Quantitative Risk Management Project 1

Capital Allocation

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1 Introduction

Capital allocation is relevant to risk management because it dictates how companies and financial institutions allocate capital to different exposures to risk. Capital allocation plays a central role in determining the pricing, stability, and profitability of financial services. This study examines two capital allocation methods, Euler-based allocation under Expected Shortfall (ES) a local approximation method under Value-at-Risk (VaR). It also investigates how different dependence structures affect diversification benefits across the portfolio.

The analysis is based on a portfolio of four risk components, each modeled using a probability distribution. Both Gaussian and Clayton copulas introduce dependence among risks. The study compares the allocation of capital and diversification benefit under these different dependence models, and checks for sensitivity of the results to the parameters of the Clayton survival copula. Results are presented in tables and scatter plots to show the relationships and allocation effects between the components.

2 Theory

2.1 Euler Principle and The Tasche Theorem

2.1.1 Euler Principle

Euler principle [1] is a capital allocation method based on the Euler theorem and its differentiation properties on homogeneous functions. Let

$$Z \;=\; \sum_{i=1}^{n} X_i$$

be the total portfolio profit or loss composed of n sub-risks X_i. If $\rho(Z)$ is a risk measure that is positive homogeneous and differentiable, the Euler principle will allocate each sub-risk X_i a marginal contribution given by:

$$\rho\big(X_i \mid Z\big) \;=\; \frac{\mathrm{d}}{\mathrm{d}h}\,\rho\big(Z + h\,X_i\big)\bigg|_{h=0}.$$

You can say that one adds a small weight h on the sub-risk X_i and then measures how the total risk $\rho(Z)$ changes. This marginal change, is exactly the portion of capital that gets allocated to the risk. By doing this for every sub-risk in the portfolio Z, the sum of all individual allocations equals to the total capital requirement, this is called full allocation [1].

2.1.2 The Tasche Theorem

The Tasche theorem [1] gives the solution to the problem of finding a capital allocation rule that is consistent. Assume that $\rho(Z(u))$ is continuously differentiable in sub-portfolio weights

$$u = (u_1, \ldots, u_n)$$

Suppose you are tasked to find an allocation rule $a_i(u)$ that (1) sums exactly to $\rho(Z)$ i.e full allocation, (2) treats risk-free positions neutrally, (3) never allocates more than each standalone risk, and (4) is RoRAC compatible. Which means that if a sub-risk X_i has a higher return on risk capital than the portfolio average, then slightly increasing its weight should improve the portfolio's overall RoRAC. According to Tasches Theorem [1], the only continuous allocation $a(u)$ satisfying all above properties is:

$$a_i(u) \;=\; \frac{\partial\,\rho\big(Z(u)\big)}{\partial u_i} \;=\; \frac{\mathrm{d}}{\mathrm{d}h}\,\rho\big(Z(u) + h\,X_i\big)\bigg|_{h=0}.$$

i.e the Euler allocation. The Tasche theorem states that if a capital allocation method is continuously differentiable and satisfies the four key properties previously mentioned, then it must be the Euler allocation [1]. In the context of risk measures such as Expected Shortfall (ES), Tasche also showed that the Euler capital allocation to a sub-risk S can be expressed as:

$$a_i = -\mathbb{E}[X_i \mid Z \le F_Z^{-1}(\alpha)],$$

which is the expected loss from X_i given that the total portfolio loss Z exceeds its α-quantile.This formula make sure that capital is allocated based on the contribution of a risk component to tail risk [6].

2.2 Coherence Properties of Risk Measure

A good capital allocation principle should fulfill the following four properties[3, 2], which are closely related to the coherence axioms for risk measures.

2.2.1 (1) Full Allocation and Positive Homogeneity

A method must divide all the total risk capital $\rho(Z)$ between individual sub-risks X_i so that:

$$\sum_{i=1}^{n} \rho\big(X_i \mid Z\big) \;=\; \rho(Z).$$

This ensures that there is no capital left that is not allocated [7]. Positive homogeneity,

$$\rho(hX) = h\,\rho(X) \quad \text{for } h > 0,$$

ensures that scaling the portfolio by a constant factor h also scales the total capital accordingly. This property allows for the breakdown of $\rho(Z)$ into smaller parts that add up to the total risk, ensuring risk is fully allocated [2].

2.2.2 (2) Fairness and Sub-additivity

Each unit's allocated capital should not exceed the risk it would require on a standalone basis:

$$\rho(X_i \mid Z) \leq \rho(X_i) \quad \forall i.$$

This property ensures that business units retain the benefits of diversification [7]. Sub-additivity,

$$\rho(X + Y) \leq \rho(X) + \rho(Y),$$

formalizes this diversification gain by stating that the risk of a combined portfolio should not exceed the sum of individual risks. Together, these properties uphold fairness in the capital distribution [3, 2].

2.2.3 (3) Risk-less Allocation and Translation Invariance

If a risk-free asset is added to the portfolio, it should not receive a capital allocation and should not affect the other allocations [7]. Translation invariance,

$$\rho(X + a) = \rho(X) - a,$$

means that adding a constant gain a reduces the total risk capital by the same amount, without affecting risk contributions from the other positions. This ensures that risk-free components are treated neutrally [3, 2].

2.2.4 (4) RoRAC Compatibility and Monotonicity

For each business unit i, return on risk-adjusted capital (RoRAC) is defined as:

$$\text{RoRAC}_i = \frac{\mathbb{E}[X_i]}{\rho(X_i \mid Z)}, \quad \text{RoRAC}_{\text{port}} = \frac{\mathbb{E}[Z]}{\rho(Z)}.$$

An allocation is RoRAC compatible if, whenever the RoRAC of an individual unit exceeds that of the overall portfolio, increasing its weight improves the total RoRAC of the portfolio [7]. Monotonicity

$$X \leq Y \implies \rho(X) \leq \rho(Y),$$

means that units with better performance or lower risk are assigned less capital, which makes the allocation fair [2]. This idea is a key reason why the Euler allocation rule is widely used, as emphasized by Tasche [5].

2.3 Clayton, Survival Clayton and Gaussian Copula

Copulas are mathematical functions that aggregate marginal distributions of single variables to a multivariate distribution, allowing the structure of the dependence to be characterized independently of the marginals. The copulas allow preserving patterns of dependency[5].

2.3.1 The Clayton Copula

The Clayton copula is an Archimedean copula and is used to model asymmetric dependence, suitable for situations where a strong dependence occurs in the lower tail. Definition, For two uniform variables $u, v \in [0, 1]$ and a parameter $\theta \in (-1, \infty) \setminus \{0\}$, the Clayton copula is defined as:

$$C_\theta(u, v) \;=\; \bigl(u^{-\theta} + v^{-\theta} - 1\bigr)^{-1/\theta}.$$

The Clayton copula has an asymmetric tail dependence, which means that it shows a strong link when low values occur at the same time. The parameter θ is used to describe how strong this link is, i.e, a higher θ means a strong dependence. The Clayton copula captures lower tail dependency efficiently. [5].

