CSE 380	Discrete Mathematics II
More About Infinity	
January 13, 2021	*Learn*

Learn About Infinite Sets

A Famous Set-Theoretic Paradox?

Russell's paradox

Which says

Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$.

Contradictory!

- Show that the assumption that S is a member of S leads to a contradiction.
- Why?
  - By definition $S \in S$ means S is **not** a member of S.
  - Paradoxical!
    - Show that the assumption that S is **not** a member of S leads to a contradiction.
    - Why?
      - By definition $S \notin S$ means $S \in S$.

An Analogy

Divide the set of words into two sets:

1

- self-describing words
- For example small, polysyllabic, awkwardnessful

2

- NON-self-describing
- For example long, monosyllabic, edible
- Non-hyphenated names

  - Call set 1 the autological words.
  - Call set 2 the heterological words.
- The Question Is *heterological* heterological?

Get Technical

Learn how to Compare Set Sizes

Let S and T be (finite or infinite) sets.

Recall that |S| and |T| denote the **cardinality** of S and T, respectively.

Define |S| = |T| (the cardinalities of S and T are **equal** iff there is a *one-to-one correspondence* from S to T. (Recall bijection?)

Define |S| $\leq$ |T| (the cardinality of S is **less than or equal to** the cardinality of T) iff there is a one-to-one *function* from S to T. (Recall injection?)

If $|S| \leq |T|$ but $|S| \neq |T|$

then $|S| < |T|$.

(the cardinality of S is **strictly less** than the cardinality of T).

- Also works for $\subseteq$
  - if A $\subseteq$ B but A $\neq$ B then A $\subset$ B.

Squeeze play

$$|S| = |T| \leftrightarrow |S| \leq |T| \wedge |T| \leq |S|.$$

In first-order logic

- p $\leftrightarrow$ q $\equiv$ (p $\rightarrow$ q) $\wedge$ (p $\leftarrow$ q)

Squeeze Play Example: Legal C Programs

- Exactly how many legal C programs are there?
- Note: *legal* does not mean useful, or meaningful, or even *compilable*!
- Infinitely many, but how do we show that it is a *countable* infinity?
- Let LCP = {x | x is a string containing a legal C program}.
- Let N = {0, 1, 2, 3, ... } (the set of all natural numbers).

What is an injection from N to LCP?

- f_1 : N → LCP
- One of many mappings
  - 0 → "main(){}"
  - 1 → "main(){;}"
  - 2 → "main(){;;}"
  - ...
  - n → "main(){;;; ... ;}"
  - ...

What is an injection from LCP to N?

- f_2 : LCP → N

f_2 looks at each program in LCP as a bitstring, and returns the equivalent unsigned binary integer (converted to decimal).

- LCP_0 → 1398
- LCP_1 → 62308
- LCP_2 → 730129
- ...
- LCP_n → 987131734445134773
- ...
- Since by definition, the existence of f_1 guarantees that $|N| \leq |LCP|$,
- and the existence of f_2 guarantees that $|LCP| \leq |N|$,
- it follows that $|N| = |LCP|$.

QED

- Thus, there are as many legal C programs as there are natural numbers.

Ponder me this

Does a proper subset always have a smaller cardinality?

If we deal with finite sets, then $S \subset T$ always implies $|S| < |T|$.

But for infinite sets, *this is not necessarily the case!*

Let E be the set of even natural numbers. How does its cardinality relate to the cardinality of N (the set of all natural numbers)?

Let $f : N \to E$ be defined by $f(n) = 2n$.

Then f is a one-to-one correspondence, therefore $|E| = |N|$.

N:	0	1	2	3	4	5	6	7	...
E:	0	2	4	6	8	10	12	14	...

Ponder me this too

Do all infinite sets have the same cardinality?

The answer is

- **NO!** It would be incredibly uninteresting if they did.
- But finding sets of different infinite sizes is another matter.

It's fairly easy to see that the cardinality of the positive integers is equal to the cardinality of the positive rationals.

1/1	1/2	1/3	1/4	...
2/1	2/2	2/3	2/4	...
3/1	3/2	3/3	3/4	...
.	.	.	.	...
.	.	.	.	...

It's a little harder to show that $|Z^+| = |Q|$.

However

- there *are* cardinalities strictly greater than $|Z^+|$.
- In particular,

$$|R| > |Z^+|$$

(there are "more" real numbers than positive integers)!

Define the difference

- Sets that are finite or have the same cardinality as Z^+ (or N, it doesn't matter) are called **countable**.
- Countable sets are sets whose elements can be **listed**.
- $e_1, e_2, e_3, \ldots, e_n$ (if the set is finite) or
- $e_1, e_2, e_3, \ldots$ (if the set is infinite).
- By Contrast Uncountable sets are sets whose elements **CANNOT** be listed.

Learn how to Generate Them

Power Setting One

	1	2	3	4	5	...
1	0	0	0	0	0	...
2	0	1	0	1	0	...
3	1	0	1	0	1	...
4	0	1	1	0	1	...
5	0	0	1	0	0	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Generates the Next

	1	2	3	4	5	...
1	1	0	0	0	0	...
2	0	0	0	1	0	...
3	1	0	0	0	1	...
4	0	1	1	1	1	...
5	0	0	1	0	1	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

To Summarize

- Use N instead of Z^+
- $|P(N)| > |N|$
- $|S| \leq |T|$ if there is a one-to-one *mapping* between S and T.
- $|N| \neq |P(N)|$ must be argued for and proven to conclude $|N| < |P(N)|$
- $\aleph_0$ (Aleph Null) is the cardinality of the **smallest** infinite set.
- $\mathbb{Z}^+$

Inverting a Pairing Function

$$f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$$

With $y = f(m,n)$ and $x = m+n$,

$$y = \frac{x^2 - 3x + 2}{2} + m.$$

Move y to the other side:

$$0 = \frac{x^2 - 3x + 2}{2} + m - y$$

Multiply both sides by 2:

$$0 = x^2 - 3x + 2 + 2(m - y).$$

Pretend m and y are constants and solve the quadratic equation

$$ax^2 + bx + c = 0$$

with $a = 1, b = -3, c = 2(m - y + 1)$:

$$x = \frac{3 \pm \sqrt{9 - 8(m - y + 1)}}{2} = \frac{3 \pm \sqrt{8y - 8m + 1}}{2}$$

Taking the positive root and letting $m = 0$ (which it never is, but just pretend!):

$$x = \frac{3 + \sqrt{8y + 1}}{2}$$

Finish by justifying this Python implementation:

```
x = ceil((3 + sqrt(8 * y + 1)) / 2) - 1
```

Or even shorter (thanks Kyle!):

```
x = int((3+sqrt(8*y))/2)
```