Random Matrix

Spectrum estimation for large dimensional covariance matrices using random matrix theory

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Content

Background introduction

For i.i.d random vector $X_1 \ldots, X_n \in R^p$,and that the covariance of X_i is Σ_p. We call X the data matrix whose rows are the X_i's. How to estimate the eigenvalues of the population covariance matrix?

When p is fixed, n goes to ∞:

(Anderson,1963):Eigenvalues of the sample covariance matrix $S_p = (X - \overline{X})'(X - \overline{X})/(n-1)$ are good estimators of eigenvalues of Σ_p

when comes to to "large n, large p":

For example, we consider p/n $\rightarrow$ r, $\Sigma_p = Id_p$, if X_i i.i.d and have a forth moment, we know that

$$l_1 \rightarrow (1+\sqrt{r})^2 a.s.$$

Random Matrix: From vectors to measures

Suppose we have a vector $(y_1, \ldots, y_p)$ in R_p. We can associate to it the following measure:

$$dG_p(x) = \frac{1}{p}\sum_{i=1}^{p} \delta_{y_i}(x)$$

We denote by H_p the spectral distribution of the population covariance matrix Σ_p, i.e the measure associated with the vector of eigenvalues of Σ_p.
Similarly, we denote by F_p the measure associated with the eigenvalues of the sample covariance matrix S_p.

Random Matrix: Convergence

Notion of convergence: Weak convergence of probability measures.
The **Stieltjes transform** of a measure G on R is defined as

$$m_G(z) = \int \frac{dG(x)}{x - z}, z \in C^+$$

Properties:
If G_n is a sequence of probability measures and $m_{Gn}(z)$ has a (pointwise) limit m(z) for all $z \in C^+$, then there exists a probability measure G with Stieltjes transform $m_G = m$ if and only if $\lim_{y\to\infty} -iym(iy) = 1$. If it is the case, G_n converges weakly to G.

Random Matrix: the Marčenko-Pastur equation

X:n×p data matrix. $S_p = X'X/n$ H_p: the population spectral distribution
m_{F_p} :the Stieltjes transform of the spectral distribution, F_p, of S_p.
$v_{F_p}(z) = (1 - p/n)\frac{-1}{z} + \frac{p}{n} m_{F_p}(z)$,which is actually the Stieltjes transform of the spectral distribution XX'/n

Theorem

Suppose the data matrix X can be written $X = Y\Sigma_p^{\frac{1}{2}}$, where Σ_p is a p×p positive definite matrix and Y is an n×p matrix whose entries are i.i.d (real or complex), with $E(Y_{i,j}) = 0$, $E(|Y_{i,j}|^2) = 1$ and $E(|Y_{i,j}|^4) < \infty$. Assume that H_p converges weakly to a limit denoted H_∞.

Random Matrix: the Marčenko-Pastur equation

Then, when $p, n \to$, and $p/n \to r$, $r \in (0, \infty)$,
1.$v_{F_p}(z) \to v_\infty(z)$ a.s, where $v_\infty(z)$ is a deterministic function.
2. $v_\infty(z)$ satisfies the equation

$$-\frac{1}{v_\infty(z)} = z - r\int \frac{\lambda dH_\infty(\lambda)}{1+\lambda v_\infty(z)}, \forall z \in C^+$$

3. The previous equation has one and only one solution which is the Stieltjes transform of a measure.

Formulation of the estimation problem

Goal:
estimate the population eigenvalues
Strategy:
1. use the Marčenko-Pastur equation to estimate the measure $H_\infty.(\hat{H})$
2. estimate λ_i as the i-th quantile of our estimated distribution $\hat{H}_\infty$.

Fact:
since we are considering fixed distribution asymptotics, our estimate of H_∞ will serve as our estimate of H_p, so $\hat{H}_\infty = \hat{H}_p$.

Formulation of the estimation problem

$$-\frac{1}{v_\infty(z)} = z - r\int \frac{\lambda dH_\infty(\lambda)}{1+\lambda v_\infty(z)}, \forall z \in C^+$$

Step 1:Estiamte H_∞ from F_p.

