Capital Allocation, Diversification Benefit and Dependence

Research Project for the Course *Quantitative Risk Management*

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Use R or Python for the programing and Latex to write the project (output of Python for latex does not count. It should be a report you write in Latex). The report must be limited to 15 pages; submissions exceeding this length will be disregarded. It should include written text, tables, and figures, but not computer code, which may be placed in an appendix if desired. The document should be structured as a presentation for individuals (your future manager) unfamiliar with the issue, thus it needs to be self-contained and incorporate a theoretical introduction. As much as possible, use table with values expressed as percentages to two decimal places (e.g., 12.34%).

Work can be done by a group of 2 or 3 (maximum), but the writing of the project must be individual (Do not forget to put your name on your report as well as to indicate the names of the other students who worked with you). I will control that you wrote your report personally. If not, the exam will be failed.

This project aims to examine capital allocation methodologies, with particular focus on the Euler principle and its application across different risk measures. Additionally, we investigate how the dependence structure between risks within a portfolio influences the resulting diversification benefits.

I – Setup

Let us consider a portfolio containing four distinct risks, where the potential losses[1] associated with each risk are mathematically represented by random variables (rv's) U, V, X, and Y, which are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. When referring to specific realizations of these loss random variables, we will use the corresponding lowercase letters u, v, x, and y. We will aggregate the risks in two steps (hierarchical tree):

1. First, aggregating U and $V : U + V = W$,
   as well as Y and $X : X + Y = Z$
2. Then, construct the portfolio P by aggregating W and Z: $P = W + Z$

See Figure 1, for an explanation. We allocate the capital, C, to a risk, say X, in a portfolio

[1] Here the losses are positive values representing the amount.

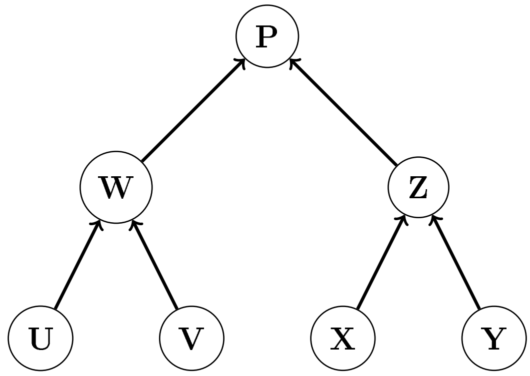

Figure 1: The aggregation tree for the study

P, according to the Euler principle, using Tasche's theorem (see chapter one, "the Price of Risk", of the course):

$$C(X \mid P) = \mathbb{E}\left[X \mid P \geq F_P^{-1}(\alpha)\right] - \mathbb{E}[X]$$

where the second term is the conditional expected shortfall (where the rv X represents a positive loss). In this case: $C(X) = \mathbb{E}\left[X \mid X \geq F_X^{-1}(\alpha)\right] - \mathbb{E}[X]$.

We can use another allocation principle based instead on Value-at-Risk:

$$C(X \mid P) = \mathbb{E}\left[X \mid P = \mathrm{VaR}_\alpha(P)\right] - \mathbb{E}[X]$$

where the Value-at-Risk, for a rv X, is defined as:

$$\mathrm{VaR}_\alpha(X) = \mathrm{Inf}\{x \in \mathbb{R} \mid \mathbb{P}(X > x) \leq 1 - \alpha\}$$

Here the allocated capital is obtained as the average of those X values for which the corresponding total loss P is equal to the α-quantile of the portfolio distribution (in practice, you take values of P between $\mathrm{VaR}_{\alpha-0.1\%}(P) \leq P \leq \mathrm{VaR}_{\alpha+0.1\%}(P)$, which represents about 200 values in the case of 100,000 simulations) and $C(X)$ is simply $\mathrm{VaR}_\alpha(X) - \mathbb{E}[X]$.

