Summer Internship Project

Report

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Abstract

In this report our main goals would be to familiarize ourselves with the methods of forming dynamical systems and analysing Dynamical systems after which we will conduct a investigation on the Nonlinear Schrodinger Equation which physically models the propagation of optical pulses in a nonlinear media.We will explore interesting solutions like solitons which arises as a consequence of the non-linearity and the dispersive properties completely neuralising each other.

Contents

1 Introduction

1.1 Differential Equations

Differential equations are used to model dynamical systems evolving through continuous time. Differential equations come in two flavours, ones involving only derivatives of a single independent variables are termed as Ordinary Differential equations and Differential equations involving partial derivatives of more than one independent variable is termed as partial differential equations . Our study is restricted to Ordinary differential equations(ODEs).

For ODEs the order of the equation is the highest derivative present in the equation. In general , analytically solving a differential equation proves to be a challenging task;we create a workaround to this problem by using computers to give a solution of desired accuracy using numerical methods like euler's method, runge kutta method and obtain a qualitative idea about the nature of solutions by constructing a space of all the dynamical variables which is called the phase plane.A specific solution of an ODE can be visualised as a phase trajectory in the phase plane.A point in the phase plane of n dynamical variables is denoted by $(x_1, x_2, x_3, ..., x_n)$.

Here ,the velocity of each variable is a function of all the dynamical variables

$$\dot{x}_1 = f_1(x_1, x_2, ...x_n)$$

$$.$$

$$.$$

$$\dot{x}_n = f_n(x_1, x_2, ...x_n)$$

Systems can have explicit time dependence(*Non-autonomous systems*) which are generally harder to solve than systems with no explicit time dependence(*Autonomous systems*) Non-autonomous systems can be made autonomous by assigning time as a new dependent variable which increases the dimension of the phase space by 1

For a n-Dimensional system we define:

$$x_{n+1} = t$$

$$\dot{x}_{n+1} = 1$$

1.2 Flows in 1D

First order ODEs are the simplest dynamical systems, where there is only one variable dependent on the independent variable.

$$x = f(x)$$

1.2.1 Fixed points and stability

for any point x in the domain the movements are to the left($f(x) < 0$) , to the right ($f(x) > 0$) or zero movement($f(x) = 0$). These movements are usually understood as flows; Fixed points are phase points where the magnitude of the flow is zero.

Further fixed points can be classified based on the nature of flows in the neighbourhood of the fixed point.

- **Unstable Fixed Points:** $\frac{df}{dx} > 0$ any disturbance from the fixed point will grow exponentially hence the nature of the fixed point is unstable.
- **Stable Fixed Points:** $\frac{df}{dx} < 0$ any disturbance from the fixed point decays exponentially towards the fixed point hence the nature of the fixed point is stable.
- **Semi-stable or Inconclusive Fixed points:** $\frac{df}{dx} = 0$ The flow is unidirectional indicating disturbances grow exponentially in one direction and decay exponentially into the fixed point on the other. It is also possible that the test may be inconclusive , then we resort to the n^{th} derivative test to understand the nature of flows

1.2.2 Linear stability analysis

Any function can be approximated as a polynomial function by Taylor series approximation. Taylor series around x^* and all derivatives evaluated at $x = x^*$:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x^*)^n}{n!} \frac{d^n f}{dx^n}$$

Let's define

$$\eta(x) = x - x^* \tag{1}$$

For smaller perturbations we neglect the higher order terms and only contributions by the first power of η is significant

$$f(\eta) \approx f(x^*) + \eta \frac{df}{dx}$$

Also since this expansion is around a fixed point $f(x^*) = 0$,then

$$f(\eta) \approx \eta \frac{df}{dx}$$

Our classification of Fixed points feels more natural through the lens of Linear approximation of Taylor series.

1.2.3 Existence and Uniqueness

Solutions from a given initial conditions need not be unique unless the function and it's derivative are continuous.

For example $\dot{x} = x^{1/3}$ starting from x=0 does not have an unique solution

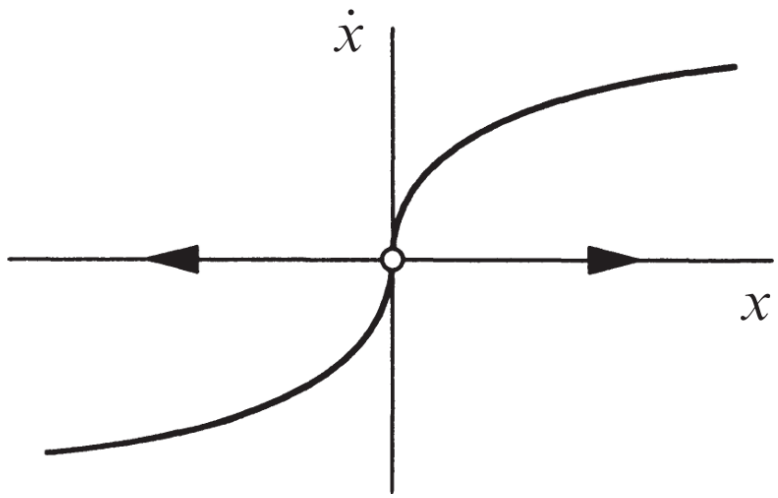

Figure 1: $\dot{x} = x^{-1/3}$ [1]

- $x = 0 \implies \dot{x} = 0$, hence x=0 is a fixed point
- we can also integrate to obtain: $\int x^{-1/3}\, dx = \int\, dt \implies x^{2/3} + C = t \implies x = t^{3/2}$ (since x(0)=0)

This Non-uniqueness arises due to the slope of f(x) being infinite at x=0 which makes the fixed point very unstable.

We further restrict our study to functions which are continuous and have a continuous first derivative.

1.2.4 Impossibility of oscillations

Oscillations are possible only if we obtain a closed orbit in the phase space . But this will not be possible in case of first order systems as this requires some $\dot{x}$ to have two outputs for a single input which isn't possible In other words phase points never reverses in direction; all phase trajectories either settle towards a fixed point or diverge to $\pm\infty$

1.2.5 Numerical Methods

We have discussed earlier that solving differential equations analytically often time proves to be extremely difficult, So we resort towards Numerical methods to spit out an approximate solution.Smaller and smaller values of time steps yield more and more accurate results.We will discuss about **Euler method, improved Euler method,Runge Kutta Method**

- *Euler method:*

  Euler's method is the simplest method to numerically solve ODEs By first order approximation:

  $$\Delta x \approx f(x)\Delta t$$

  This gives us an algorithm for obtaining x(t)

  $$x_{n+1} = x_n + f(x_n)\Delta t \tag{2}$$

  $$\implies x_{n+1} = x(t_n + \Delta t)$$

  by first order approximation we get:

  $$x_{n+1} = x_n + f(x)|_{x=x_n}\Delta t$$

  since we start neglecting from powers of two the order of the error term is of order 2 denoted by $\mathcal{O}((\Delta t)^2)$ It is observed that the deviation compounds quickly in Euler's method. The algorithm is modified to obtain the improved Euler's method.