Compute eigenvalue of $S_p \rightarrow F_p \rightarrow v_{F_p}(z) \xrightarrow{\{z_j\}_{j=1}^{J_n}} \{v_{F_p}(z_j)\}_{j=1}^{J_n}$

$$\hat{H}_\infty = \arg\min_H L(\{\frac{1}{v_{F_p}(z_j)} + z_j - \frac{p}{n}\int \frac{\lambda dH(\lambda)}{1+\lambda v_{F_p}(z_j)}\}_{j=1}^{J_n})$$

Discretization

Step 2:Discretization.

Naturally, dH can be simply approximated by a weighted sum of point masses:

$$dH(x) \approx \sum_{k=i}^{K} w_k \delta_{t_k}(x)$$

where $\{t_k\}_{k=1}^{K}$ is a grid of points, chosen by us, and w_k's are weights, which satisfies

$$\sum_{k=1}^{K} w_k = 1, w_k \geq 0$$

Discretization

Hence finding a measure that approximately satisfies Equation (M-P) is equivalent to finding a set of weights $\{w_k\}_{k=1}^{K}$,for which we have

$$-\frac{1}{v_{F_\infty}(z_j)} \approx z_j - \frac{p}{n}\sum_{k=1}^{K}\frac{w_k t_k}{1+t_k v_{F_\infty}(z_j)}, \forall j$$

Replace v_∞ by v_{F_p}. Our problem is thus to find $\{w_k\}_{k=1}^{K}$ such taht,

$$-\frac{1}{v_{F_p}(z_j)} \approx z_j - \frac{p}{n}\sum_{k=1}^{K}\frac{w_k t_k}{1+t_k v_{F_p}(z_j)}, \forall j$$

Convex Optimization formulation

Approximation errors

$$e_j = \frac{1}{v_{F_p}(z_j)} + z_j - \frac{p}{n}\sum_{k=1}^{K}\frac{w_k t_k}{1 + t_k v_{F_p}(z_j)}$$

Step 3:Formulating our problem as a convex optimization problem.
"L_∞" version: Find w_k's to

$$Minimize \max_{j=1,\dots,J_n} max\{|Re(e_j)|, |Im(e_j)|\}$$

Convex Optimization formulation

The "translation" of the problem into a convex optimization problem is

$$\min_{(w_1,\ldots,w_K,u)} \quad u$$

$$\forall j, -u \leq Re(e_j) \leq u$$

$$\forall j, -u \leq Im(e_j) \leq u$$

$$\sum_{k=1}^{K} w_k = 1$$

$$w_k \geq 0, \forall j$$

Consistency

Our estimate is consistent in L^∞ sense. The general result is as follows.

Theorem

Under setups in previous thm, $H_p \Rightarrow H_\infty$ *and* $\frac{p}{n} \to \gamma$. $J_1 \cdots \in \mathbb{Z}$ *with limit* ∞ *and* $z_1, \cdots \in \mathbb{C}^+$ *bounded and convergent,* $\hat{H}_p$ *is the solution to*

$$\widehat{H}_p = \arg\min_H \max_{j \leq J_n} \left| \frac{1}{v_{F_p}(z_j)} + z_j - \frac{p}{n} \int \frac{\lambda dH(\lambda)}{1 + \lambda v_{F_p}(z_j)} \right|$$

Then

$$\hat{H}_p \Rightarrow H_\infty \quad a.s.$$

Same thing holds for our estimation which is made over a mixture of diracs.

Idea of Proof

$$\Delta = \frac{1}{v_{F_p}(z_j)} + z_j - \frac{p}{n}\int \frac{\lambda dH(\lambda)}{1+\lambda v_{F_p}(z_j)}$$

Recall Thm1, when $H_p \Rightarrow H_\infty$

- $v_{F_p} \to v_\infty$ a.s. (Thm 1)
- $v_{F_p} \to v_\infty$ uniformly (Prop 1)
- $Im(v_{F_p})$ bounded away from 0 (Prop 2)
- $|\Delta| \downarrow 0$ (Prop 3)
- $\hat{H}_p \Rightarrow H_\infty$ (Lemma)
- $\hat{H}_p$ is fine to be sum of atoms (Cor)

Lemma

Find convergent $\hat{H}_p$ over constraint over convergent sequence of z_i.