2.3.2 Generate Realization of The Clayton Copula

To sample (U_1, U_2) from a Clayton copula with parameter $\theta > 0$, First, draw and independent uniform random variable:

$$U_1 \sim \text{Uniform}(0, 1).$$

Then, draw another independent uniform random variable that is truly independent form U_1:

$$V \sim \text{Uniform}(0, 1).$$

For the Clayton copula, the conditional CDF of U_2 given $U_1 = u$ has a closed form inverse, so to draw U_2, invert it by setting:

$$U_2 \;=\; \bigl(u^{-\theta}\bigl(V^{-\frac{\theta}{\theta+1}} - 1\bigr) + 1\bigr)^{-\frac{1}{\theta}}.$$

The resulting pair (U_1, U_2) has exactly the Clayton copula C_θ, i,e.

$$C_\theta(u_1, u_2) \;=\; \bigl(u_1^{-\theta} + u_2^{-\theta} - 1\bigr)^{-1/\theta}.$$

To get two variables X and Y with distributions F_X and F_Y, transform your copula draws (U_1, U_2) by:

$$X = F_X^{-1}(U_1), \quad Y = F_Y^{-1}(U_2).$$

This method of simulating the Clayton copula is described in[5].

2.3.3 The Survival Clayton Copula

The main difference between the Clayton copula and the Survival Clayton copula lies in the dependence of the tail. As mentioned above, the standard Clayton copula captures lower-tail dependence, which makes it suitable for modeling situations where extremely low outcomes occur at the same time. However, the Clayton survival copula is modeling the opposite side, which means that it captures the upper-tail dependence. This copula is useful for modeling extreme upper-tail dependence, capturing simultaneous occurrences of large values, such as simultaneous substantial gains or losses. [5].The survival Clayton copula is obtained by applying a transformation to the Clayton copula according to the following relation:

$$C_\theta^{\text{survival}}(u, v) \;=\; u + v - 1 + C_\theta\bigl(1 - u,\, 1 - v\bigr).$$

Here $C_\theta(u, v)$, denotes the standard Clayton copula with dependence parameter $\theta > 0$. The survival Clayton copula is simply the standard Clayton reflected in the upper right corner, capturing upper-tail dependence, whereas the usual Clayton models lower-tail dependence [5].

2.3.4 Generate Realization of The Survival Clayton Copula

To simulate one draw (U, V) from a survival Clayton copula with parameter $\theta > 0$, and then map these variables to any continuous marginal distribution, first, draw two independent uniforms:

$$U_1,\ W \;\sim\; \text{Uniform}(0, 1).$$

Secondly, compute the Clayton-conditional quantile:

$$U_2^* \;=\; \bigl(U_1^{-\theta}\bigl(W^{-\frac{\theta}{\theta+1}} - 1\bigr) \;+\; 1\bigr)^{-\frac{1}{\theta}}.$$

To switch from the standard Clayton copula $C(u_1, u_2)$ to its survival version C^{surv} set $U = 1 - u_1$ and $V = 1 - u_2^*$. This reflection will create the same tail-dependence into the upper-right corner of the unit square, matching heavy dependence in large-loss scenarios. Given target CDFs F_X and F_Y, set

$$X = F_X^{-1}(U), \quad Y = F_Y^{-1}(V).$$

from Sklar's theorem yields (X, Y) with the survival Clayton dependence[5].

2.3.5 The Gaussian Copula

The Gaussian copula models dependence between variables using the correlation structure of a multivariate normal distribution. It is often associated with elliptical distributions, since applying it to normal marginals yields an elliptical joint distribution. It describes dependence using correlation and can capture symmetric dependence structures[4]. Definition: Let Φ be the standard Normal CDF and Φ_P the joint CDF of a d-variate Normal $\mathcal{N}(0, P)$ with correlation matrix P. Then the Gaussian copula is defined by the formula:

$$C_P^{\text{Gauss}}(u_1, \ldots, u_d) \; = \; \Phi_P\big(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\big),$$

The gaussian has several key properties. First, the Gaussian copula has an elliptical shape and depends only on the correlation matrix P, that means that any two Gaussian vectors with the same P also have same copula. Secondly, the Gaussian copula exhibits zero tail dependence for all correlations $|\rho| < 1$ [4]. This means that even if one variable takes an extreme value, there is no positive probability that the other variable is extreme at the same time. Only in the degenerate limits $\rho = \pm 1$ does one obtain perfect co-movement in the tails. Thirdly, the Gaussian copula is invariant under any strictly increasing transformation of its marginals [4]. In other words, applying a monotonic function to each variable does not change the copula.

2.3.6 Generate Realization of The Gaussian Copula

In order to generate realizations of two dependent random variables according to the gaussian copula, one starts by build the copula correlation matrix:

$$P \; = \; \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

By using cholesky-factorizeing, find a matrix A such that $P = A\,A^{\top}$ for a 2×2 matrix you can write:

$$A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}.$$

Then, simulate two independent random numbers Z_1 and Z_2, each following a normal distribution with mean zero and variance one $(Z_1, Z_2) \sim \mathcal{N}(0, I_2)$ and with no correlation between them. Then introduce correlation by computing:

$$\begin{pmatrix} X_1^* \\ X_2^* \end{pmatrix} = A \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

Then (X_1^*, X_2^*) has the desired Gaussian dependence.Then, transform (X_1^*, X_2^*) into a uniform $[0, 1]$ variable, set:

$$U_i = \Phi\big(X_i^*\big), \quad i = 1, 2.$$

where Φ is the standard normal CDF. Now (U_1, U_2) has the Gaussian copula. Finally, pull back the marginal, let:

$$X = F_X^{-1}(U_1), \quad Y = F_Y^{-1}(U_2).$$

Then $X \sim F_X$ and $Y \sim F_Y$, and their dependence is the Gaussian copula with the parameter ρ [4].

3 Methodology

3.1 Data Generation

3.1.1 Portfolio construction

The simulations were done in R using built-in functions qlnorm and qfrechet to simulate lognormal and fréchet distributions. For every component risks, $n = 100{,}000$ independent realizations were simulated. The lognormal risks U and V were simulated using parameters $(\mu, \sigma) = (2.00, 0.40)$ and $(2.20, 0.50)$, respectively. Fréchet risks X and Y were simulated using parameters $\alpha = 1.60$ and $\alpha = 1.40$ and $m = 5.5$, 5.4. The resulting composite risks were then defined as:

$$W = U + V, \quad Z = X + Y, \quad P = W + Z.$$

A fixed random seed 7 was used to ensure reproducibility.

3.1.2 Alternative Method: Inverse Transform Sampling

As an alternative to the built-in functions, apply the inverse transform method to sample the Fréchet- and lognormal distributed risk components of the portfolio. Here create random uniform variables and applying them via the related quantile functions to get samples from the lognormal and Fréchet distributions.

(a) Generate independent uniforms: $u_i, v_i, x_i, y_i \sim \text{Uniform}(0, 1)$, for $i = 1, \ldots, n$.

