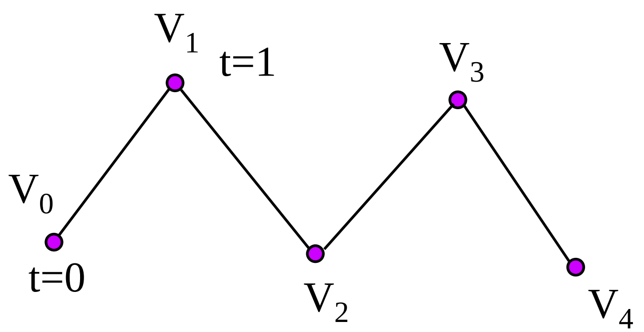
# 6. Linearna interpolacija, krivulje

## 6.1. LINEARNA INTERPOLACIJA

- parametarska jednadžba pravca kroz dvije točke

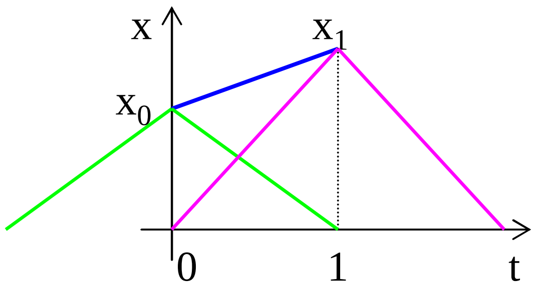
$$\mathbf{V} = \mathbf{V}_0 + (\mathbf{V}_1 - \mathbf{V}_0)\,t = (1-t)\mathbf{V}_0 + t\,\mathbf{V}_1 = f_0(t)\mathbf{V}_0 + f_1(t)\mathbf{V}_1$$

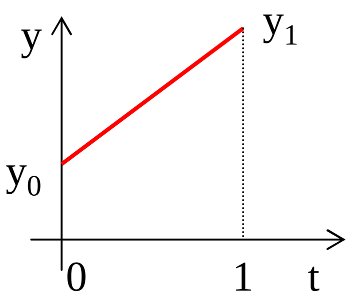



```
lerp (t, v0, v1) {
            return v0 + t*(v1-v0);
    //  ili   return (1-t)*v0 + t*v1;
}
```

$f_0(t),\ f_1(t)$ težinske funkcije

$f_0(t) + f_1(t) = 1 \Rightarrow$ baricentrične koordinate



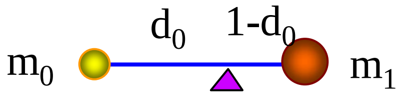




za zadane udaljenosti $d_0$ i $(1-d_0)$

odrediti težinske funkcije $m_i$

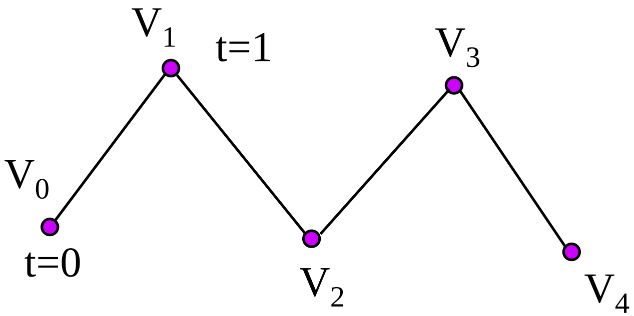
težište $m_0 d_0 = m_1(1-d_0)$

$\rightarrow$ $m_0 = (1-d_0)$, $m_1 = d_0$



LINEARNA INTERPOLACIJA

- parametarska jednadžba pravca kroz dvije točke

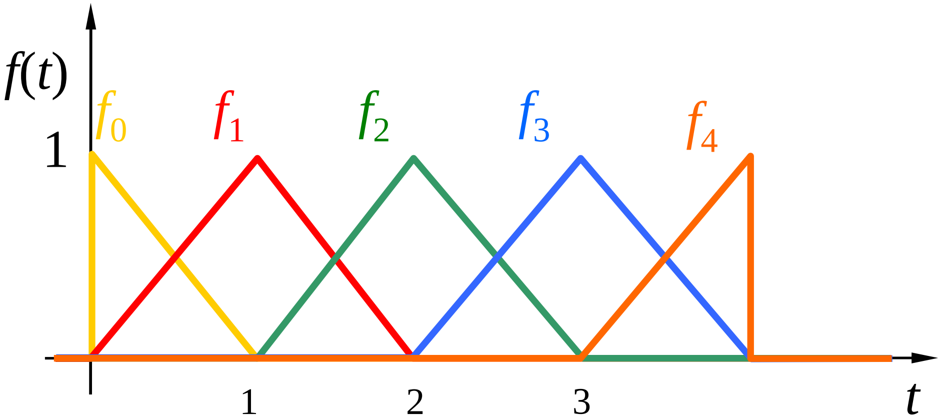



- po odsječcima linearna interpolacija

$$\mathbf{V} = \sum_i f_i(t)\mathbf{V}_i$$

$f_i(t)$ težinske funkcije

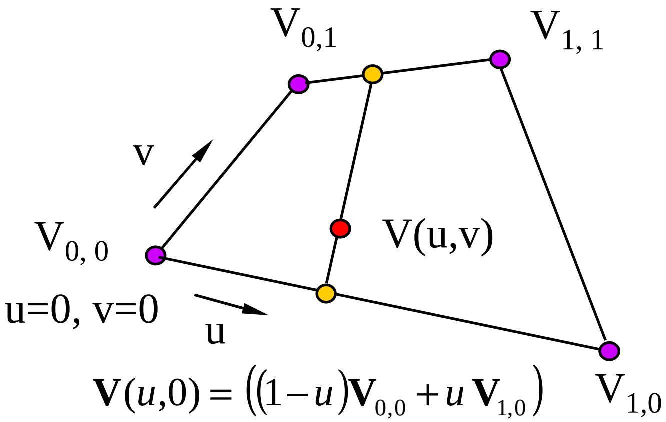
$$\sum_i f_i(t) = 1$$



BILINEARNA INTERPOLACIJA

- parametarska jednadžba kroz četiri točke

$$\mathbf{V}(u,1) = \left(\left(1-u\right)\mathbf{V}_{0,1} + u\,\mathbf{V}_{1,1}\right)$$




$$\mathbf{V}(u,0) = \left(\left(1-u\right)\mathbf{V}_{0,0} + u\,\mathbf{V}_{1,0}\right)$$

$$\mathbf{V}(u,v) = \mathbf{V}_{u,0}\,\left(1-v\right) + \mathbf{V}_{u,1}\,v$$

$$\mathbf{V}(u,v) = \left(\left(1-u\right)\mathbf{V}_{0,0} + u\,\mathbf{V}_{1,0}\right)\left(1-v\right) + \left(\left(1-u\right)\mathbf{V}_{0,1} + u\,\mathbf{V}_{1,1}\right)v$$

$$\mathbf{V}(u,v) = (1-u)(1-v)\mathbf{V}_{0,0} + u\,(1-v)\mathbf{V}_{1,0} + (1-u)v\mathbf{V}_{0,1} + u\,v\mathbf{V}_{1,1}$$

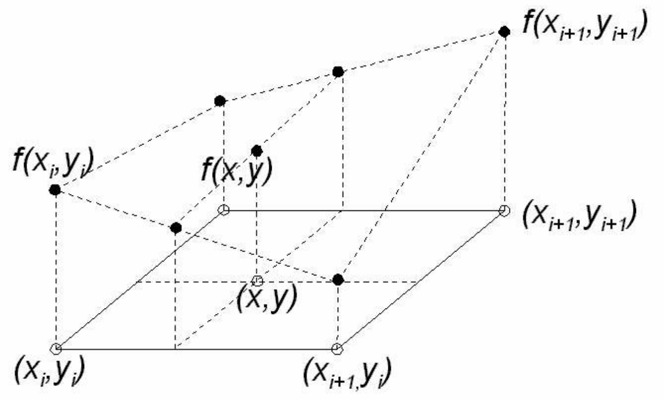
http://www.cs.technion.ac.il/~cs234325/Applets/NewApplets/experiments/interpolation.html

Primjena npr. interpolacija boje http://micro.magnet.fsu.edu/primer/java/digitalimaging/processing/panscrollzoom/index.html

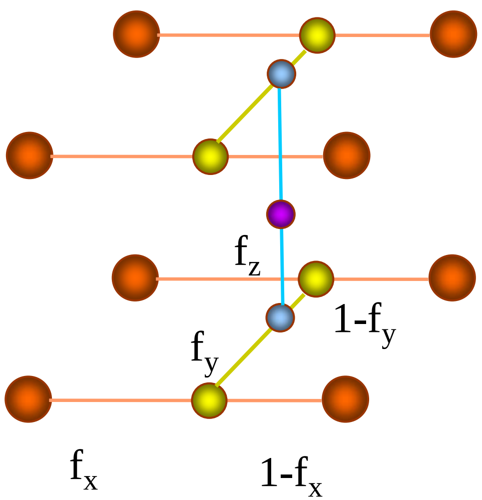
# BILINEARNA INTERPOLACIJA

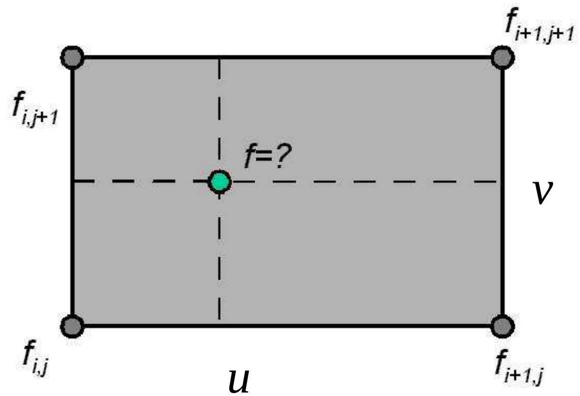
– bilinearna interpolacija ***nije*** linearna (nije ravnina)

$$\mathbf{V}(u,v) = (1-u)(1-v)\mathbf{V}_{0,0} + u(1-v)\mathbf{V}_{1,0} + (1-u)v\mathbf{V}_{0,1} + u\,v\mathbf{V}_{1,1}$$