Lemma

With $\epsilon_n \downarrow 0$ *and*

$$\forall j \leq J_n \quad \left| \frac{1}{v_{F_p}(z_j)} + z_j - \frac{p}{n} \int \frac{\lambda d\tilde{H}_p(\lambda)}{1 + \lambda v_{F_p}(z_j)} \right| < \varepsilon_n$$

Also, $v_{F_p}(z_j) \to v_\infty(z_j)$ *and both analytic as well as bounded away from reals, then we have convergence of* $\tilde{H}_p(\lambda)$ *to* H_∞.

Proposition

Proposition

Stieltjes transform S_H is Lipschitz by $\frac{1}{u_{min}^2}$ on $\mathbb{C}^+ \cap \{Im(z) > u_{min}\}$

Proposition

Population spectral distribution H_p has a limit H_∞, where all spectra uniformly bounded, then for a ball in $\mathbb{C}^+$,

$$\exists N \quad n > N \quad \Rightarrow \quad \inf_{n, z \in B(z_0, r)} \operatorname{Im}\left(v_{F_p}(z)\right) = \delta > 0$$

where v_{F_p} is the Stieltjes transform of F_p.

Proposition

Proposition

If $\exists$ *N s.t. any n¿N,* $|v_{F_p}(z) - v_\infty(z)| < \epsilon$*, and* $|Im(v_\infty(z))| > u_{min}$*, then for* $\epsilon < \frac{u_{min}}{2}, \exists N' \in \mathbb{N}, \forall z \in B(z_0, r), \forall n > N'$.

$$\left| \frac{1}{v_{F_p}(z)} + z - \frac{p}{n} \int \frac{\lambda dH_\infty(\lambda)}{1 + \lambda v_{F_p}(z)} \right| < 2\varepsilon \frac{1 + 2\gamma}{u_{\min}^2}$$

Consistency of Proposed Algorithm

Only need to show Proposition 3 holds for sum of atoms.

Corollary

Restrict $\hat{H}_p$ over measures which are sums of atoms, the locations of which are restricted to belong to a grid (depending on n) whose step size is going to 0 as $n \to \infty$

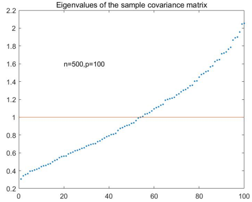
Simulations

Case 1:Identity covariance:Σ=Id

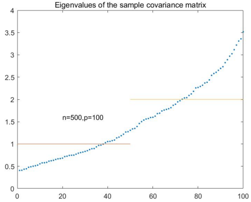
Case 2:"Two spike" spectrum: Σ has d/2 eigenvalues equal to 1 and d/2 equal to 2

Case 2:covariance $\Sigma = T$ where $T_{i,j} = 0.3^{|i-j|}$is a d×d Toeplitz matrix

