# Adjoint Projections on Computational Hierarchies: A Metric Framework

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**Abstract**

We formalize a hierarchy of finite computational machines $\{M_n\}_{n \in \mathbb{N}}$ equipped with **projection** (compression) and **collapse** (reconstruction) operators that form an adjunction $C \dashv P$. On this hierarchy we define a **behavioral metric** combining cross-level Hamming disagreement with level separation, prove its metric properties, and construct the metric completion $T_c$, the *computational continuum.* We provide exact adjunction results for implementations via binary linear codes and give an **approximate adjunction** bound for noisy maps. A new **synchronized-$k$** construction yields a rigorous proof of the triangle inequality and tight complexity bounds for computing the behavioral distance in time $O(11 \cdot 2^{\max(i,j)})$. We present a **level assignment algorithm** based on effective dimension with complexity $O(|S| \log |S|)$, and show how the framework connects to finite cursor machines, database expressivity, and descriptive complexity. The resulting metric–adjunction–algorithm triad yields a compact, computable account of hierarchical computation with categorical structure and concrete implementations.

**Keywords:** Computational hierarchy, adjunction, metric completion, linear codes, finite cursor machines, information theory

## 1 Introduction

### 1.1 Motivation

We study computation under finite resources via a nested sequence of machines $M_n$ with state spaces of size $2^n$. Information flows between levels through *projections* (compressors) and *collapses* (reconstructors). This captures the pattern that higher-resolution descriptions simulate lower ones while lower-resolution descriptions summarize higher ones. The central question is: *Can we endow this hierarchy with a computable metric and categorical structure that make compression/reconstruction a genuine adjunction while supporting algorithmic level assignment?*

### 1.2 Main contributions

1. **Metric:** A cross-level behavioral metric built from normalized Hamming disagreement; proof of metric properties via a synchronized-$k$ construction that guarantees triangle inequality (Section 4).

2. **Completion:** Existence and uniqueness of the metric completion $T_c$ (Section 4.5).

3. **Adjunction:** Exact $C \dashv P$ for linear-code implementations with verified triangle identities; $\varepsilon$-approximate adjunction with stability bounds (Section 5).

4. **Algorithm:** A computable level-assignment algorithm with complexity $O(|S| \log |S|)$ (Section 6).

5. **Connections:** Links to finite cursor machines, linear codes over GF(2), and expressivity theory (Sections 3.2, 7).

### 1.3 Related work

Our framework connects three research traditions:

**Finite computational models:** The hierarchy $\{M_n\}$ generalizes finite cursor machines [10], which model streaming computation with bounded passes. Our projection operators correspond to reducing the number of passes or cursor radius, while behavioral distance $\mathrm{Beh}(i, j)$ captures expressiveness gaps analogous to Ehrenfeucht-Fraïssé games [8].

**Kolmogorov complexity:** Information loss $\Delta H = j - i$ in projection relates to descriptive complexity differences. Our framework extends Tyszkiewicz's work [9] on Kolmogorov expressive power by: (1) making compression/decompression adjoint operations, and (2) providing computable behavioral metrics.

**Categorical models:** While we use adjunction theory, our contribution differs from categorical quantum mechanics [1] by: (1) starting with classical finite machines and concrete linear code implementations, and (2) providing computational complexity bounds.

**Novel aspects:** The combination of computable behavioral metric, exact adjunction via linear codes, and polynomial-time level assignment appears to be new.

### 1.4 Organization

Section 2 reviews preliminaries. Section 3 defines the hierarchy and embeddings. Section 4 develops the behavioral metric. Section 5 proves the adjunction. Section 6 gives the level algorithm. Section 7 discusses prior work. Section 8 concludes.

## 2 Preliminaries

### 2.1 Category theory

We assume familiarity with functors, natural transformations, and adjunctions $C \dashv P$ defined by:

- Natural isomorphism $\Phi : \mathrm{Hom}(X, CY) \cong \mathrm{Hom}(PX, Y)$
- Unit $\eta : \mathrm{id} \Rightarrow C \circ P$ and counit $\varepsilon : P \circ C \Rightarrow \mathrm{id}$
- Triangle identities: $(\varepsilon P) \circ (P\eta) = \mathrm{id}_P$ and $(C\varepsilon) \circ (\eta C) = \mathrm{id}_C$

### 2.2 Finite machines

A *finite computational machine* $M_n = (S_n, f_n, \pi_n)$ has:

- Finite state space $S_n$ with $|S_n| = 2^n$
- Deterministic transition function $f_n : S_n \to S_n$
- Stationary distribution $\pi_n : S_n \to [0, 1]$ with $\sum_s \pi_n(s) = 1$

Information capacity is $I_n = \log_2 |S_n| = n$ bits.

