Image registration

Maureen van Eijnatten

Today:

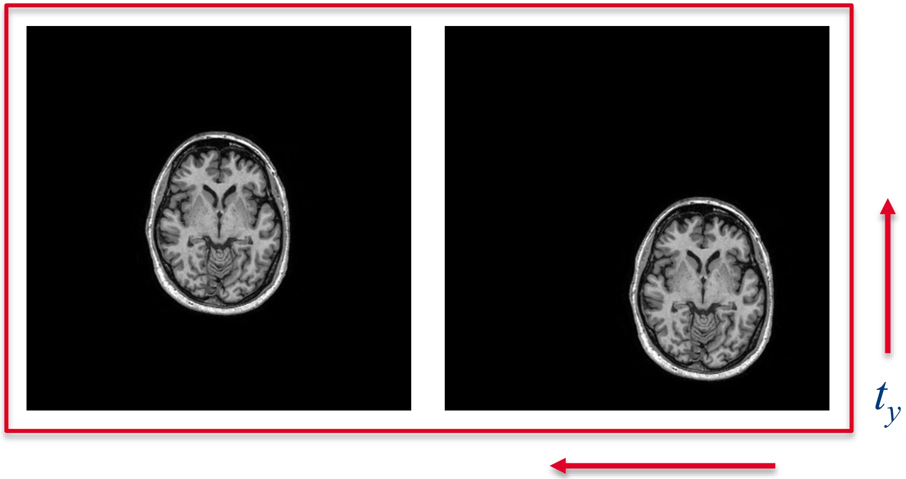
- Introduction of medical image registration
- Recap of linear algebra
- Geometrical transformations

Learning outcomes

The student can:

- name possible causes of misalignment in medical images
- name different applications of medical image registration
- classify medical image registration methods using eight different criteria
- apply the basic principles of linear algebra (i.e., matrix-vector, vector-matrix products, transpose, norms, orthogonality, determinant) to image registration tasks
- use the determinant of a transformation matrix T to predict the orientation and magnification of an object transformed with T
- compose and combine rigid and affine transformations in 2D and 3D (and rewrite them using homogeneous coordinates)
- explain the difference between affine and non-linear registrations

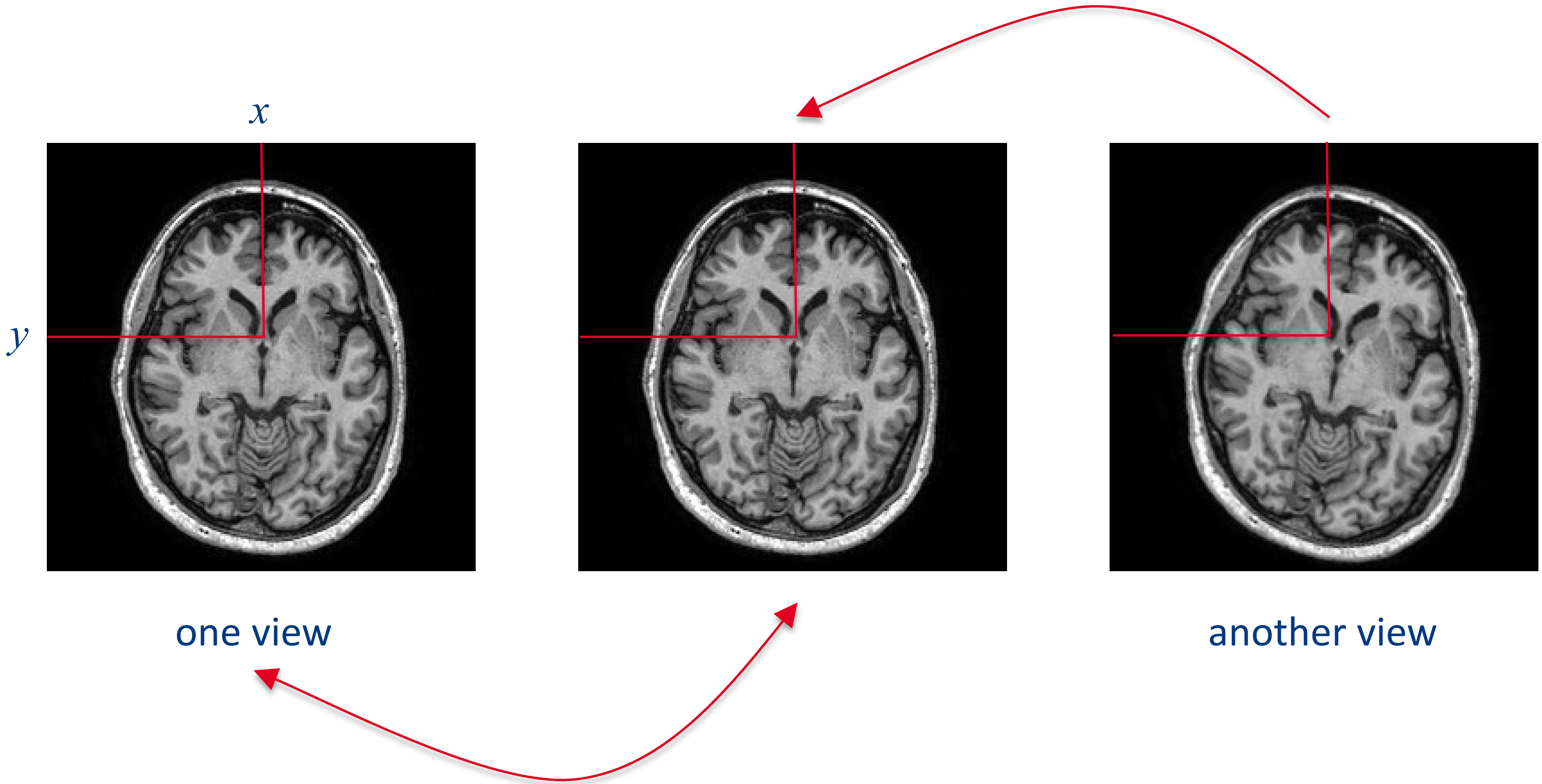
Image registration: Introduction



r/pics post: “I went to Milan to create a frame for this photo. Live frame.”

Image registration: determination of a **geometrical transformation** that aligns **one view** of an object with **another view** of that object or another object.