(b) Lognormal risks are obtained by applying the inverse standard normal transformation followed by exponentiation:

$$U_i = \exp(\mu_U + \sigma_U\, \Phi^{-1}(u_i)), \quad V_i = \exp(\mu_V + \sigma_V\, \Phi^{-1}(v_i))$$

with $\mu_U = 2.00$, $\sigma_U = 0.40$ and $\mu_V = 2.20$, $\sigma_V = 0.50$.

(c) Fréchet risks are generated by applying the inverse of the Fréchet CDF:

$$X_i = m_X + s\,(-\ln x_i)^{-1/\alpha_X}, \quad Y_i = m_Y + s\,(-\ln y_i)^{-1/\alpha_Y}$$

where $\alpha_X = 1.60$, $m_X = 5.5$, $\alpha_Y = 1.40$, $m_Y = 5.4$, and $s = 1$.

(d) Aggregation is defined as:

$$W_i = U_i + V_i, \quad Z_i = X_i + Y_i, \quad P_i = W_i + Z_i.$$

3.2 Constructing Dependence Scenarios

Plug in the four base risks into three copula-based dependence structures and one independent dependence structure. Consider the following four scenarios:

Scenario	Copula type	Parameters
1. Independent	—	—
2. Survival–Clayton	Survival Clayton	$\theta_{UV} = 2.0$, $\theta_{XY} = 1.0$, $\theta_{WZ} = 0.5$
3. Gaussian	Gaussian copula	ρ from empirical rank correlations in Scenario 2
4. Weak Survival-Clayton	Survival Clayton	$\theta_{UV} = 1.0$, $\theta_{XY} = 0.5$, $\theta_{WZ} = 0.1$

Table 1: Dependence scenarios and their copula parameters.

3.2.1 Independent Scenario

In the independent scenario, all base risks are generated independently using built-in distribution functions. Variables U and V are drawn from lognormal distributions with given mean and standard deviation parameters, X and Y follow Fréchet distributions. There is no dependence introduced between the variables; all marginal draws are independent. The composite risks are constructed hierarchically as described in the portfolio construction part of the methodology.

3.2.2 Tail-dependent scenarios via Survival–Clayton copulas

To generate upper-tail dependence between risk components, use the Survival–Clayton copula. This copula is derived from the standard Clayton copula by changing its lower-tail dependence into a upper tail. In practice, this is done by sampling from a Clayton copula and subtracting the resulting uniforms from 1.

(i) **Base-level dependence:**
For each pair of base risks (U, V) and (X, Y), we generate dependent uniforms using the built-in function `claytonCopula(param, dim)` in R, with parameters θ_1 and θ_2 respectively. These uniforms are built to show lower-tail dependence. To obtain upper-tail dependence, each uniform variable is transformed as $(1-u, 1-v)$. The transformed samples are then translated into lognormal and Fréchet-distributed variables, resulting in U, V and X, Y with upper-tail dependence at the base level.

(ii) **Aggregate-level dependence using a second copula:**
A third Survival–Clayton copula with parameter θ_3 is simulated to generate new dependent uniform pairs (w_i, z_i). These pairs are intended to introduce dependence between the aggregate sums $W = U + V$ and $Z = X + Y$, without affecting the marginal distributions constructed in the previous step.

(iii) **Rank-matching to preserve marginals and impose dependence:**
To inject dependence at the aggregate level without altering the marginal distributions of U, V, X, and Y, we apply a rank-matching approach. Specifically, the sorted values of each marginal are reassigned based on the ranks of the generated copula uniforms (w_i) and (z_i). For example, the smallest w_i is mapped to the smallest U, the second smallest to the second smallest, and so on. This is done similarly for V, X, and Y. This ensures:

- The original marginal distributions remain unchanged.
- Dependence is introduced between the sums $W = \tilde{U} + \tilde{V}$ and $Z = \tilde{X} + \tilde{Y}$.
- The identity $P = U + V + X + Y$ is preserved.

(iv) **Portfolio construction:**
The rank-matched variables are aggregated as $W_i = \tilde{U}_i + \tilde{V}_i$ and $Z_i = \tilde{X}_i + \tilde{Y}_i$. The total portfolio loss is then computed as $P_i = W_i + Z_i$, which now reflects upper-tail dependence at both base and aggregate levels.

This approach allows us to create the upper-tail dependence in the system. By adjusting the copula parameter θ, create both strong $(\theta_1, \theta_2, \theta_3 = 2.0, 1.0, 0.5)$ and weak $(\theta_1, \theta_2, \theta_3 = 1.0, 0.5, 0.1)$ dependence structures for (U, V), (X, Y), and (W, Z).

3.2.3 The Gaussian copula

The Gaussian copula captures symmetric dependence with constant correlation. In this set-up use the Gaussian copulas: one for each base-risk pair (U, V) and (X, Y), and one for the aggregates $W = U + V$ and $Z = X + Y$. The Gaussian copulas are parameterized using the rank correlations (Spearman), calculated in the survival Clayton copula task between the pairs (U, V), (X, Y) and (W, Z). The dependence is controlled by the correlation parameters:

$$\rho_{UV} = 0.684, \quad \rho_{XY} = 0.479, \quad \rho_{WZ} = 0.296.$$

(i) **Base-level dependence:** For each pair of base risks (U, V) and (X, Y), we generate dependent uniforms using the built-in function `normalCopula(param, dim)` in R, with parameters ρ_{UV} and ρ_{XY} respectively. These uniforms are then transformed using the lognormal and Fréchet quantile functions to obtain the marginal distributions for U, V and X, Y. This step introduces symmetric dependence between the base-level risk components.

(ii) **Aggregate-level dependence using a second copula:** A third Gaussian copula with parameter ρ_{WZ} is simulated to generate new dependent uniform pairs (w_i, z_i). These pairs are intended to impose dependence between the aggregate sums $W = U + V$ and $Z = X + Y$, while keeping the same marginal distributions as before.

(iii) **Rank-matching to preserve marginals and impose dependence:** To inject dependence at the aggregate level without altering the original marginals of U, V, X, and Y, we apply a rank-matching approach. Specifically, the sorted values of each marginal are reassigned according to the rank order of the new copula uniforms (w_i). That is, the smallest w_i maps to the smallest U, the second smallest to the second smallest, and so on. The same is done for V, X, and Y using w_i and z_i. This ensures:

- The original marginal distributions remain unchanged.
- Dependence is introduced between the sums $W = \tilde{U} + \tilde{V}$ and $Z = \tilde{X} + \tilde{Y}$.
- The identity $P = U + V + X + Y$ holds exactly.

(iv) **Portfolio construction:** The rank-matched variables are aggregated as $W_i = \tilde{U}_i + \tilde{V}_i$ and $Z_i = \tilde{X}_i + \tilde{Y}_i$. The total portfolio loss is then computed as $P_i = W_i + Z_i$, now exhibiting both base-level and aggregate-level dependence.

3.3 Capital Allocation and Diversification Metrics

Compute capital allocation for each component $C \in \{U, V, X, Y, W, Z, P\}$ using two risk measures: Expected Shortfall (ES) at 99% (Euler Principle), and Value-at-Risk (VaR) at 99.5%. Each component's capital is computed both on a standalone basis and in portfolio context.