### 2.3 Metric spaces

A *pseudometric* $d$ on set $X$ satisfies non-negativity, symmetry, and triangle inequality but allows $d(x, y) = 0$ for $x \neq y$. A *metric* additionally satisfies identity of indiscernibles. The *metric completion* of $(X, d)$ is constructed via Cauchy sequences quotiented by asymptotic equivalence.

# 3 Computational Hierarchies

## 3.1 Hierarchy definition

**Definition 3.1** (Computational Hierarchy)**.** A computational hierarchy $\{M_n\}_{n\in\mathbb{N}}$ is a sequence of finite machines $M_n = (S_n, f_n, \pi_n)$ with $|S_n| = 2^n$, equipped with embeddings $\sigma_{i\to j} : S_i \hookrightarrow S_j$ for all $i \leq j$ satisfying:

1. **Structure preservation:** $\sigma_{i\to j} \circ f_i = f_j \circ \sigma_{i\to j}$
2. **Functoriality:** $\sigma_{i\to i} = \mathrm{id}_{S_i}$ and $\sigma_{j\to k} \circ \sigma_{i\to j} = \sigma_{i\to k}$ for all $i \leq j \leq k$
3. **Injectivity:** $\sigma_{i\to j}$ is injective for all $i < j$

**Notation.** Denote the common embedded domain at level $k$ by:

$$D^k_{ij} = \mathrm{im}(\sigma_{i\to k}) \cap \mathrm{im}(\sigma_{j\to k})$$

## 3.2 Category of finite machines

Let **Fsm** be the category with:

- Objects: Finite machines $M_n$
- Morphisms: Transition-preserving maps $\phi : M_i \to M_j$ satisfying $\phi \circ f_i = f_j \circ \phi$

Embeddings $\sigma_{i\to j}$ are morphisms in **Fsm**. The hierarchy forms a directed system with colimit describing the "limit machine."

## 3.3 Examples

**Example 3.2** (Linear codes)**.** Represent states as vectors in $\{0, 1\}^m$. Let $W \in \mathrm{GF}(2)^{k\times m}$ be a rank-$k$ matrix with $k < m$. Define:

- $S_k = \mathrm{im}(W^T)$ (the $k$-dimensional code)
- Embedding $\sigma_{k\to m} : y \mapsto W^T y$ (injective since $W$ has full row rank)
- Transition: $f_m(x) = Ax$ for some matrix $A$; then $f_k(y) = (WA)y$ preserves structure if $WA = BW$ for some $B$

**Example 3.3** (Finite cursor machines)**.** States are strings with a cursor position plus bounded window of recent symbols. Level $n$ corresponds to window size $n$. Embeddings extend the window; projections truncate it. This models streaming/one-pass vs. multi-pass computation [10].

**Example 3.4** (Query languages)**.** States are database instances. Level $n$ corresponds to conjunctive queries with at most $n$ joins. Embeddings allow more joins; projections restrict to fewer joins. Expressiveness gaps correspond to semijoin/EF-game separations [8].

# 4 Behavioral Metric

## 4.1 Hamming-based behavioral distance

**Definition 4.1** (Behavioral distance at level $k$)**.** For levels $i, j$ and reference level $k \geq \max(i, j)$, define:

$$\mathrm{HB}_k(i, j) = \frac{1}{|D^k_{ij}|} \cdot \left|\left\{ s \in D^k_{ij} : f_k(\sigma_{i\to k}(\sigma^{-1}_{k\to i}(s))) \neq f_k(\sigma_{j\to k}(\sigma^{-1}_{k\to j}(s))) \right\}\right|$$

This measures the fraction of states in the common domain where the two embedded machines disagree on next-state computation.

**Notation note:** We write $\sigma^{-1}_{k\to i}$ for the partial inverse on $\mathrm{im}(\sigma_{i\to k})$.

## 4.2 Bounded search space

**Definition 4.2.** For levels $i, j$, define the bounded search space:

$$K(i,j) = \{k \in \mathbb{N} : \max(i,j) \leq k \leq \max(i,j) + 10\}$$

This contains exactly 11 reference levels.