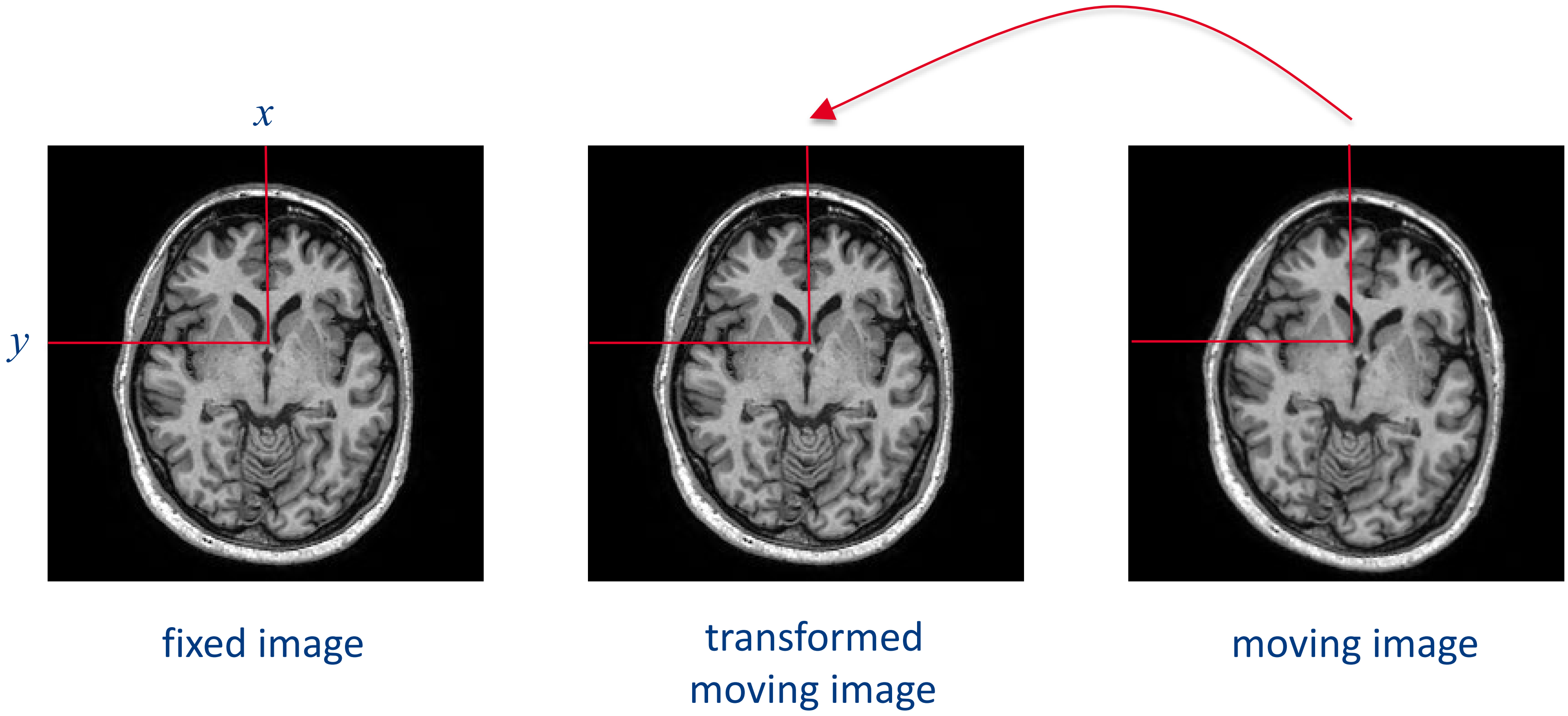
geometrical transformation

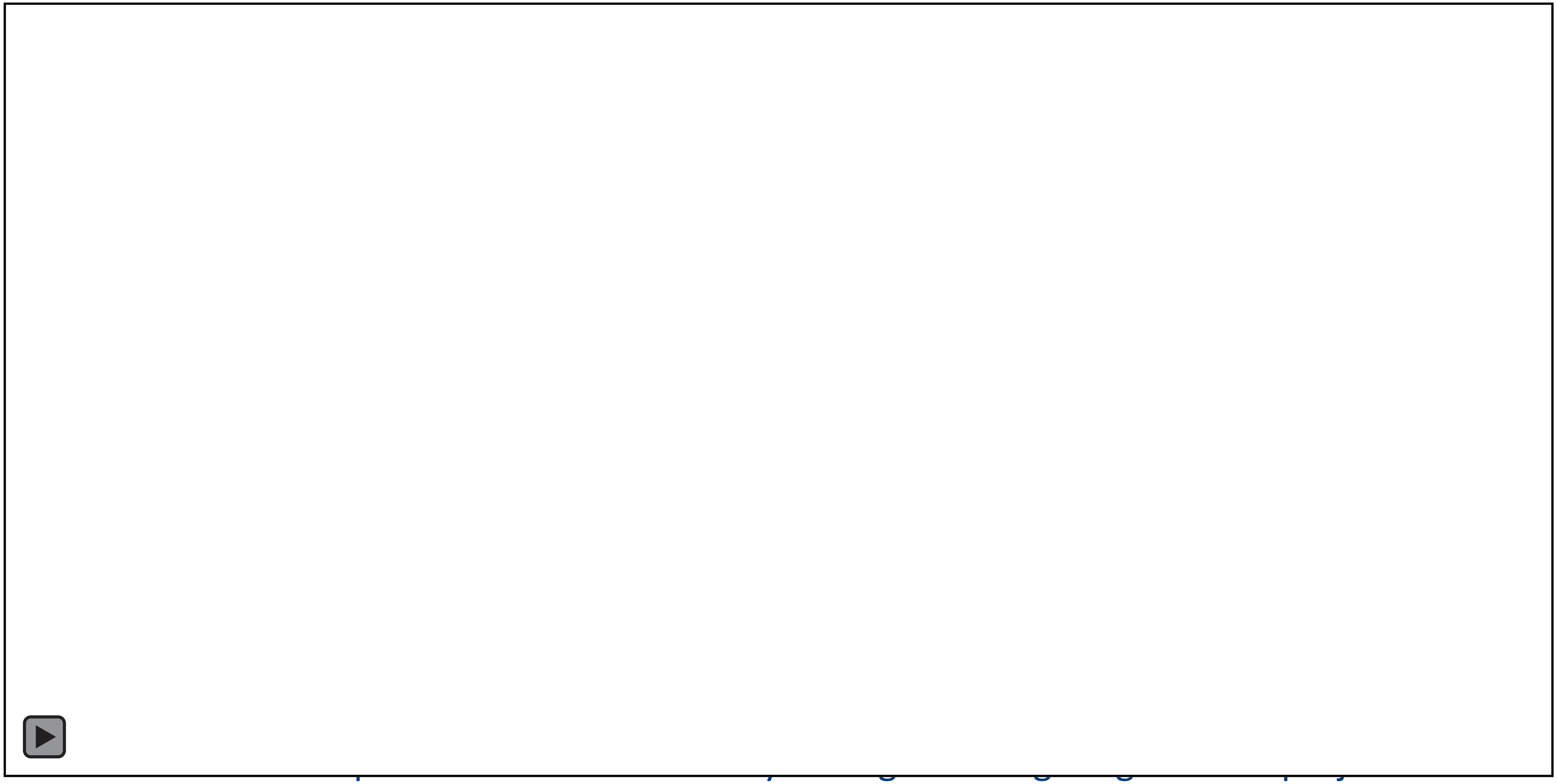


one view

another view

T aligns the first with the second view


x
y
fixed image
transformed moving image
moving image



Causes of misalignment:

- Different positioning of patient in scanner
- Movements of organs due to physiology
- Movements of patient
- Distortions caused by imaging system
- Changes caused by interventions (e.g. surgery, chemotherapy) in between acquisition of the images

Applications of image registration:

- Combining information from different sources
- Comparison: differences in (groups of) subjects
- Comparison: monitoring changes in a single subject
- Segmentation
- Motion correction
- Image-guided treatment
- Atlas, model of average anatomy

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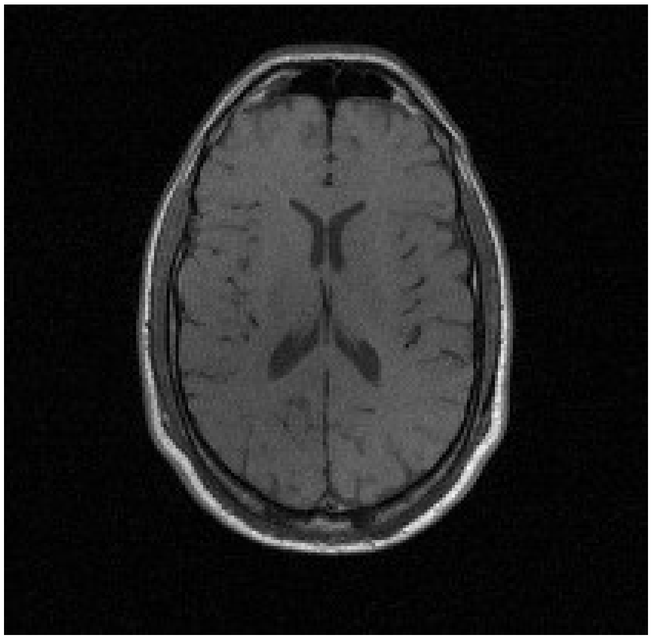
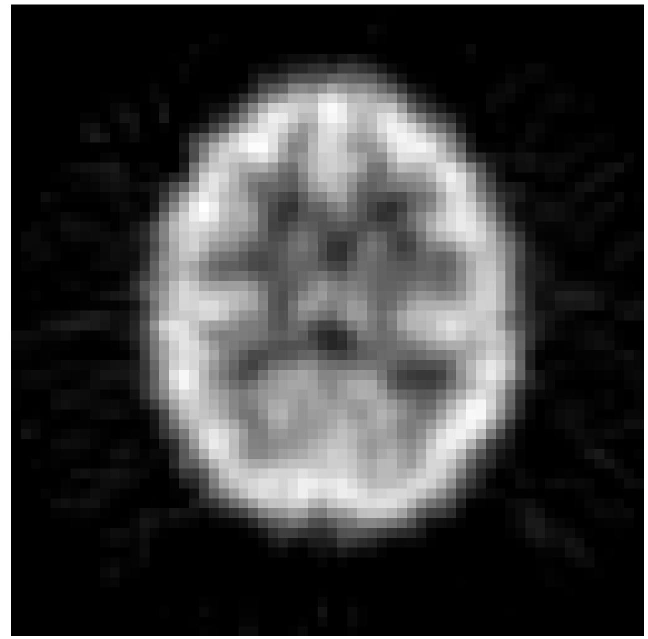