Standalone Capital

$$K_{\text{ES}}^{\text{stand}}(C) = \mathbb{E}[C \mid C \geq \text{VaR}_{0.99}(C)] - \mathbb{E}[C],$$
$$K_{\text{VaR}}^{\text{stand}}(C) = \text{VaR}_{0.995}(C) - \mathbb{E}[C].$$

Portfolio Capital Allocation

(1) **Euler principle based on ES:**

$$K_{\text{ES}}(C \mid P) = \mathbb{E}[C \mid P \geq \text{VaR}_{0.99}(P)] - \mathbb{E}[C],$$

where the conditional expectation reflects marginal contribution to portfolio ES.

(2) **VaR-based allocation:**

$$K_{\text{VaR}}(C \mid P) \approx \mathbb{E}[C \mid P \in [\text{VaR}_{0.995-\delta},\ \text{VaR}_{0.995+\delta}]] - \mathbb{E}[C],$$

where $\delta = 0.1\%$ defines a narrow space around the portfolio VaR threshold to approximate conditional expectation.

Reported Metrics

For each component and method $\rho \in \{\text{ES}, \text{VaR}\}$, compute:

- **Av. Loss**: $\mathbb{E}[C]$
- **Standalone risk (ES or VaR)**: the relevant risk measure for the isolated component.
- **Capital Standalone (CS)**: risk measure minus average loss.
- **Capital Allocated (Capt.)**: $K_\rho(C \mid P)$, i.e. the capital contribution under dependence.
- **Capt. (%)**: share of total portfolio capital:

$$100 \times \frac{K_\rho(C \mid P)}{K_\rho(P \mid P)}.$$

- **Diversification Benefit**:

$$D_\rho(C) = 1 - \frac{K_\rho(C \mid P)}{K_\rho^{\text{stand}}(C)},$$

Portfolio Diversification

Overall diversification benefit at the portfolio level is given by:

$$D_\rho(P) = 1 - \frac{K_\rho(P \mid P)}{\sum_{C \in \{U,V,X,Y\}} K_\rho^{\text{stand}}(C)}.$$

3.4 Convergence Analysis

To assess error in our capital allocation estimates under the independence scenario, we perform a convergence analysis across increasing simulation sizes:

$$n \in \{10^2,\ 10^3,\ 10^4,\ 10^5,\ 10^6\}.$$

in increments of 1,000. For each n, generate Uniform$(0, 1)$ samples and transform them into lognormal and Fréchet marginals. Aggregates are formed as $W = U + V$, $Z = X + Y$, and $P = W + Z$. Using both the Euler method and the VaR-based approach, compute capital allocations for each component. Plot a figure displaying the estimated capital contributions for all components $C_i \in \{U, V, X, Y\}$, across simulation sizes, separately for Euler and VaR-based allocation method.

4 Results

Description of Table Columns

The following table summarizes the abbreviations used in the capital allocation tables for Euler principle method under (ES) and Value-at-Risk (VaR). All values are expressed as percentages with two decimal places for readability and comparability.

Abbreviation	Description
Av. Loss	Average loss for each risk component.
ES 99%	Standalone Expected Shortfall at 99%.
ES 99% P	Contribution to portfolio ES at 99%, via Euler principle.
VaR 99.5%	Standalone Value-at-Risk at 99.5%.
VaR 99.5% P	Contribution to portfolio VaR at 99.5%.
CS	Capital Standalone = Risk measure − Avg. loss.
Capt. (ES)	Capital allocated via ES-based Euler principle.
Capt. (VaR)	Capital allocated via VaR-based methodology.
Capt. (%)	Share of total portfolio capital allocated.
Div Benefit	Diversification benefit: capital saved due to diversification.

4.1 Risk Profile and Allocation Metrics under Independence Assumption

Table 2: Capital Allocation - Euler principle under (ES) (Task 1) - Independent

Variable	Av. Loss	ES 99%	ES 99% P	CS	Capt. (ES)	Capt. (%)	Div Benefit(%)
U	7.99	21.49	8.48	13.49	0.48	0.46	96.43
V	10.23	34.06	13.45	23.82	3.22	3.06	86.49
X	7.87	52.36	36.73	44.49	28.86	27.45	35.43
Y	8.49	92.35	81.09	83.86	72.60	69.04	13.43
W	18.23	43.63	21.93	25.39	3.70	3.52	90.08
Z	16.36	120.39	117.82	104.03	101.46	96.48	20.96
P	34.59	139.75	139.75	105.16	105.16	100.00	36.53

Table 3: Capital Allocation - Value at Risk (VaR) method (Task 1) - Independent

Variable	Av. Loss	VaR 99.5%	VaR 99.5% P	CS	Capt. (VaR)	Capt. (%)	Div Benefit(%)
U	7.99	20.78	8.56	12.78	0.56	1.01	95.60
V	10.23	32.37	13.39	22.13	3.16	5.69	85.71
X	7.87	33.85	26.98	25.98	19.11	34.39	26.58
Y	8.49	49.50	41.22	41.01	32.73	58.90	20.23
W	18.23	42.96	21.96	23.73	3.73	6.71	89.33
Z	16.36	70.81	68.19	54.45	51.84	93.29	22.62
P	34.59	89.62	89.62	55.56	55.56	100.00	45.47

4.2 Risk Profile and Allocation Metrics under Survival Clayton Copula Dependence

Table 4: Capital Allocation - Euler principle under (ES) (Task 2) - Survival Clayton Copula

Variable	Av. Loss	ES 99%	ES 99% P	CS	Capt. (ES)	Capt. (%)	Div Benefit(%)
U	7.99	21.53	17.76	13.52	9.75	5.65	27.89
V	10.23	34.58	27.53	24.36	17.31	10.04	28.94
X	7.87	55.45	54.28	47.56	46.39	26.90	2.46
Y	8.49	109.57	7.65	100.92	98.99	57.41	1.90
W	18.23	56.12	45.29	37.87	27.06	15.69	28.57
Z	16.36	165.03	161.94	148.48	145.39	84.31	2.08
P	34.59	207.23	207.23	172.44	172.44	100.00	7.46

Table 5: Capital Allocation - Value at Risk (VaR) method (Task 2) - Survival Clayton Copula

Variable	Av. Loss	VaR 99.5%	VaR 99.5% P	CS	Capt. (VaR)	Capt. (%)	Div Benefit(%)
U	7.99	20.73	17.21	12.72	9.19	10.37	27.77
V	10.23	32.58	26.47	22.35	16.25	18.32	27.32
X	7.87	33.20	31.90	25.31	24.01	27.08	5.13
Y	8.49	49.72	47.88	41.07	39.23	44.24	4.50
W	18.23	53.31	43.68	35.07	25.44	28.69	27.46
Z	16.36	82.92	79.79	66.38	63.25	71.31	4.73
P	34.59	123.46	123.46	88.69	88.69	100.00	12.59

4.3 Risk Profile and Allocation Metrics under Gaussian Copula Dependence

Table 6: Capital Allocation - Euler principle under (ES) (Task 3) - Gaussian Copula