– trilinearna interpolacija proširenje bilinearne



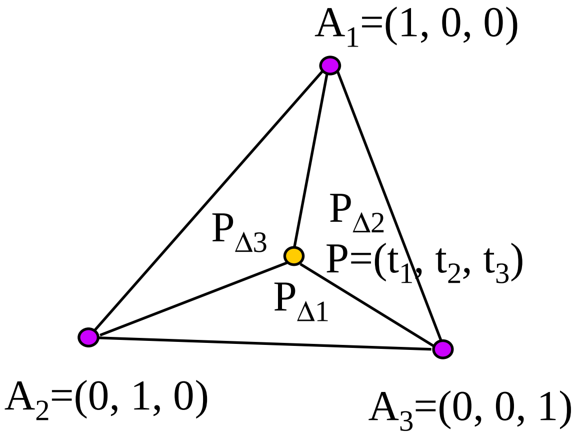




$$u = \frac{x - x_i}{x_{i+1} - x_i}, \quad v = \frac{y - y_i}{y_{i+1} - y_i}$$

## BARICENTRIČNE KOORDINATE

– točka P ima normalizirane baricentrične koordinate $P(t_1, t_2, t_3)$ - Baricentar,

$t_1 : t_2 : t_3 = P_{\Delta 1} : P_{\Delta 2} : P_{\Delta 3}$ $t_1 + t_2 + t_3 = 1$ (ako su u $t_i$ mase u $A_i$, $P_i$ je centar masa)

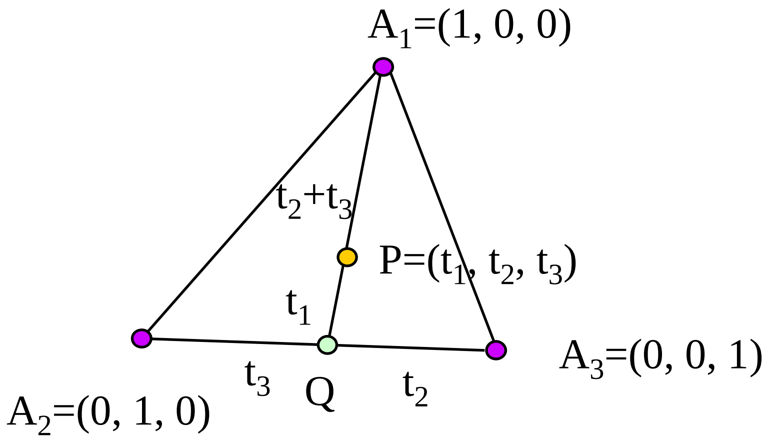



- točka je unutar trokuta ako su $0 \le t_1, t_2, t_3 \le 1$
- ako je neki $t_i$=0 točka P pada na brid

- određivanje baricentričnih koordinata

  kroz točke $A_1$ i P povučemo pravac $\rightarrow t_3, t_2$

– http://www.vis.uni-stuttgart.de/~kraus/LiveGraphics3D/cagd/chap3fig5.html



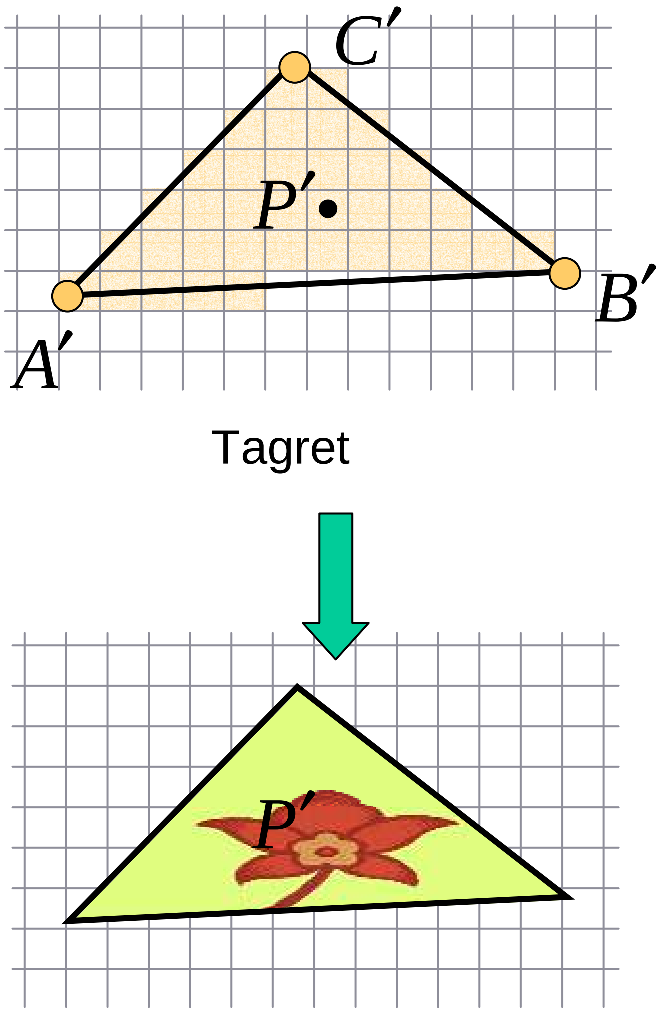

$$t_1 = \frac{P_{\Delta 1}}{P_{A_1 A_2 A_3}}$$

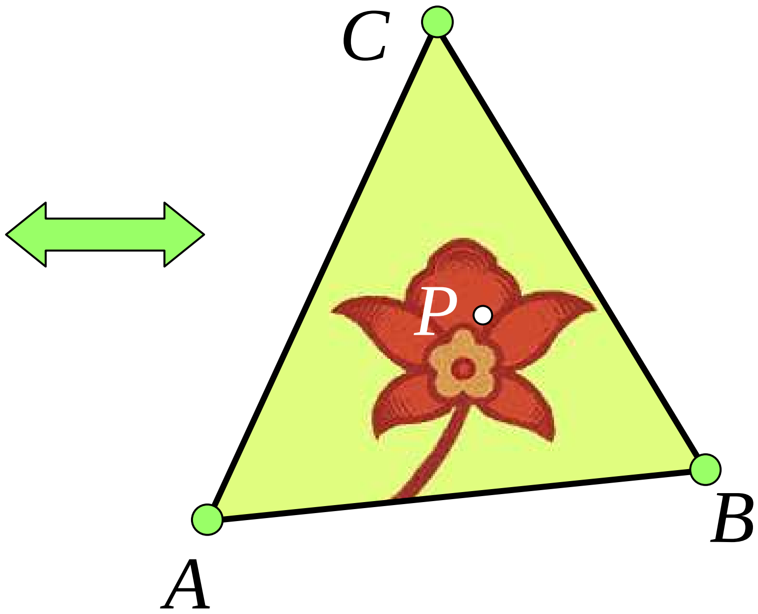
$$t_2 = \frac{P_{\Delta 2}}{P_{A_1 A_2 A_3}}$$

$$t_3 = \frac{P_{\Delta 3}}{P_{A_1 A_2 A_3}}$$

$$P = t_1 A_1 + t_2 A_2 + t_3 A_3$$

Primjer primjena baricentričnih koordinata: ispravno preslikavanje teksture, deformacija konveksnih objekata – preobražaj (mesh deformation, morphing)




baricentrične koordinate $t_1$, $t_2$, $t_3$ točke P’ obzirom na A’, B’, C’ određuju boju Color(P’) = Color($t_1$ A + $t_2$ B + $t_3$ C)

# 6.2. KRIVULJE

- postupak projektiranja krivulje

  – analitički izraz izvorne krivulje u pravilu je nepoznat

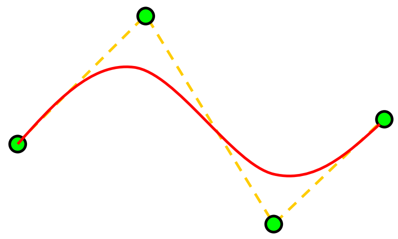
  – poznato je
    - koordinate u nekim točkama
    - nagibi, zakrivljenost ili izvijanje u nekim točkama

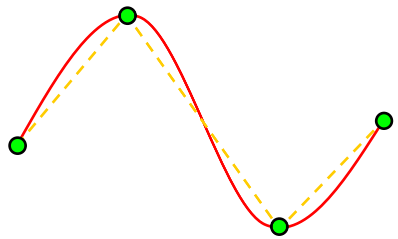
    $\Rightarrow$ modeliranje

  – opis segmenta krivulje
  – segmentiranje
    - povezivanje segmenata uz ostvarivanje kontinuiteta između segmenata

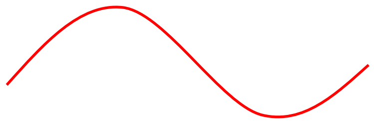
## PODJELA KRIVULJA

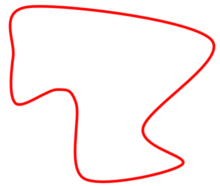
- aproksimacijske
- interpolacijske





- otvorene
- zatvorene





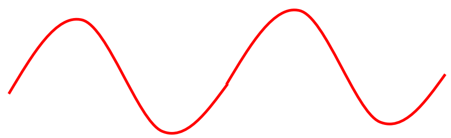
- razlomljene
- nerazlomljene

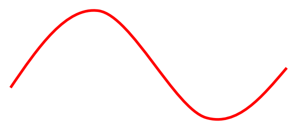
$$x(t)=\frac{a_1t^3+b_1t^2+c_1t+d_1}{at^3+bt^2+ct+d}$$

$$x(t)=a_1t^3+b_1t^2+c_1t+d_1$$

- periodične
- neperiodične

(periodičnost težinskih funkcija)