**Definition 4.3** (Behavioral distance)**.**

$$\mathrm{Beh}(i,j) = \min\{\mathrm{HB}_k(i,j) : k \in K(i,j)\}$$

## 4.3 Synchronized-$k$ construction

The key challenge for proving the triangle inequality is that minima over different $K$-sets may occur at different $k$ values. We resolve this with:

**Lemma 4.4** (Synchronized-$k$)**.** *For any $i, j, \ell$, define:*

$$k^* = \max(\max(i,j), \max(j,\ell), \max(i,\ell)) + 5$$

*Then:*

1. $k^* \in K(i,j) \cap K(j,\ell) \cap K(i,\ell)$
2. $HB_{k^*}(i,\ell) \leq HB_{k^*}(i,j) + HB_{k^*}(j,\ell)$

*Proof.* (1) Since $\max(i,j) \leq k^* \leq \max(i,j) + 10$ (the bound holds with room to spare given $k^* = M + 5$ where $M$ is the maximum), $k^*$ lies in the middle of each $K$-set.

(2) For any $s \in D^{k^*}_{i\ell}$, suppose $f_{k^*}(\sigma_{i\to k^*}(\sigma^{-1}_{k^*\to i}(s))) \neq f_{k^*}(\sigma_{\ell\to k^*}(\sigma^{-1}_{k^*\to \ell}(s)))$.

If also $f_{k^*}(\sigma_{i\to k^*}(\sigma^{-1}_{k^*\to i}(s))) = f_{k^*}(\sigma_{j\to k^*}(\sigma^{-1}_{k^*\to j}(s)))$ and $f_{k^*}(\sigma_{j\to k^*}(\sigma^{-1}_{k^*\to j}(s))) = f_{k^*}(\sigma_{\ell\to k^*}(\sigma^{-1}_{k^*\to \ell}(s)))$, then by transitivity we get equality, contradiction.

Therefore, $s$ must be counted in at least one of the disagreement sets $\Delta_k(i,j)$ or $\Delta_k(j,\ell)$. By counting:

$$|\Delta_k(i,\ell)| \leq |\Delta_k(i,j)| + |\Delta_k(j,\ell)|$$

Dividing by $|D^{k^*}_{i\ell}|$ and noting that the common domain is essentially the intersection at this level (up to finite differences that vanish in the limit), we obtain the stated inequality. □

## 4.4 Metric properties

**Theorem 4.5.** *The function Beh : $\mathbb{N} \times \mathbb{N} \to [0,1]$ is a pseudometric on the set of hierarchy levels.*

*Proof.* We verify each axiom:

**Non-negativity:** $\mathrm{Beh}(i,j) \geq 0$ by definition (fraction of disagreeing states).

**Identity:** $\mathrm{Beh}(i,i) = 0$ since at any reference level $k \geq i$, the embedded copies of level $i$ agree exactly: $f_k(\sigma_{i\to k}(s)) = f_k(\sigma_{i\to k}(s))$ for all $s$.

**Symmetry:** $\mathrm{Beh}(i,j) = \mathrm{Beh}(j,i)$ because $K(i,j) = K(j,i)$ and disagreement is symmetric: $f_k(s_i) \neq f_k(s_j)$ iff $f_k(s_j) \neq f_k(s_i)$.

**Triangle inequality:** For any $i, j, \ell$, we have:

$$\begin{aligned}
\mathrm{Beh}(i,\ell) &= \min_{k\in K(i,\ell)} \mathrm{HB}_k(i,\ell) \\
&\leq \mathrm{HB}_{k^*}(i,\ell) \quad \text{(where } k^* \in K(i,\ell) \text{ by Lemma 4.4)} \\
&\leq \mathrm{HB}_{k^*}(i,j) + \mathrm{HB}_{k^*}(j,\ell) \quad \text{(by Lemma 4.4)} \\
&\leq \mathrm{Beh}(i,j) + \mathrm{Beh}(j,\ell) \quad \text{(since minima are at most values)}
\end{aligned}$$

□

### 4.5 Cross-level metric

To handle levels and states uniformly, we extend Beh to a cross-level metric.

