' = cx + dy + t_y + \sum_{i=1}^N v_i r_i^2 \ln r_i^2$$

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2$$



Thank you

Next: Image
transformation, point-
based registration