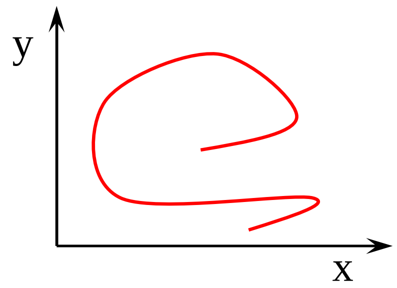




# POŽELJNA SVOJSTVA KRIVULJA

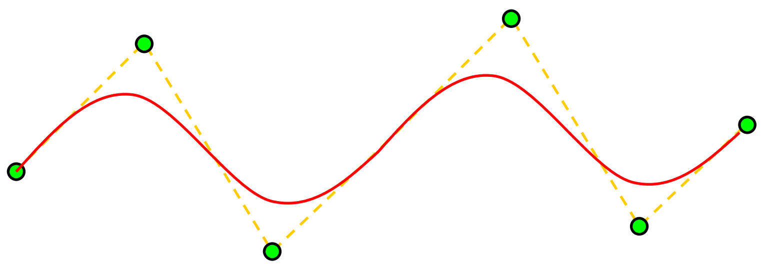
- višestruke vrijednosti

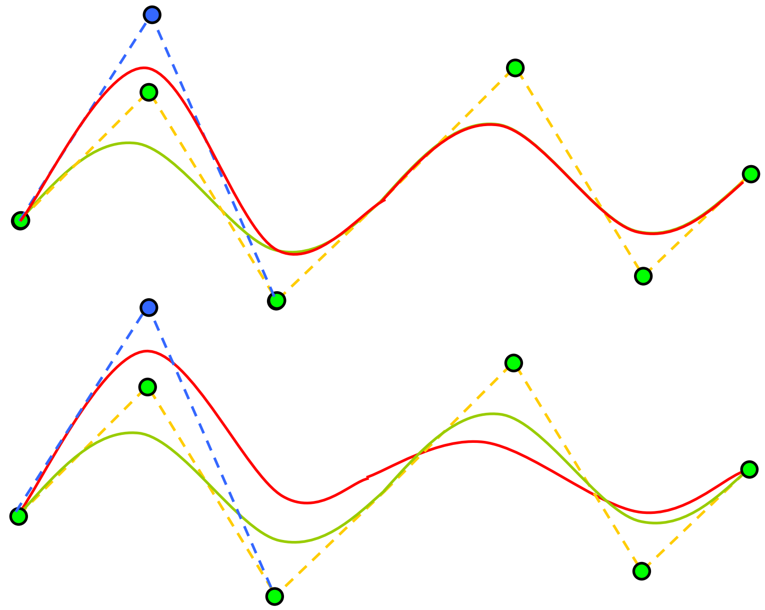
http://www.math.aau.dk/~raussen/VIDIGEO/GEOLAB/3Dparametrization.html



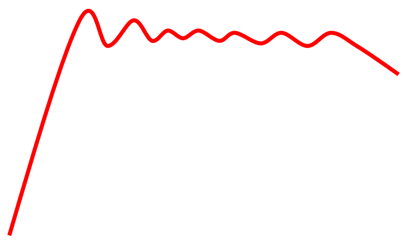

- neovisnost o koordinatnom sustavu (Kartezijev, polarni)

- lokalni nadzor

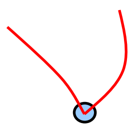




- smanjenje varijacije - kod visokog stupnja polinoma može se javiti titranje krivulje

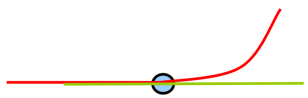


- kontrola reda neprekinutosti
- http://www.slu.edu/classes/maymk/Applets/Derivatives2.html



$C^0$

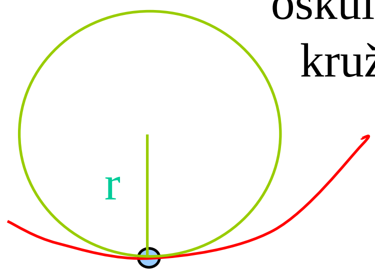
ista vrijednost koordinate



$C^1$

ista tangenta

oskulatorna kružnica

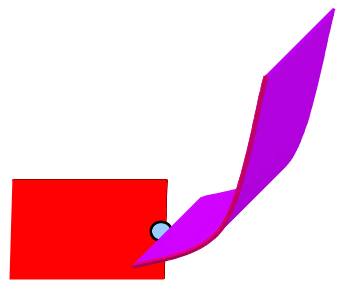



$C^2$

kontinuitet radijusa zakrivljenosti

http://www.ies.co.jp/math/java/calc/curve/curve.html

http://www.vis.uni-stuttgart.de/~kraus/LiveGraphics3D/cagd/chap10fig4.html



$C^3$

kontinuitet u izvijanju

$C^0$ - ista vrijednost koordinate $f(t) = g(t)$

$C^1$ - ista vrijednost derivacije $f'(t) = g'(t)$

$C^2$ - ista vrijednost druge derivacije $f''(t) = g''(t)$

Zakrivljenost krivulje obrnuto je proporcionalna radijusu oskulatorne kružnice. Ako je radijus velik zakrivljenost je mala (i obrnuto).

$C^3$ - ista vrijednost treće derivacije $f'''(t) = g'''(t)$

Osim C kontinuiteta postoje i G kontinuiteti koji zahtijevaju proporcionalnost.

G (geometrijski)

$G^1$ - proporcionalna vrijednost derivacije $f'(t) = k_1 g'(t), k_1>0$

$G^2$ - proporcionalna vrijednost druge derivacije $f''(t) = k_2 g''(t), k_2>0$

$G^3$ - proporcionalna vrijednost treće derivacije $f'''(t) = k_3 g'''(t), k_3>0$

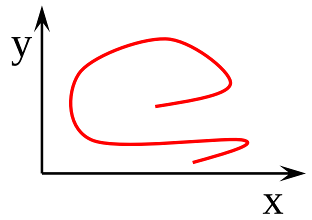
$C^1$ kontinuitet implicira $G^1$ kontinuitet osim kada je vektor tangente [0 0 0]

kod $C^1$ kontinuiteta može doći do promjene smjera, kod $G^1$ ne može.

# ANALITIČKI OPIS PROSTORNIH KRIVULJA

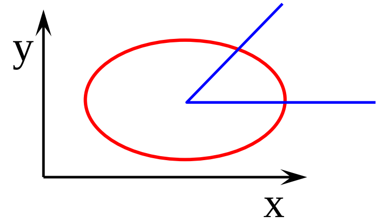
a) eksplicitni oblik - nemogućnost prikaza višestrukih vrijednosti

$$y = f(x), \qquad z = g(x)$$




b) implicitni oblik - za prikaz dijela krivulje trebaju dodatni uvjeti

$$F(x,\ y,\ z) = 0$$

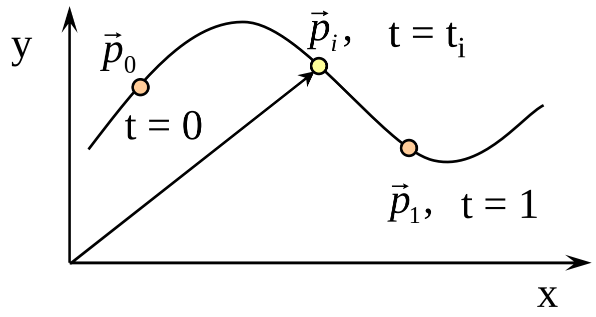



c) parametarski oblik

$$x = \mathrm{x}(t), \quad y = \mathrm{y}(t), \quad z = \mathrm{z}(t).$$

točka na krivulji - vektorska funkcija

$$\mathrm{V}(t) = [\mathrm{x}(t) \quad \mathrm{y}(t) \quad \mathrm{z}(t)].$$




vektor tangente

$$\mathrm{V}'(t) = [\mathrm{x}'(t) \quad \mathrm{y}'(t) \quad \mathrm{z}'(t)].$$

$$\mathrm{V}(t_i) = \vec{p}(t_i) = \vec{p}_{t_i}.$$

# 6.3. SEGMENT KRIVULJE

## 6.3.1. KRIVULJA BEZIERA

Postupak poznat pod imenom krivulje Beziera nezavisno su razvili

- BEZIER 1962. Rénault
- DE CASTELJAU 1959. Citroën

kao polaznu osnovu u CAD sustavima. De Casteljau direktno koristi Bernsteinove polinome.

1970. R. Forest otkriva vezu Bezierovog rada i Bernsteinovih polinoma.

P. Bezier objavljuje svoj rad i krivulje dobivaju ime po njemu.

- aproksimacijske krivulje Beziera
- interpolacijske krivulje Beziera

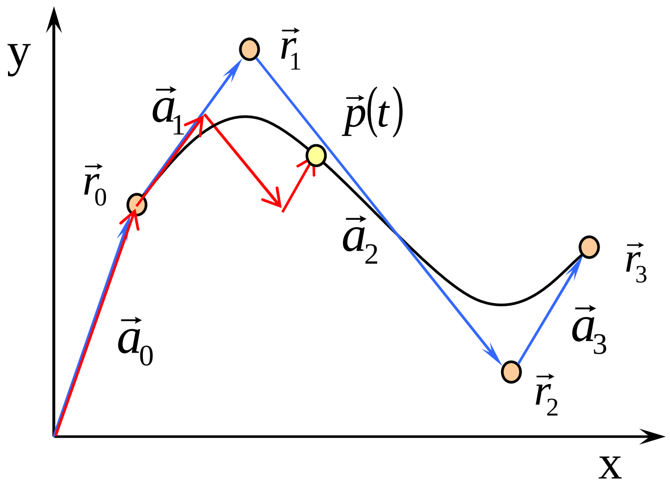
- Bezierove težinske funkcije (Bezier)
- Bernsteinove težinske funkcije (De Casteljau)

APROKSIMACIJSKE KRIVULJE BEZIERA

Prolaze početnom i krajnjom točkom, a ostalima se samo približava.

a) BEZIEROVE TEŽINSKE FUNKCIJE

Korištenje gibanja vrha sastavljenog otvorenog poligona.




$$\boxed{\vec{p}(t) = \sum_{i=0}^{n} \vec{a}_i \, f_{i,n}(t) \qquad t \in [0, 1]}$$

$$\vec{a}_0 = \vec{r}_0 \,,$$

$$\vec{a}_i = \vec{r}_i - \vec{r}_{i-1}, \qquad i = 1..\, n$$

$n+1$ .. broj točaka.