Variable	Av. Loss	ES 99%	ES 99% P	CS	Capt. (ES)	Capt. (%)	Div Benefit(%)
U	7.99	21.58	13.76	13.58	5.76	4.16	57.60
V	10.23	34.37	20.28	24.16	10.07	7.28	58.31
X	7.87	48.73	46.83	40.90	39.00	28.18	4.65
Y	8.49	95.24	92.06	86.72	83.55	60.38	3.66
W	18.23	55.96	34.04	37.74	15.83	11.44	58.06
Z	16.36	143.97	138.89	127.63	122.56	88.56	3.98
P	34.59	172.94	172.94	138.40	138.40	100.00	16.32

Table 7: Capital Allocation - Value at Risk (VaR) method (Task 3) - Gaussian copula

Variable	Av. Loss	VaR 99.5%	VaR 99.5% P	CS	Capt. (VaR)	Capt. (%)	Div Benefit (%)
U	7.99	20.76	13.63	12.76	5.64	8.60	55.83
V	10.23	32.53	20.15	22.32	9.94	14.92	55.47
X	7.87	32.99	30.98	25.18	23.16	29.16	8.01
Y	8.49	49.29	46.40	40.79	37.89	47.32	7.11
W	18.23	53.29	33.78	35.08	15.57	23.52	55.61
Z	16.36	82.30	77.38	65.96	61.05	76.48	7.45
P	34.59	111.16	111.16	76.62	76.62	100.00	24.17

4.4 Risk Profile and Allocation Metrics under Weaker Survival Clayton Copula Dependence

Table 8: Capital Allocation - Euler principle under (ES) (Task 4) - Weaker Survival Clayton copula

Variable	Av. Loss	ES 99%	ES 99% P	CS	Capt. (ES)	Capt. (%)	Div Benefit(%)
U	7.99	21.53	12.98	13.52	4.97	3.09	63.18
V	10.23	34.63	19.13	24.41	8.90	5.53	63.52
X	7.87	55.46	53.17	47.56	45.28	28.12	4.80
Y	8.49	114.36	110.59	105.66	101.88	63.26	3.57
W	18.23	56.16	32.11	37.92	13.88	8.62	63.39
Z	16.36	169.82	163.76	153.23	147.17	91.38	3.95
P	34.59	195.87	195.87	161.05	161.05	100.00	15.75

Table 9: Capital Allocation - Value at Risk (VaR) method (Task 4) - Gaussian copula

Variable	Av. Loss	VaR 99.5%	VaR 99.5% P	CS	Capt. (VaR)	Capt. (%)	Div Benefit(%)
U	7.99	20.73	13.40	12.72	5.39	7.29	57.65
V	10.23	32.75	20.13	22.52	9.91	13.41	56.00
X	7.87	33.20	30.19	25.31	22.30	30.19	11.91
Y	8.49	49.86	44.98	41.15	36.27	49.11	11.86
W	18.23	53.48	33.53	35.25	15.30	20.71	56.60
Z	16.36	83.06	75.16	66.47	58.57	79.29	11.88
P	34.59	108.69	108.69	73.86	73.86	100.00	27.37

4.5 Distributions of the Risk Components U, V, X, and Y

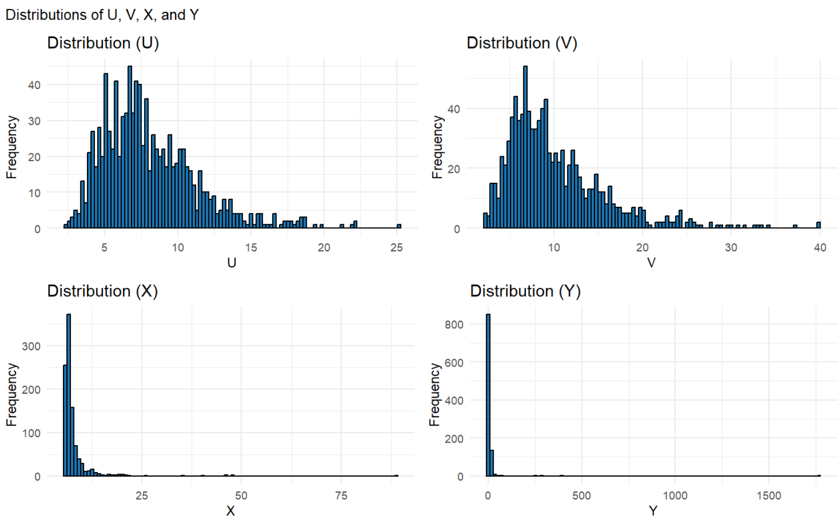

Figure 1: Histogram plots of the marginal distributions of the four base risk components. U and V are lognormally distributed with moderate skewness, while X and Y are Fréchet-distributed. This plot is produced based on 1,000 simulations for more visual effectiveness; a version based on 100,000 simulations is presented in Appendix 7.

4.6 Convergence Analysis of Capital Allocation Methods

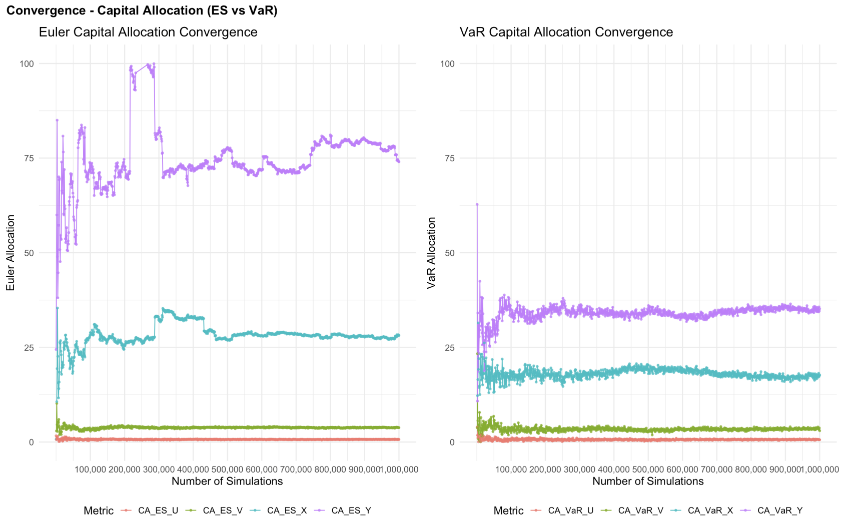

Figure 2: Convergence analysis of capital allocation under Expected Shortfall (ES) and Value-at-Risk (VaR) methods. The left panel shows Euler capital allocation convergence for each risk component (U, V, X, Y) as the number of simulations increases. The right panel displays corresponding convergence using VaR-based allocation.