We define the diversification benefit as:

$$D_{\rho,\alpha,n} = 1 - \frac{C_{\rho,\alpha,n}(X \mid P)}{C_{\rho,\alpha,n}(X)}$$

where ρ is the chosen risk measure (here either ES or VaR) α is the threshold of the risk measure and n the number of risks (here 4). For the portfolio, the diversification benefit reads:

$$D_{\rho,\alpha,n} = 1 - \frac{C_{\rho,\alpha,n}(P)}{\sum_{i=1}^{n} C_{\rho,\alpha,n}\left(X_i\right)}$$

II – Theory

Present the Euler principle and the Tasche Theorem. Briefly expose the properties of a good capital allocation method (see slides of chapter III: "Capital Management") and discuss its relationship with the coherence properties of risk measure.

Briefly present the Gauss and the Clayton copulas, discuss the difference between Clayton and mirrored or survival Clayton, then explain how to generate realizations of two dependent random variables according to those copulas (see slides of chapter II: "Aggregation of Risks").

III – Problem

Simulate $100,000$ realizations of the four random variables U, V, X, Y where:

- U and V are lognormally distributed with parameters μ and σ (related to the mean and standard deviation):

$$U \sim \text{Lognormal}(\mu = 2.00, \sigma = 0.40) \tag{1}$$
$$V \sim \text{Lognormal}(\mu = 2.20, \sigma = 0.50) \tag{2}$$

- X and Y are Fréchet distributed with shape parameter α, location parameter m, and scale parameter s:

$$X \sim \text{Fréchet}(\alpha = 1.60, m = 5.5, s = 1) \tag{3}$$
$$Y \sim \text{Fréchet}(\alpha = 1.40, m = 5.4, s = 1) \tag{4}$$

Set the threshold α at 99% for ES. One could consider the leaves U, V, X, Y as business units (e.g. property, motor, motor liability, and natural catastrophe coverage). In the following, we will use the terms "leaves" and "business units" interchangeably to refer to the initial risks U, V, X, Y, before aggregation.

1. Compute the capital allocated to the four risks in the portfolio P and compute the resulting diversification benefits, using the two different capital allocation methods. Check the convergence of the allocation by changing the number of simulations from 100 to 1'000'000 simulations.
2. Now take U and V linked via a mirrored (also called survival) Clayton copula of parameter $\theta = 2.0$, and then X and Y linked via a mirrored Clayton copula of parameter $\theta = 1.0$, while W and Z are linked via a mirrored Clayton copula with parameter $\theta = 0.5$. Then, repeat question 1 with this new set up.
(**Important note:** When ordering W and Z, ensure that you perform the ordering on the pairs of realizations (u, v) and (x, y) respectively. This approach preserves the critical relationship: $p = x + y + u + v$).
3. Repeat question 2 while using instead a Gauss copula. To parametrize the Gauss copulas, use the rank correlation that you can compute from question 2 between U and V, between X and Y, and between W and Z.

4. Change the parameters of the survival Clayton copula between U and $\vee$ by using the $\theta = 1.0$, and then X and Y linked via a survival Clayton copula of parameter $\theta = 0.5$, and a survival copula with $\theta = 0.1$ between W and Z and repeat question 2.

5. Create two tables (one for each allocation method) displaying, for each variable, U, V, X, Y, W, Z, P:

   - Average loss
   - corresponding risk measure (ES at 99% or VaR at 99.5%), standalone and in the portfolio
   - Capital standalone
   - Allocated capital in the portfolio
   - Diversification benefit

   To facilitate clear comparison, present all numerical values as percentages with two decimal places where appropriate (e.g., 12.78%). Analyze how the dependence structure influences the outcomes by examining:

   - The effect of changing between different dependence models (such as transitioning from Clayton survival copula to Gaussian copula)
   - The sensitivity of results to parameter variations within the Clayton survival model
   - The comparative advantages and limitations of both capital allocation methodologies

   Additionally, create square-formatted rank scatter plots for the pairs (U, V), (X, Y), and (W, Z) under all four dependence scenarios (independent, survival Clayton, Gaussian copula, and weaker survival Clayton). Provide a detailed discussion of the patterns and relationships revealed in these visualizations.