$n$ .. stupanj krivulje.

$a_i$ .. kontrolni poligon.

$p(t)$ .. točka na krivulji - linearna kombinacija $f_{i,n}(t)$ i $a_i$.

$f_{i,n}(t)$ .. težinska funkcija - njena vrijednost pokazuje koliko $i$-ti element poligona pridonosi pripadnoj točki za parametar $t$.

$f_{i,n}(t)$ - težinska funkcija je općenita i mora zadovoljiti niz posebnih uvjeta:

1. <u>početna točka</u> $p(0) = a_0 \;\Rightarrow\; f_{0,n}(0) = 1,$

$$f_{i,n}(0) = 0, \quad i = 1 \,..\, n$$

2. <u>završna točka</u> $p(1) = \Sigma a_i \;\Rightarrow\; f_{i,n}(1) = 1, \quad i = 0 \,..\, n$ zbroj svih vektora

3. <u>osnovni vektor</u> $a_1$ treba biti paralelan s tangentom u početnoj točki

$$p'(0) = k_1 a_1 \Rightarrow f'_{1,n}(0) \neq 0,$$

$$f'_{i,n}(0) = 0, \quad i \neq 1$$

4. <u>osnovni vektor</u> $a_n$ treba biti paralelan s tangentom u završnoj točki.

$$p'(1) = k_n a_n \Rightarrow f'_{i,n}(1) = 0, \quad i = 0 \,..\, n\text{-}1,$$

$$f'_{n,n}(1) \neq 0.$$

5. <u>oskulatorna ravnina</u> u početnoj točki treba biti paralelna s $a_1$ i $a_2$

$$\Rightarrow f''_{1,n}(0) \neq 0,\ f''_{2,n}(0) \neq 0,$$
$$f''_{i,n}(0) = 0,\ \text{inače.}$$

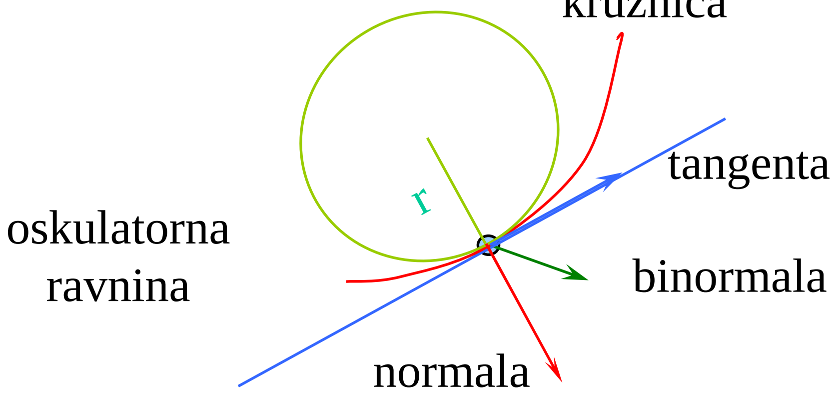
http://www.math.aau.dk/~raussen/VIDIGEO/GEOLAB/apposcplane.html

6. <u>oskulatorna ravnina</u> u završnoj točki treba biti paralelna s $a_{n\text{-}1}$ i $a_n$

$$\Rightarrow f''_{i,n}(1) = 0, \quad i = 0\ ..\ n\text{-}2,$$
$$f''_{n\text{-}1,n}(1) \neq 0,\ f''_{n,n}(1) \neq 0.$$

7. <u>simetričnost</u> težinske funkcije - zamjena početne i završne točke povlači promjenu smjera i redoslijeda vektora.

$$\Rightarrow f_{i,n}(t) = 1\text{-}\ f_{n\text{-}i+1,n}(1\text{-}t)\ , \quad i = 1\ ..\ n.$$



$\Rightarrow$ BEZIEROVE TEŽINSKE FUNKCIJE

$$f_{i,n}(t) = \frac{(-t)^i}{(i-1)!} \frac{d^{(i-1)}\Phi_n(t)}{d^{(i-1)}t}, \quad \Phi_n(t) = \frac{1-(1-t)^n}{-t},$$

gdje $d^{(i-1)}$ je $(i-1)$ derivacija $\qquad i = 1..n$

rekurzivni oblik pogodan za implementaciju na računalu:

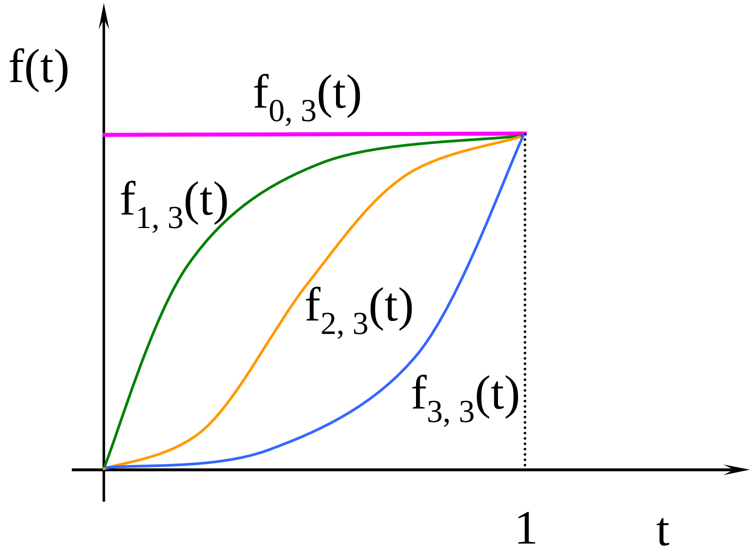
$$f_{i,n}(t) = (1-t)\, f_{i,n-1}(t) + t\, f_{i-1,n-1}(t),$$

uvjeti zaustavljanja rekurzije

$$f_{0,0}(t) = 1, \quad f_{k+1,k}(t) = 0, \quad f_{-1,k}(t) = 1.$$

* PRIMJER

Odrediti Bezierove težinske funkcije ako su zadane četiri točke.




$$\Phi_3(t) = \frac{1-(1-t)^3}{-t} = -3+3t-t^2,$$

$$f_{i,3}(t) = \frac{(-t)^i}{(i-1)!} \frac{d^{(i-1)}\Phi_3(t)}{d^{(i-1)}t},$$

$$f_{0,3}(t) = 1,$$

$$f_{1,3}(t) = 3t-3t^2+t^3,$$

$$f_{2,3}(t) = 3t^2-2t^3,$$

$$f_{3,3}(t) = t^3.$$

$$\vec{p}(t) = \sum_{i=0}^{3} \vec{a}_i\, f_{i,3}(t),$$

$$\vec{p}(t) = \vec{a}_0 + \left(3t-3t^2+t^3\right)\vec{a}_1 + \left(3t^2-2t^3\right)\vec{a}_2 + t^3\vec{a}_3$$

Provjera postavljenih uvjeta na težinsku funkciju:

1. početna točka $\vec{p}(0) = \vec{a}_0$ ,
2. završna točka $\vec{p}(1) = \vec{a}_0 + \vec{a}_1 + \vec{a}_2 + \vec{a}_3$,

$$\vec{p}'(t) = \left(3 - 6t + 3t^2\right) \vec{a}_1 + \left(6t - 6t^2\right) \vec{a}_2 + 3t^2 \vec{a}_3,$$

3. derivacija u početnoj točki $\vec{p}'(0) = 3\vec{a}_1$ ,
4. derivacija u završnoj točki $\vec{p}'(1) = 3\vec{a}_3$ ,

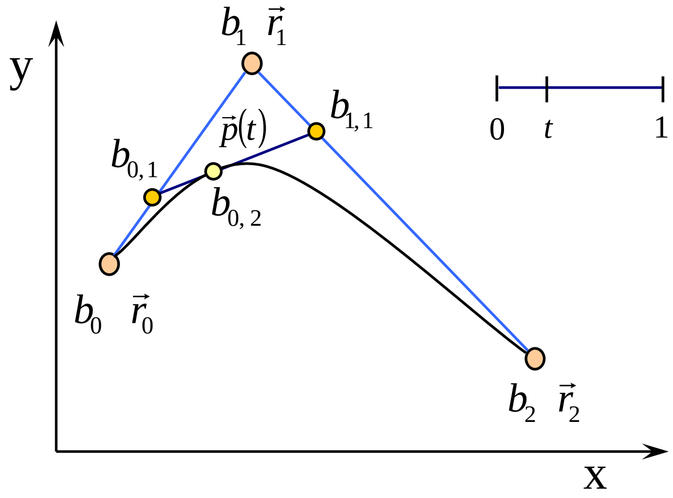
$$\vec{p}''(t) = (-6 + 6t)\, \vec{a}_1 + (6 - 12t)\, \vec{a}_2 + 6t\vec{a}_3,$$