MRI, information about form



PET, information about function

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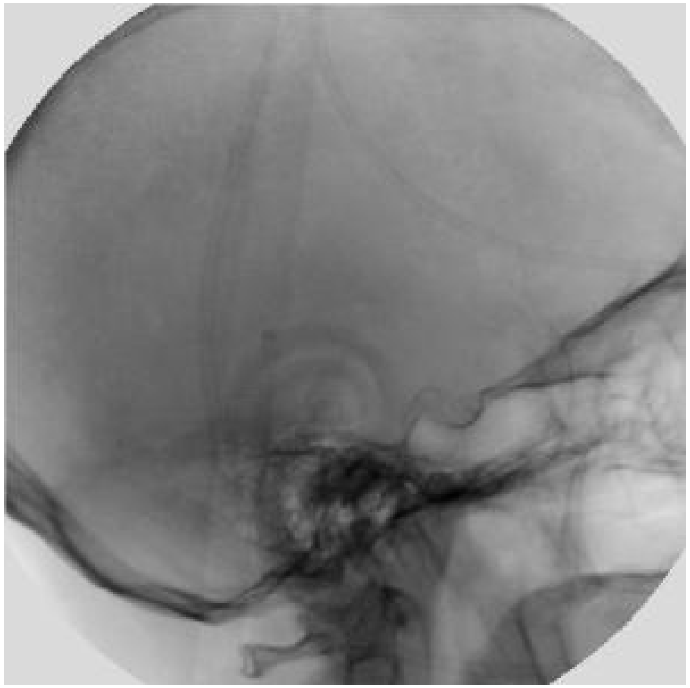
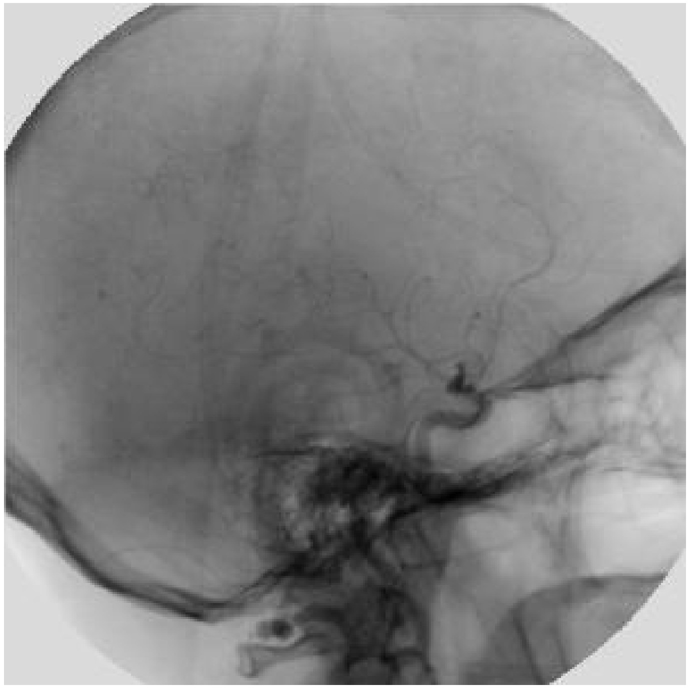
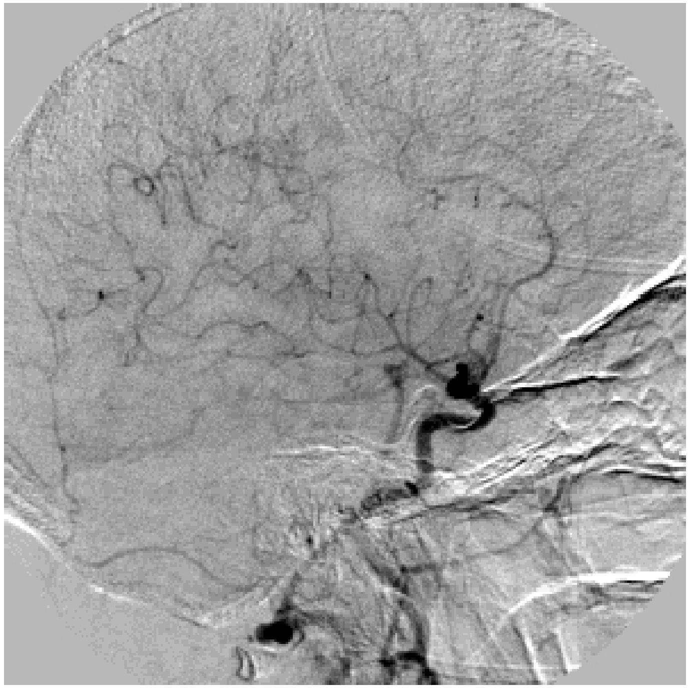
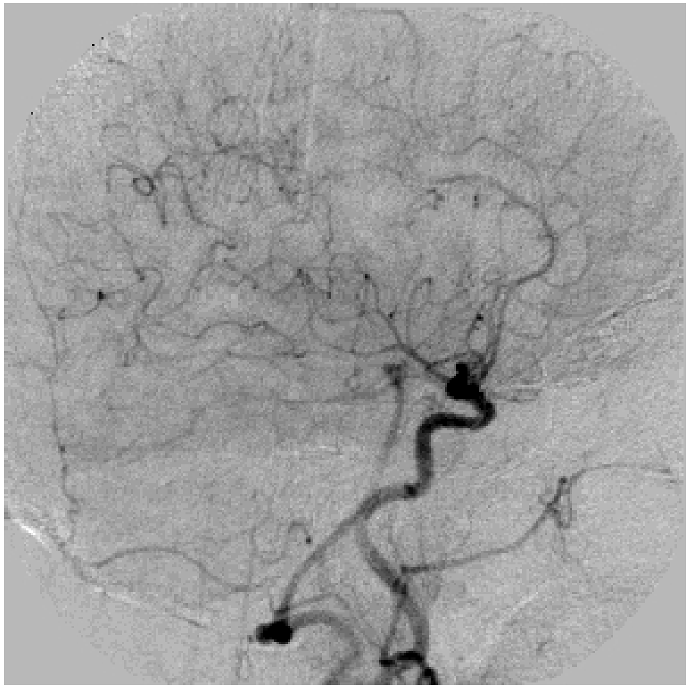



Digital subtraction angiography



Without registration



With registration

Digital subtraction angiography

Classification of image registration:

- Image dimensionality: 2D, 3D, 3D + time…
- Registration basis: point sets, intensity…
- Geometrical transformations: rigid, affine, nonlinear…
- Degree of interaction: automatic, semi-automatic
- Optimization procedure: closed-form solution, iterative
- Modalities: multi-modal, intra-modal
- Subject: inter-patient, intra-patient, atlas
- Object: head, vertebra, liver…

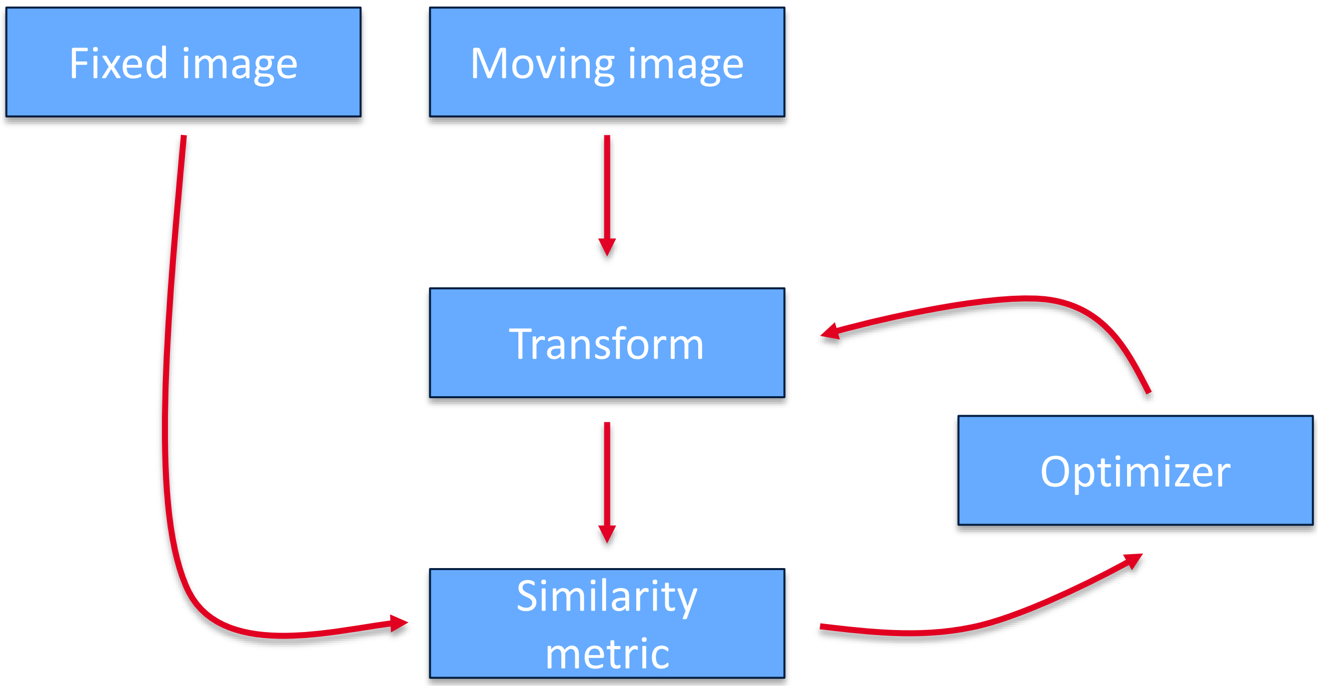
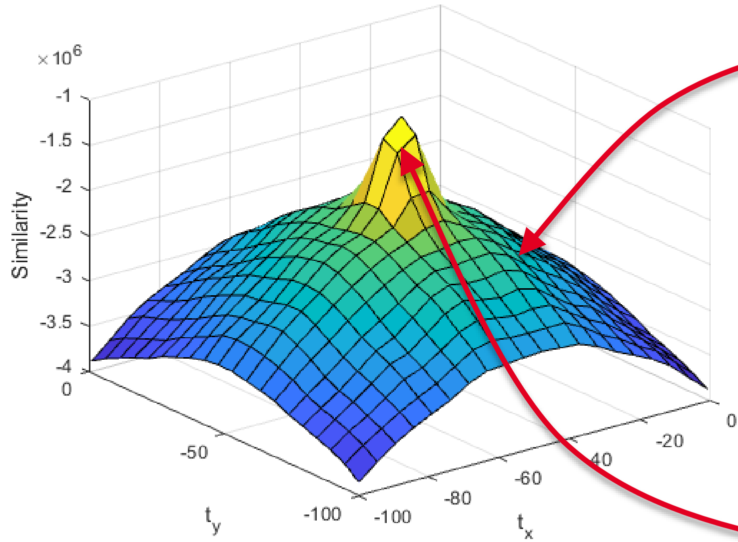
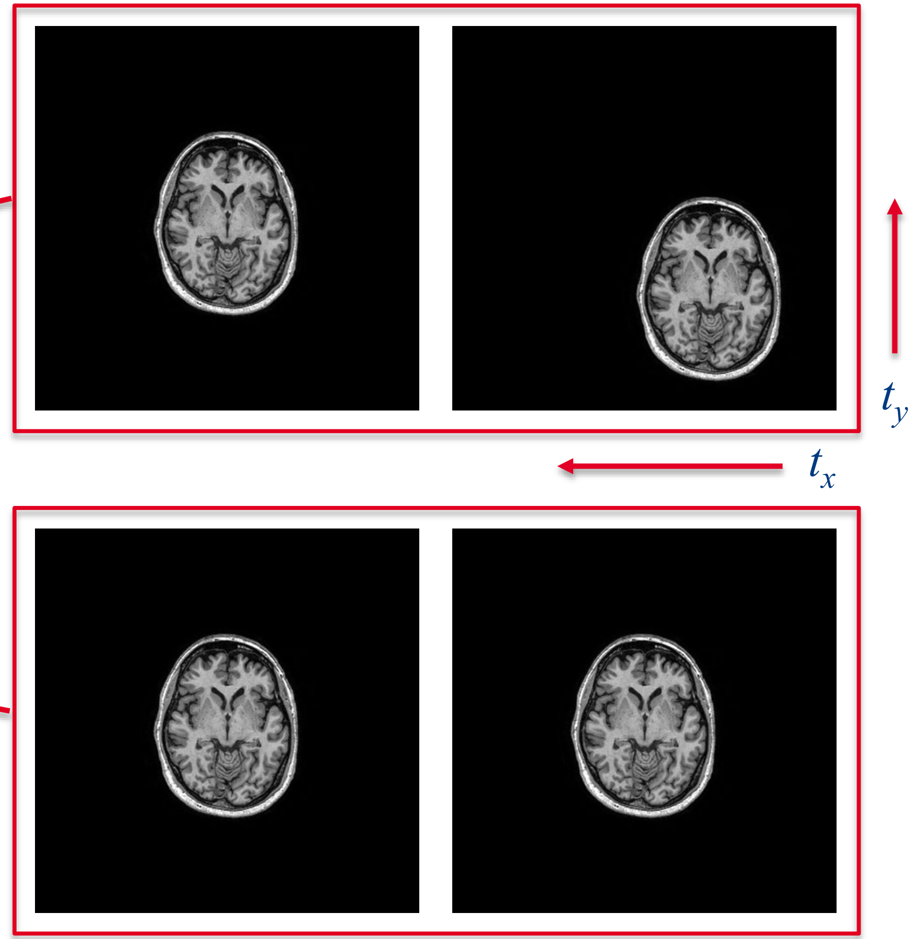