- *Improved Euler's method:* This involves taking a trial step using the Euler's method:

  $$\tilde{x_{n+1}} = f(x_n)\Delta t + x_n$$

  Now the actual step :

  $$x_{n+1} = \frac{(f(x_n) + f(\tilde{x_{n+1}}))\Delta t}{2} + x_n$$

  By doing a Linear expansion we can conclude that the error term is $\mathcal{O}(\Delta t^3)$

- *Runge kutta method:* Runge kutta method is extensively used to numerically solve Differential Equations as it strikes the sweet spot of time complexity and accuracy. Finding x_{n+1}:

  $$k_1 = f(x_n)\Delta t$$

  $$k_2 = f(x_n + \frac{k_1}{2})\Delta t$$

$$k_3 = f(x_n + \frac{k_2}{2})\Delta t$$
$$k_4 = f(x_n + k3)\Delta t$$
$$x_{n+1} = x_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

We can work out the error term to be $\mathcal{O}(\Delta t^5)$

1.3 Bifurcations

Bifurcations is the analysis of fixed points in a dynamical system as one or more independent parameters are varied.Bifurcations in 1D involving a single independent parameter can be represented as

$$\dot{x} = f(x, r)$$

here r is a parameter that is not influenced by the system dynamics.However,r can be tweaked by factors that can be controlled. There are certain special values of r_s where the dynamics of the system is very different for $r < r_s$ and $r > r_s$.The dynamics around these fixed points will be similar to certain prototypical forms called Normal forms; These Normal forms can be derived by Taylor expanding any function of interest.Often times it's useful to plot the fixed points against r for understanding,these are called Bifurcation diagram.Broadly all Bifurcations can be segregated into 4 categories

1.3.1 Saddle Node Bifurcation

Normal form:$\dot{x} = r \pm x^2$ For$\dot{x} = r + x^2$, There exists no fixed points for $r > 0$, at $r = 0$ a semi-stable fixed point appears at the origin and a stable and an unstable fixed points emerges for $r < 0$.

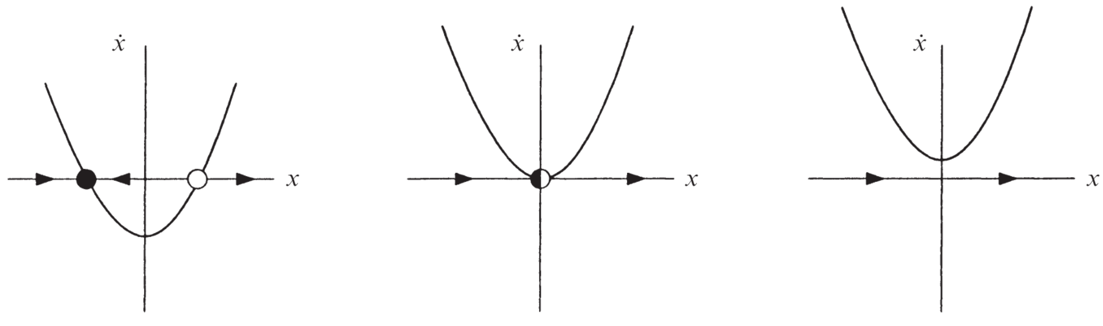

Figure 2: $\dot{x} = r + x^2$ for different r values [1]

Fixed points as function of r:
$\dot{x} = x^2 + r = 0$
$\implies x^* = \pm\sqrt{-r}$

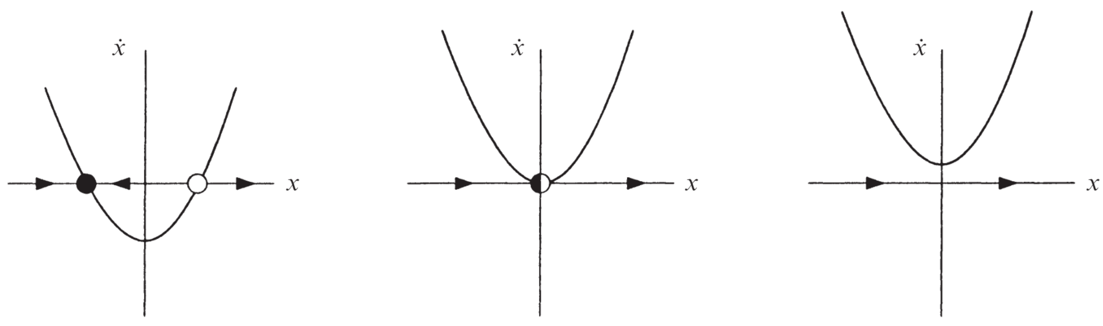

Figure 3: x^* vs r [1]

1.3.2 Transcritical Bifurcation

Normal form: $\dot{x} = rx - x^2$ Here the origin remains a fixed point for all r but the nature of this fixed point changes;Another Fixed point vanishes only for r=0 and moves to other side of the origin as sign of r changes along-with a change of nature of fixed points from stable to unstable or vice-versa.

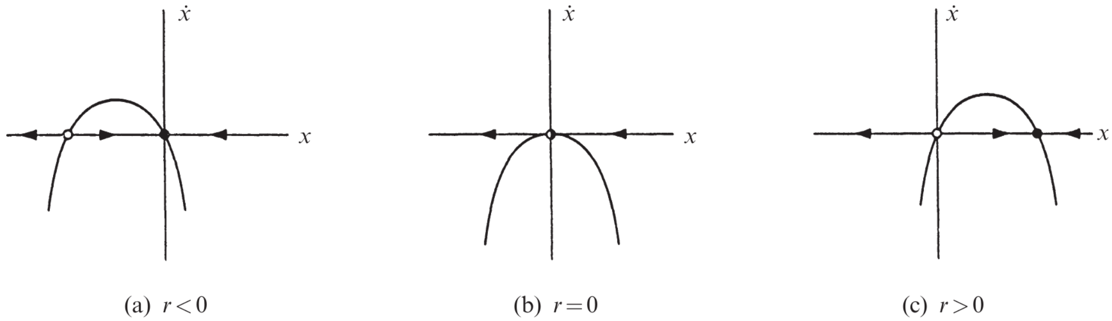

Figure 4: $\dot{x} = rx - x^2$ for different r values [1]

Fixed points as a function of r :
$\dot{x} = rx - x^2 = 0 \implies x^* = r$ and $x^* = 0$

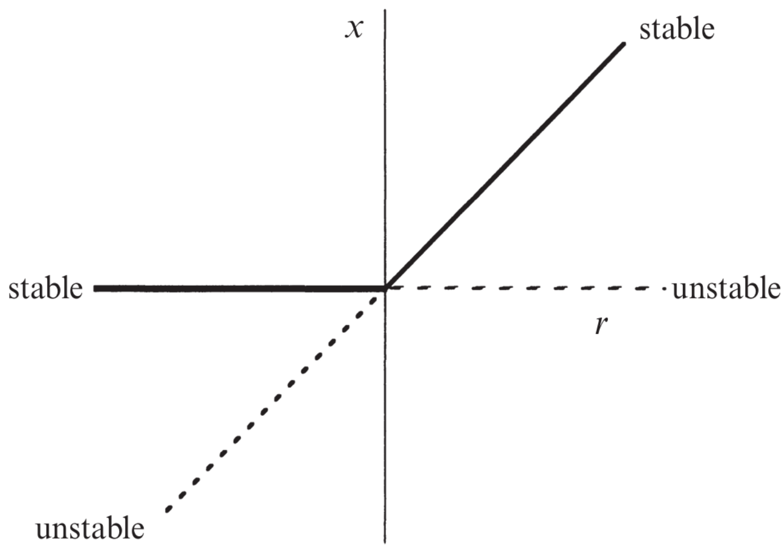

Figure 5: x^* vs r [1]

1.3.3 Pitchfork Bifurcation

These sort of Bifurcations are common in systems with symmetry as fixed points that appear on either side of origin have the same nature and are equidistant from the origin

1. **Supercritical Pitchfork Bifurcation:** Normal form:$\dot{x} = rx - x^3$ here the x^3 term acts as the stabilising term . for $r < 0$ the origin is a stable fixed points and is the only fixed point as r tends to zero the attractiveness of the fixed point reduces and for $r > 0$ the origin becomes unstable and two stable fixed points symmetrically appear on either sides of origin.