5. derivacija u početnoj točki $\vec{p}''(0) = 6\left(\vec{a}_2 - \vec{a}_1\right)$,
6. derivacija u završnoj točki $\vec{p}''(1) = 6\left(\vec{a}_3 - \vec{a}_2\right)$,

7. simetričnost $f_{1,3}(t) = 1 - f_{3,3}(1 - t)$.

b) BERNSTEINOVE TEŽINSKE FUNKCIJE

De Casteljau - intuitivna geometrijska konstrukcija

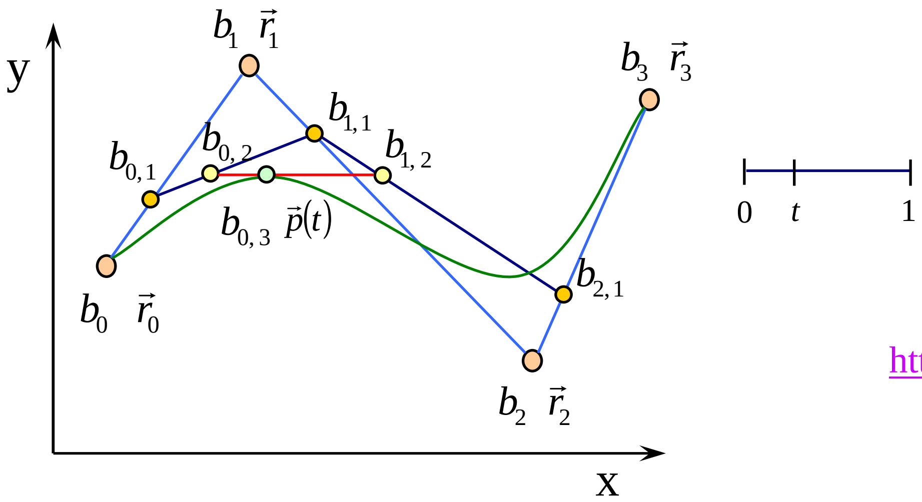



- uzastopne linearne interpolacije:

$$\left.\begin{aligned} b_{0,1} &= (1-t)b_0 + t\,b_1, \\ b_{1,1} &= (1-t)b_1 + t\,b_2, \end{aligned}\right\} \quad b_{0,2} = (1-t)b_{0,1} + t\,b_{1,1}\ ,$$

- uvrstimo:

$$b_{0,2} = (1-t)^2\, b_0 + 2t(1-t)b_1 + t^2 b_2$$




http://saltire.com/applets/spline.htm
http://www.cs.technion.ac.il/~cs234325/

- poopćenje ovog postupka daje De Casteljau-ov algoritam

$$b_{i,r} = (1-t)\, b_{i,r-1} + t\, b_{i+1,r-1}(t), \quad r = 1..n,\ i = 0..n-r,$$

$$b_{i,0}(t) = b_i \qquad \vec{r}_i \ \text{vrhovi kontrolnog poligona,}$$

$$b_{0,n}(t) \qquad \vec{p}(t) \ \text{točka na krivulji.}$$

$$\boxed{\vec{p}(t) = \sum_{i=0}^{n} \vec{r}_i\, b_{i,n}(t) \qquad t \in [0, 1]} \qquad \text{vrijedi} \qquad \sum_{i=0}^{n} b_{i,n}(t) = 1 \qquad t \in [0, 1]$$

$b_{i,n}(t)$ – bazne funkcije – Bernsteinovi polinomi stupnja $n$

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} = \frac{n!}{i!(n-i)!}\, t^i (1-t)^{n-i}$$

Diskretna binomna razdioba:

t – vjerojatnost događaja u svakom od $n+1$ pokušaja

$b_{i,n}$ – vjerojatnost postizanja točno $i$ događaja u $n+1$ pokušaja

* PRIMJER

Odrediti Bernsteinove težinske funkcije ako su zadane četiri točke.

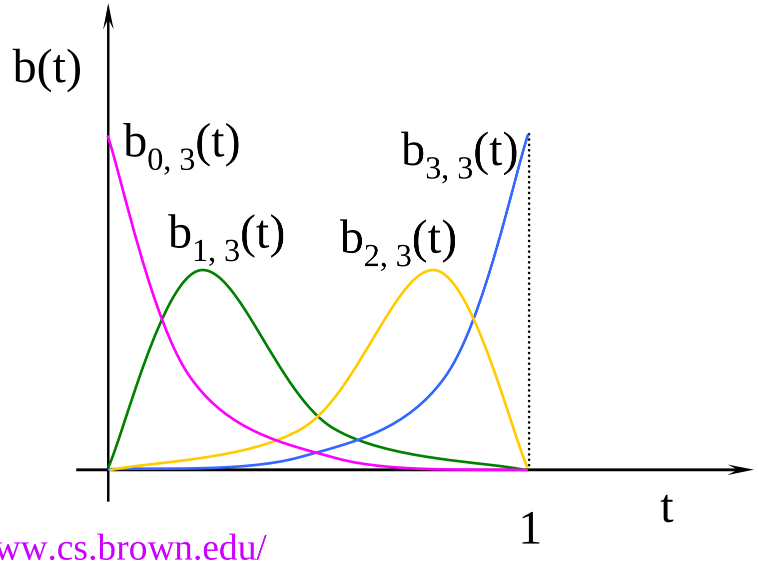
$$b_{i,3}(t) = \frac{3!}{i!(3-i)!} t^i (1-t)^{3-i},$$

$$b_{0,3}(t) = (1-t)^3,$$

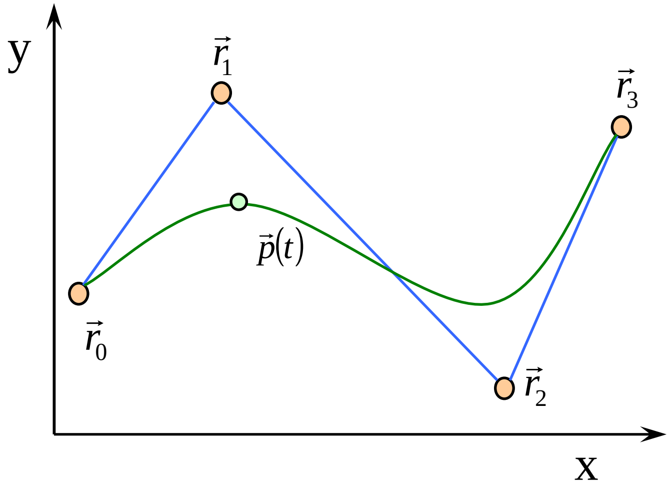
$$b_{1,3}(t) = 3t(1-t)^2,$$

$$b_{2,3}(t) = 3t^2(1-t),$$

$$b_{3,3}(t) = t^3.$$




http://www.cs.brown.edu/




$$\vec{p}(t) = \sum_{i=0}^{3} \vec{r}_i b_{i,3}(t),$$

$$\vec{p}(t) = \vec{r}_0 (1-t)^3 + 3t(1-t)^2 \vec{r}_1 + 3t^2(1-t)\vec{r}_2 + t^3 \vec{r}_3$$

$$\vec{p}'(0) = 3(\vec{r}_1 - \vec{r}_0),$$

$$\vec{p}'(1) = 3(\vec{r}_3 - \vec{r}_2).$$

Matrično:

$$\vec{p}(t)=\begin{bmatrix} t^3 & t^2 & t & t \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \vec{r}_0 \\ \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}.$$

Za tangentu na Bezierovu krivulju opisanu preko Bernsteinovih težinskih funkcija vrijedi:

$$\vec{p}'(0)=n(\vec{r}_1-\vec{r}_0),$$

$$\vec{p}'(1)=n(\vec{r}_n-\vec{r}_{n-1}). \quad n\text{..stupanj krivulje}$$

Veza Bezierovih i Bernsteinovih težinskih funkcija:

$$f_{i,n}(t)=\sum_{j=i}^{n} b_{j,n}(t), \quad i=0..n.$$

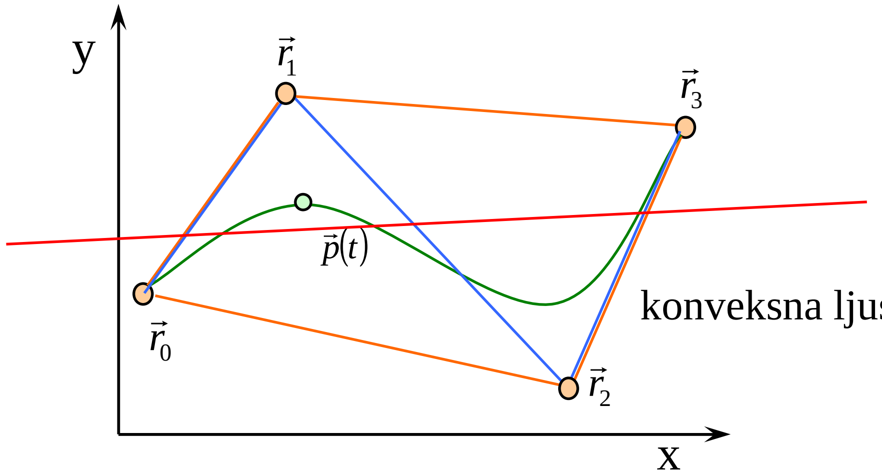
$$\vec{a}_0=\vec{r}_0, \quad \vec{a}_i=\vec{r}_i-\vec{r}_{i-1}, \quad i=1..n,$$

*ili*

$$\vec{r}_0=\vec{a}_0, \quad \vec{r}_i=\vec{a}_i+\vec{r}_{i-1}, \quad i=1..n$$

SVOJSTVA APROKSIMACIJSKIH BEZIEROVIH KRIVULJA

- postoji konveksna ljuska




suma težinskih funkcija je 1

važno kod ispitivanja sjecišta krivulje

(težinske funkcije su nenegativne

imaju točno jedan maksimum)