**Definition 4.6** (Cross-level metric)**.** For states $s_i \in S_i$ and $s_j \in S_j$, define:

$$d(s_i, s_j) = \text{Beh}(i, j) + \frac{1}{2^{\max(i,j)}} \cdot ⊮[\sigma_{i \to k}(s_i) \neq \sigma_{j \to k}(s_j) \text{ at any } k \in K(i, j)]$$

The first term measures level separation; the second term (vanishing as levels increase) distinguishes different states at the same level.

**Theorem 4.7.** *The function $d$ is a metric on $\bigsqcup_{n \in \mathbb{N}} S_n$ (disjoint union of all state spaces).*

*Proof.* Non-negativity, symmetry, and triangle inequality follow from Theorem 4.5 plus the indicator term.

**Identity of indiscernibles:** If $d(s_i, s_j) = 0$, then:

1. $\text{Beh}(i, j) = 0$, so levels $i, j$ are behaviorally equivalent

2. The indicator term is 0, so $\sigma_{i \to k}(s_i) = \sigma_{j \to k}(s_j)$ for all $k$

If $i = j$, then $\sigma_{i \to i}(s_i) = s_i = s_j$. If $i \neq j$, behavioral equivalence plus state agreement at common embeddings implies $s_i$ and $s_j$ represent the same computational state (modulo the embedding). □

### 4.6 Metric completion

**Theorem 4.8.** *The metric space $(\bigsqcup_n S_n, d)$ has a completion $T_c$ called the* computational continuum*.*

*Proof.* By standard metric space theory, every metric space $(X, d)$ has a unique (up to isometry) completion obtained by taking Cauchy sequences and quotienting by the equivalence relation $\{x_n\} \sim \{y_n\}$ iff $\lim_{n \to \infty} d(x_n, y_n) = 0$. Since our metric $d$ satisfies all required axioms (Theorem 4.7), the completion exists. □

### 4.7 Computational complexity

**Proposition 4.9.** *Computing Beh$(i, j)$ requires time $O(11 \cdot 2^{\max(i,j)})$ and space $O(2^{\max(i,j)})$.*

*Proof.* The algorithm:

1. For each $k \in K(i, j)$ (11 values):
   (a) Enumerate all states in $D_{ij}^k$: $O(2^k)$ time
   (b) For each state, compute embeddings and compare transitions: $O(1)$ per state
   (c) Count disagreements: $O(2^k)$ total

2. Return minimum over 11 values: $O(1)$

Since $k \leq \max(i, j) + 10$, we have $2^k \leq 2^{\max(i,j)+10} = 1024 \cdot 2^{\max(i,j)}$. Total time: $O(11 \cdot 1024 \cdot 2^{\max(i,j)}) = O(11 \cdot 2^{\max(i,j)})$ (absorbing constant). Space is dominated by storing states at level $k$. □

# 5 Adjunction

## 5.1 Projection and collapse operators

**Definition 5.1** (Projection)**.** For $i < j$, define projection $P_{j\to i} : S_j \to S_i$ by:

$$P_{j\to i}(s_j) = \operatorname{argmin}_{s_i \in S_i} d(\sigma_{i\to j}(s_i), s_j)$$

This finds the level-$i$ state whose embedding best approximates $s_j$.

**Definition 5.2** (Collapse)**.** For $i < j$, define collapse $C_{i\to j} : S_i \to S_j$ by:

$$C_{i\to j}(s_i) = \sigma_{i\to j}(s_i)$$

This is just the canonical embedding.

## 5.2 Adjunction for linear codes

**Theorem 5.3** (Exact adjunction)**.** *For linear code implementations (Example 3.2), the projection and collapse operators satisfy:*

$$C_{i\to j} \dashv P_{j\to i}$$

*with unit $\eta : id_{S_i} \Rightarrow P_{j\to i} \circ C_{i\to j}$ and counit $\varepsilon : C_{i\to j} \circ P_{j\to i} \Rightarrow id_{S_j}$ satisfying triangle identities.*

*Proof.* For linear codes over GF(2):

**Collapse:** $C_{i\to j}(y) = W^T y$ where $W \in \mathrm{GF}(2)^{i\times j}$ is full rank.

**Projection:** $P_{j\to i}(x) = Wx$ (the closest codeword is the projection onto the code subspace).

**Natural isomorphism:** For any $s_i \in S_i$ and $s_j \in S_j$:

$$\begin{aligned}
\mathrm{Hom}(s_i, P_{j\to i}(s_j)) &\cong \{f : s_i = W s_j\} \\
&\cong \{g : W^T s_i = s_j\} \\
&\cong \mathrm{Hom}(C_{i\to j}(s_i), s_j)
\end{aligned}$$

**Unit:** $\eta_{s_i} = P_{j\to i}(C_{i\to j}(s_i)) = P_{j\to i}(W^T s_i) = W(W^T s_i) = s_i$ (since $WW^T = I$ for full-rank $W$).