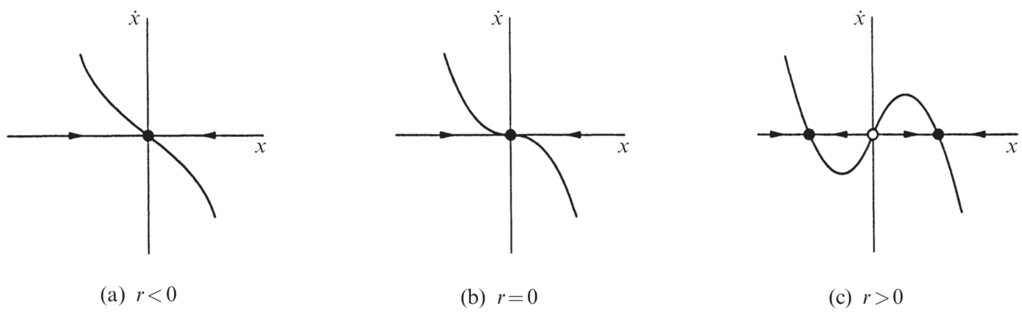

Figure 6: $\dot{x} = rx - x^3$ [1]

Fixed points as function of r: $\dot{x} = rx - x^3 = 0 \implies x^* = 0$ and $x^* = \pm\sqrt{r}$ where x^* is a fixed point

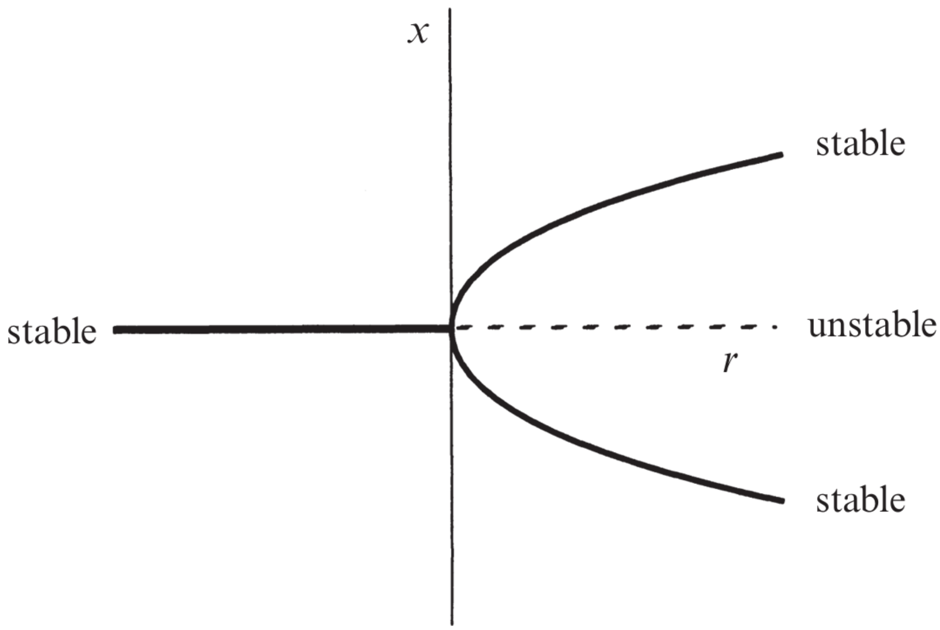

Figure 7: x^*vsr for different r values [1]

2. **Subcritical Pitchfork Bifurcation:** Normal Form: $\dot{x} = rx + x^3$ Here the x^3 acts as a destabilising factor. for $r < 0$ the origin is an unstable fixed points and is the only fixed point, as r tends to zero the repulsiveness of the fixed point reduces and for $r > 0$ the origin becomes stable and two unstable fixed points symmetrically appear on either sides of origin.

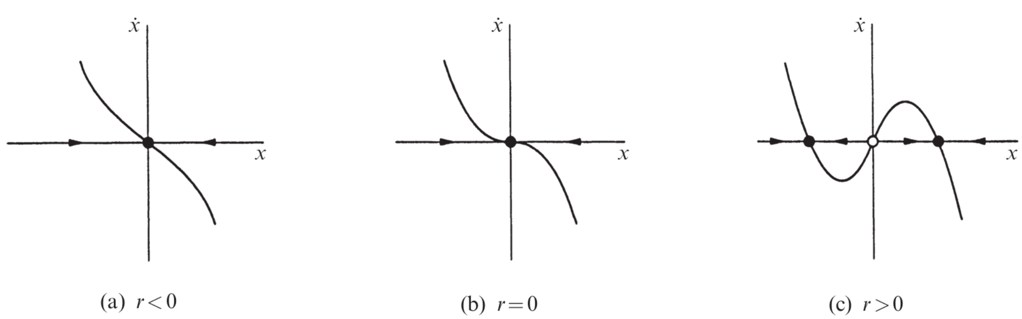

Figure 8: x^*vsr for different r values [1]

1.4 Flows in 2D

General equation for flow in 2D:

$$x = f_1(x, y)$$

$$y = f_2(x, y)$$

Most of the classical physical systems can be modelled by 2 Dimensional Phase space as Newton's Laws of motion involve 2nd order differential equations

1.4.1 Linear Systems

$$x = ax + by$$

$$y = cx + dy$$

This can be more compactly written using Matrix notation

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \, and \, \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This notation makes the Linearity property of the System more visible. If x_1 and x_2 are solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ then,

$$\dot{\mathbf{x}} = \mathbf{Ax} \implies \mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

Eigenvalues and it's corresponding *Eigenvectors* of the linear system constitute solutions which remain on a line. In general we would have two unique Eigenvectors with unitary magnitude , this allows us to represent all the vectors in phase space as linear combination of said eigenvectors which leads to a simpler way of representing the general solution as permitted by the linearity property of matrix multiplication.

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0 \implies 0 = det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

$$\implies \lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

Here τ and Δ are the trace and Determinant of the system.

- $\Delta < 0$: Irrespective of the value of τ, We will have two eigenvalues of different signs suggesting flows decaying exponentially into the origin on one direction and growing exponentially from the origin on the other.Origin here is called a *Saddle point*

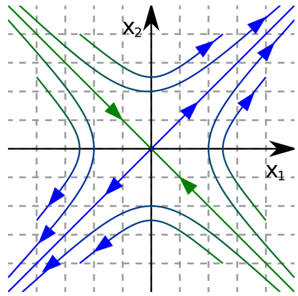

Figure 9: $\lambda_1 > 0$ and $\lambda_2 < 0$

- $\Delta > 0$**:**

  1. $\tau^2 > 4\Delta$:Here both the eigenvalues are of the same sign,hence we get unstable nodes for $\tau > 0$ and stable nodes for $\tau < 0$

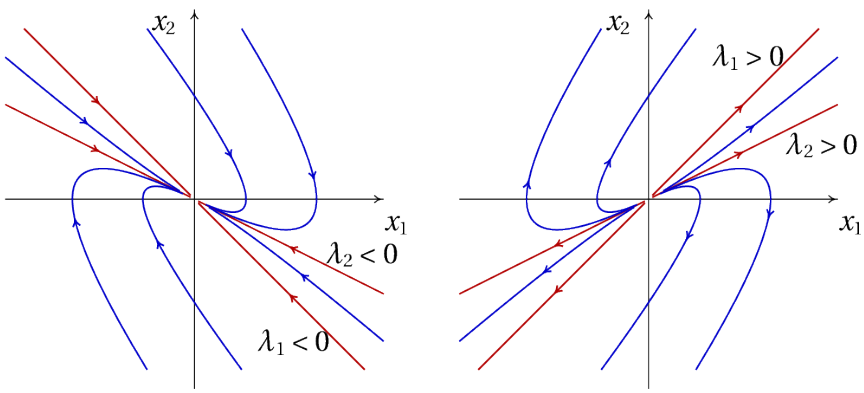

Figure 10: $\lambda_1.\lambda_2 > 0$

  2. $\tau^2 < 4\Delta$: Here we get imaginary eigenvalues signifying a rotating behaviour;$\tau =$ 0 yields purely imaginary eigenvalues resulting in closed orbits around the Origin these sort of fixed points are common in Conservative systems.