4.7 Dependence Structure: Independent Copula (Task 1)

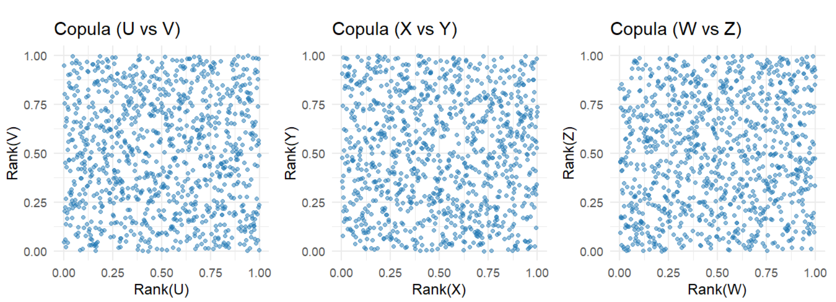

Figure 3: Ranked scatter plots of independent dependence structures for variable pairs: (U, V) (left), (X, Y) (middle), and (W, Z) (right). This graph is created by 1,000 simulations for more visual effectiveness; for an equivalent graph with 100,000 simulations, see Appendix 8.

4.8 Dependence Structure: Survival Clayton Copula (Task 2)

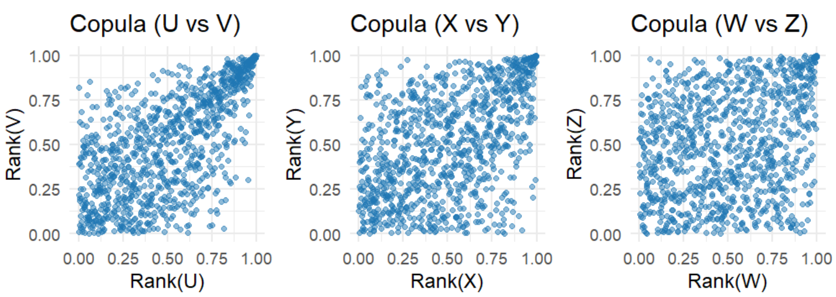

Figure 4: Scatter plots of ranked data representing the dependence structure between pairs of variables according to the Survival Clayton copula: (U, V) (left), (X, Y) (center), and (W, Z) (right). This plot uses 1,000 simulations for easier visual readability; a 100,000 simulation version is shown in Appendix 9.

4.9 Dependence Structure: Gaussian Copula (Task 3)

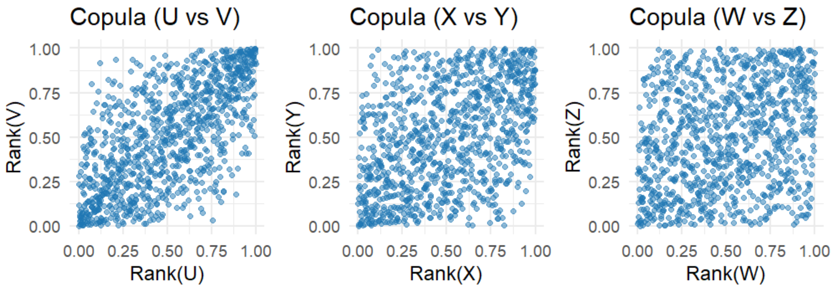

Figure 5: Ranked scatter plots of dependence structures under the Gaussian copula: (U, V) (left), (X, Y) (middle), and (W, Z) (right). This plot uses 1,000 simulations for improved visual clarity; a version with 100,000 simulations is provided in Appendix 10.

4.10 Dependence Structure: Weaker Survival Clayton Copula (Task 4)

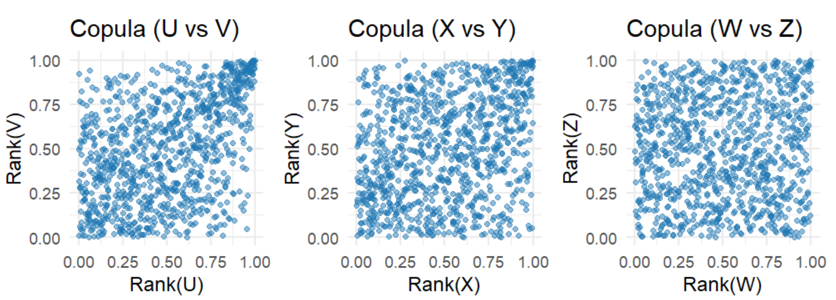

Figure 6: Ordered scatter plots showing dependence structure under the weaker Survival Clayton copula setting: (U, V) (left), (X, Y) (center), and (W, Z) (right). This figure is based on 1,000 simulations for illustration purposes; for the corresponding plot based on 100,000 simulations, see Appendix 11.

5 Discussion

5.1 Comparison of Dependence Models

Our results demonstrate that the dependence structure among risk components has a significant impact on both capital allocation and diversification benefits. When comparing the independent scenario with the three copula based scenarios, we quickly find that the dependence that is introduced by the copulas increases portfolio capital requirements and reduces diversification gains. Under the Euler allocation method for instance, total capital in the independent case was 105.16, with the portfolio diversification benefit of 36.53%. When tail dependence is introduced through the survival Clayton copula, capital allocation rise to 172.44 and diversification benefit drops to 7.46%. The Gaussian copula, requires under the Euler allocation method a total capital of 138.40 and a diversification benefit of 16.32%. These results confirm that while independence maximizes diversification, real world examples can erode the diversification benefits and lead to higher capital requirements. A similar pattern holds under the Value-at-Risk (VaR) allocation method. Although the absolute capital levels differ, the relative ordering of scenarios is the same as the Euler method.

When only comparing survival Clayton copula to the Gaussian copula, we noticed that the diversification benefits are higher under the Gaussian model. As mentioned in the previous example, the capital requirements for the Gaussian copula are lower then for the survival Clayton copula, and the diversification benefit is higher for the Gaussian copula. At first glance, this might means that the Gaussian copula allows for better diversification. In reality, this is due to the absence of the tail dependence in the Gaussian model.

This improvement in diversification when transitioning from the survival Clayton copula to the Gaussian copula is primary due to differences in how dependence is modeled. The Gaussian copula assumes symmetric dependence and therefore exhibits asymptotic independence in both tails for correlations less than one. This means that as one component experiences an extreme loss, the probability that other components also experience large losses approaches zero. This may lead to an overestimation of the diversification benefits in scenarios where extreme losses across multiple components are likely to occur at the same time.

In contrast, the survival Clayton copula captures upper-tail dependence, meaning it reflects that there actually is a real chance of multiple large losses happening together. This results in a more conservative capital allocation and a smaller diversification benefit, but also a more realistic assessment of risk in a portfolio where extreme co-movements are relevant.

Our results show that diversification benefits decline as dependence increases. The independent case provides the highest diversification (36.53%), followed by the Gaussian (16.32%), while the survival Clayton shows the lowest of (7.46%) but also the most conservative capital estimate. Although Gaussian appears more capital efficient, it likely underestimates joint risk. The survival Clayton copula captures better the extreme co-movements, making it the most realistic model for portfolios vulnerable to extreme market events.

5.2 Sensitivity Analysis

We observed that even small changes in the θ-parameter of the Survival Clayton copula led to noticeable shifts in how capital was allocated. By comparing the standard survival Clayton copula from task two with the weaker variant from task number four where $\theta_{UV} = 1$, $\theta_{XY} = 0.5$, and $\theta_{WZ} = 0.1$, we observed a significant increase in diversification benefits. Under the Euler allocation method, the diversification benefits went from 6.29% to 15.75%. Similarly, under the Value-at-Risk (VaR) method, diversification benefits increased from 7.47% to 24.94%. This shows that a reduction in tail dependence can improve capital efficiency by enhancing diversification.