Image registration overview





Similarity as a function of translation

Medical image registration in 8DC00 in a nutshell:

- **Week 1, day 1**: Course introduction, introduction into medical image registration, geometrical transforms
- **Week 1, day 2**: Image transformation, point-based image registration, least-squares

- **Week 2, day 1**: Intensity-based image similarity metrics, gradient descent, intensity-based registration
- **Week 2, day 2**: Validation; active shape models

- **Week 3, day 1**: Guest lecture on image-guided treatments
- **Week 3, day 2**: *no lecture*

- **Week 7, day 1**: Deep learning for image registration

Study materials:

- Primary: lecture slides, exercises, virtual reader
- Recommended reading – relevant sections from: Fitzpatrick, J.M., Hill, D.L. and Maurer Jr, C.R., Image registration.

Review of linear algebra

Kolter, Z. Do, C., Linear Algebra Review and Reference (http://cs229.stanford.edu/section/cs229-linalg.pdf)

Topics to review:

- Matrix-vector, vector-matrix products
- Transpose
- Norms
- Orthogonality
- Determinant

Scalars

- A scalar is a single number
- Integers, real numbers, rational numbers, etc.
- We denote it with italic font:

$$a, n, x$$

Vectors

- A vector is a 1-D array of numbers:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{2.1}$$

- Can be real, binary, integer, etc.
- Example notation for type and size:

$$\mathbb{R}^n$$

Goodfellow 2016
deeplearningbook.org

Matrices

- A matrix is a 2-D array of numbers:

Row

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}. \tag{2.2}$$

Column

- Example notation for type and shape:

$$\boldsymbol{A} \in \mathbb{R}^{m \times n}$$

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Matrix Transpose

$$(\boldsymbol{A}^\top)_{i,j} = A_{j,i}. \tag{2.3}$$

$$\boldsymbol{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix} \Rightarrow \boldsymbol{A}^\top = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

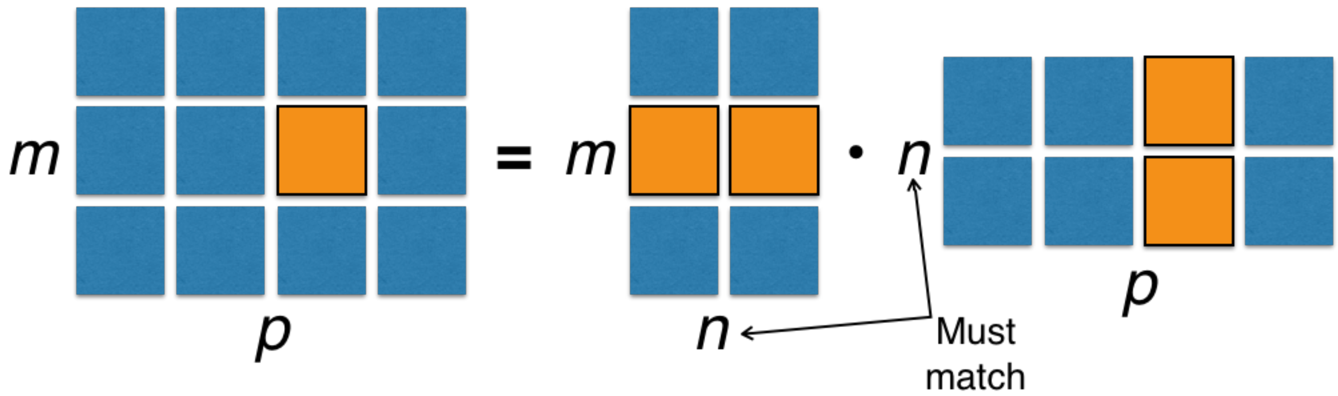
Figure 2.1: The transpose of the matrix can be thought of as a mirror image across the main diagonal.

$$(\boldsymbol{AB})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top. \tag{2.9}$$

Matrix (Dot) Product

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}. \tag{2.4}$$

$$C_{i,j} = \sum_k A_{i,k} B_{k,j}. \tag{2.5}$$



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Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 2.2: *Example identity matrix*: This is $\boldsymbol{I}_3$.

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{I}_n \boldsymbol{x} = \boldsymbol{x}. \qquad (2.20)$$

Systems of Equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{2.11}$$

expands to

$$\boldsymbol{A}_{1,:}\boldsymbol{x} = b_1 \tag{2.12}$$

$$\boldsymbol{A}_{2,:}\boldsymbol{x} = b_2 \tag{2.13}$$

$$\dots \tag{2.14}$$

$$\boldsymbol{A}_{m,:}\boldsymbol{x} = b_m \tag{2.15}$$

System of Linear Equation

2.0x + 4.0y +6.0z = 18

4.0x + 5.0y +6.0z = 24

3.0x + 1y - 2.0z = 4

Matrix representation

$$A = \begin{bmatrix} 2.0 & 4.0 & 6.0 \\ 4.0 & 5.0 & 6.0 \\ 3.0 & 1.0 & -2.0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 18.0 \\ 24.0 \\ 4.0 \end{bmatrix}$$

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Matrix Inversion

- Matrix inverse:

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n. \tag{2.21}$$

- Solving a system using an inverse:

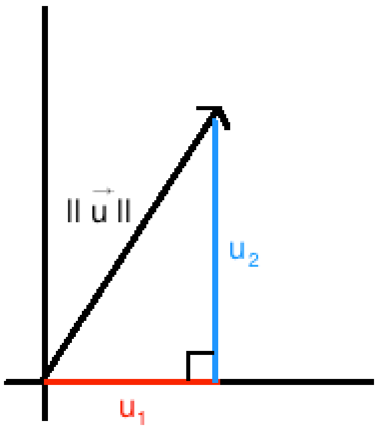
$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{2.22}$$

$$\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} \tag{2.23}$$

$$\boldsymbol{I}_n\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} \tag{2.24}$$

- Numerically unstable, but useful for abstract analysis

Norms



- Functions that measure how "large" a vector is
- Similar to a distance between zero and the point represented by the vector
  - $f(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = \boldsymbol{0}$
  - $f(\boldsymbol{x} + \boldsymbol{y}) \leq f(\boldsymbol{x}) + f(\boldsymbol{y})$ (the *triangle inequality*)
  - $\forall \alpha \in \mathbb{R}, f(\alpha \boldsymbol{x}) = |\alpha| f(\boldsymbol{x})$

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Special Matrices and Vectors

- Unit vector:

$$||\boldsymbol{x}||_2 = 1. \tag{2.36}$$

- Symmetric Matrix:

$$\boldsymbol{A} = \boldsymbol{A}^\top. \tag{2.35}$$

- Orthogonal matrix:

$$\boldsymbol{A}^\top \boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I}. \tag{2.37}$$