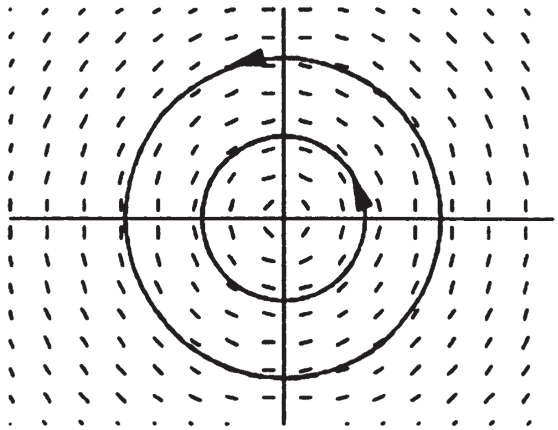

Figure 11: λ_1 and λ_2 purely imaginary [1]

$\tau > 0$ results in phase trajectories spiraling out from the origin ,hence named Unstable spiral.Using similar reasoning
,$\tau < 0$ results in stable spirals.

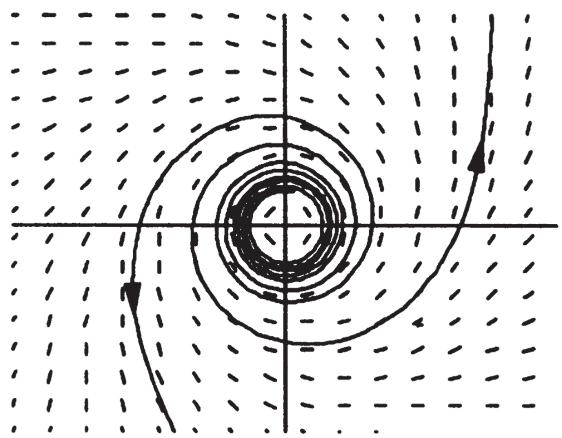

Figure 12: $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0$ [1]

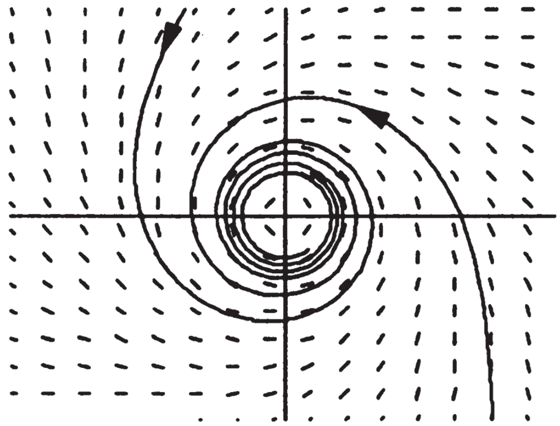

Figure 13: $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$ [1]

3. $\tau^2 = 4\Delta$: This results in a radially symmetric flows called star nodes.It is easy to see $\tau < 0$ and $\tau > 0$ results in stable and unstable star nodes respectively.

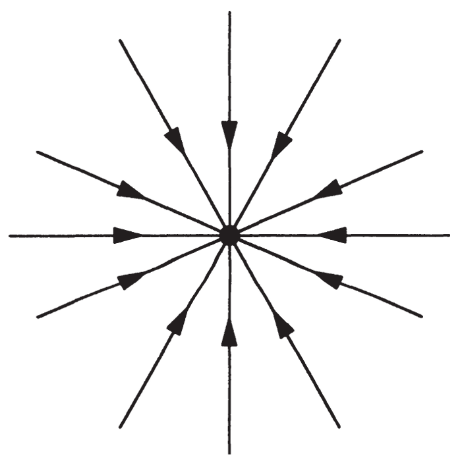

Figure 14: λ_1=$\lambda_2 > 0$ and $\Delta \neq 0$

- $\Delta = 0$**:** The fixed point is called a degenerate node. A good way to think about the degenerate node is to imagine that it has been created by deforming an ordinary node. The ordinary node has two independent eigen directions; all trajectories are parallel to the slow eigen direction as $\lim_{t\to\infty}$ and to the fast eigendirection as $\lim_{t\to-\infty}$ Now suppose we start changing the parameters of the system in such a way that the two eigendirections are scissored together. Then some of the trajectories will get squashed in the collapsing region between the two eigendirections, while the surviving trajectories get pulled around to form the degenerate node

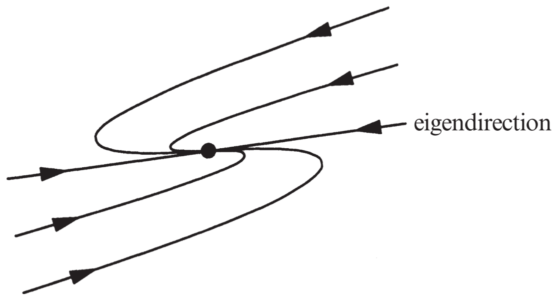

Figure 15: $\Delta = 0$

Here is a plot summarising the points discussed above

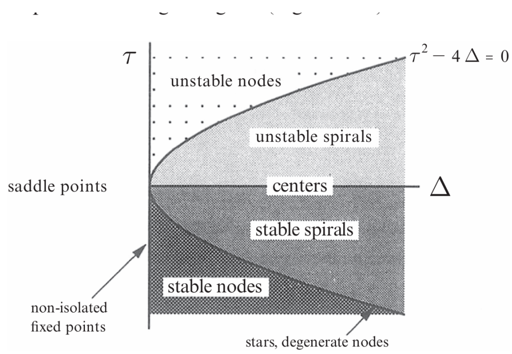

Figure 16: Nature of Fixed points

1.4.2 Nonlinear flows in 2D

Flows in 2D in general looks like

$$\dot{x} = f_1(x, y)$$

$$\dot{y} = f_2(x, y)$$

Our aim will be to linearize the matrix around Fixed points to understand the nature of flows in the neighbourhood of the point Let's linearize f(x,y) around (x_*, y_*) where

$f(x_*, y_*) = 0$

$$f(x,y) \approx f(x_*, y_*) + \left.\frac{\partial f}{\partial x}\right|_{x_*,y_*}(x - x_*) + \left.\frac{\partial f}{\partial y}\right|_{x_*,y_*}(y - y_*)$$

Now for 2D:

$$f_1(x,y) = \left.\frac{\partial f_1}{\partial x}\right|_{x_*,y_*}(x - x_*) + \left.\frac{\partial f_1}{\partial y}\right|_{x_*,y_*}(y - y_*)$$

$$f_2(x,y) = \left.\frac{\partial f_2}{\partial x}\right|_{x_*,y_*}(x - x_*) + \left.\frac{\partial f_2}{\partial y}\right|_{x_*,y_*}(y - y_*)$$

To tidy up this notational framework we define:

$$u = x - x_*$$

$$v = y - y_*$$

the time derivative remains unchanged due to the addition of a constant

This can be represented compactly using the Matrix notation

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x_*,y_*)} \begin{pmatrix} u \\ v \end{pmatrix}$$

1.4.3 Conservative Systems

Newton's law F=ma is the source of many important second-order systems. For example, consider a particle of mass m moving along the x-axis, subject to a non linear force F (x). Then the equation of motion is

$$m\ddot{x} = F(x)$$

Notice that we are assuming that F is independent of both x and t; hence there is no damping or friction of any kind, and there is no time-dependent driving force. Under these assumptions, we can show that energy is conserved, as follows. Let V (x) denote the potential energy, defined by $F(x) = -\frac{dV}{dx}$. Then

$$m\ddot{x} + \frac{dV}{dx} = 0$$

Multiplying both sides by $\dot{x}$ we obtain,

$$m\ddot{x}\dot{x} + \frac{dV}{dx}\dot{x} = 0$$

$$\implies \frac{d(\frac{m\dot{x}^2}{2} + V(x))}{dt} = 0$$

This constant will be referred to as E due to influence from Physics. We understand that all phase points in a phase trajectory have the same E value. Also we introduce an additional constraint that open sets cannot have constant E value. This might seem unnecessary at first glance but a closer look reveals that we can assign a trivial value like **E(x)**=0 and this would mean all the dynamical systems would be conservative.Hence our constraint proves to be useful. The phase trajectories can be visualised as contour plots of $E(x, \dot{x})$ as Phase trajectories have constant E value. Conservative systems heavily restricts the nature of Fixed points.