The survival Clayton copula captures asymmetric tail dependence, meaning that portfolios are more vulnerable to joint extreme losses, particularly when θ is high. A high θ means that there is a strong dependence in the upper-tail, causing risk to be concentrated under extreme scenarios. When θ is reduce, the dependency between components in the upper-tail weakens allowing the portfolio to benefit from more diversification. This change not only gives a better picture of the portfolios risk, but also shows how important it is to adjust the copula parameters to reflect how the risks are actually connected.

5.3 Capital Allocation Methodologies: Euler Principle (ES) vs VaR

In this study, we compare two capital allocation methodologies, based in different risk measures and allocation principles: one based on the Euler principle applied to Expected Shortfall (ES), and the other one based on Value-at-Risk (VaR) using local conditioning. While both methods are commonly used in practice, they differ in how they handle extreme losses and in how reliable they are when capital must be allocated across dependent risks.

From a theoretical point of view, ES is a what is called a coherent risk measure, and therefore satisfying properties such as *monotonicity*, *positive*, *homogeneity*, *translation invariance and subadditivity*. These features ensure that ES encourages diversification and allows portfolio allocation based on the Euler principle where marginal contributions would exactly equal total portfolio risk. This makes ES compatible with performance measures like Return on Risk-Adjusted Capital (RoRAC), as described by Tasche's theorem.

VaR, on the other hand, is not coherent in general, failing to be sub-additive for heavy-tailed or dependent distributions, and is also non-differentiable, so Euler-based allocation is unavailable. In our results, however, sub-additivity held. The VaR-based capital allocation method uses a numerical approximation instead: capital is allocated by conditioning on a small range around the VaR cutoff of the portfolio. While this approach is practical, it can lead to numerical instability or sample-size sensitivity.

Our results consistently confirm the theoretical differences between ES and VaR in all four scenarios. In every case, ES allocates more capital to the portfolio than VaR, showing the sensitivity to losses beyond the quantile threshold. This difference is the most significant under the survival Clayton copula. Under this dependence scenario, ES assigned total capital of 172.44, while VaR allocated only 88.69, thereby significantly underestimating the required capital in a extreme loss scenario. In addition, the diversification benefit under ES was 7.46%, compared to a higher and potentially overstated 12.59% under VaR. These findings show that while ES consistently provides more conservative capital estimates, the divergence from VaR becomes especially significant in the presence of tail dependence, aligning with theoretical expectations.

We can also see this difference in how the methods behave in our convergence analysis (Figure 2), where ES fluctuates more with sample size than VaR. When the number of simulation increases, capital allocation under the ES-based Euler method shows more fluctuation and slower convergence for heavy -tailed components such as Y. In contrast, the VaR-based method displays more stable behavior across the different simulations sizes. While ES provides theoretically superior and more risk-sensitive allocation, it is more demanding in terms of simulation size and may suffer in estimation instability. Therefore, when having limited data, VaR may appear more stable even if it potentially underestimates tail risk.

In contrast, VaR is computationally efficient, globally accepted, and deeply embedded in regulatory frameworks such as Basel III and Solvency II. For portfolios of low dependence and light-tailed risks, VaR risk estimates can be operationally viable for reporting or compliance purposes. However, this simplicity comes at the cost of reduced stress-scenario and capital optimization function reliability.

In conclusion, while VaR remains useful for communicative and regulatory reasons, ES coupled with Euler allocation is stronger for risk-sensitive capital allocation.Thanks to its strong theoretical foundation and sensitivity to tail risk, ES with Euler allocation is the preferred method for accurate capital allocation in portfolios exposed to systemic or joint extreme events.

5.4 Dependence Structure Visualization and Interpretation

In an effort to better understand how different dependence structures influence the joint behavior of risk components, (Figure 3 to Figure 6) presents ranked scatter plots for the pairs (U, V), (X, Y), and (W, Z) for every dependence scenario: independent, survival Clayton, Gaussian and weaker survival Clayton.

In the independent scenario (Figure 3), the scatter plots is uniformly scattered across the square, and there is no visible pattern or clustering. This confirms the assumption of that the components are uncorrelated with no dependence. Extreme values in one variable are completely unrelated to those in the other. As a result, joint extreme events are unlikely and diversification potential is maximized.

Under the survival Clayton copula (Figure 4), concentration emerges in the upper-right corners of the plots. This pattern indicates the upper-tail dependence, meaning that large values in one variable are likely to coincide with large values in the other. The differences in dependence strength between the variable pairs are also visible and correspond to the parameter θ. The pair (U,V), in particular with highest $\theta = 2$ shows the most pronounced clustering in the upper tail. The (X, Y) pair, with $\theta = 1.0$, have moderate upper-tail dependence, while (W, Z), linked with the weakest dependence $\theta = 0.5$ shows mild concentration. A similar pattern can be observed under the weaker survival Clayton copula in figure 6, but with less clustering in the upper tail in all variable pairs. The overall dependence is weaker and the upper-tail co-movement is present but to the same degree as the (Figure 4). This is evidence that lower θ values reduce the probability of extreme outcomes happen at the same time.

In the (Figure 5), the Gaussian copula exhibit an elliptical pattern that reflect the dependence across the distribution. This means that both low and high values in one variable tend to occur with similarly extreme values in the other. The dependence is spread evenly, without a specific focus on either lower or upper tail. The strength of dependence is visually evident in these plots in the shape of the ellipses. (U, V) plot, based on a correlation of ρ=0.684 shows the most pronounced elliptical structure. The (X, Y) pair, with ρ=0.479 exhibits a weaker pattern, and (W, Z), with the lowest correlation ρ=0.296, appears cloud-like with only small dependence.

5.5 Own findings

One observation during the analysis was how the Frechet distributed components, particularly Y, consistently receive a large share of the total capital in all four dependence scenario. This outcome can be explained by the statistical properties of the Frechet distribution, which has a heavy right tail and no defined variance when the shape parameter $\alpha < 2$. In our case, $X \sim \text{Fréchet}(\alpha = 1.6)$ and $Y \sim \text{Fréchet}(\alpha = 1.4)$, so both can produce very large loss values. Because Y has a lower α, it has a heavier tail than X, and is more likely to generate extreme losses.

This Pattern is confirmed in the results. For example, under survival Clayton copula, Euler-based capital allocation assigns 57.41% of the total capital to Y, compared to only 26.90% to X. Similarly in the Gaussian copula, Y still receives 60.38% of the capital while X receives 28.18% despite both being heavy-tailed. This shows how a small differences in how heavy the tails are can have a big impact on how much capital each component gets.

Both Value-at-Risk and Expected Shortfall are tail-based risk measures. For heavy-tailed distributions like Frechet, tail values can be very large which then can lead to elevated risk contributions. When copula based dependency are introduced, these extreme losses are more likely to happen at the same time, which increases how much Y contributes to the total portfolio losses.