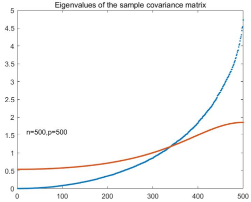
eigenvalues of sample covarience matrix



(a) Case1

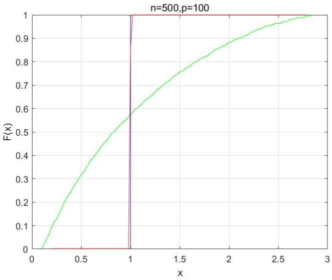


(b) Case2

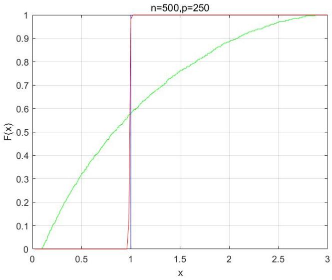


(c) Case3

Case1

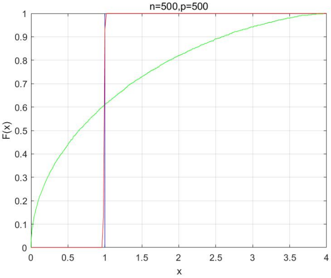


(d) n=500,p=100

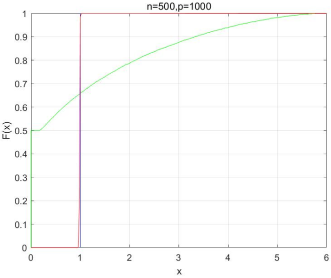


(e) n=500,p=250

Case1

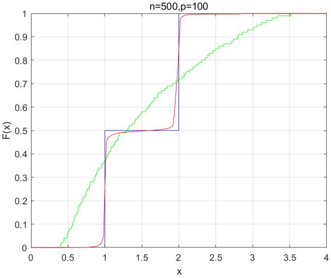


(f) n=500,p=500

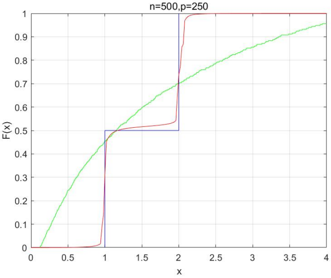


(g) n=500,p=1000

Case2

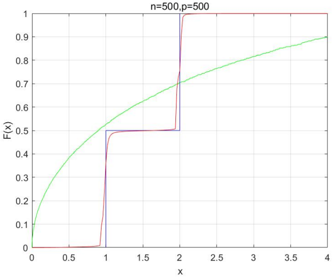


(h) n=500,p=100

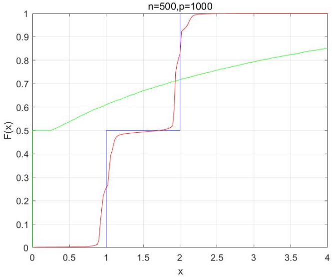


(i) n=500,p=250

Case2

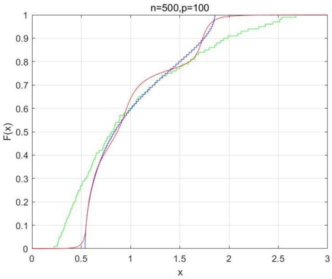


(j) n=500,p=500

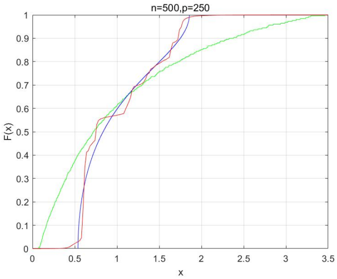


(k) n=500,p=1000

Case3

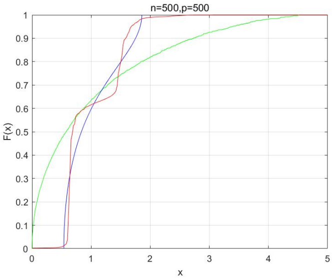


(l) n=500,p=100

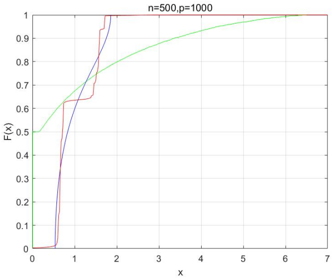


(m) n=500,p=250

Case3

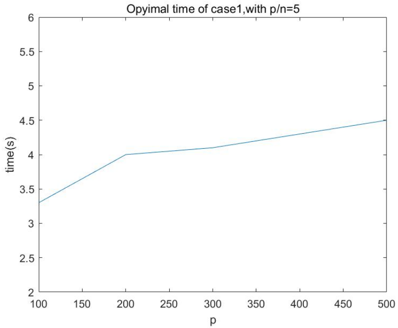


(n) n=500,p=500



(o) n=500,p=1000

Optimal time



(p) case1,Average optimal time

Conclusion

Advantages:

1.The method turn the problem of estimating the population spectrum distribution to a convex optimal problem, which is much easier to solve.

2.The algorithm is free to add some regularization and constraints to make the estimator have more good properties.

3. The method provide us a non-linear shrinkage of eigenvalues.

4. This method does not requie strong assumptions, so it is capable to deal with nonstructured issues.

Conclusion

Limitations:

1.The algorithm implement over $z \in C^+$.The deeper we go into C^+,the more "smoothed-out" is the Stieltjes transform, as it is an analytic function; therefore, the more information is lost. And another availiable method which optimized over the Marcenko–Pastur law on the real line was propose by Ledoit and Wolf(2012).

2.

Thanks for watching