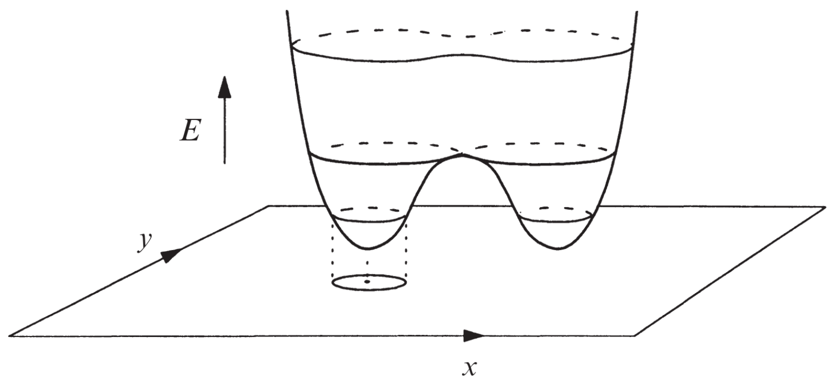

Figure 17: Conserved quantity E(x,y) [1]

For starters, we cannot have any attracting or repelling fixed point as this would imply we have an open set with a constant E value which contradicts our system is conservative. So Fixed points we observe in Conservative systems can only be centers or saddles.

1.4.4 Reversible Systems

Reversible Systems treat both flow directions of time as equal. In other words the system dynamics remain unchanged under the transformation

$$t \longleftrightarrow -t$$

We practically know that all the mechanical systems have time reversal symmetry , let's have a closer look

$$\dot{x} = y$$

$$\dot{y} = \frac{F(x)}{m}$$

Here y is the velocity,x is the displacement from origin, m is mass of the system and F(x) is the force.

we see that the change $t \to -t$ and $y \to -y$ leaves the system unchanged. This implies if (x(t),y(t)) is a solution then (x(-t),-y(-t)) is also a solution. For a general system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ The transformation $t \to -t$ and $y \to -y$ must imply

$$f(x, -y) = -f(x, y)$$

$$g(x, -y) = g(x, y)$$

for the system to be reversible.

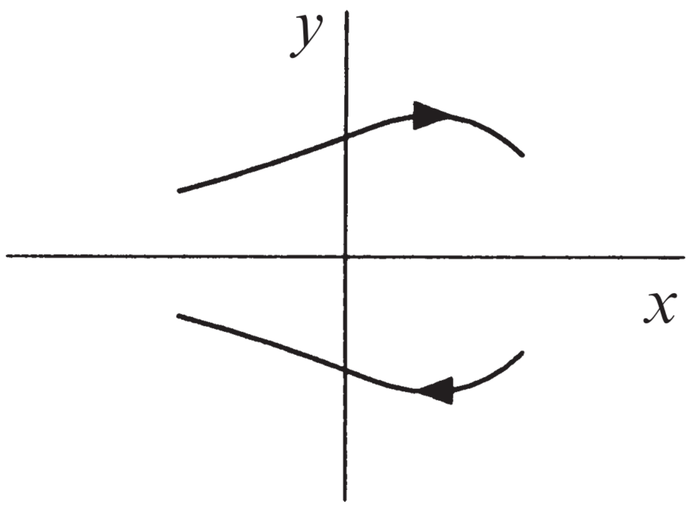

Figure 18: Reversible System [1]

2 Methods and Results

The paper by Kamyar Hosseini,Evren Hincal and Mousa Ilie [2] attempts to analytically solve the Non-linear $Schrödinger Equation (NLSE)$ by placing certain constraints which will also limit our solutions to a subset of the general solutions.The NLSE physically models the propagation of optical pulses in a non-linear media(Non-linear in this context implies that the Polarization is not linearly proportional to the external Electric Field).Physical interpretations of the equations and Formulations will be disclosed whenever possible.

2.1 NLSE

The Nonlinear Schrodinger equation under examination:

$$i\frac{\partial u}{\partial y} + \frac{1}{2}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2}) + \alpha|u|^2 u = 0 \tag{3}$$

here,$\alpha|u|^2 u$ represents the non-linear potential of the media and the double derivatives packaged in the curly brackets account for the dispersion of the pulses in the media

In order to solve the Non-linear partial differential equation we formulate a trial solution

$$u(x, y, z, t) = U(\epsilon)e^{i(\lambda_2 x+\mu_2 y+\kappa_2 z-\omega_2 t)} \tag{4}$$

Here,

$$\epsilon = \lambda_1 x + \mu_1 y + \kappa_1 z - \omega_1 t$$

Physical interpretation: The $U(\epsilon)$ acts as the modulation function and determines properties like direction of propagation , intensity . The exponential with imaginary exponent models the carrier wave which represents the oscillatory component of the pulse.

Now let's plug in the trial solution in eq(3)

$$\frac{\partial u}{\partial y} = (i\mu_2 U(\epsilon) + \mu_1\frac{dU}{d\epsilon})e^{i(\lambda_2 x+\mu_2 y+\kappa_2 z-\omega_2 t)} \tag{5}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} &= ((\lambda_1^2 + \kappa_1^2 - \omega_1^2)\frac{d^2U}{d\epsilon^2} \\ &+ (2i\kappa_1\kappa_2 + 2i\lambda_1\lambda_2 - 2i\omega_1\omega_2)\frac{dU}{d\epsilon} \\ &+ (-\kappa_2^2 - \lambda_2^2 + \omega_2^2)U(\epsilon))e^{i(\lambda_2 x+\mu_2 y+\kappa_2 z-\omega_2 t)}\end{aligned} \tag{6}$$

$$\alpha|u|^2 u = \alpha U^3(\epsilon)e^{i(\lambda_2 x+\mu_2 y+\kappa_2 z-\omega_2 t)} \tag{7}$$

Substituting eq(5),eq(6),eq(7) in eq(3) we get,

$$
\begin{aligned}
&(\lambda_1^2+\kappa_1^2-\omega_1^2)\frac{d^2U}{d\epsilon^2} \\
&+\frac{dU}{d\epsilon}(i(\kappa_1\kappa_2+\lambda_1\lambda_2-\omega_1\omega_2+\mu_2)) \\
&+U(\epsilon)(\frac{1}{2}(-\kappa_2^2-\lambda_2^2+\omega_2^2+2\mu_2))=0
\end{aligned}
\tag{8}
$$

Here an additional constraint is introduced:

$$\mu_2=\omega_1\omega_2-\lambda_1\lambda_2-\kappa_1\kappa_2$$

which leaves us with,

$$
\begin{aligned}
&(\lambda_1^2+\kappa_1^2-\omega_1^2)\frac{d^2U}{d\epsilon^2} \\
&+U(\epsilon)(\frac{-\kappa_2^2-\lambda_2^2+\omega_2^2+2\mu_2}{2})+\alpha U^3=0
\end{aligned}
\tag{9}
$$

This will facilitate us to analytically solve the Differential Equation which will be explored in the next section.