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^\top$$

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The Determinant

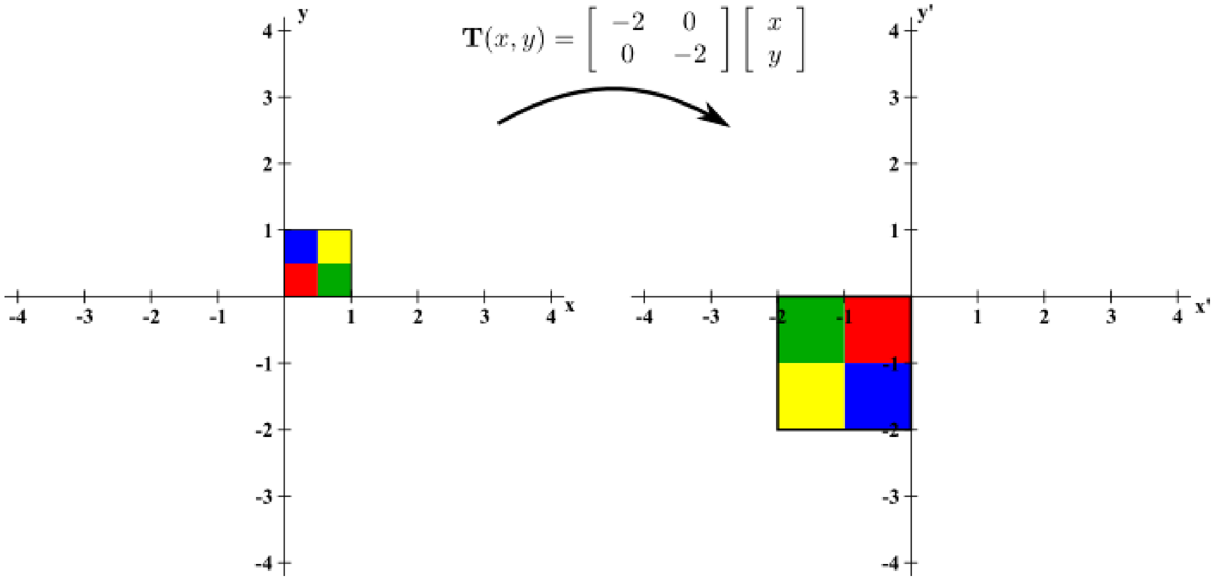
Symbol of Determinant

$$\det(\boldsymbol{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- The determinant of a square matrix maps matrices to real scalars

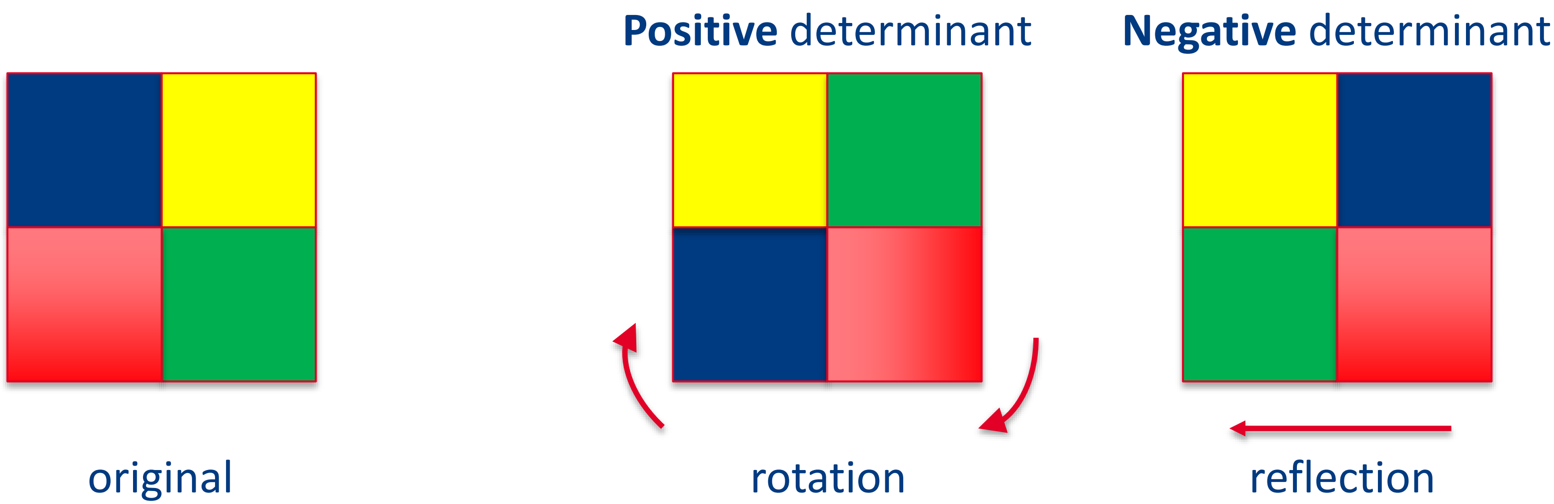
Question: What is the determinant of the identity matrix $\det(\boldsymbol{I})$?

Determinant of a transformation matrix **T**: the signed area of a unit square shape after transforming with **T**. The sign reflects whether the orientation has changed.



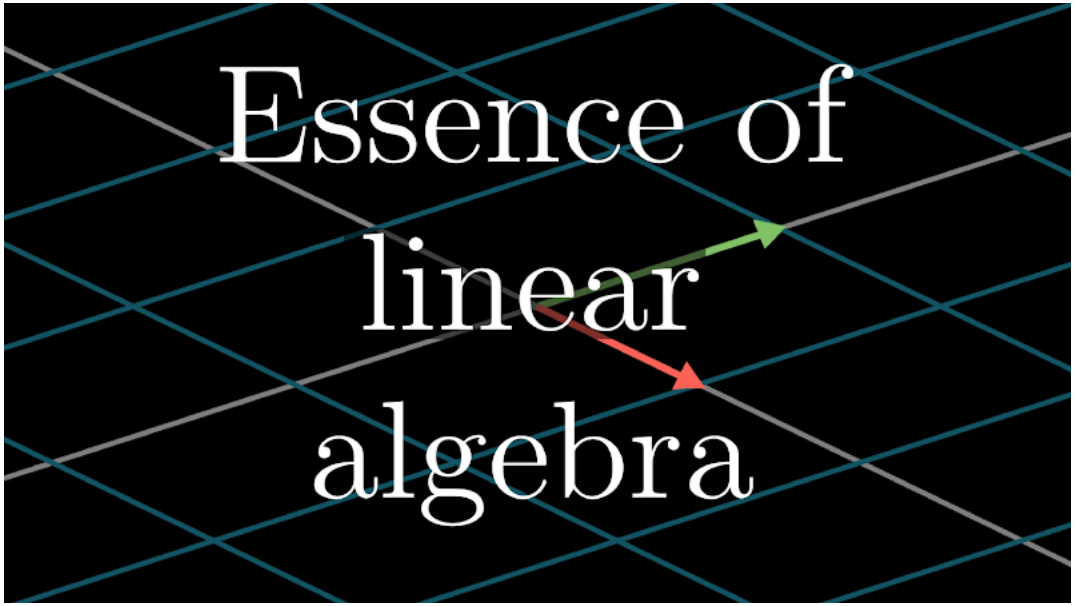
https://mathinsight.org/determinant_linear_transformation

Determinant of a transformation matrix $\mathbf{T}$: the signed **area** of a unit square shape after transforming with $\mathbf{T}$. The **sign** reflects whether the orientation has changed.



$|T|$

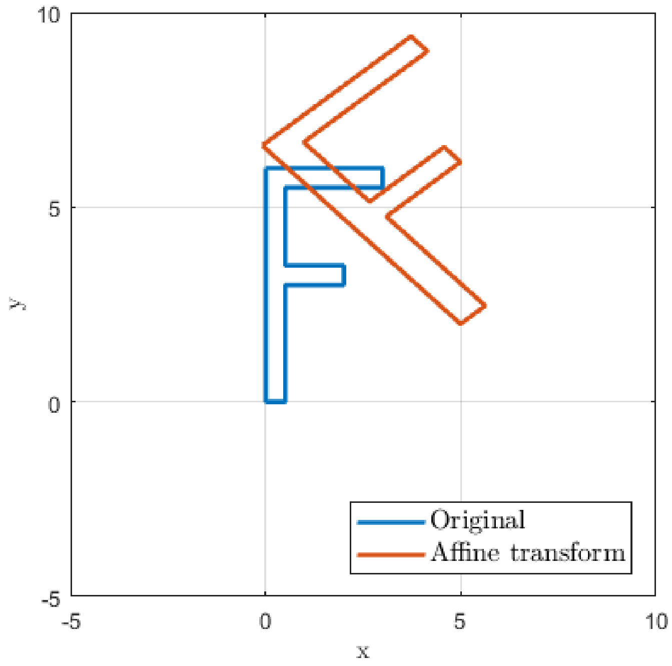
- $= 1$ → no magnification
- > 1 → the matrix has magnification property
- < 1 → the matrix has shrinking property
- $= 0$ → shrink any object to a dot / matrix is not invertable