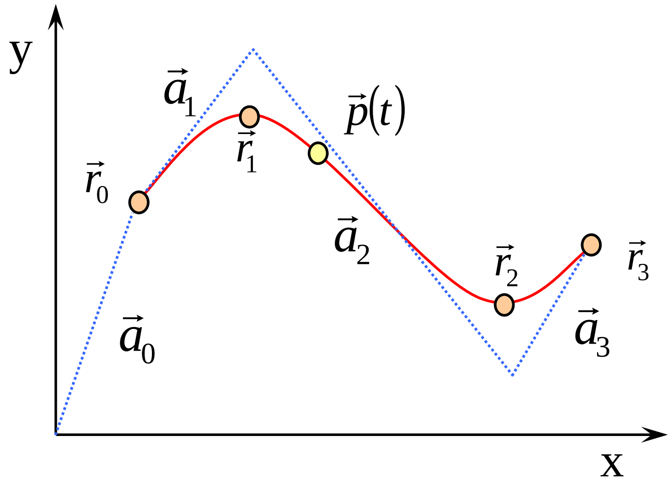
- krivulja nema više valova od kontrolnog poligona

broj sjecišta ravnine i kontrolnog poligona <= br. sjec. ravnine i krivulje

- lokalni nadzor - nije ispunjeno
- broj kontrolnih točaka je u direktnoj vezi sa stupnjem krivulje
- neovisnost o transformacijama (translacija, rotacija, skaliranje)
- simetričnost - kod uvrštenja možemo simetrično zamijeniti popis točaka
- http://www.cs.unc.edu/~mantler/research/bezier/index.html
- http://i33www.ira.uka.de/applets/mocca/html/noplugin/curves.html
- http://www.ibiblio.org/e-notes/VRML/Anim/Morph.wrl

## 6.3.2. INTERPOLACIJSKE KRIVULJE BEZIERA

Prolaze svim zadanim točkama.




$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^{n} \vec{a}_i f_{i,n}(t) \qquad t \in [0, 1]$$

$f_{i,n}$ – poznato na osnovi broja točaka

$\vec{a}_i$ – nepoznato – određuje se na temelju nečega poznatog ili željenog o krivulji

Potrebno je poznavati $n+1$ uvjet.

POZNATO

1. $n$+1 točka krivulje s vrijednošću parametra $\vec{p}_i(t_i), \quad t_i = \dfrac{i}{n}, \ i = 0..n.$

ili

2. tangente u pojedinim točkama $\vec{p}_i'(t_i) = \sum_{i=1}^{n} \vec{a}_i f_{i,n}'(t)\ .$

ili

3. oskulatorne ravnine, položaji centara zakrivljenosti $\vec{p}_i''(t_i) = \sum_{i=1}^{n} \vec{a}_i f_{i,n}''(t)\ .$

INTERPOLACIJSKA KRIVULJA KROZ *n*+1 TOČKU:

neka su poznate točke $\vec{p}_0 = \vec{p}(t_0), \vec{p}_1 = \vec{p}(t_1), \vec{p}_2 = \vec{p}(t_2), \ldots, \vec{p}_n = \vec{p}(t_n),$

uz parametar $t_i = \dfrac{i}{n}$ gdje $i = 0 .. \, n.$

$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^{n} \vec{a}_i f_{i,n}(t) \qquad t \in [0, 1]$$

$$\begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \ldots \\ \vec{p}_n \end{bmatrix} = \begin{bmatrix} 1 & f_{1,n}(0) & f_{2,n}(0) & \ldots & f_{n,n}(0) \\ 1 & f_{1,n}(t_1) & f_{2,n}(t_1) & \ldots & f_{n,n}(t_1) \\ 1 & f_{1,n}(t_2) & f_{2,n}(t_2) & \ldots & f_{n,n}(t_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & f_{1,n}(1) & f_{2,n}(1) & \ldots & f_{n,n}(1) \end{bmatrix} \begin{bmatrix} \vec{a}_0 \\ \vec{a}_1 \\ \vec{a}_2 \\ \ldots \\ \vec{a}_n \end{bmatrix}.$$

uvrstili smo : $t_0 = 0, \; t_n = 1.$

uvrstit ćemo:

za početnu točku je $\vec{p}(0)=\vec{a}_0$, tj. $f_{0,n}(0)=1,\ f_{i,n}(0)=0,\ i=1..n,$

za završnu točku je $\vec{p}(1)=\sum_{i=0}^{n}\vec{a}_i$, tj. $f_{i,n}(1)=1,\ i=0..n.$

$$\begin{bmatrix}\vec{a}_0\\ \vec{a}_1\\ \vec{a}_2\\ \dots\\ \vec{a}_n\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & \dots & 0\\ 1 & f_{1,n}(t_1) & f_{2,n}(t_1) & \dots & f_{n,n}(t_1)\\ 1 & f_{1,n}(t_2) & f_{2,n}(t_2) & \dots & f_{n,n}(t_2)\\ \dots & \dots & \dots & \dots & \dots\\ 1 & 1 & 1 & \dots & 1\end{bmatrix}^{-1}\begin{bmatrix}\vec{p}_0\\ \vec{p}_1\\ \vec{p}_2\\ \dots\\ \vec{p}_n\end{bmatrix}$$

kada odredimo nepoznate vektore $\vec{a}_i$ možemo do pojedine točke krivulje doći na osnovi Bezijer-ovih ili Bernstein-ovih težinskih funkcija.

* PRIMJER

Odrediti Interpolacijsku Bezierovu krivulju kroz četiri točke korištenjem Bezierovih težinskih funkcija.

Neka su poznate točke $\vec{p}_0 = \vec{p}(0), \vec{p}_1 = \vec{p}(1/3), \vec{p}_2 = \vec{p}(2/3), \vec{p}_3 = \vec{p}(1).$

$$\vec{p}(t) = \vec{a}_0 + \sum_{i=1}^{3} \vec{a}_i f_{i,3}(t) \qquad t \in [0, 1].$$

Iz prethodnog primjera za aproksimacijske Bezijerove krivulje poznate su težinske funkcije.

$$f_{0,3}(t) = 1,$$
$$f_{1,3}(t) = 3t - 3t^2 + t^3,$$
$$f_{2,3}(t) = 3t^2 - 2t^3,$$
$$f_{3,3}(t) = t^3.$$

$$\begin{bmatrix} \vec{a}_0 \\ \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{19}{27} & \frac{7}{27} & \frac{1}{27} \\ 1 & \frac{26}{27} & \frac{20}{27} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \vec{p}_0 \\ \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{bmatrix}$$

$$\vec{p}(t) = \vec{a}_0 + \left(3t - 3t^2 + t^3\right) \vec{a}_1 + \left(3t^2 - 2t^3\right) \vec{a}_2 + t^3 \vec{a}_3$$

# 6.3.3. RAZLOMLJENE FUNKCIJE

## PRIKAZ KRIVULJA POMOĆU KVADRATNIH RAZLOM. FUNKCIJA

- pogodan oblik za prikaz krivulja drugog reda
- homogena koordinata omogućava prikaz koničnih krivulja

(presjek ravnine i stošca) http://www.slu.edu/classes/maymk/banchoff/CriticalPoints.html

- invarijantnost na transformaciju perspektivne projekcije (nerazlomljene krivulje su invarijantne samo na translaciju, rotaciju, skaliranje)

$$\left.\begin{aligned} x &= \frac{x_1}{x_4} = \frac{a_1 t^2 + b_1 t + c_1}{a t^2 + b t + c}, \\ y &= \frac{x_2}{x_4} = \frac{a_2 t^2 + b_2 t + c_2}{a t^2 + b t + c}, \\ z &= \frac{x_3}{x_4} = \frac{a_3 t^2 + b_3 t + c_3}{a t^2 + b t + c} \end{aligned}\right\} \text{u radnom prostoru}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & b \\ c_1 & c_2 & c_3 & c \end{bmatrix} \quad \text{matrični oblik}$$

**K** - karakteristična matrica kvadratne krivulje, $0 \leq t \leq 1$,

$$\boxed{\mathbf{X} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \mathbf{K}}$$

- derivacije vektora $[x_1\ x_2\ x_3\ x_4]$ po parametru t - u homogenom prostoru
- matrica **K** određuje i derivacije duž krivulje

$$x_1' = \frac{d\,x_1}{d\,t} = 2a_1\,t + b_1,$$

$$x_2' = \frac{d\,x_2}{d\,t} = 2a_2\,t + b_2,$$

$$x_3' = \frac{d\,x_3}{d\,t} = 2a_3\,t + b_3,$$

$$x_4' = \frac{d\,x_4}{d\,t} = 2a\,t + b.$$

$$\mathbf{X}' = \begin{bmatrix} x_1' & x_2' & x_3' & x_4' \end{bmatrix} = \begin{bmatrix} 2t & 1 & 0 \end{bmatrix} \mathbf{K}$$

$$\mathbf{X}'' = \begin{bmatrix} x_1'' & x_2'' & x_3'' & x_4'' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{K}$$

$$\boxed{\mathbf{X}' = \begin{bmatrix} 2t & 1 & 0 \end{bmatrix} \mathbf{K}}$$

$$\boxed{\mathbf{X}'' = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{K}}$$

♣ kvadratna razlomljena krivulja određena je s tri točke

$$\mathbf{V}_0, \quad t_0 = 0, \quad \mathbf{X}_0 = [\mathbf{V}_0 \quad 1]$$

$$\mathbf{V}_1, \quad t_1 = \frac{1}{2}, \quad \mathbf{X}_1 = [\mathbf{V}_1 \quad 1]$$

$$\mathbf{V}_2, \quad t_2 = 1, \quad \mathbf{X}_2 = [\mathbf{V}_2 \quad 1]$$

tri točke uvrstimo u jednadžbu krivulje, uzmimo da su poznati iznosi parametra

$$\mathbf{K} = [t^2 \quad t \quad 1]^{-1}\, \mathbf{X}$$

$$\mathbf{K} = \begin{bmatrix} t_0^2 & t_0 & 1 \\ t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$$