2.2 Creating a Dynamical system

Now we have a 2^{nd} order ODE which can be solved by breaking it into 2 1^{st} order ODEs

$$
\begin{aligned}
\frac{dU(\epsilon)}{d\epsilon}&=y(\epsilon) \\
\frac{dy(\epsilon)}{d\epsilon}&=-F_1U^3(\epsilon)+F_2U(\epsilon)
\end{aligned}
\tag{10}
$$

where $F_1=\frac{2\alpha}{\lambda_1^2+\kappa_1^2-\omega_1^2}$ and $F_2=\frac{\kappa_2^2+\lambda_2^2-\omega_2^2+2\mu_2}{\lambda_1^2+\kappa_1^2-\omega_1^2}$ The non-linear nature of the system is to be noted.

physical interpretation: for t=0 , we get a series of infinite planes perpendicular to $(\lambda_1,\mu_1,\kappa_1)$ as ϵ is varied . U varies along these planes and $\frac{dU}{d\epsilon}$ measures the rate of change of U. Since the envelope function is translated in time,ϵ is modified to account for it.

2.3 Bifurcation analysis

A closer inspection reveals the conservative nature of the dynamical system.Note this has no relevance in deciding the conservative properties of the optical pulses. Here we introduce the Hamiltonian function for the system, dividing the first order ODEs we get:

$$\begin{aligned}\frac{dU}{dy} &= \frac{y}{-F_1U^3 + F_2U} \\ \implies & \int(-F_1U^3 + F_2U)dU = \int ydy \\ \implies & \frac{1}{2}y^2 + \frac{1}{4}F_1U^4 - \frac{1}{2}F_2U^2 = h\end{aligned} \tag{11}$$

where h is the hamiltonian constant Now to obtain the fixed points we need to equate f($\mathbf{x}$) to 0,

$$y = 0$$

$$-F_1U^3 + F_2U = 0$$

Our fixed points would be $(0, 0)$,$(\sqrt{\frac{F_2}{F_1}}, 0)$,

$(-\sqrt{\frac{F_2}{F_1}}, 0)$. It's evident that F_1 and F_2 must be of the same sign for all three fixed points to coexist but the origin will be a fixed point regardless of the parameters. **Physical interpretation:**

$$\frac{F_2}{F_1} = \frac{\kappa_2^2 + \lambda_2^2 - \omega_2^2 + 2\mu_2}{2\alpha} \tag{12}$$

Here,Surprisingly the fixed points don't depend on the direction of propagation of the pulse.This indicates that the equilibrium amplitude of the wave is solely determined by the direction of propagation of the carrier wave and the Kerr nonlinearity constant.Also we see that for higher values of α the stable amplitude is lower which makes sense physically since stable amplitudes can be maintained only if the non-linearity doesn't suppress or expand the envelope.As α increases it becomes increasingly difficult to maintain a particular amplitude hence the stable amplitude

reduces.This line of reasoning is valid only for $\alpha > 0$. The nature of flows could be determined by evaluating the **Jacobian** around the fixed points

$$J(U, y) = \det \begin{bmatrix} 0 & 1 \\ -3F_1U^2 + F_2 & 0 \end{bmatrix}$$

Here are the possible outcomes by varying the parameters involved:

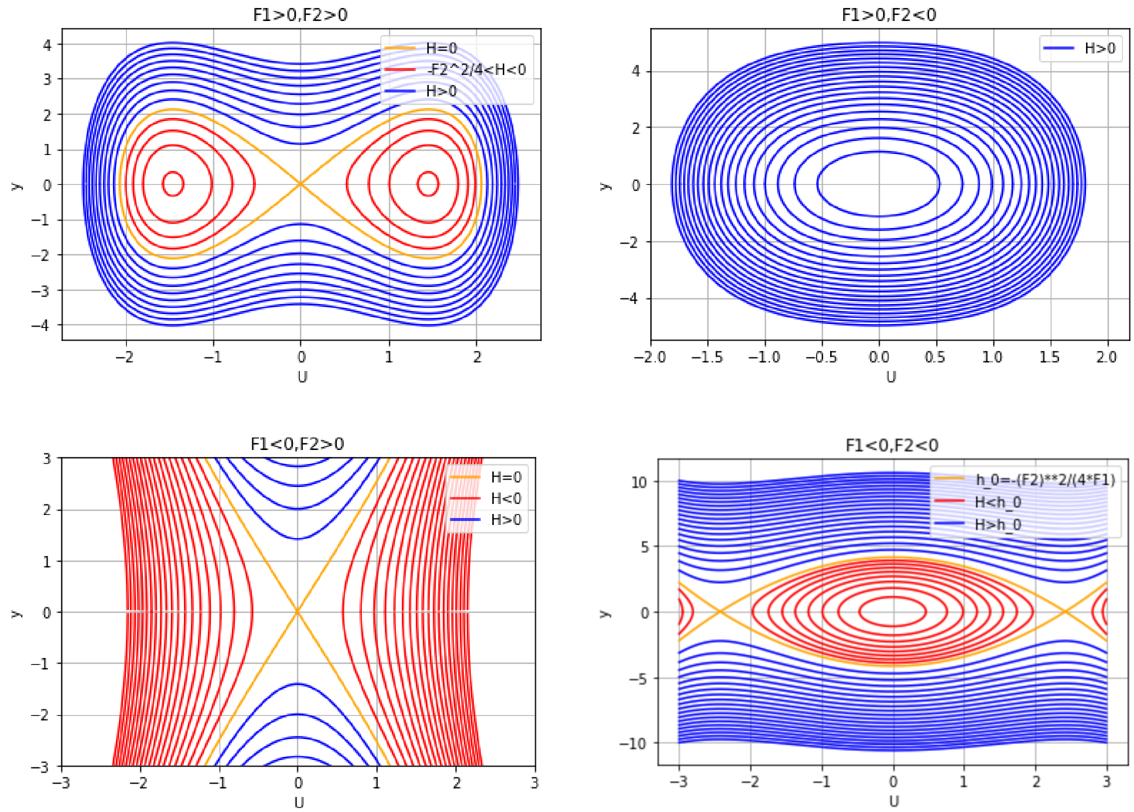

Figure 19: For $F_1F_2 > 0$,along the seperatices(marked in yellow),solitons can be derived.

2.4 Sensitivity analysis

In this subsection, the sensitivity analysis of the dynamical system is accomplished using the Runge–Kutta method. To this end, the following dynamical system

$$\frac{dU(t)}{dt} = y(t)$$

$$\frac{dy(t)}{dt} = -F_1U^3 + F_2U \tag{13}$$

is solved by the runge kutta method for $\lambda_1 = 1, \kappa_1 = 1, \lambda_2 = 1, \kappa_2 = 0.5, \mu_2 = 2, \omega_1 = 1, \omega_2 = 1$ and $\alpha = 1$. Looking at the figures, it is clear that small changes in the initial conditions don't affect the stability of the solution very much

2.5 Bright and dark solitons of the governing equation

Soliton solutions arise on phase plane with homoclinic or heteroclinic trajectories ,hence we will analytically solve the ODEs in these phase trajectories.We will also solve for periodic phase trajectories

$F_1 > 0$ and $F_2 > 0$: for $h \in (-F_2^2/4F_1, 0)$ the hamiltonian can be rewritten as,

$$y^2 = \frac{F_1}{2}(-U^4 + \frac{2F_2}{F_1}U^2 + \frac{4h}{F_1})$$

$$y^2 = \frac{F_1}{2}(U^2 - \sigma_{1h}^2)(\sigma_{2h}^2 - U^2) \tag{14}$$

where $\sigma_{1h}^2 = \frac{F_2 - \sqrt{F_2^2 + 4hF_1}}{F_1}$ and

$\sigma_{2h}^2 = \frac{F_2 + \sqrt{F_2^2 + 4hF_1}}{F_1}$

$$y = \pm\sqrt{\frac{F_1}{2}(U^2 - \sigma_{1h}^2)(\sigma_{2h}^2 - U^2)} \tag{15}$$

Now we can integrate along a closed orbit by substituting (14) in $\frac{dU}{d\epsilon} = y$ to obtain analytic solutions for closed curves inside the separatix

$$\int_{-\sigma_{2h}}^{U} \frac{d\zeta}{\sqrt{(\zeta^2 - \sigma_{1h}^2)(\sigma_{2h}^2 - \zeta^2)}} = \pm\sqrt{\frac{F_1}{2}}(\epsilon - \epsilon_0) \tag{16}$$