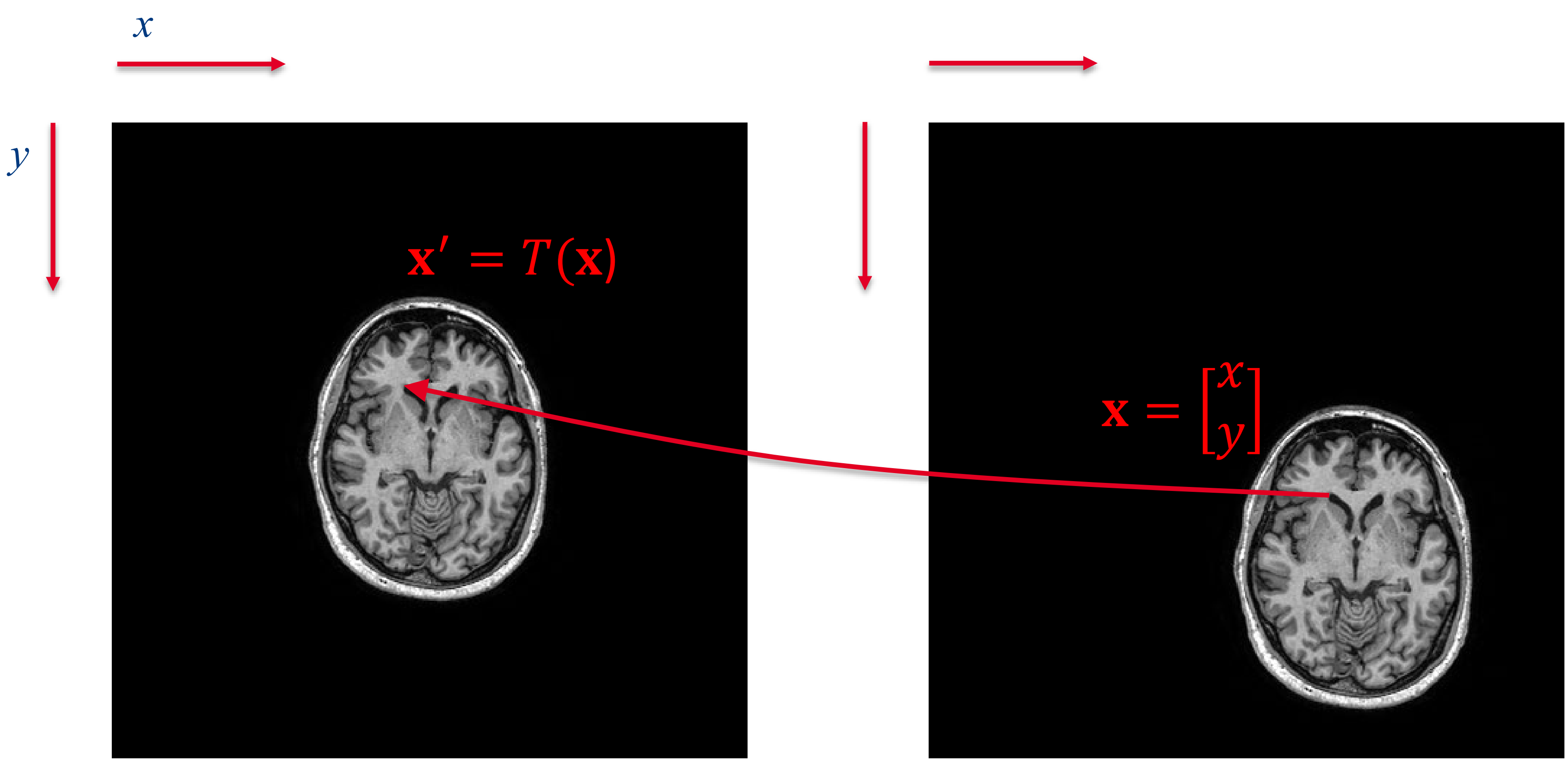


[Essence of linear algebra, 3Blue1Brown channel]

Geometrical transformations

Maureen van Eijnatten




x
y
$\mathbf{x}' = T(\mathbf{x})$
$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

All examples will be for 2D geometrical shapes and images, but they can be easily generalized to 3D.

Translation:

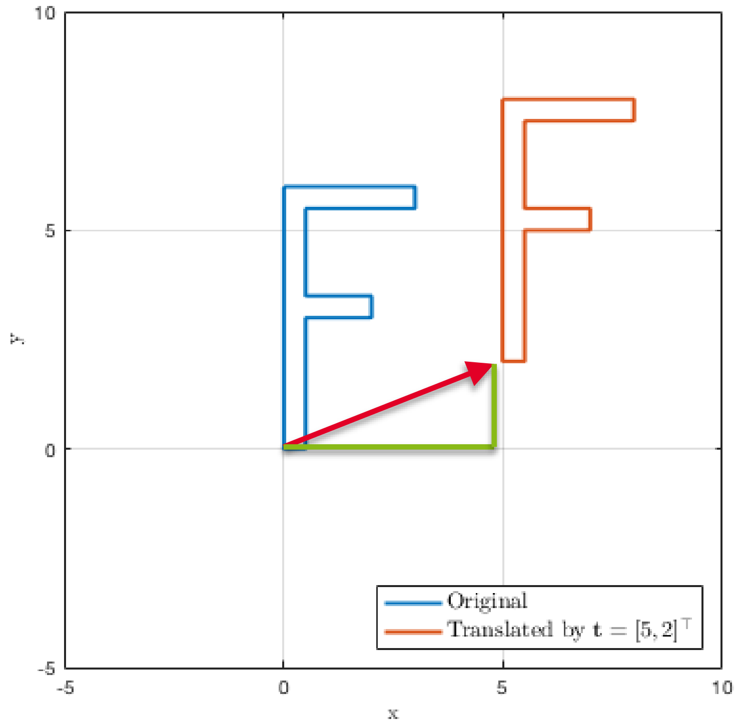
$$x = x + t_x$$

$$y = y + t_y$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



Distance between two points in 2D:

$$D(\mathbf{x}, \mathbf{x}') = \sqrt{(x - x')^2 + (y - y')^2} = \sqrt{(\mathbf{x} - \mathbf{x}')^\top}$$

Rotation:

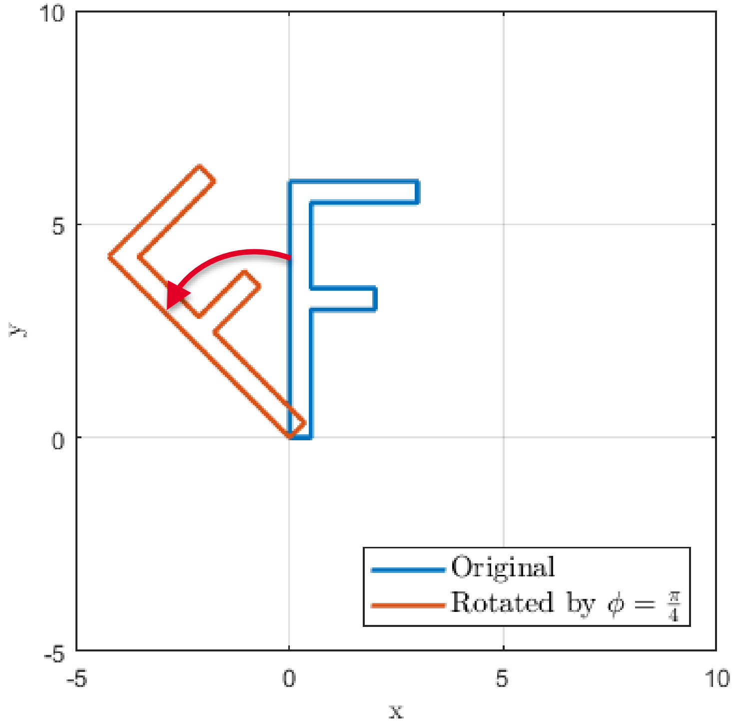
$$x' = \cos(\phi)x - \sin(\phi)y$$

$$y' = \sin(\phi)x + \cos(\phi)y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$



Not every matrix can be considered a rotation matrix.

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Rotation matrices:

- Are orthogonal:

$$\mathbf{R}\mathbf{R}^{-1} = \mathbf{R}\mathbf{R}^{\intercal} = \mathbf{I}$$

- Have determinant equal to 1:

$$\det(\mathbf{R}) = 1$$

Scaling:

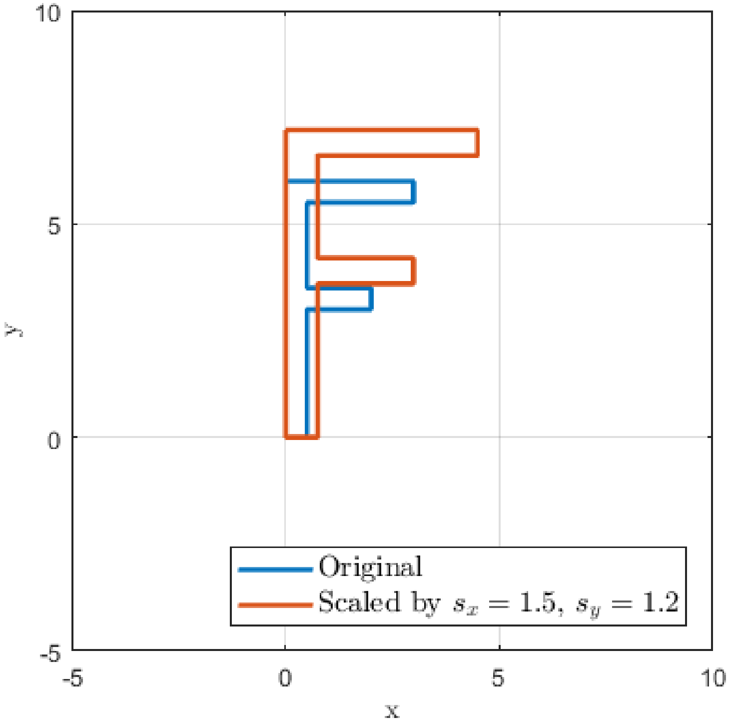
$$x' = s_x x$$

$$y' = s_y y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{S}\mathbf{x}$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