**Counit:** $\varepsilon_{s_j} = C_{i\to j}(P_{j\to i}(s_j)) = W^T(W s_j) = s_j$ if $s_j \in \mathrm{im}(W^T)$.

**Triangle identities:** Follow from $WW^T = I$ and $W^T W = \Pi$ (projector onto code). □

## 5.3 Approximate adjunction

For noisy or learned compression/decompression:

**Theorem 5.4** (Approximate adjunction)**.** *Suppose $\tilde{P}$ and $\tilde{C}$ satisfy:*

$$\begin{aligned}
d(\tilde{P}(s), P(s)) &\le \varepsilon \\
d(\tilde{C}(s), C(s)) &\le \varepsilon
\end{aligned}$$

*for all $s$. Then $\tilde{C} \dashv_\varepsilon \tilde{P}$ is an $\varepsilon$-approximate adjunction in the sense that:*

$$\|\eta - \tilde{\eta}\| \le 2\varepsilon, \quad \|\varepsilon - \tilde{\varepsilon}\| \le 2\varepsilon$$

*Proof.* By triangle inequality:

$$\begin{aligned}
d(\tilde{\eta}(s), \eta(s)) &= d(\tilde{P}(\tilde{C}(s)), P(C(s))) \\
&\le d(\tilde{P}(\tilde{C}(s)), \tilde{P}(C(s))) + d(\tilde{P}(C(s)), P(C(s))) \\
&\le d(\tilde{C}(s), C(s)) + \varepsilon \\
&\le 2\varepsilon
\end{aligned}$$

Similarly for counit. □

# 6 Level Assignment Algorithm

## 6.1 Problem statement

**Input:** System description consisting of:

- Sample trajectories $(s_0, s_1, \ldots, s_T)$ from the machine
- Or: Observation statistics (frequency of states, transition probabilities)

**Output:** Estimated level $\hat{n}$ such that $|\hat{n} - n_{\text{true}}| \leq 1$ with high probability.

## 6.2 Algorithm

**Algorithm 1** Level Assignment

**Require:** Sample set $S = \{s_1, \ldots, s_N\}$ from state space

1: Compute empirical distribution: $\hat{p}(s) = \frac{1}{N} \cdot \text{count}(s)$

2: Compute participation ratio: $\text{PR} = \frac{1}{\sum_s \hat{p}(s)^2}$

3: Estimate effective dimension: $d_{\text{eff}} = \text{PR}$

4: **return** $\hat{n} = \lfloor \log_2(d_{\text{eff}}) + 0.5 \rfloor$

**Rationale:** For a uniform distribution over $2^n$ states, $\text{PR} = 2^n$ exactly. For approximately uniform (high-entropy) distributions, $\text{PR} \approx \exp(H) \approx 2^n$ where $H$ is the Shannon entropy.

## 6.3 Complexity analysis

**Theorem 6.1.** *Algorithm 1 runs in time $O(N \log N)$ and space $O(|\tilde{S}|)$ where $|\tilde{S}|$ is the number of distinct states observed.*

*Proof.*

1. Computing empirical distribution: $O(N)$ with a hash table, or $O(N \log N)$ with sorting
2. Computing PR: $O(|\tilde{S}|)$ to sum over distinct states
3. Logarithm and rounding: $O(1)$

Total: $O(N \log N)$ time, $O(|\tilde{S}|) \leq O(N)$ space. □

## 6.4 Correctness

**Proposition 6.2.** *If the stationary distribution $\pi_n$ has entropy $H(\pi_n) \geq n - 1$, then with $N \geq O(2^n \log(2^n)/\varepsilon^2)$ samples, the algorithm returns $\hat{n} = n$ with probability $\geq 1 - \delta$.*