This integral yields the solution for closed curves on left side of the y-axis

$$\int_{U}^{\sigma_{2h}} \frac{d\zeta}{\sqrt{(\zeta^2 - \sigma_{1h}^2)(\sigma_{2h}^2 - \zeta^2)}} = \mp\sqrt{\frac{F_1}{2}}(\epsilon - \epsilon_0) \tag{17}$$

This yields solutions for curves on right side of the y-axis Integrals obtained in (16) and (17) are elliptic integrals and can be written as

$$u_{1,2}(x, y, z, t) = \pm\sigma_{2h} dn \left(\sigma_{2h}\sqrt{\frac{1}{2}F_1}(\lambda_1 x + \mu_1 y + \kappa_1 z - \omega_1 - \epsilon_0), \frac{\sqrt{\sigma_{2h}^2 - \sigma_{1h}^2}}{\sigma_{2h}}\right) e^{(\lambda_2 x + \mu_2 y + \kappa_2 z - \omega_2 t)}$$

Here the first parameter passed into the function dn(u,k) will be the arc length and the second parameter denotes the eccentricity of the ellipse on which the elliptic function is performed.Here the output will be the radius of the ellipse for the u value.

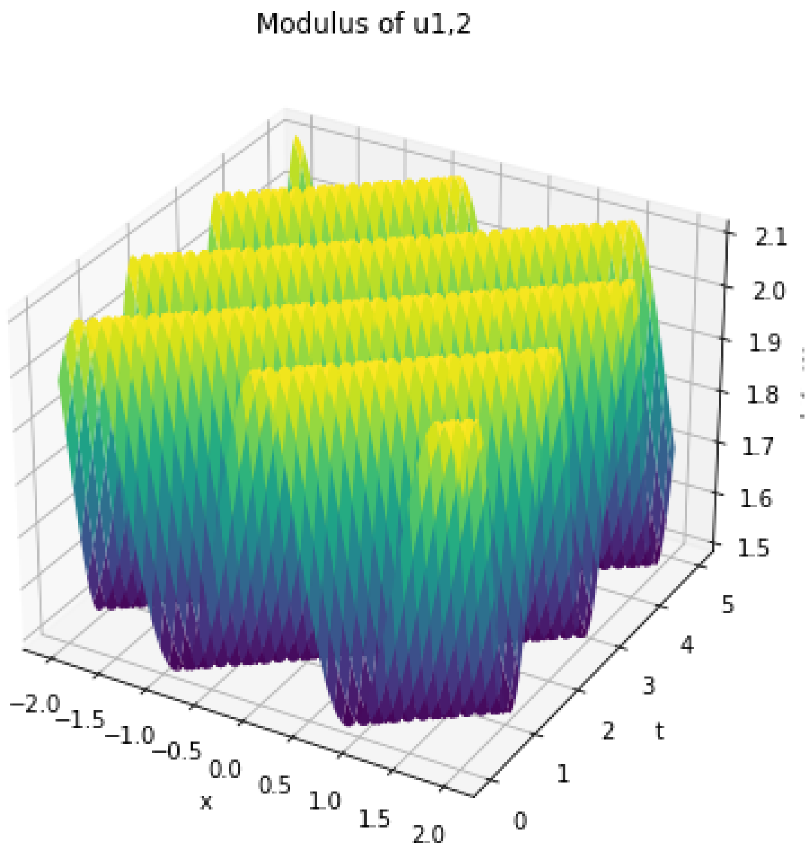

Figure 20: $|u_{1,2}|$ for $\alpha_1 = 1$, $\lambda_1 = 1$, $\kappa_1 = 1$, $\omega_1 = 1$,λ_2=1, $\omega_2 = 1$, $\kappa_2 = 0.5$, $\mu_2 = 0.5$ with y=z=0 and h=1

considering h=0 leads to $\sigma_{1h}^2 = 0$ and $\sigma_{2h}^2 = \frac{2F_2}{F_1}$ Thus the following bright solitons to the governing model are constructed

$$u_{3,4}(x, y, z, t) = \pm\sqrt{\frac{2F_2}{F_1}} sech \\ (\sqrt{F_2}(\lambda_1 x + \mu_1 y + \kappa_1 z - \omega_1 - \epsilon_0)) \\ e^{(\lambda_2 x + \mu_2 y + \kappa_2 z - \omega_2 t)} \tag{18}$$

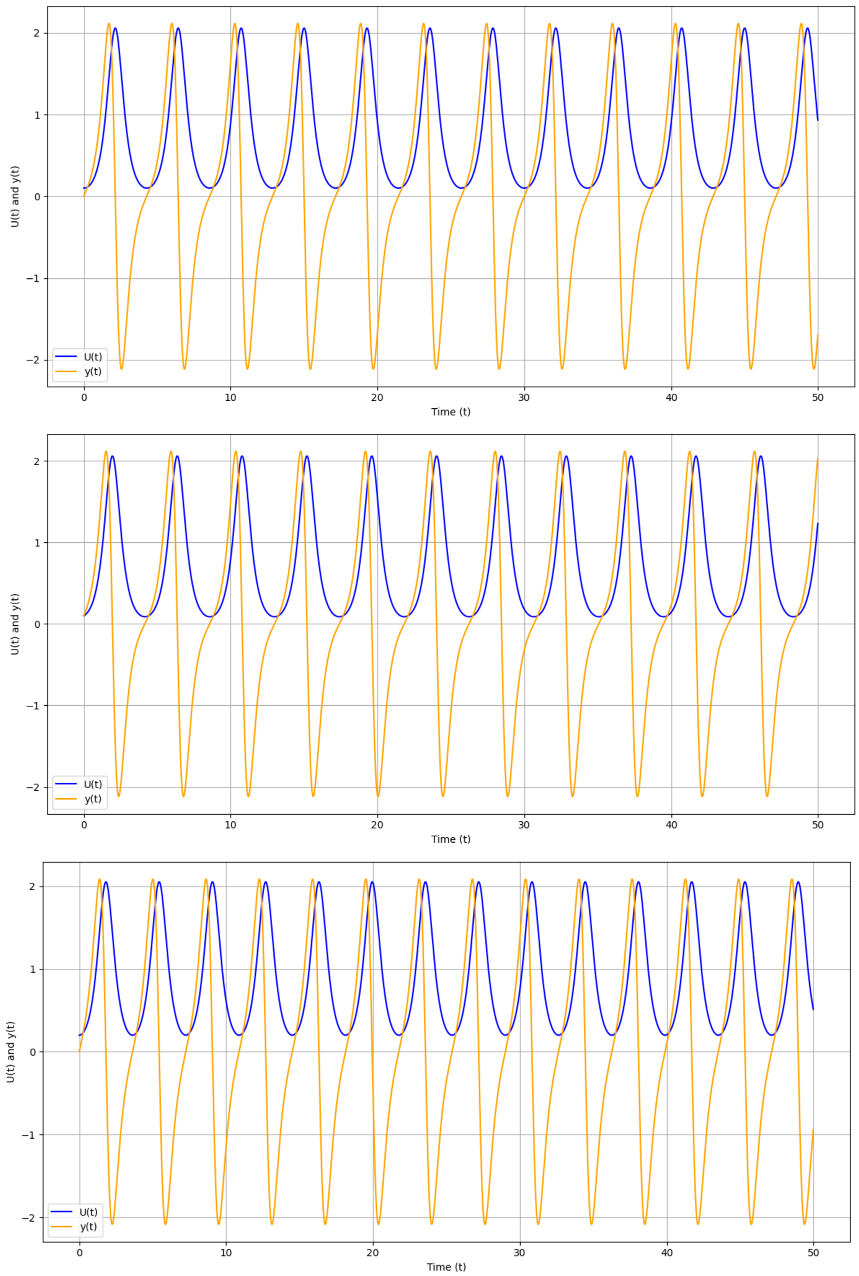

Figure 21: Sensitivity Analysis using Runge-Kutta method.
$\alpha_1 = 1$, $\lambda_1 = 1$, $\kappa_1 = 1$, $\omega_1 = 1$, $\omega_2 = 1$,λ_2=1 $\kappa_2 = 0.5$, $\mu_2 = 0.5$
for initial conditions:
a) $U = 0.1, y = 0$ b) $U = 0.1, y = 0.1$ c) $U = 0.2, y = 0$