Scaling (on whiteboard)

Rotation (on whiteboard)

Shearing:

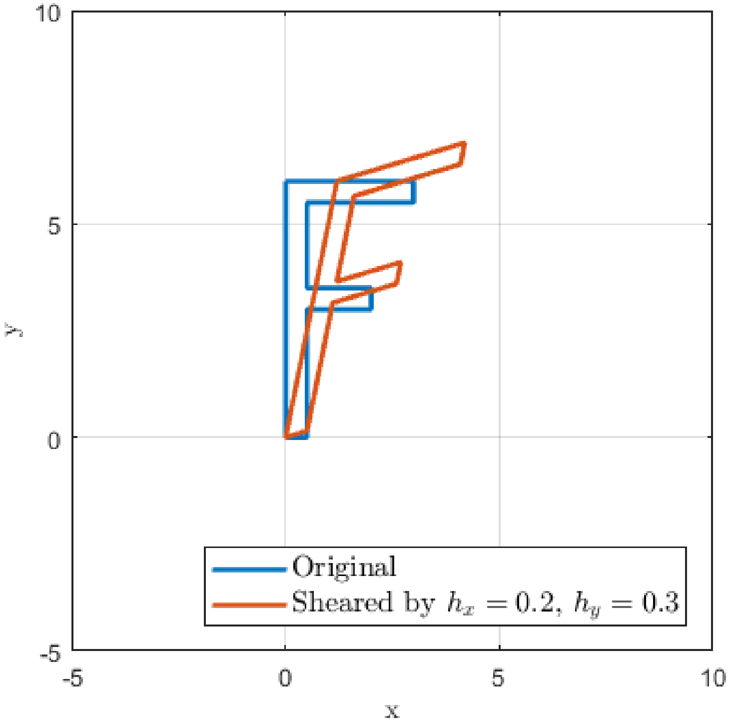
$$x' = x + h_x y$$

$$y' = h_y x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{H} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix}$$



Reflection:

Horizontal:

$$x' = -x$$

$$y' = y$$

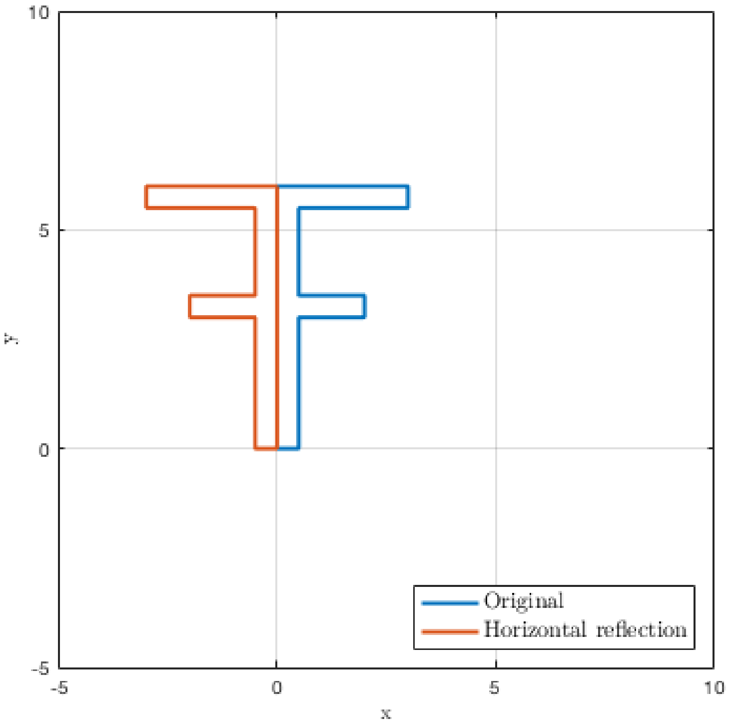
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Vertical:

$$x' = x$$

$$y' = -y$$

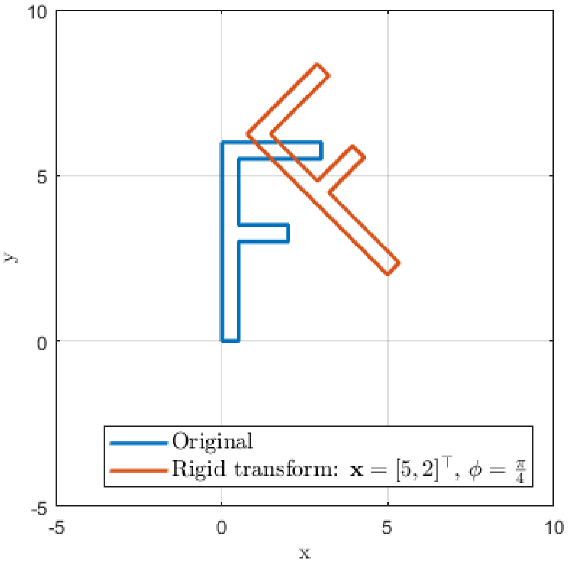
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Composition of transformations:

Rotation + translation (rigid):

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

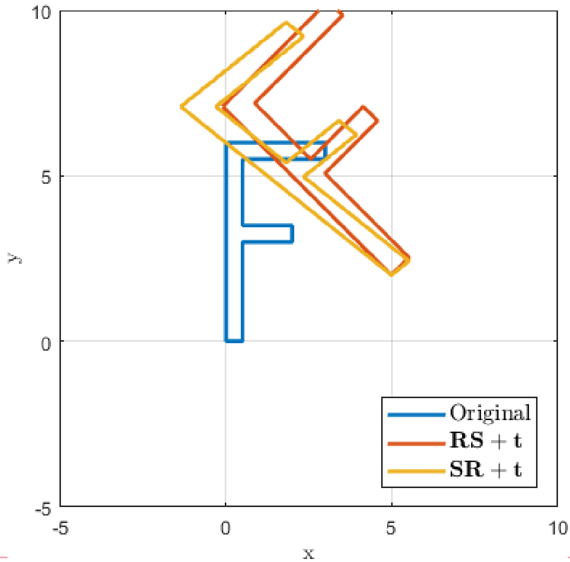


Transformations can be combined by multiplying the transformation matrices.

Rotation, scaling + translation:

$$\mathbf{x}' = \mathbf{R}\mathbf{S}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \mathbf{S}\mathbf{R}\mathbf{x} + \mathbf{t}$$



Note that matrix multiplication is not commutative:

$$\mathbf{T_1 T_2 x} \neq \mathbf{T_2 T_1 x}$$

First scaling, then rotation, then translation:

$$\mathbf{x}' = \mathbf{RSx} + \mathbf{t}$$

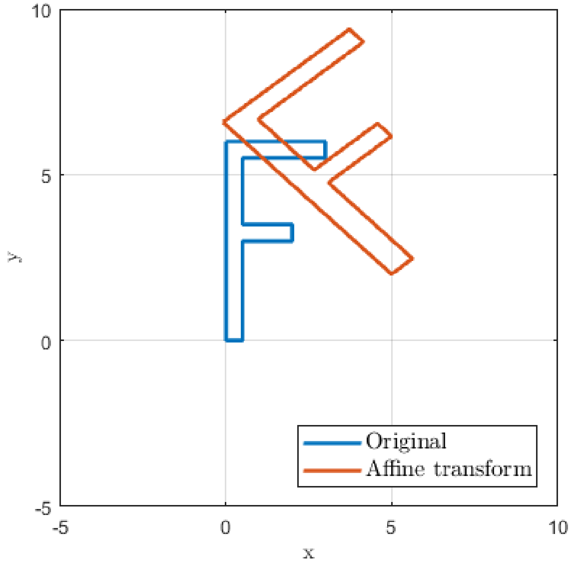
First rotation, then scaling, then translation:

$$\mathbf{x}' = \mathbf{SRx} + \mathbf{t}$$

Affine transformation (no restriction on the transformation parameters):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}$$



It can be considered as a composition of any combination of rotations, scalings, shearings, reflections + translations.

Note that the affine transformation has **6 parameters**: 2×2 transformation matrix and 2×1 translation vector.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

The combination of rotation, scaling, shearing, reflection + translation has **9 parameters**: 1 rotation angle, 2 scaling parameters, 2 shearing parameters, 2 reflection parameters and 2×1 translation vector.

$$\begin{bmatrix} \phi & s_x & s_y & h_x & h_y & r_x & r_y & t_x & t_y \end{bmatrix}$$

However, the first 7 parameters are not **independent**.

The first parameterization is more compact, the second more human-readable.

Affine transformation in 2D has only **6 degrees of freedom**.

In medical image registration, **reflections do not usually occur,** and it can be very problematic if two images are incorrectly registered with a reflection (e.g. can cause a surgical procedure to be performed on the wrong side of the body).

Thus, reflections should be excluded from affine registration.

$$\begin{bmatrix} \phi & s_x & s_y & h_x & h_y & t_x & t_y \end{bmatrix}$$

When using the unrestricted transformation matrix, a check for reflection can be made by examining $\det(\mathbf{A})$. If a reflection has occurred $\det(\mathbf{A}) < 0$.

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}$$

A transformation matrix and a translation vector can be combined when using homogeneous coordinates:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This largely simplifies the notation and implementation of complex transformations.