* PRIMJER

Neka su zadane tri točke
i pripadni iznosi parametra.
Odrediti kvadratnu razlomljenu krivulju.

$$\mathbf{V}_0, \quad t_0 = 0, \quad \mathbf{X}_0 = \begin{bmatrix} r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{V}_1, \quad t_1 = \frac{1}{2}, \quad \mathbf{X}_1 = \begin{bmatrix} 0 & r & 0 & 1 \end{bmatrix}$$

$$\mathbf{V}_2, \quad t_2 = 1, \quad \mathbf{X}_2 = \begin{bmatrix} -r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r & 0 & 0 & 1 \\ 0 & r & 0 & 1 \\ -r & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4r & 0 & 0 \\ -2r & 4r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 0 & -4r & 0 & 0 \\ -2r & 4r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

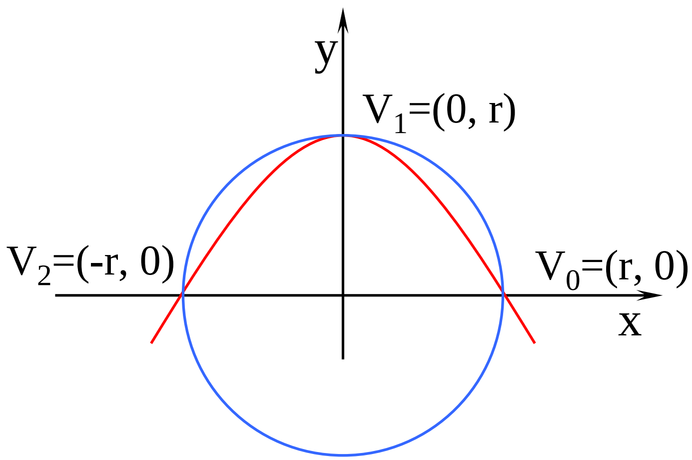
$$x_1 = -2rt + r = r(1-2t),$$

$$x_2 = -4rt^2 + 4rt = 4r\left(t - t^2\right),$$

$$x_3 = 0,$$

$$x_4 = 1.$$

po komponentama




Rezultat je parabola, to je opća krivulja drugog reda.

Ako želimo načiniti kružnicu potrebno je upotrijebiti analitičke poznate izraze za kružnicu. http://i33www.ira.uka.de/applets/mocca/html/noplugin/curves.html

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ \hline t &= \operatorname{tg}\frac{\varphi}{2} \end{aligned}\right\} \quad \left.\begin{aligned} x &= r\frac{1-t^2}{1+t^2} \\ y &= r\frac{2\,t}{1+t^2} \end{aligned}\right\} \quad \begin{aligned} x_1 &= r\left(1-t^2\right), \\ x_2 &= 2\,r\,t, \\ x_3 &= 0, \\ x_4 &= 1+t^2. \end{aligned}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -r & 0 & 0 & 1 \\ 0 & 2r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

## PRIKAZ KRIVULJA POMOĆU KUBNIH RAZLOMLJENIH FUNKCIJA

- kvadratnim razlomljenim funkcijama ne možemo prikazati infleksiju i ostale pojave višeg reda

$$
\left.
\begin{aligned}
x &= \frac{x_1}{x_4} = \frac{a_1 t^3 + b_1 t^2 + c_1 t + d_1}{a t^3 + b t^2 + ct + d}, \\
y &= \frac{x_2}{x_4} = \frac{a_2 t^3 + b_2 t^2 + c_2 t + d_2}{a t^3 + b t^2 + ct + d}, \\
z &= \frac{x_3}{x_4} = \frac{a_3 t^3 + b_3 t^2 + c_3 t + d_3}{a t^3 + b t^2 + ct + d}
\end{aligned}
\right\} \text{u radnom prostoru}
$$

$$
\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a \\ b_1 & b_2 & b_3 & b \\ c_1 & c_2 & c_3 & c \\ d_1 & d_2 & d_3 & d \end{bmatrix} \quad \text{matrični oblik}
$$

**A** - karakteristična matrica kubne krivulje, $0 \le t \le 1$,

$$\boxed{\mathbf{X} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{A}}$$

- derivacije vektora [$x_1$ $x_2$ $x_3$ $x_4$] po parametru $t$ - u homogenom prostoru
- matrica **A** određuje i derivacije duž krivulje

$$x_1' = \frac{d\,x_1}{d\,t} = 3a_1 t^2 + 2b_1 t + c_1,$$

$$x_2' = \frac{d\,x_2}{d\,t} = 3a_2 t^2 + 2b_2 t + c_2,$$

$$x_3' = \frac{d\,x_3}{d\,t} = 3a_3 t^2 + 2b_3 t + c_3,$$

$$x_4' = \frac{d\,x_4}{d\,t} = 3a t^2 + 2b t + c.$$

$$\boxed{\mathbf{X}' = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \mathbf{A}}$$

$$\boxed{\mathbf{X}'' = \begin{bmatrix} 6t & 2 & 0 & 0 \end{bmatrix} \mathbf{A}}$$

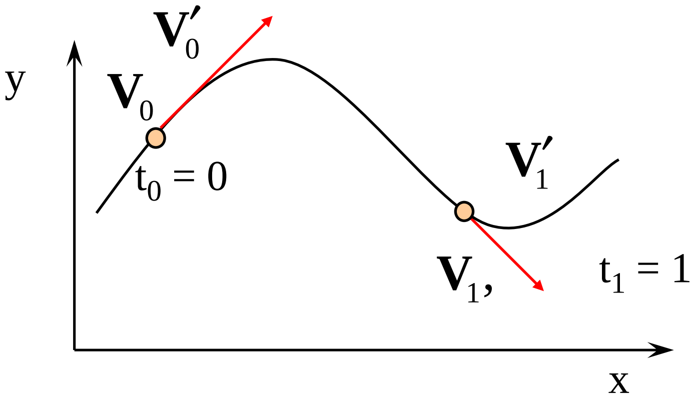
$$\boxed{\mathbf{X}''' = \begin{bmatrix} 6 & 0 & 0 & 0 \end{bmatrix} \mathbf{A}}$$

- za određivanje kubne razlomljene krivulje potrebna su četiri uvjeta

(kako bi mogli invertirati matricu)

To mogu biti 4 točke ili na primjer 2 točke i 2 derivacije.

♣ kubna razlomljena krivulja određena s dvije rubne točke i derivacije




$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} = \begin{bmatrix} x_4 x & x_4 y & x_4 z & x_4 \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\mathbf{A}$$

$$\mathbf{X}' = \begin{pmatrix} x_1' & x_2' & x_3' & x_4' \end{pmatrix} = \begin{bmatrix} (x_4 x)' & (x_4 y)' & (x_4 z)' & x_4' \end{bmatrix} = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix}\mathbf{A}$$

$$t_0 = 0,\quad \mathbf{X}_0 = \begin{bmatrix} x_{40}\mathbf{V}_0 & x_{40} \end{bmatrix} = \begin{bmatrix} t_0^{\,3} & t_0^{\,2} & t_0 & 1 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{A}$$

$$t_1 = 1,\quad \mathbf{X}_1 = \begin{bmatrix} x_{41}\mathbf{V}_1 & x_{41} \end{bmatrix} = \begin{bmatrix} t_1^{\,3} & t_1^{\,2} & t_1 & 1 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\mathbf{A}$$

$$t_0 = 0,\quad \mathbf{X}_0' = \begin{bmatrix} x_{40}'\mathbf{V}_0 + x_{40}\mathbf{V}_0' & x_{40}' \end{bmatrix} = \begin{bmatrix} 3t_0^{\,2} & 2t_0 & 1 & 0 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{A}$$

$$t_1 = 1,\quad \mathbf{X}_1' = \begin{bmatrix} x_{41}'\mathbf{V}_1 + x_{41}\mathbf{V}_1' & x_{41}' \end{bmatrix} = \begin{bmatrix} 3t_1^{\,2} & 2t_1 & 1 & 0 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix}\mathbf{A}$$

$$\begin{bmatrix} x_{40}\mathbf{V}_0 & x_{40} \\ x_{41}\mathbf{V}_1 & x_{41} \\ x'_{40}\mathbf{V}_0 + x_{40}\mathbf{V}'_0 & x'_{40} \\ x'_{41}\mathbf{V}_1 + x_{41}\mathbf{V}'_1 & x'_{41} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x_{40}\mathbf{V}_0 & x_{40} \\ x_{41}\mathbf{V}_1 & x_{41} \\ x'_{40}\mathbf{V}_0 + x_{40}\mathbf{V}'_0 & x'_{40} \\ x'_{41}\mathbf{V}_1 + x_{41}\mathbf{V}'_1 & x'_{41} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{40} & 0 & 0 & 0 \\ 0 & x_{41} & 0 & 0 \\ x'_{40} & 0 & x_{40} & 0 \\ 0 & x'_{41} & 0 & x_{41} \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}'_0 & 0 \\ \mathbf{V}'_1 & 0 \end{bmatrix}$$

$\boxed{\mathbf{A} = \mathbf{MHV}}$ $\mathbf{M}$.........univerzalna transformacijska matrica

- Segment krivulje određen rubnim točkama i derivacijama u njima

$$\mathbf{X} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{A}$$

$$\mathbf{A} = \mathbf{MHV}$$

**M -** ne ovisi o obliku krivulje već o izboru točaka (derivacija)

**H -** krivulja prolazi početnom i krajnjom točkom uz zadane derivacije,

a derivacija homogene komponente određuje kako će prolaziti

- ako je $x'_{40} = x'_{41} = 0$ dobit ćemo specijalan slučaj odnosno običnu parametarsku kubnu krivulju koja se zove HERMITOVA KRIVULJA