6 Conclusion

Our results confirm the theoretical expectations discussed in this report. Dependence structure plays an important role in capital allocation and diversification outcomes. In particular, survival Clayton copulas leads to substantially higher capital requirements and lower diversification benefits. Gaussian copulas, while more capital-efficient, underestimate join tail risk due to their lack of tail dependence.

Expected Shortfall (ES), combined with Euler-based allocation, produced more conservative and risk-sensitive capital estimates compared to Value-at-Risk (VaR), especially unders trong tail dependence. VaR underestimated risk in scenarios involving heavy tails and strong dependence.

Finally, our findings highlight how extreme loss contributes from heavy-tailed components, such as Fréchet-distributed Y, can dominate portfolio capital.

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A Additional Graphs

A.1 Distributions of the Risk Components U, V, X, and Y

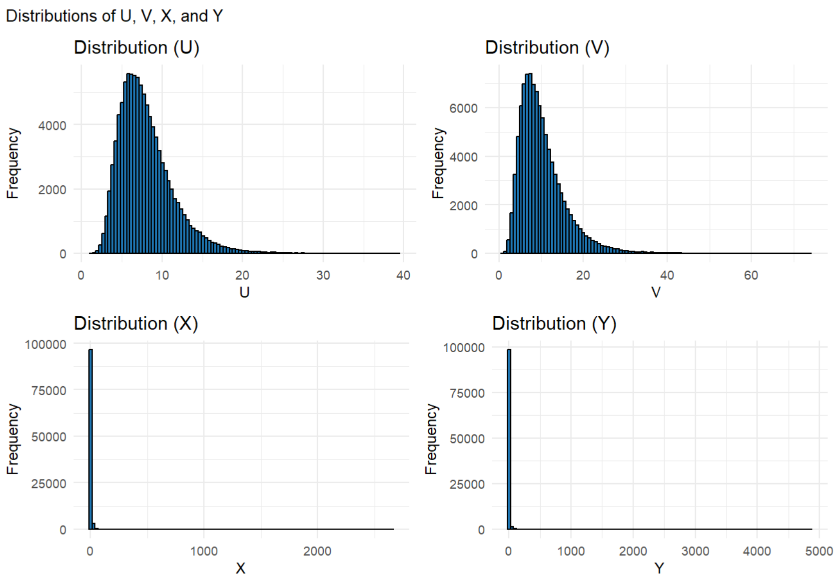

Figure 7: Histogram plots of the marginal distributions of the four base risk components. U and V are lognormally distributed with moderate skewness, while X and Y are Fréchet-distributed. This plot is produced based on 100,000 simulations.

A.2 Dependence Structure: Independent Copula (Task 1)

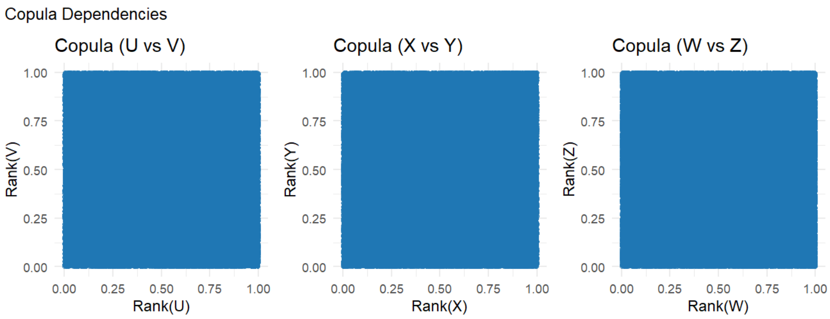

Figure 8: Ranked scatter plots of independent dependence structures for variable pairs: (U, V) (left), (X, Y) (middle), and (W, Z) (right). This graph is created by 100,000 simulations.

A.3 Dependence Structure: Survival Clayton Copula (Task 2)

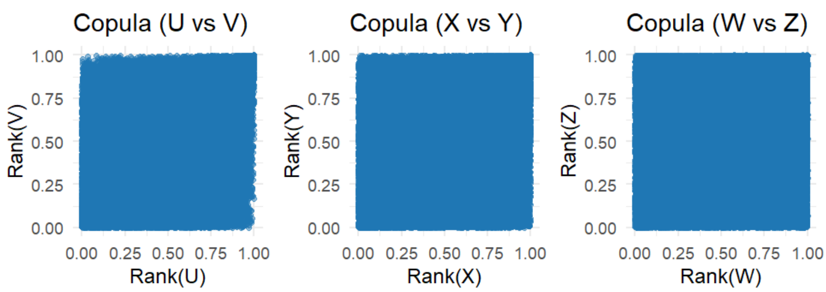

Figure 9: Scatter plots of ranked data representing the dependence structure between pairs of variables under the Survival Clayton copula: (U, V) (left), (X, Y) (middle), and (W, Z) (right). This version is based on 100,000 simulations.

A.4 Dependence Structure: Gaussian Copula (Task 3)

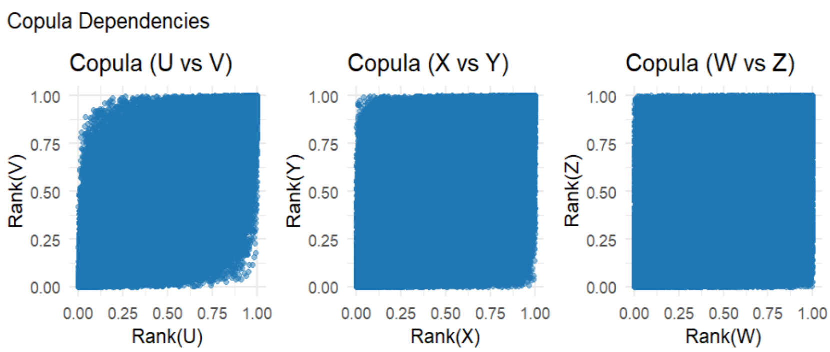

Figure 10: Ranked scatter plots of dependence structures under the Gaussian copula for variable pairs: (U, V) (left), (X, Y) (middle), and (W, Z) (right). This version is based on 100,000 simulations.

A.5 Dependence Structure: Weaker survival Clayton copula (Task 4)

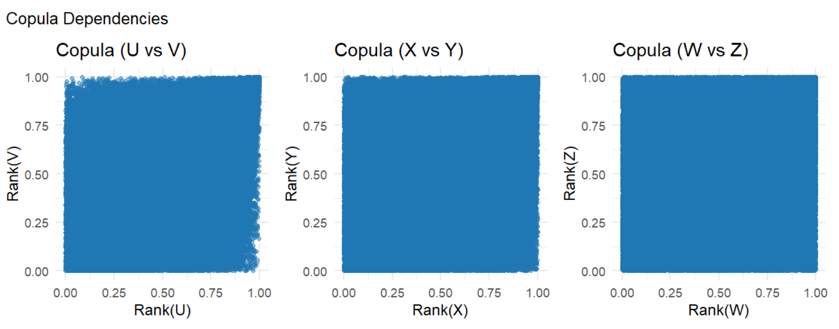

Figure 11: Ranked scatter plots of dependence structures under the weaker Survival Clayton copula for variable pairs: (U, V) (left), (X, Y) (middle), and (W, Z) (right). This version is based on 100,000 simulations.