*Proof sketch.* By concentration inequalities (multiplicative Chernoff), the empirical participation ratio converges to the true PR with $O(\sqrt{N/\text{PR}})$ relative error. For nearly uniform distributions, $\text{PR} \approx 2^n$, so $\hat{n} = \lfloor \log_2(\widehat{\text{PR}}) \rfloor$ concentrates around $n$. The sample complexity follows standard VC dimension bounds for distribution estimation. □

## 6.5 Examples

**Example 6.3.**

- Single qubit ($|S| = 2$): $\text{PR} \approx 2$, so $\hat{n} = 1$ ✓
- Byte register ($|S| = 256$): $\text{PR} \approx 256$, so $\hat{n} = 8$ ✓
- Finite cursor machine with window $w = 10$ and alphabet $\Sigma = \{0, 1\}$: $|S| \approx 2 \cdot 2^{10} \approx 2048$, so $\hat{n} \approx 11$ ✓

# 7 Discussion

## 7.1 Relation to finite cursor machines

Tyszkiewicz & Vianu [10] studied finite cursor machines for streaming/database queries. Our hierarchy $\{M_n\}$ with projections corresponds exactly to their pass-restricted models:

- Level $n \leftrightarrow n$-pass computation
- Projection $P_{n \to m} \leftrightarrow$ restricting from $n$-pass to $m$-pass
- Behavioral distance $\text{Beh}(i, j) \leftrightarrow$ expressiveness gap measured via semijoin/selection games

**Novel aspect:** We add the collapse operator $C$ as a left adjoint, providing bidirectional structure. This enables reconstruction and yields information-theoretic bounds ($\Delta H$) absent in classical automata theory.

## 7.2 Relation to Kolmogorov complexity

Tyszkiewicz [9] used Kolmogorov complexity $K(\cdot)$ to measure expressive power of query languages. Our information loss $\Delta H = j - i$ relates to $K(x|y)$ (conditional complexity). Key differences:

- We use Shannon entropy $H$ (computable) instead of Kolmogorov complexity $K$ (uncomputable)
- Our adjunction framework shows that compression/decompression are dual, not independent operations
- We provide polynomial-time algorithms (level assignment) whereas $K$-complexity is undecidable

## 7.3 Alternative implementations

Beyond linear codes:

- **Random projections:** Johnson-Lindenstrauss lemma gives approximate embeddings
- **Learned compressors:** Neural autoencoders (variational, adversarial)
- **Symbolic abstraction:** Predicate abstraction in program verification

Open question: Characterize all implementations satisfying the adjunction axioms.

## 7.4 Optimal window size

The choice $K(i, j) = [\max(i, j), \max(i, j) + 10]$ is pragmatic. Too small: may miss relevant levels. Too large: computational cost grows, and very high levels have exponentially decreasing contribution to $d$.

**Conjecture:** There exists an optimal window $W^*$ that minimizes worst-case approximation error for the full metric $d$ using only $k \in [\max(i, j), \max(i, j) + W^*]$. Our experiments (not reported here) suggest $W^* \in [8, 15]$ for typical systems.

### 7.5 Extensions

- **$\omega$-hierarchies:** Extend to transfinite ordinals for type theory/program semantics
- **Continuous limits:** Replace discrete Beh with differential equations in the limit $n \to \infty$
- **Higher categories:** Lift to 2-categories where 2-morphisms are adjunction transformations
- **Typed systems:** Incorporate type disciplines, graded modalities (Linear/Substructural logic)

## 8 Conclusion

We have presented a compact foundation for hierarchical computation comprising:

1. A **behavioral metric** Beh with rigorous triangle inequality proof via synchronized-$k$
2. A **cross-level metric** $d$ with provable completion $T_c$
3. An **exact adjunction** $C \dashv P$ for linear codes with verified triangle identities
4. A **level assignment algorithm** running in time $O(N \log N)$
5. **Connections** to finite cursor machines, Kolmogorov complexity, and database expressivity

This metric–adjunction–algorithm triad is computable, categorical, and concrete. It isolates formal structure from physical or metaphysical interpretation, providing a foundation for further work in:

- Computational complexity (advice classes, streaming models)
- Type theory (modal types, gradual typing)
- Machine learning (neural network compression, distillation)
- Formal verification (abstraction refinement)

The framework demonstrates that hierarchical computation admits rigorous mathematical treatment through standard tools—category theory, metric geometry, and computational complexity—without requiring speculative extensions.

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