**V -** zadane točke i derivacije koje određuju segment krivulje u radnom prostoru

$$\mathbf{M} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} x_{40} & 0 & 0 & 0 \\ 0 & x_{41} & 0 & 0 \\ x'_{40} & 0 & x_{40} & 0 \\ 0 & x'_{41} & 0 & x_{41} \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}'_0 & 0 \\ \mathbf{V}'_1 & 0 \end{bmatrix}$$

- HERMITOVA KRIVULJA

$$\mathbf{X}=\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{V}_0 & 1 \\ \mathbf{V}_1 & 1 \\ \mathbf{V}_0' & 0 \\ \mathbf{V}_1' & 0 \end{bmatrix}$$

- VEZA HERMITOVE I BEZIEROVE KRIVULJE (preko Bernsteinovih polinoma)

$$\vec{p}(t)=\begin{bmatrix} t^3 & t^2 & t & t \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \vec{r}_0 \\ \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}. \qquad \begin{aligned} \mathbf{V}_0 &= \vec{r}_0 \\ \mathbf{V}_1 &= \vec{r}_3, \\ \mathbf{V}_0' &= 3(\vec{r}_1-\vec{r}_0) \\ \mathbf{V}_1' &= 3(\vec{r}_3-\vec{r}_2) \end{aligned}$$

$\Rightarrow$ radi se o istoj krivulji

* PRIMJER

Neka su zadane dvije točke
i derivacije u njima.
Odrediti kubnu razlomljenu krivulju.

$$\mathbf{V}_0 = \begin{bmatrix}0 & 0 & 0\end{bmatrix}, \quad t_0 = 0,$$
$$\mathbf{V}_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \quad t_1 = 1,$$
$$\mathbf{V}_0' = \begin{bmatrix}1 & 1 & 0\end{bmatrix}, \quad t_0 = 0,$$
$$\mathbf{V}_1' = \begin{bmatrix}1 & -1 & 0\end{bmatrix}, \quad t_1 = 1.$$

$$\begin{bmatrix}x_{40} & x_{41} & x_{40}' & x_{41}'\end{bmatrix} = \begin{bmatrix}1 & 1 & a & b\end{bmatrix}$$

$$\mathbf{A} = \mathbf{MHV} =$$

$$\mathbf{A} = \begin{bmatrix}2 & -2 & 1 & 1\\ -3 & 3 & -2 & -1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ a & 0 & 1 & 0\\ 0 & b & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\end{bmatrix} = \begin{bmatrix}b & 0 & 0 & a+b\\ -b & -1 & 0 & -(2a+b)\\ 1 & 1 & 0 & a\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix} = \begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\mathbf{A}$$

$$\left.\begin{aligned} x &= \frac{x_1}{x_4} = \frac{bt^3 - bt^2 + t}{(a+b)t^3 - (2a+b)t^2 + at + 1}, \\ y &= \frac{x_2}{x_4} = \frac{-t^2 + t}{(a+b)t^3 - (2a+b)t^2 + at + 1}, \\ z &= \frac{x_3}{x_4} = 0 \end{aligned}\right\} \; a, b = ?$$

Uvodimo dodatnu točku $V_2=(1/2\ 1/2\ 0)$, $t_2=1/2$. $\Rightarrow$ a=-2, b=2

http://www.rose-hulman.edu/~finn/courses/MA323GeomModel/TestApplets/RationalC2Spline.html +/- A/Z

$$x = \frac{2t^3 - 2t^2 + t}{2t^2 - 2t + 1},$$

$$y = \frac{-t^2 + t}{2t^2 - 2t + 1},$$

$$z = 0$$

# VEZA KOORDINATA I PARAMETARSKIH DERIVACIJA IZMEĐU RADNOG I HOMOGENOG PROSTORA

- radni prostor:

$$\mathrm{V}(t) = [\mathrm{x}(t) \quad \mathrm{y}(t) \quad \mathrm{z}(t)].$$

$$\frac{d\,\mathrm{V}(t)}{dt} = \begin{bmatrix} \frac{\mathrm{x}(t)}{dt} & \frac{\mathrm{y}(t)}{dt} & \frac{\mathrm{z}(t)}{dt} \end{bmatrix}.$$

- homogeni prostor

$$\mathbf{X}(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)].$$

$$\frac{d\,\mathbf{X}(t)}{dt} = \begin{bmatrix} \frac{\mathrm{x}_1(t)}{dt} & \frac{\mathrm{x}_2(t)}{dt} & \frac{\mathrm{x}_3(t)}{dt} & \frac{\mathrm{x}_4(t)}{dt} \end{bmatrix}.$$

- VEZA KOORDINATA

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad z = \frac{x_3}{x_4}$$

$$x_1 = x\,x_4, \quad x_2 = y\,x_4, \quad x_3 = z\,x_4$$

$$\mathbf{X} = (x_1 \quad x_2 \quad x_3 \quad x_4) = [x_4 x \quad x_4 y \quad x_4 z \quad x_4] = x_4[x \quad y \quad z \quad 1] = x_4[\mathbf{V} \quad 1]$$

$$\boxed{\mathbf{X} = x_4[\mathbf{V} \quad 1]}$$

- VEZA PRVE DERIVACIJE - homogena komponenta nije konstanta

$$\mathbf{X}' = \begin{pmatrix} x_1' & x_2' & x_3' & x_4' \end{pmatrix} = \begin{bmatrix} (x_4 x)' & (x_4 y)' & (x_4 z)' & x_4' \end{bmatrix} =$$

$$\left[ (x_4' x + x_4 x') \quad (x_4' y + x_4 y') \quad (x_4' z + x_4 z') \quad x_4' \right] = \left[ x_4' \quad x_4 \right] \begin{bmatrix} x & y & z & 1 \\ x' & y' & z' & 0 \end{bmatrix}$$

$$\boxed{\mathbf{X}' = \left[ x_4' \quad x_4 \right] \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \end{bmatrix}}$$

- VEZA DRUGE DERIVACIJE

$$\mathbf{X}'' = (\mathbf{X}')' = \left( \left[ x_4' \quad x_4 \right] \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \end{bmatrix} \right)' = \left( (x_4' \mathbf{V} + x_4 \mathbf{V}') \quad x_4' \right)' = \begin{bmatrix} (x_4 \mathbf{V})'' & x_4'' \end{bmatrix} =$$

$$= \left[ \left( x_4'' \mathbf{V} + 2\, x_4' \mathbf{V}' + x_4 \mathbf{V}'' \right) \quad x_4'' \right]$$

$$\boxed{\mathbf{X}'' = \left[ x_4'' \quad 2x_4' \quad x_4 \right] \begin{bmatrix} \mathbf{V} & 1 \\ \mathbf{V}' & 0 \\ \mathbf{V}'' & 0 \end{bmatrix}}$$

* PRIMJER

Odrediti prvu derivaciju u homogenom i radnom prostoru na kružnicu.

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -r & 0 & 0 & 1 \\ 0 & 2r & 0 & 0 \\ r & 0 & 0 & 1 \end{bmatrix}$$

$\underline{\text{homogeni prostor}}:$

$$x_1 = r\left(1 - t^2\right),$$
$$x_2 = 2rt,$$
$$x_4 = t^2 + 1.$$

$$x_1' = -2rt,$$
$$x_2' = 2r,$$
$$x_4' = 2t.$$

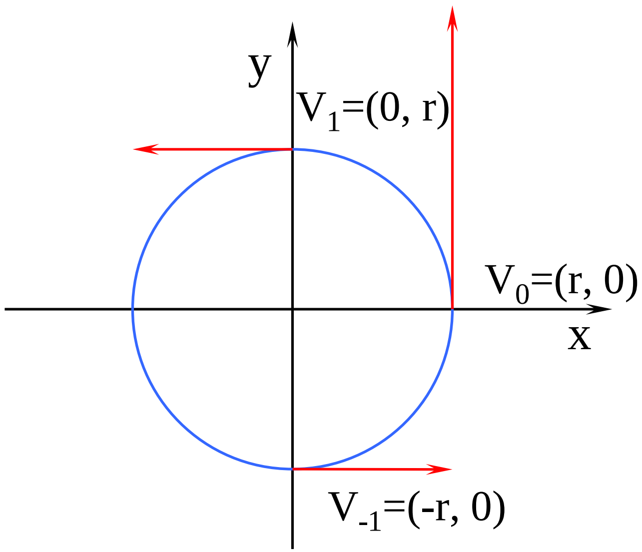
$\underline{\text{radni prostor}}:$

$$x = \frac{r\left(1 - t^2\right)}{1 + t^2},$$
$$y = \frac{2rt}{1 + t^2}.$$

$$x' = \left(\frac{x_1}{x_4}\right)' = \frac{-4rt}{1 + 2t^2 + t^4} \neq \frac{x_1'}{x_4'},$$

$$y' = \left(\frac{x_2}{x_4}\right)' = \frac{2r\left(1 - t^2\right)}{1 + 2t^2 + t^4} \neq \frac{x_2'}{x_4'}.$$

| t | $x_1$ | $x_2$ | $x_4$ | x | y | $x'_1$ | $x'_2$ | $x'_4$ | x' | y' |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | r | 0 | 1 | r | 0 | 0 | 2r | 0 | 0 | 2r |
| 1 | 0 | 2r | 2 | 0 | r | -2r | 2r | 2 | -r | 0 |
| -1 | 0 | -2r | 2 | 0 | -r | 2r | 2r | -2 | r | 0 |




$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{nagib tangente}$$

http://www.math.aau.dk/~raussen/VIDIGEO/GEOLAB/speed.html