Example:

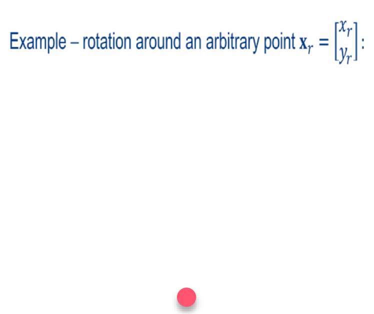
$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix}$$

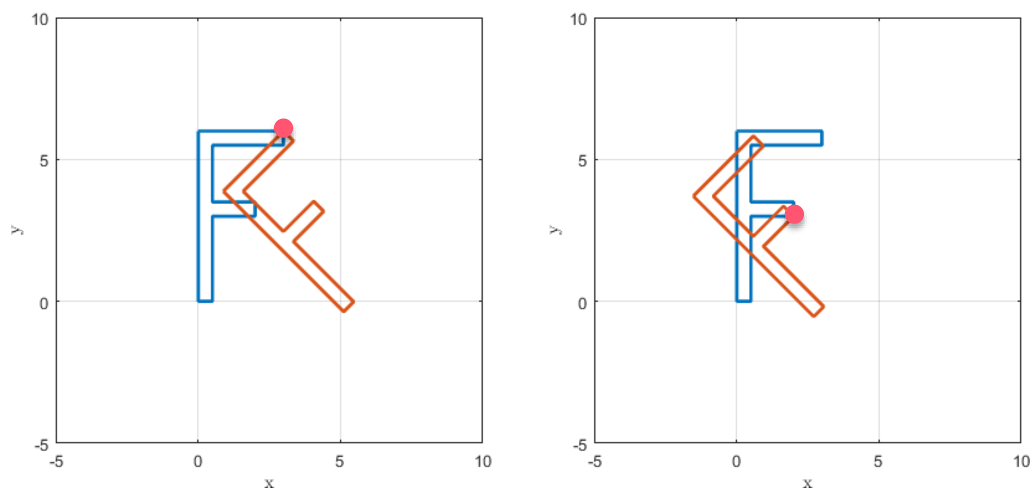
$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_1\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x,1} + t_{x,2} \\ 0 & 1 & t_{y,1} + t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

Example – rotation around an arbitrary point $\mathbf{x}_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$





Inverse transformation can be achieved by taking the inverse of the transformation matrix:

$$\mathbf{T} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Affine transformation in 3D:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How many rotation angles in 3D?

How many degrees of freedom?

Non-linear transformations:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

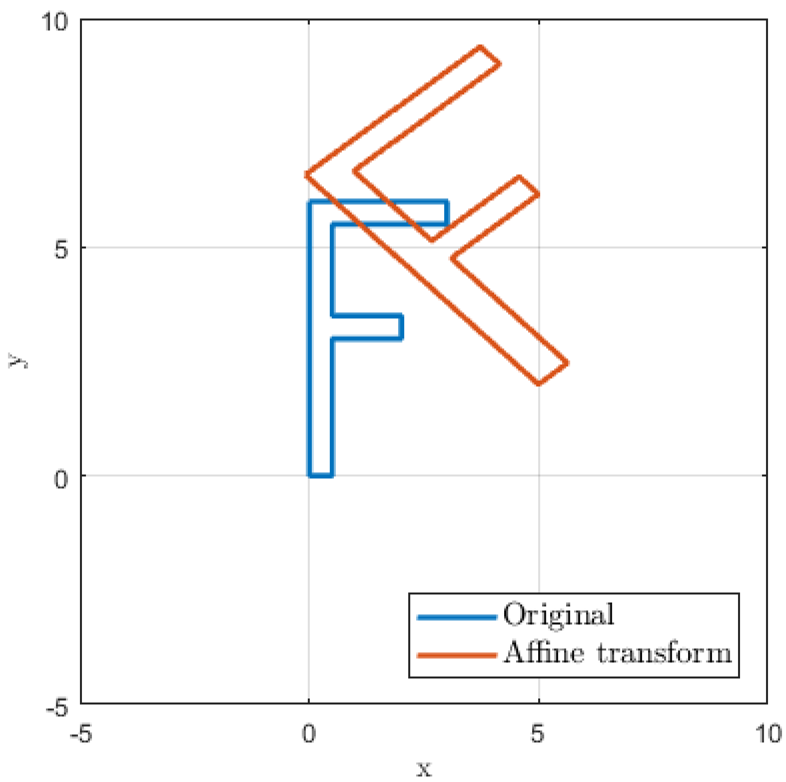
$$x' = ax + by + t_x$$
$$y' = cx + dy + t_y$$

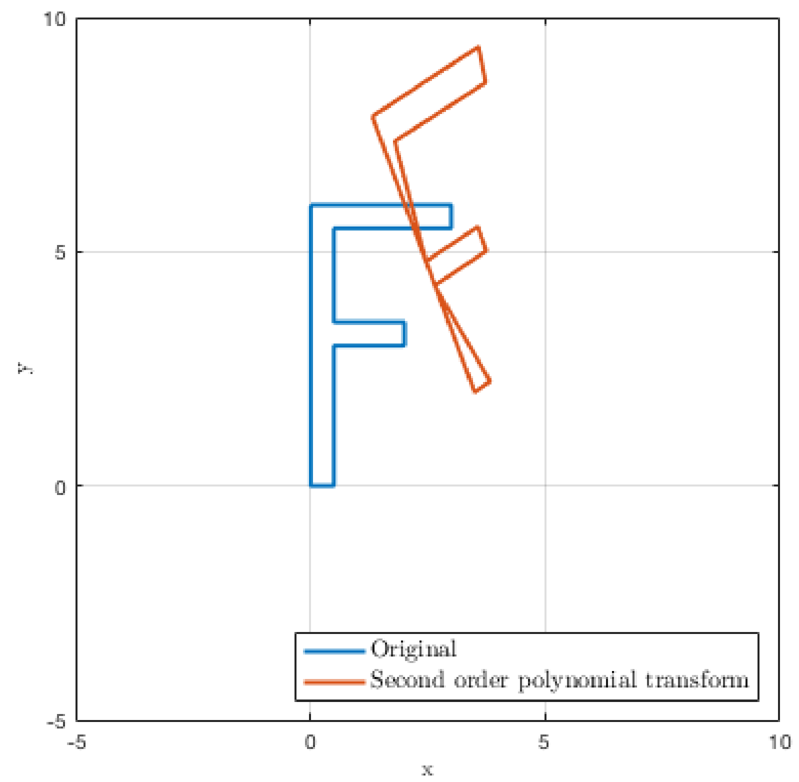
Linear polynomial

$$x' = ax + by + t_x + u_1x^2 + u_2y^2 + u_3xy \ldots$$
$$y' = cx + dy + t_y + v_1x^2 + v_2y^2 + v_3xy \ldots$$

Higher order polynomial

Non-linear transformations:



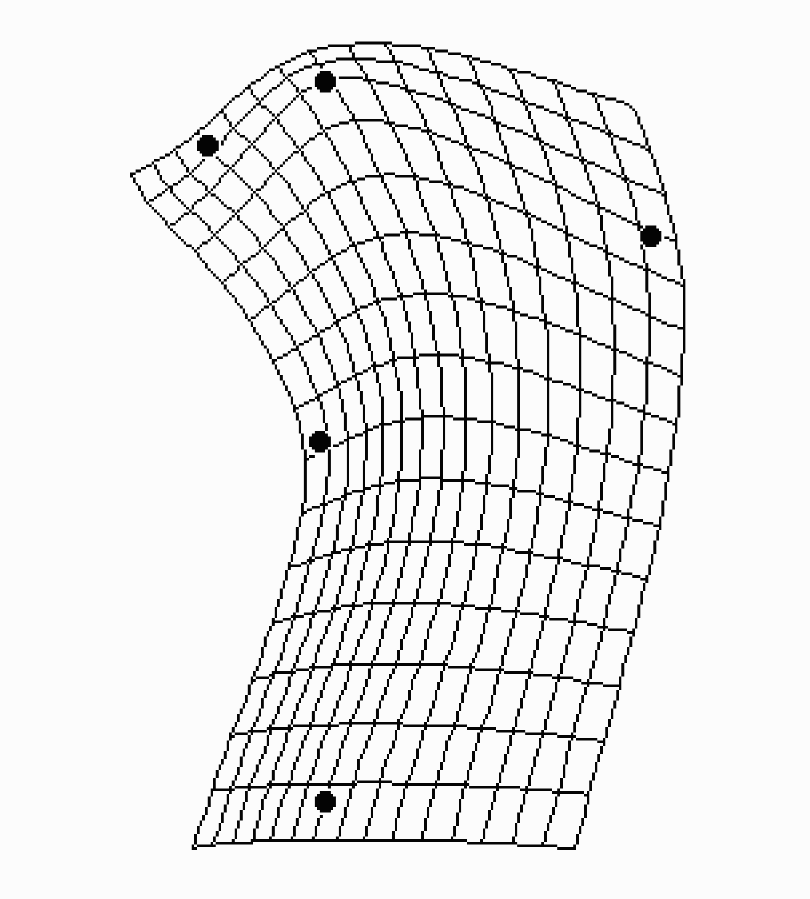


Non-linear transformations: thin-plate spline

$$x' = ax + by + t_x + \sum_{i=1}^{N} u_i r_i^2 \ln r_i^2$$

$$y' = cx + dy + t_y + \sum_{i=1}^{N} v_i r_i^2 \ln r_i^2$$

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2$$



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Thank you

Next: Image transformation, point-based registration