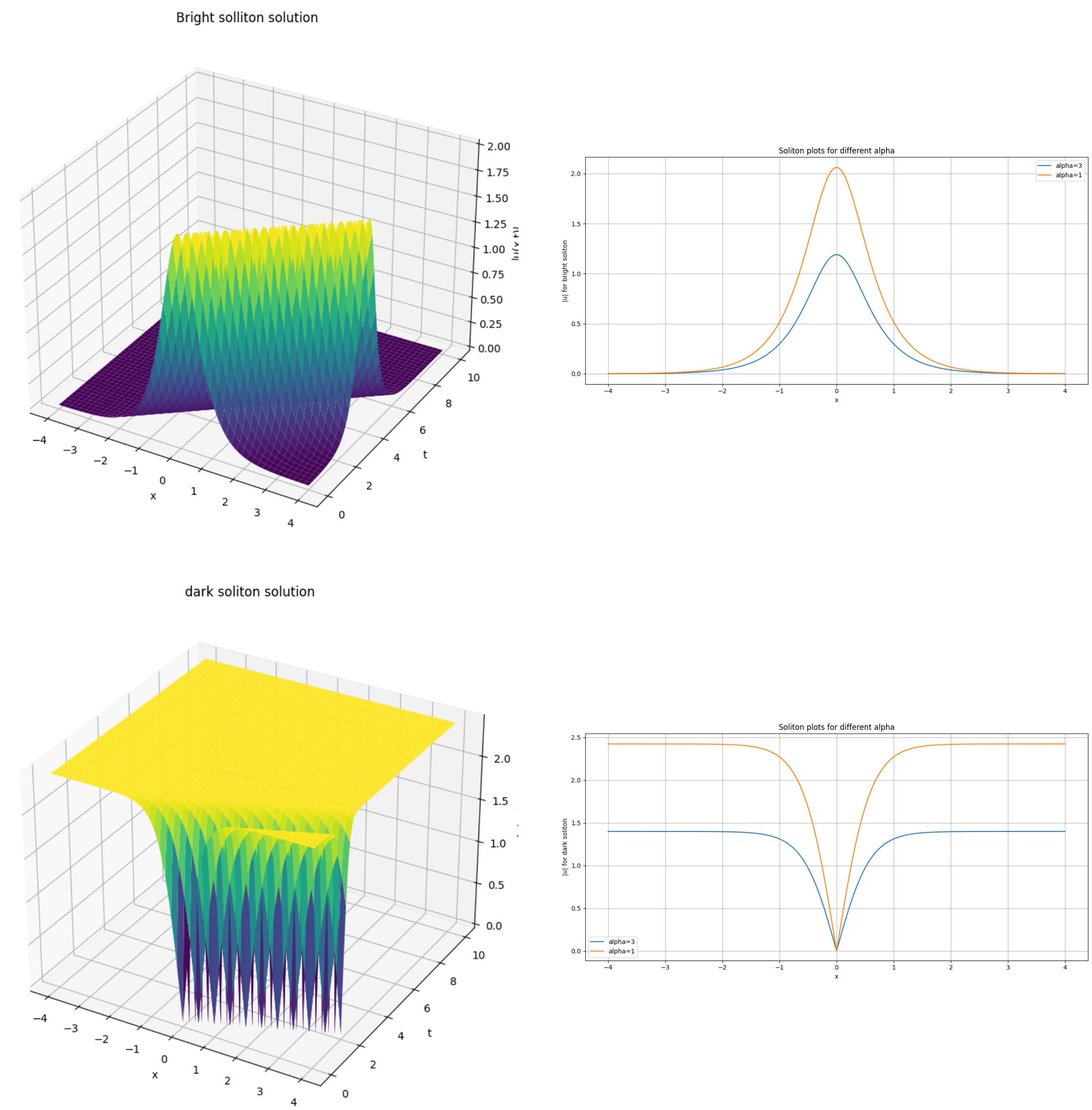

Figure 22: Bright soliton(surface plot): $\alpha_1 = 1$, $\lambda_1 = 1$, $\kappa_1 = 1$, $\omega_1 = 1$,λ_2=1, $\omega_2 = 1$, $\kappa_2 = 0.5$, $\mu_2 = 0.5$
with y=z=0
(alpha plot):same parameters with t=0 and α=$1, 3$
Dark soliton(surface plot): $\alpha_1 = 1$, $\lambda_1 = 1$, $\kappa_1 = 1$, $\omega_1 = 1$, $\omega_2 = 1$, λ_2=2,$\kappa_2 = 2$, $\mu_2 = 2$
with y=z=0
(alpha plot):same parameters with α=1 and α=3

$\mathbf{F_1 < 0}$ **and** $\mathbf{F_2 < 0}$**:** In this case we are able to rewrite the Hamiltonian as

$$y^2 = -\frac{F_1}{2}(U^4 - \frac{2F_2}{F_1}U^2 - \frac{4h}{F_1})$$

$$y^2 = -\frac{F_1}{2}(\sigma_{3h}^2 - U^2)(\sigma_{4h}^2 - U^2) \tag{19}$$

by substituting (19) in $\frac{dU}{d\epsilon} = y$ and integrating we get,

$$\int_0^U \frac{d\zeta}{\sqrt{(\sigma_{3h}^2 - \zeta^2)(\sigma_{4h}^2 - \zeta^2)}}$$
$$= \pm\sqrt{\frac{F_1}{2}}(\epsilon - \epsilon_0)$$

The above integral is in form of a jacobi integral,

$$u_{5,6}(x,y,z,t) = \pm\sigma_{3h}sn$$
$$(\sigma_{4h}\sqrt{\frac{-F_1}{2}}(\lambda_1 x + \mu_1 y + \kappa_1 z - \omega_1 - \epsilon_0), \frac{\sigma_{3h}}{\sigma_{4h}})$$
$$e^{(\lambda_2 x + \mu_2 y + \kappa_2 z - \omega_2 t)}$$

Assuming $h = -\frac{F_2^2}{4F_1}$ results in $\sigma_{3h}^2 = \sigma_{4h}^2 = \frac{F_2}{F_1}$,Thus the dark solitons to the governing model are derived

$$u_{7,8} = tanh$$
$$(\sqrt{\frac{-F_2}{2}}(\lambda_1 x + \mu_1 y + \kappa_1 z - \omega_1 - \epsilon_0))$$
$$e^{(\lambda_2 x + \mu_2 y + \kappa_2 z - \omega_2 t)}$$

3 Conclusion

We have investigated the Non linear Schrodinger equation that describes optical pulses propagating through a nonlinear media.The partial differential equation was transformed into a 2nd order ODE with the help of few constraints and the Galilean Transformation was utilized to form the conservative dynamical system .We found that the Fixed points are independent of the direction of propagation of the envelope function. We can also conclude for **Dark solitons** ,increasing the value of α results in a decrease in the height of the soliton and broadening of the width of the pulse. In the case of **Bright Solitons**,increasing the value of α decreases both the width and height of the function.

References

[1] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.* Cambridge: Westview Press, 1st ed., 2000.

[2] K. Hosseini, E. Hinçal, and M. Ilie, "Bifurcation analysis, chaotic behaviors, sensitivity analysis, and soliton solutions of a generalized schrödinger equation," *Nonlinear Dynamics*, vol. 111, pp. 17455–17462, Sept. 2023.